

# Participating life annuities incorporating longevity risk sharing arrangements\*

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## Abstract

In this paper we develop a conceptual framework for the payout phase in which annuity providers and policyholders share longevity and investment risks in a flexible way. To be more precise, we develop an participating life annuity product in which systematic longevity risk, i.e., the risk associated with systematic deviations from mortality rates extracted from prospective life tables derived for the Portuguese population, is shared between both counterparties. This will address some of the main demand and supply constraints in annuity markets, namely the inexistence of prospective life tables for the Portuguese population, the perception of unfair pricing, the consideration of bequest motives, adverse selection problems or the lack of financial instruments to hedge against longevity risk. Contrary to traditional GSA's, in which surviving policyholders bear both systematic and unsystematic longevity risk, we devise a contract in which, in exchange for a relatively small premium, annuitants will bear only the part of longevity that exceeds pre-determined thresholds.

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## 1 Introduction

Longevity risk, i.e., the risk that members of some reference population might live longer, on average, than anticipated, has recently emerged as one of the largest sources of risk faced by individuals, life insurance companies, pension funds and annuity providers. This risk is amplified by the current problems in defined benefit (DB) pension systems (either PAYGO financed or funded and public or private), in which the amount of retirement benefits is determined largely by years of service, that will inevitably force the systems to moderate benefit promises in the future. Additionally, the international pension environment is shifting towards defined contribution (DC) systems, in which retirement wealth depends on how much individuals save and how successfully they allocate their assets accumulated in DC plans, forcing individuals to be much more aware and active in managing this risk. The efficient allocation of these assets requires the managing of a number of risks, such as the investment risk, timing of annuitization and longevity risk (i.e. the possibility of exhausting assets before passing away). It also depends on the existence of solutions to manage both financial and demographic risks, on the regulatory environment and on type of options and products available.

If in recent years pension discussions were mainly focused on questions such as system design or on ways to encourage saving in the accumulation phase, the design of the payout phase and the different retirement options for DC plans will soon be at the center of the debate. The main problem will be on how to encourage certain retirement payout options in order to guarantee that people will have appropriate retirement income and longevity protection, while considering liquidity, health-care costs or bequest motives. The regulatory implications of the guarantees offered to retirees will need to be balanced so that capital reserves required (e.g., life insurance companies) in excess of what can be fairly and profitably delivered by private providers don't result in the lack of products or in inefficient allocation of resources.

Assets accumulated in DC pension plans may be allocated in the payout phase in three alternative ways: lump-sum payments, programmed withdrawals, and annuities, although we can envisage mixed arrangements involving any combination of these. With lump-sums, individuals receive the entire value of the assets accumulated for retirement as a single payment. Under programmed withdrawals, individuals establish a schedule of periodic fixed or variable payments. Finally, “plain vanilla” life annuities involve a constant stream of income paid at some regular interval for as long as the individual lives. The main factors that differentiate

between these options are the degree of flexibility and exposure to investment risk versus the degree of protection against longevity risk.

The main purpose of this project is to develop a conceptual framework for the payout phase in which annuity providers and policyholders share longevity and investment risks in a flexible way. To be more precise, we develop an participating life annuity product in which systematic longevity risk, i.e., the risk associated with systematic deviations from mortality rates derived using prospective life tables for the Portuguese population, is shared between both counterparties. This will address some of the main demand and supply constraints in annuity markets, namely the inexistence of prospective life tables for the Portuguese population, the perception of unfair pricing, the consideration of bequest motives, adverse selection problems or the lack of financial instruments to hedge against longevity risk. We hope the results of this project will contribute to the development of an efficient annuity market in Portugal.

The paper is organized in four main parts. In Section 1 we briefly review the main demographic trends observed worldwide and discuss the macroeconomics and financial implications of longevity risk. Next, we discuss the type of retirement payout options for accumulated assets in savings accounts or DC pensions, emphasizing the importance of annuity markets in protecting individuals from longevity risk. Next, we discuss the main demand and supply constraints undermining the development of annuity markets in Portugal and in most OECD countries. Finally, we briefly introduce traditional and stochastic mortality approaches in mortality modelling.

In Section 2 we analyse in detail the main features of a special type of participating life annuity called Guaranteed (or pooled) Self Annuitization (GSA) annuity fund. The advantages and limitations of this contract in hedging longevity risk are highlighted in comparison with standard “plain vanilla” annuities. In Section 3 we derive the first prospective lifetables for the Portuguese population. This provides us with new tools for the analysis of mortality trends, namely the possibility to investigate the evolution of mortality not only in terms of calendar time but also in terms of year of birth or cohort. In Section 4 we use stochastic differential equations to model the random evolution of survival probabilities. Specifically, we propose (and calibrate) a new SDE for the force of mortality. The model is then embedded into an affine-jump term structure framework in order to derive closed-form solutions for the survival probability, a key element when pricing life insurance contracts. In Section 5 we develop a new participating life annuity with a longevity risk sharing mechanism. Section 6 concludes.

## 1.1 Demographic trends

It is well documented that the population of the industrialized world underwent a major mortality transition over the last decades. Improved hygiene and living standards, breakthrough medical progresses, generally healthier lifestyles, the absence of both major pandemic crisis and global military conflicts have created the conditions for individuals to enjoy raising life expectancy at all ages. Based on all available demographic databases, historical trends show that both average and the maximum lifetime have increased gradually during the last century, with human life span showing no signs of approaching a fixed limit imposed by biology. In Portugal, life expectancy at birth increased from 48.08 (52.12) years in 1930-31 to 75.49 (81.74) in 2006-08 for the male (female) population.

As in other developed countries, the mortality decline has been dominated by two major trends: a huge reduction in mortality due to infectious diseases affecting mainly young ages, more evident during the first half of the century, and a decrease in mortality at older ages, more pronounced during the second half. As a consequence, the number of those surviving up to older ages (e.g., 80 years and above) has increased significantly representing, in 2006, 4.9% (2.9%) of the Portuguese female (male) population. Additionally, the number of deaths of the oldest-old accounts for an increasing proportion of all deaths, with reductions of mortality beyond these ages having a growing contribution to future gains in life expectancy. Decreasing mortality at old ages raised longevity to values considered impossible in the past. In Portugal, life expectancy at age 65 raised from 11.49 (13.09) years in 1930-31 to 16.25 (19.61) years in 2006-08.

The general downward trend in mortality rates at almost all ages means that an increasing proportion of the members of a given generation lives up to very old ages (around 75-85 years), shifting the survival function upwards and to the right to a more rectangular shape in what is known in the literature as the rectangularization phenomena. At the same time, we can observe that the age of maximum mortality gradually shifted towards older ages, in what is sometimes called the expansion phenomenon of the survival curve.

At the same time, fertility rates are declining. Recent data shows that while in 1960 each Portuguese woman gave birth to 3.1 children on average, nowadays the ratio is only 1.4, far below the threshold of around 2.1 necessary to keep the population of a developed country constant. In fact, in Portugal as in many developed countries low fertility rates are, together with increasing life expectancy, the main drivers of an ageing population.

The immediate consequence of higher life expectancy and low fertility rates

in unambiguous. According to the United Nations, in 2050 27% of the European population will be older than 65 years (16% in 2005) and around 10% will be older than 85 (compared with 3.5% in 2005). This has important consequences in terms of population mix, as can be seen, for example, by looking at the evolution of young-age and old-age dependency ratios. In Portugal, the young-age dependency ratio has been cut by more than half from 46.0 in 1960 to 22.8 in 2007. In opposite direction, the old-age dependency ratio has increased steadily from 15.6 in 1970 to 25.9 in 2007. Considering the ageing (or vitality) index, while in 1970 there were 34 old people for each young people, in 2007 this relation has dramatically shifted to 114 old for 100 young people.

## 1.2 Financial implications of longevity risk

Mortality improvements are naturally viewed as a positive change for individuals and as a substantial social achievement for developed countries. Nonetheless, the combination of longer life and low fertility rates poses a huge challenge to both societies and individuals since they are now exposed to increasing longevity risk. Macroeconomics effects of population ageing range from impacts on labour supply and its rate of utilization to investment, productivity and saving/consumption patterns, external balances and cross-border capital flows, consumer preferences and corporate strategies, health-care and social security systems. In the insurance market, mortality improvements have an obvious impact on the pricing and reserving for any kind of long-term living benefits, particularly on annuities.

The demographic scenario described above is also driving to important changes in the income mix of retirees. First, as a consequence of a rising old-age dependency ratio, the number of wage and salary earners is becoming insufficient to fund a growing number of retirees. Traditional PAYGO social security systems will progressively become unsustainable and will require substantial reforms. Alternative solutions involve an increase in the contribution rates, a reduction in pension/salary replacement rates, an increase in retirement age, a search for new funding sources. Changes in public pension systems are likely to imply, *ceteris paribus*, a noteworthy reduction in the retirement income relative to wages. i.e., a relative reduction in state-provided pension income.

Second, there is a clear market trend away from defined-benefit (DB) corporate pension schemes to defined-contribution (DC) schemes. In these arrangements, retirement benefits are largely determined by how much workers save and how successfully they allocate their assets accumulated in DC plans. The efficient allocation of these assets requires the managing of risks, such as the timing of

annuitization and longevity risk, i.e., the possibility of outliving one's retirement income. It also depends on the type of options and products available and on the regulatory environment. This means that employer-related pension benefits could equally become more uncertain in the future.

Third, the extended mobility of the workforce has broken down traditional family networks, thus reducing in practice the ability of younger members of a family to take care of the older ones, the main source of intergenerational solidarity mechanism in the past. The changing pattern observed in labour markets towards more flexible and less stable arrangements will probably induce erratic social security contribution patterns, essentially dependent on salaries profiles over time.

This said, individuals will have to become in a near future more self-reliant and will want to supplement and diversify their sources of income in retirement, assigning greater weight to private solutions and increasing the flow of saving allocated to fund retirement. In addition, increases in life expectancy will probably not be followed by an equivalent upward adjustment in the retirement age and thus individuals will have to put aside an increasing proportion of their lifetime income in order to fund their extended lifetime.

Moreover, increases in life expectancy have consistently exceeded forecasts, i.e., individuals are faced with longevity risk, something that must also be considered in order to ensure that the elderly do not experience drops in consumption.

### 1.3 Options for the Payout-Phase

Given the importance of addressing retiree's needs in their financial needs, both in their accumulation and decumulation (or payout) phases, in this section we briefly review the main retirement options available for the payout-phase. Individuals with assets accumulated in DC plans or individual saving accounts have roughly three main options for the payout phase: lump-sum payments, programmed withdrawals and annuities. Combined solutions involve any possible combination between these three alternatives are of course possible.

With lump-sums, individuals simply receive the entire value of the assets accumulated for retirement as a single payment. That amount can then be freely allocated, for example, to buy discretionary items, to pay down debts, to buy annuities, to cover for contingencies (e.g., medical expenses). Under programmed withdrawals, individual agree on a set of periodic payments (fixed or variable), which can be determined on different ways (e.g., by dividing the accumulated capital by a fixed number of years) and allow for some flexibility, for example to adjust for unexpected contingency payments. Finally, a traditional whole life

annuity is a stream of income payments paid at some regular interval for as long as individual lives.

The main factors that differentiate between these options are the degree of flexibility versus the degree of protection from longevity risk. Lump-sum payments are fully flexible and provide complete liquidity, allowing individuals to dispose and allocate their wealth as they wish, including the option to leave bequests.

However, lump-sum payments do not provide protection from outliving one's own resources, i.e., individuals bear in this case all longevity risk. According to life-cycle theory, in a world with no uncertainty rational individuals would save optimally and, on retirement, would merely allocate their wealth by spreading assets over their remaining years of life, so as to ensure optimal retirement consumption (and cover bequest motives, if any). In a dynamic environment, future life expectancy is unknown and as such individuals are faced with the prospect of outliving their expected life spans. In a scenario of unknown longevity, individuals rely heavily on financial discipline to manage their resources. Retirees can reduce the risk of exhausting assets before passing away by consuming less per year, but such a tactic then increases the chance that they might die with too much wealth left unconsumed. In other words, dying with too little wealth is undesirable, but having too much wealth is also undesirable, since it represents foregone consumption opportunities.

Programmed withdrawals provide more financial discipline than lump-sums, while maintaining some degree of flexibility, access to liquidity and the possibility to cover bequest motives, but fail once again to provide any kind of protection from longevity risk.

Finally, life annuities offer full protection against longevity risk, but at least in their "plain vanilla" form, are inflexible and illiquid and do not provide for bequest motives. Nevertheless, in some countries annuity markets offer today a wide range of complex annuity products, including embedded guarantees that protect against interest rate, inflation, market volatility, and early death, accommodate liquidity and contingency payments and offer tax advantages. However, up to now little attention has been devoted to the development of annuity products in which mortality and longevity risks are shared and payouts linked to the evolution of demographic variables. In this paper tackle this problem and develop an annuity product in which mortality and longevity risks are shared between annuitants and life insurance companies.

The decision as to which of these three main retirement payout options is preferred relies mostly on individual preferences, the type of pension arrangements

in place, the “generosity” of PAYGO pension systems, as measured for example by the replacement rates, the availability of other sources of income in retirement, tax incentives, the existence of individual account type systems, financial education, mandatory annuitization constraints or the level of development of insurance markets.

Overall, the life insurance industry should be prepared to help retirees to meet their financial needs, both in the accumulation and payout phases. In the accumulation phase, companies should help individuals to build up a desired level of savings throughout their working years in a flexible and efficient way. In a flexible way, assisting individuals to choose the amount and timing of their contributions to the capitalisation plan. In an efficient way via, for example, investment diversification strategies, gradual adjustment of the risk/return profile according to age, tax incentives.

As to the decumulation phase, life insurance companies have a crucial role in allowing individuals to have access and run their asset pool in a flexible and smooth way, while offering protection against longevity, inflation and investment risks. This can be done by offering various types of annuities, with alternative payout mechanisms (fixed or variable, inflation-linked, equity-linked, participating arrangements, additional embedded guarantees), through health care and long-term care insurance or through wealth monetisation (e.g., reverse mortgages) for those whose assets are not in liquid form.

#### 1.4 Main constraints facing annuity markets

Life annuity products have been sold in the past primarily as retirement accumulation vehicles, rather than decumulation products (Brown *et al.*, 2001). This may explain why annuity markets in Portugal and in most OECD countries have been relatively underdeveloped to date. However, annuity markets suffer from a wide range of demand and supply constraints<sup>1</sup>. On the demand side, limitations to the development of annuity markets include, first, the level of annuitization from PAYGO-financed pensions, i.e., the degree on which annuities are crowded out by social security provision and the degree on which they are crowded out by other forms of pension saving such as DB occupational schemes. Second, annuities are perceived to be unfairly priced, mostly because life insurance companies do not fully disclose information on the technical basis used to calculate annuity premiums. Third, the motive to bequest assets on death to dependents is not covered by “plain vanilla” annuities. Fourth, the demand for annuities is determined to

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<sup>1</sup>For a detailed discussion on this subject see, for example, Stewart (2007) and Rusconi (2008).

some extent by personal considerations such as family support, the need to cover the costs of unexpected medical expenses, the inexistence of sufficient liquid assets to purchase an annuity or liquidity concerns. For example, for older people, the risk of having to pay large medical bills or cover special health care costs induces them to retain at least a fraction of their assets instead of annuitising them.

Fifth, fiscal incentives are considered insufficient to stimulate insurance protection against longevity risk. In modern competitive markets, individual financial decisions are also driven by people's perceptions about the appeal of alternative investments, both during their working lifetime and after retirement. For instance, some individuals may avoid annuitisation on the grounds that they can manage their assets better than institutional fund managers. In this scenario, introducing tax incentives (or tax-favoured competing assets) could undermine saving decisions in favour of buying annuity protection. Finally, in some cases there is a general mistrust of institutions providing annuities.

On the supply side, the type and scope of the limitations to the development of annuity markets is also significant. First, high-quality information on mortality tables depicting a particular group's distribution of expected remaining lifetime is required. Projected mortality tables should take into consideration the stochastic nature of the remaining lifetime and encompass cohort effects. Uncertainty regarding mortality tables can cause insurance companies to price annuities conservatively, exacerbating adverse selection problems and lowering the access to the market. Additionally, uncertainty regarding mortality data can cause individuals to seriously underestimate their survival prospects, which, in turn, can lead them to undervalue the importance of longevity insurance. Dissemination of mortality should, in this sense, be considered a matter of public interest and form part of a clear supervision policy. In Portugal, there are not regulatory lifetables (neither contemporaneous lifetables nor prospective lifetable) either for the Portuguese overall population or for life insured populations. As a result, life insurance companies are forced to use as their technical basis lifetables adopted in other countries. Although this practice is authorized by the supervising authorities, using a survival law drawn up from other population's experience, potentially biased when compared to the demographic conditions observed in Portugal, involves significant basis risks, in particular the risk of overestimating the mortality risk of the population. In Section 3 we address this issue and derive the first prospective lifetables for the Portuguese general population.

Second, annuity markets are often affected by strong adverse-selection problems. This arises if buyers of annuities prove to be live longer than average,

inducing insurance companies to devise separate mortality tables for annuitants as opposed to those for the general population. The existence of adverse selection problems induces companies to include significant margins when pricing for annuity contracts. Whether adverse selection is quantitatively important may depend on whether annuitisation is considered optional or mandatory. In this sense, increasing compulsory annuitisation can significantly reduce adverse-selection problems.

Third, the potential for growth in annuity markets cannot be fully accommodated if insurance companies lack assets with which to back the long-term promises represented by annuities. Appropriate asset types either do not exist or are available in insufficient quantity. Insurance companies offering annuity products are faced with two major risk sources: interest-rate risk and longevity risk. Standard immunisation theory suggests that in order to protect themselves from small changes in the term structure of interest rates, insurance companies should back their annuity portfolios with assets whose respective durations equal those of the annuity liabilities, and whose respective convexities are larger than those of the annuity liabilities. This is difficult in practice, since long-term bonds are not available in most bond markets. Moreover, if real annuities are to be provided, real long-term bonds will have to be issued as well. This means that annuity markets would definitely benefit from the issuance of long-term government bonds. Moreover, recent events in Argentina and Russia have shown that the quality of assets considered is important, since the possibility of default is real.

On the other hand, longevity risk, i.e., the chance that entire cohorts live longer than anticipated in projected mortality tables, remains a real concern for insurance companies selling annuity products, since substantial changes in mortality patterns could seriously challenge their profitability. Insurance companies can, for example, hedge longevity risk with offsetting life insurance contracts, reducing (but not eliminating completely) the impact of negative mortality scenarios. Some advocate that governments (or private companies) should issue cohort “survivor bonds” (or longevity bonds), i.e., bonds whose future coupons payments depend on a survivorship index (for example, the percentage of the whole population of retirement age - say 65 - on the issue date still alive on the future coupon payment dates).<sup>2</sup>

Although survivor bonds are good candidates for hedging aggregate mortality risk, they do not provide a perfect hedge against the particular characteristics of a company’s pool of annuitants. In this sense, there is basis risk between the

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<sup>2</sup>See, e.g., Blake and Burrows (2001) and Blake et al. (2006a).

reference population mortality and the mortality experienced by any individual pool of annuitants. Other problems related to the issuance of survivor bonds include: i) the ability of dealing with a business involving huge amounts of capital, ii) pricing complications related to the adoption of a particular (stochastic) representation of mortality uncertainty and the estimation of the market price of longevity risk and iii) the importance of attractive contract design in order to boost market liquidity for traded securities and reduce credit risk. Once a well organised and liquid market for survivor bonds is in place, a whole new avenue is open for the development of survivor-derivative products (for example, mortality options based on a certain mortality index, futures contracts based on survival forecasts, survivor swaps interchanging cash flows based on two different mortality experiences, longevity forwards<sup>3</sup>).

Finally, alternative methods of hedging longevity risk include the use of traditional reinsurance methods, or through risk-sharing in the capital markets, which are particularly attractive for investors because of the low or negative correlation with traditional risk factors such as financial market indexes, or through the option of annuity securitisation, which would benefit insurance companies by providing them with alternative means for offloading their mortality improvements risk exposure.

Fourth, traditional annuity markets are incomplete in the sense that do not offer protection against inflation, they lack equity market exposure, they are illiquid and do not insure against multiple shocks. Finally, there are concerns regarding regulatory capital requirements or with the strength of existing providers that would make it difficult for new entrants to survive.

In order to address these problems, many policy options exist to encourage and promote annuity markets. Examples include mandating annuitization, improving financial literacy, dealing with longevity risk or producing longevity indexes.

## 1.5 Modeling mortality and longevity risk

One of the key conditions for the development of longevity-linked products and markets and for the hedging of longevity risk is the development of generally agreed market models for risk measurement. Whereas traditional market risks such as equity market, interest rate, exchange rate, credit and commodity risks have well consolidated methodologies for quantifying risk-based capital and for establishing market prices, longevity and mortality risk has historically been a very opaque risk. For a long time, only demographers, actuaries and insurance

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<sup>3</sup>For a detailed discussion see, e.g., Blake *et al.* (2006a,b) and Bravo (2007).

companies showed any interest in measuring and managing this risk, mainly for pricing purposes. A number of explanations can be given for this, particularly the fact that it is a non-financial risk that has been measured and analyzed in a different way from financial risks, generally adopting deterministic or scenario based approaches.

Historically, actuaries have been calculating premiums and mathematical reserves using a deterministic approach, by considering a deterministic mortality intensity, which is a function of the age only, extracted from available (static) lifetables and by setting a flat (“best estimate”) interest rate to discount cash flows over time. Since neither the mortality intensity nor interest rates are actually deterministic, life insurance companies are exposed to both financial and mortality (systematic and unsystematic) risks when pricing and reserving for any kind of long-term living benefits, particularly on annuities. In particular, the calculation of expected present values requires an appropriate mortality projection in order to avoid significant underestimation of future costs.

In order to protect the company from mortality improvements, actuaries have different solutions, among them to resort to projected (dynamic or prospective) lifetables, i.e., lifetables including a forecast of future trends of mortality instead of static lifetables. Static lifetables are obtained using data collected during a specific period (1 to 4 years) whereas dynamic lifetables incorporate mortality projections. In a situation where longevity is increasing over time, static lifetables underestimate lifelengths and thus premiums relating to life insurance contracts. Conversely, dynamic lifetables will project mortality into the future accounting for longevity improvements.

Since the future mortality is actually unknown, there is enormous likelihood that future death rates will turn out to be different from the projected ones, and so a better assessment of longevity risk would be one that consists of both a mean estimate and a measure of uncertainty. Such assessment can only be performed using stochastic models to describe both demographic and financial risks. In the following sections, we review both the traditional “dynamic approach” and the new “stochastic mortality approach”.

## 2 Group Self Annuitization life annuities

### 2.1 Risk pooling principle

Through “plain vanilla” annuities, life insurers offer their policyholders protection against two broad classes of risk: biometric risks, such as longevity and mortality

risks, and macroeconomic and financial market risks, such as interest rate, inflation, equity market or credit market risk. In this kind of product insurers bear all risk, both systematic (e.g., longevity risk, the risk that people systematically live longer than predicted) and unsystematic or idiosyncratic risk (e.g., financial market volatility, mortality deviations around predicted values,...).

To introduce a special type of participating life annuity called Guaranteed (or pooled) Self Annuitization (GSA) annuity fund consider the following simple example. Let us take a group of ten 90-year-old Portuguese women, who are concerned about outliving their financial wealth over the next year. Statistically, the latest estimations show that there is an approximately 20% probability of death in the next year. To protect against longevity risk, they agree to contribute EUR 100 to a common fund, which will redistribute the capital and investment return (say 5% yield pa) amongst survivors. At the end of the year each of them will get between EUR 105 (if no-one dies) and EUR 1050 (if nine out of ten die), based on actual mortality experience.

What this example highlights is that by pooling mortality risk and ceding bequest, individuals seem to all gain. In fact, ex-ante all fund participants receive some protection against longevity risk over the duration of the contract. If the agreement between the ten old ladies were to be intermediated by an insurance company, involved a large number of people, and lasted for the remaining lifetime of participants' lives, it would constitute a special type of participating life annuity called Guaranteed Self Annuitization annuity fund. Through this kind of arrangement, with a large investment pool, and assuming that longevity risk is null, the funds contributed by those who die earlier than expected on the basis of expected mortality rates are "inherited" by those who survive and supplement the pool's capital market gains, offering thus a larger benefit than could be achieved through individual investments.

Stated more formally, consider a standard GSA annuity fund.<sup>4</sup> Without loss of generality, the pool starts (at time  $t = 0$ ) with an initial size of  $l_x$  homogeneous insured persons in the sense of identical age, gender and cohort, identical monetary amounts and identical risk exposures. We assume that contracts are sold to policyholders in exchange for a single upfront premium  $P_0$ , given exogenously throughout the entire analysis. The contract provides a flat benefit  $B_0$  paid once a year. Given these assumptions and the best estimate of future mortality, the

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<sup>4</sup>For a comprehensible introduction GSA's see, for instance, Piggott and Detzel (2004) and Richter and Weber (2009).

starting total fund is

$$F_0 = l_x B_0 \ddot{a}_x \quad (1)$$

where  $\ddot{a}_x$  is the standard actuarial notation for the present value of whole life annuity-due, determined using the mortality information and projections available at time  $t = 0$  and assuming a constant discrete interest rate of  $i$  per period (we will use the year as the period, any generalizations can be made using the usual approaches, e.g., interpolation). Such a pure annuity provides a unit payment for the remaining lifetime of an insured person initially aged  $x$ , i.e., contingent on the insured's survival. Using the equivalence premium principle,  $\ddot{a}_x$  is given by

$$\ddot{a}_x = \mathbb{E} \left( \sum_{t=0}^{K(x)} v^t \right) = \sum_{t=0}^{\infty} \{ v^t \cdot_t p_x \} \quad (2)$$

where  $K = K(x) = [T(x)]$  is the number of completed future years lived by  $(x)$ , also denominated the curtate future lifetime of  $x$  (see, for instance, Gerber [1997]), and where  $v = (1 + i)^{-1}$  denotes the standard discount factor. This starting total fund can also be considered the initial total reserve, i.e.,  $F_0 = V_0$ .

In a GSA, the future value of annuity benefits remains constant over the whole contract unless deviations from expected mortality rates are observed. If that is not the case, i.e., if the number of those surviving up to higher ages is different from expected, the remaining reserves have to be redistributed among the remaining survivors. Assuming that realized investment rates will be as expected<sup>5</sup>, the total fund at time  $t = 1$  comprises the initial value less annuity payments accrued at the technical interest rate

$$F_1 = V_1 = (F_0 - l_x B_0) (1 + i) \quad (3)$$

Redistributing this reserve among the actual  $l_{x+1}^o$  remaining survivors for their expected future lifetime, including the reserves "inherited" from non-surviving members, the future value of annuity benefits becomes, after some algebra

$$B_1 = \frac{V_1}{l_{x+1}^o} = B_0 \left( \frac{p_x}{p_x^o} \right) \quad (4)$$

where  $p_x$  and  $p_x^o$  denotes, respectively, the expected and realized survival probabilities for an individual aged  $x$  at time  $t = 0$  in the time interval  $(t, t + 1)$ .

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<sup>5</sup>The extension to the case where the realized investment earnings pattern is different from the assumed constant rate  $i$  is straightforward (see, e.g., Piggott *et al.*, 2004)

Proceeding inductively, at any time  $t$  in the future the benefit payment will be determined by

$$B_t = B_0 \left( \frac{tp_x}{tp_x^*} \right) \quad (5)$$

From (5) it is clear that future annuity payments depend on the ratio of survivorship rates. In a scenario of longevity risk, i.e., in a scenario where the number of those surviving to age  $x + t$  is systematically higher than initially expected benefit payments will inevitably drop in order to prevent fund imbalance. This contrasts with traditional life annuity contracts that guarantee a level payment for the remainder of recipient's lifetime independently of future mortality developments.

From (5) it is also clear that benefit payments at time  $t$  can be expressed as

$$f_t(B_t) = \begin{cases} \min(B_0, B_t), & \frac{tp_x}{tp_x^*} < 1 \\ B_0, & \frac{tp_x}{tp_x^*} = 1 \\ \max(B_0, B_t), & \frac{tp_x}{tp_x^*} > 1 \end{cases} \quad (6)$$

For instance, in a scenario of longevity risk the benefit payment is given by the current value of the reference fund distributed among the actual  $l_{x+1}^o$  remaining survivors capped by its inception value  $B_0$ . This benefit can be expressed in terms of the final (maturity) payoff of an European put option with strike equal to the annuity benefit at inception, i.e.,

$$f_t(B_t) = B_0 - \max\{B_0 - B_t; 0\} \quad (7)$$

If, for the contrary, actual remaining survivors are less than initially estimated, benefit payments at time  $t$  are floored by the annuity benefit at inception and can be expressed in terms of the final payoff of an European call option with strike equal to  $B_0$ , i.e.,

$$f_t(B_t) = B_0 + \max\{B_t - B_0; 0\} \quad (8)$$

Equations (8) and (7) show that GSA annuity contracts include option features that, up to our knowledge, have never been considered in the design and pricing of these contracts. In fact, insurance companies adopt an over-simplified approach and completely ignore embedded options, resorting to consolidated actuarial techniques for pricing (and hedging) the contract. After all, in a GSA annuity fund all actual losses/profits are borne by the remaining survivors, whose benefits fluctuate according to mortality developments.

However, this solution may have a disastrous effect from a marketability point

of view. For example, in a scenario of longevity risk future benefits will decrease and this may spread discontent through those who weren't aware of the potential impact of future mortality improvements on their "apparently guaranteed" income and may feel they didn't receive any compensation for being short in put option contract. Compared to standard life annuities, policyholders in a GSA may question the fact that insurance companies don't pay a premium for the option to cut back annuity payments in case of adverse mortality improvements.

On the other hand, if actual longevity developments are worst than initially expected, benefit payments in a GSA will increase due to a higher "inheritance effect" or "survivor bonus" since the accumulated funds will be spread across a smaller surviving group. Compared to standard life annuities, in this case life insurance companies will not be compensated for the "lost" reserves. Individuals assessing the possibility of annuitizing their wealth but disbelief about their longevity prospects may feel attracted to buy an GSA annuity contract type if given the chance to increase annuity payments if their prospects confirm. Moreover, in this case life insurance companies may sell a separates call options on future benefit payments, upgrading thus the value of the overall line of business.

Although the framework of GSA is interesting at a theoretical level it as no practical interest in life insurance competitive markets for a number of reasons. Firstly, as in other variable annuity contracts the annuitant does not know in advance the rate of return of the pool, hence it carries some risk. In particular, GSA without additional guarantees are structured so that individuals share both mortality and investment risk in upside and downside times. Second, annuitants can see theirs payments dropping below a reasonable value in the presence of longevity risk. Third, there will be no payments for lives above the technical limit of the mortality table used to first price annuities, i.e., individuals might end up with no resources to fund consumption. Fourth, insurance companies (or fund's manager) does not bear any kind of risk, either actuarial or financial, either systematic of idiosyncratic. In fact, this is a simple approach to the diffusion of risk since, in the classical framework of GSA, there is no need to use a risk bearer (as an insurer or fund's manager) since the funds are periodically reallocated to the annuitants, based on the previous payment adjusted for any deviations in mortality and interest from expectations.

Fifth, in its simplest form, a GSA does not give pool member's access to the principal investment nor to any accumulated fund. This means that the product does not cover legacy motives. Finally, buyers of such product tend to be people who expect to live longer, raising once again the question of adverse selection.

The apparent advantage of GSA's over self-insuring is that the risk exposure is not immediate in that is borne by the pool and it's smoothed out by the insurance company over a long time horizon. In addition, since mortality and investment are largely uncorrelated, there is some chance that a negative return on investments may be partially offset by a positive "inheritance effect" or vice-versa. In other words, the effects of the overall risk exposure might be mitigated and postponed at the individual level when considered in a pooling structure.

To address these concerns, we propose in Section 5 a new participating life annuity contract in which mortality and longevity risks are shared between policyholders and insurance companies. The contract includes option-like features that adjust benefits if future mortality developments are significantly different from expected.

## 2.2 Annuity portfolio losses

A different way to understand the option-like features of annuity contracts is to analyse the relation between survival probabilities and annuity portfolio losses. Consider a classic life annuity contract with level payment  $B_0$ . The loss on the underlying annuity portfolio at time  $t$  is defined as

$$L_t = \sum_{i=1}^{l_x} (I_i(t)B_0 - \mathbb{E}[I_i(t)B_0])^+ \quad (9)$$

where  $I_i(t) = 1_{\tau_i > t}$  is an indicator function that jumps from 1 to 0 at the time of death  $\tau_i$  of the annuitant. Note that  $\mathbb{E}[I_i(t)] =_t p_x$ . Losses on the portfolio are the amount that the annuity payments at time  $t$  exceed the expected payments. For a given population survival probability  ${}_t p_x$ , the distribution of the number alive at time  $t$  is binomial

$$l_{x+t} \sim \text{Binomial}(l_x, {}_t p_x) | {}_t p_x \quad (10)$$

As recognized by Lin and Cox (2005), there are two sources of uncertainty in the portfolio loss at time  $t$ . This first is due to uncertain lifetimes given the actual mortality rates. The second is attributed to the stochastic nature of survival probabilities. Given this, the total variability in the portfolio is the unconditional variance of the compound binomial distribution

$$\text{Var}(l_{x+t}) = \mathbb{E}[\text{Var}[(l_{x+t}) | {}_t p_x]] + \text{Var}[\mathbb{E}[(l_{x+t}) | {}_t p_x]] \quad (11)$$

For large portfolios of annuitants, the main source of uncertainty will come from changes in mortality rates impacting all lives in the portfolio rather than the variability in the number of deaths at a particular age given the mortality rate. In other words, the randomness in  $l_{x+t}$  will mainly be due to the uncertainty in  $tp_x$ .

Life insurance companies pool mortality risks by the Law of Large Numbers, i.e.,  $\lim_{l_c \rightarrow \infty} \mathbb{E} [Var [(l_{x+t}) | tp_x]] \rightarrow 0$ . However, from (11) we can conclude that the assumption that mortality and longevity risks can be diversified away by writing a large number of policies is incorrect if we take into account the dynamics of the underlying mortality rates, i.e.,

$$Var [\mathbb{E} [(l_{x+t}) | tp_x]] \neq 0$$

Mortality dynamics is influenced by a complex setting of socioeconomic factors, biological variables, government policies, environmental effects, health conditions and social behaviours. Since the future mortality is actually unknown, there is always the likelihood that future death rates will turn out to be different from the projected ones and thus mortality shocks can destroy the insurance pooling mechanism. For example, for an annuity portfolio the risk is that the annuitants will systematically live longer than expected at the policies inception. Systematic mortality risk cannot be eliminated by diversification and thus should have a market price.

The portfolio loss in equation (9) can be written as

$$L_t = l_x B_0 [tp_x^{\circ} - t p_x]^+ \quad (12)$$

Redistributing among the remaining survivors we have

$$\begin{aligned} \frac{L_t}{l_{x+t}^{\circ}} &= \frac{l_x B_0}{l_{x+t}^{\circ}} [tp_x^{\circ} - t p_x]^+ = B_0 \max \left[ \left( \frac{tp_x^{\circ} - t p_x}{tp_x^{\circ}} \right); 0 \right] \\ &= B_0 \max \left[ \left( 1 - \frac{tp_x}{tp_x^{\circ}} \right); 0 \right] \end{aligned} \quad (13)$$

Equation (13) shows that the loss "inherited" by each surviving policyholder includes an option feature that depends on ratio of survivorship rates. To be more specific, the loss has an embedded put option with strike equal to unity and underlying equal to the ratio between estimated and actual survivorship rates. To value this option we can resort to traditional discrete-time (Binomial) or continuous-time approaches (Black-Scholes), with proper adjustments for an

incomplete markets situation. Alternatively, we can resort to Monte-Carlo simulation techniques.<sup>6</sup>

### 3 Deriving Prospective Lifetables for Portugal

In this section we derive prospective lifetables for the Portuguese general population. The results are then compared with that of classical static lifetable approach to give an indication of the longevity risk currently faced by portuguese insurance companies.

#### 3.1 Notation, assumptions and quantities of interest

The basic idea underlying projected lifetable methods is to analyse changes in mortality as a function of both age  $x$  and time  $t$ . Let  $\mu_x(t)$  denote the force of mortality at age  $x$  during calendar year  $t$ . Let  $q_x(t)$  and  $p_x(t) = 1 - q_x(t)$  represent, respectively, the one-year death probability at age  $x$  in year  $t$  and the corresponding survival probability. Let  $D_{x,t}$  denote the number of deaths recorded at age  $x$  during year  $t$ , from an exposure-to-risk (i.e., the number of person years from which  $D_{x,t}$  arise)  $E_{x,t}$ .

Consider now the classic Lexis diagram, that is, a coordinate system that has calendar time as abscissa and age as coordinate. If we assume that both time scales are divided into yearly bands, the Lexis plane is partitioned into squared segments. In this paper, we assume that the age-specific forces of mortality are constant within bands of time and age, but authorized to change from one band to the next. Formally, given any integer age  $x$  and calendar year  $t$ , we assume that

$$\mu_{x+\xi}(t+\tau) = \mu_x(t) \text{ for any } 0 \leq \xi, \tau < 1 \quad (14)$$

In other words, assumption (14) means that mortality rates are constant within each square of the Lexis diagram, but allowed to vary between squares. From (14) the calculation of the probability of an individual aged  $x$  in year  $t$ ,  $p_x(t)$ , and of the corresponding death probability  $q_x(t) = 1 - p_x(t)$  simplifies to

$$p_x(t) = \exp(-\mu_x(t)) = 1 - p_x(t) \quad (15)$$

Several markers are regularly used by demographers to measure the evolution of mortality, namely life expectancies, variance of residual lifetime, median lifetime

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<sup>6</sup>The valuation of options embedded in GSA funds is being performed in an accompanying paper.

or the entropy of a lifetable. Let  $\hat{e}_x(t)$  denote the (complete) life expectancy of an  $x$ -aged individual in year  $t$ , i.e., the average number of years he is expected to survive. This means we expect this individual will die in year  $t + \hat{e}_x(t)$  then aged  $x + \hat{e}_x(t)$ . Contrary to classic static lifetables, the use of projected lifetables allows us to estimate the “true” (diagonal) expected residual lifetime of an individual. The appropriate formula for  $\hat{e}_x(t)$  is given by

$$\begin{aligned}\hat{e}_x(t) &= \sum_{k \geq 0} \left\{ \prod_{j=0}^k p_{x+j}(t+j) \right\} \\ &= \frac{1 - \exp(-\mu_x(t))}{\mu_x(t)} \\ &\quad + \sum_{k \geq 1} \left\{ \prod_{j=0}^{k-1} \exp(-\mu_{x+j}(t+j)) \right\} \frac{1 - \exp(-\mu_{x+k}(t+k))}{\mu_{x+k}(t+k)}\end{aligned}\tag{16}$$

The actual computation of  $\hat{e}_x(t)$  requires the knowledge of  $\mu_\xi(\tau)$  (or  $p_\xi(\tau)$ ) for  $x \leq \xi \leq \omega$  and  $t \leq \tau \leq t + \omega - x$ , where  $\omega$  denotes the ultimate (maximum) age. Since these survival probabilities are knot known at time  $t$ , they have to be estimated using extrapolation methods based on past trends. The next section gives an example of how this can be done in practice.

For life insurance companies and annuity providers, the net single premium of an immediate life annuity sold to an  $x$ -aged individual in year  $t$ ,  $a_x(t)$ , is of special interest. In a dynamic approach, the appropriate formula for  $a_x(t)$  is given by

$$a_x(t) = \sum_{k \geq 0} \left\{ \prod_{j=0}^k p_{x+j}(t+j) \right\} v^{k+1}\tag{17}$$

where  $v = (1+i)^{-1}$  is the classic discount factor considering a flat term structure. As can be seen, mortality projections and projected survival probabilities are particularly important to price correctly annuity and other life insurance contracts.

### 3.2 Mortality projection method

The literature on the construction of projected lifetables is vast and growing.<sup>7</sup> The classical approach is to fit an appropriate parametric function (e.g. Makeham model) to each calendar year data. Then, each of the parameter estimates is treated as independent time series, extrapolating their behaviour to the future in order to provide the actuary with projected lifetables (see, e.g., CMIB (1976) and

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<sup>7</sup>A detailed review of mortality projection methods can be found in Tuljapurkar and Boe (1998), Pitacco (2004), Wong-Fupuy and Haberman (2004) and Bravo (2007).

Heligman e Pollard (1980)). Despite simple, this approach has serious limitations. In the first place, this approach strongly relies on the appropriateness of the parametric function adopted. Secondly, parameter estimates are very unstable a feature that undermines the reliability of univariate extrapolations. Thirdly, the time series for parameter estimates are not independent and often robustly correlated. Although applying multivariate time series methods for the parameter estimates is theoretically possible, this will complicate the approach and introduce new problems.

Lee and Carter (1992) developed a simple model for describing the long term trends in mortality as a function of a simple time index. The method models the logarithm of a time series of age-specific death rates  $\mu_x(t)$  as the sum of an age-specific component  $\alpha_x$ , that is independent of time, and a second component, expressed as a product of a time-varying parameter denoting the general level of mortality  $\kappa_t$ , and an age-specific component  $\beta_x$  that signals the sensitiveness of mortality rates at each age when the general level of mortality changes. Formally, we have

$$\ln \mu_x(t) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \quad (18)$$

where  $\epsilon_{x,t} \sim \mathcal{N}or(0, \sigma_\epsilon^2)$  is a white-noise, representing transitory shocks. Parameters  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  have to be constrained by

$$\sum_{t=t_{\min}}^{t_{\max}} \kappa_t = 0 \quad \text{and} \quad \sum_{x=x_{\min}}^{x_{\max}} \beta_x = 1, \quad (19)$$

in order to ensure model identification.

Parameter estimates are obtained by ordinary least squares, i.e., by solving the following minimization program

$$(\hat{\alpha}_x, \hat{\beta}_x, \hat{\kappa}_t) = \arg \min_{\alpha_x, \beta_x, \kappa_t} \left\{ \sum_{x=x_{\min}}^{x_{\max}} \sum_{t=t_{\min}}^{t_{\max}} (\ln \mu_x(t) - \alpha_x - \beta_x \kappa_t)^2 \right\}. \quad (20)$$

Lee and Carter (1992) solve (20) by resorting to *Singular Value Decomposition* techniques but alternative estimation procedures can be implemented considering iterative methods (see, e.g., Bravo (2007)) or Weighted Least-Squares (see, e.g., Wilmoth (1993)). The resulting time-varying parameter estimates are then modelled and forecasted using standard Box-Jenkins time series methods. Finally, from this forecast of the general level of mortality, projected age-specific death rates are derived using the estimated age-specific parameters.

There have been several extensions to the Lee-Carter model including different

error assumptions and estimation procedures.<sup>8</sup> Bell (1997), Booth *et al.* (2002a) and Renshaw and Haberman (2003c,d) include a second log-bilinear term in (18) and estimate parameters by considering the first two term in a SVD. Additionally, they adopt a multivariate setting in order to project the evolution of the two time indices  $\kappa_{t,i}$  ( $i = 1, 2$ ). Carter and Prskawetz (2001) consider the possibility of time varying parameters  $\alpha_x$  and  $\beta_x$ . Renshaw and Haberman (2003a) include additional non-linear age factors when modeling the so-called “mortality reduction factors” within a Generalized Linear Models (GLM’s) approach. Renshaw and Haberman (2006) and Currie *et al.* (2004) include a cohort factor including year of birth as a factor impacting the rate of longevity improvement. This cohort factor is found to be significant in UK mortality data.

Renshaw e Haberman (2005) and Bravo (2007) develop a version of the Lee-Carter model considering positive asymptotic mortality. This result is, for most age groups, more consistent with observed mortality patterns when compared with that of the original model. Wilmoth and Valkonen (2002) develop an extension of the Lee-Carter model aimed to investigate differential mortality by considering a number of alternative covariates other than age and calendar time. Cairns, Blake and Dowd (2006b) develop and apply a two-factor model similar to the Lee-Carter model with a smoothing of age effects using a logit transformation of mortality rates. Cairns *et al.* (2007) analyze England and Wales and US mortality data showing that models that allow for an age effect, a quadratic age effect and a cohort effect fit the data best although the analysis of error distributions in these models revealed disappointing. De Jong and Tickle (2006) formulate the Lee-Carter model in a *state space framework*.

Brouhns *et al.* (2002a) and Renshaw and Haberman (2003c) develop an extension of the Lee-Carter model allowing for Poisson error assumptions and apply it to Belgian data. This Poisson log-bilinear approach can be stated as

$$D_{x,t} \sim \text{Poisson}(\mu_x(t) E_{x,t}), \quad (21)$$

where  $D_{x,t}$  denotes the number of deaths recorded at age  $x$  during year  $t$ , from an exposure-to-risk (i.e., the number of person-years from which  $D_{x,t}$  arise),  $E_{x,t}$ , and  $\mu_x(t)$  is given once again by (18). This model has several advantages over the Lee-Carter specification. First, the model doesn’t assume that errors are homoskedastic, an unrealistic assumption since the logarithm of the force of mor-

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<sup>8</sup>See Lee (2000), Lee and Miller (2001), Tuljapurkar and Boe (1998), Brouhns *et al.* (2002a), Wong-Fupuy and Haberman (2004), Bravo (2007) and Cairns *et al.* (2007) for a detailed discussion of Lee-Carter model and extensions.

tality is much more variable at older ages than at younger ages. Second, contrary to Lee-Carter model, the Poisson log-bilinear approach doesn't require a complete matrix of observed death rates. Finally, one of the main advantages over the Lee-Carter model is that specification (21) allows us to use maximum-likelihood methods to estimate the parameters instead of resorting to least squares (SVD) methods. Formally, we estimate the parameters  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  by maximizing the log-likelihood derived from model (18)-(21)

$$\ln \mathcal{V}(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\kappa}) = \sum_{t=t_{\min}}^{t_{\max}} \sum_{x=x_{\min}}^{x_{\max}} \{D_{x,t} (\alpha_x + \beta_x \kappa_t) - E_{x,t} \exp(\alpha_x + \beta_x \kappa_t)\} + c, \quad (22)$$

where  $\boldsymbol{\alpha} = (\alpha_{x_{\min}}, \dots, \alpha_{x_{\max}})'$ ,  $\boldsymbol{\beta} = (\beta_{x_{\min}}, \dots, \beta_{x_{\max}})'$ ,  $\boldsymbol{\kappa} = (\kappa_{x_{\min}}, \dots, \kappa_{x_{\max}})'$  and  $c$  is a constant.

The presence of the log-bilinear term  $\beta_x \kappa_t$  makes it impossible to estimate the model using standard statistical packages that include Poisson regression. Because of this, we resort in this paper to an iterative method proposed by Goodman (1979). The algorithm, which is essentially a Newton-Raphson standard method, states that in iteration  $v+1$ , a single set of parameters is updated fixing the other parameters at their current estimates according to the following updating scheme

$$\hat{\theta}_j^{(v+1)} = \hat{\theta}_j^{(v)} - \frac{\partial \mathcal{L}^{(v)}}{\partial \theta_j} / \frac{\partial^2 \mathcal{L}^{(v)}}{\partial \theta_j^2} \quad (23)$$

where  $\mathcal{L}^{(v)} = \mathcal{L}^{(v)}(\hat{\theta}^{(v)})$ . Recall that in our case we have three sets of parameters, corresponding to the  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  terms.

The updating scheme is as follows: starting with a given initial vector  $(\hat{\alpha}_x^{(0)} \hat{\beta}_x^{(0)} \hat{\kappa}_t^{(0)})$ ,

then:

$$\begin{aligned}
\hat{\alpha}_x^{(v+1)} &= \hat{\alpha}_x^{(v)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \left[ d_{x,t} - E_{x,t} \exp \left( \hat{\alpha}_x^{(v)} + \hat{\beta}_x^{(v)} \hat{\kappa}_t^{(v)} \right) \right]}{- \sum_{t=t_{\min}}^{t_{\max}} \left[ E_{x,t} \exp \left( \hat{\alpha}_x^{(v)} + \hat{\beta}_x^{(v)} \hat{\kappa}_t^{(v)} \right) \right]}, \\
\hat{\beta}_x^{(v+1)} &= \hat{\beta}_x^{(v)}, \quad \hat{\kappa}_t^{(v+1)} = \hat{\kappa}_t^{(v)} \\
\hat{\kappa}_t^{(v+2)} &= \hat{\kappa}_t^{(v+1)} - \frac{\sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x^{(v+1)} \left[ d_{x,t} - E_{x,t} \exp \left( \hat{\alpha}_x^{(v+1)} + \hat{\beta}_x^{(v+1)} \hat{\kappa}_t^{(v+1)} \right) \right]}{- \sum_{x=x_{\min}}^{x_{\max}} \left( \hat{\beta}_x^{(v+1)} \right)^2 \left[ E_{x,t} \exp \left( \hat{\alpha}_x^{(v+1)} + \hat{\beta}_x^{(v+1)} \hat{\kappa}_t^{(v+1)} \right) \right]}, \\
\hat{\alpha}_x^{(v+2)} &= \hat{\alpha}_x^{(v+1)}, \quad \hat{\beta}_x^{(v+2)} = \hat{\beta}_x^{(v+1)} \\
\hat{\beta}_x^{(v+3)} &= \hat{\beta}_x^{(v+2)} - \frac{\sum_{t=t_{\min}}^{t_{\max}} \hat{\kappa}_t^{(v+1)} \left[ d_{x,t} - E_{x,t} \exp \left( \hat{\alpha}_x^{(v+2)} + \hat{\beta}_x^{(v+2)} \hat{\kappa}_t^{(v+2)} \right) \right]}{- \sum_{t=t_{\min}}^{t_{\max}} \left( \hat{\kappa}_t^{(v+2)} \right)^2 \left[ E_{x,t} \exp \left( \hat{\alpha}_x^{(v+2)} + \hat{\beta}_x^{(v+2)} \hat{\kappa}_t^{(v+2)} \right) \right]}, \\
\hat{\alpha}_x^{(v+3)} &= \hat{\alpha}_x^{(v+2)}, \quad \hat{\kappa}_t^{(v+3)} = \hat{\kappa}_t^{(v+2)}
\end{aligned} \tag{24}$$

We use as a criterion to stop the iterative procedure a very small increase of the log-likelihood function.

The maximum-likelihood estimations of the parameters generated by (24) do not match the identification constraints (??), and have thus to be adapted. This is guaranteed by changing the parameterization in the following manner:

$$\kappa_t^* = (\hat{\kappa}_t - \bar{\kappa}) K \quad \text{and} \quad \beta_x^* = \frac{\hat{\beta}_x}{\sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x} \tag{25}$$

where  $\bar{\kappa}$  denotes average value for  $\hat{\kappa}_t$ , i.e.

$$\bar{\kappa} = \frac{1}{t_{\max} - t_{\min} + 1} \sum_{t=t_{\min}}^{t_{\max}} \hat{\kappa}_t$$

and where  $K$  is given by

$$K = \sum_{x=x_{\min}}^{x_{\max}} \hat{\beta}_x$$

from which we finally calculate

$$\alpha_x^* = \hat{\alpha}_x + \hat{\beta}_x \bar{\kappa} \tag{26}$$

The new estimates  $\alpha_x^*$ ,  $\beta_x^*$  and  $\kappa_t^*$  fulfill the constraints (??) and provide the same  $\hat{D}_{x,t}$  since  $\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t = \alpha_x^* + \beta_x^* \kappa_t^*$ . Note also that differentiating the log-likelihood function with respect to  $\alpha_x$  yields the equality

$$\sum_t D_{x,t} = \sum_t \hat{D}_{x,t} = \sum_t E_{x,t} \exp \left( \hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_t \right)$$

This means that the estimated  $\kappa_t$ 's are such that the resulting death rates applied to the actual risk exposure produce the total number of deaths actually observed in the data for each age  $x$ .

### 3.3 Modelling the time-factor

In the Poisson log-bilinear methodology, the time factor  $\kappa_t$  is intrinsically viewed as stochastic process. In this sense, standard Box-Jenkins techniques are used to estimate and forecast  $\kappa_t$  within an ARIMA( $p, d, q$ ) time series model. Recall that the model takes the general form

$$(1 - B)^d \kappa_t = \mu + \frac{\Theta_q(B)}{\Phi_q(B)} \epsilon_t \quad (27)$$

where  $B$  is the delay operator (i.e.,  $B(\kappa_t) = \kappa_{t-1}$ ,  $B^2(\kappa_t) = \kappa_{t-2}, \dots$ ),  $1 - B$  is the difference operator (i.e.,  $(1 - B)\kappa_t = \kappa_t - \kappa_{t-1}$ ,  $(1 - B)^2\kappa_t = \kappa_t - 2\kappa_{t-1} + \kappa_{t-2}, \dots$ ),  $\Theta_q(B)$  is the Moving Average polynomial, with coefficients  $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_q)$ ,  $\Phi_q(B)$  is the Autoregressive polynomial, with coefficients  $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_p)$ , and  $\epsilon_t$  is white noise with variance  $\sigma_\epsilon^2$ .

The method used to derive estimates for the ARIMA parameters  $\mu$ ,  $\boldsymbol{\theta}$ ,  $\boldsymbol{\phi}$  and  $\sigma_\epsilon$  is conditional least squares. From these, forecasted values of the time parameter, denoted by  $\kappa_t^*$ , are derived. Finally, the parameter estimates of the Poisson model and the forecasts  $\kappa_t^*$  can be inserted in (??) to obtain age-specific mortality rates, prospective lifetables, life expectancies, annuities single premiums and other related markers. In the following we apply the Poisson modelling to Portugal's general population data in order to derive prospective lifetables.

### 3.4 Data

The model used in this paper is fitted to the matrix of crude Portuguese death rates, from year 1970 to 2004 and for ages 0 to 84. The data, discriminated by age and sex, refers to the entire Portuguese population and has been supplied by Statistics Portugal(INE - Instituto Nacional de Estatística).

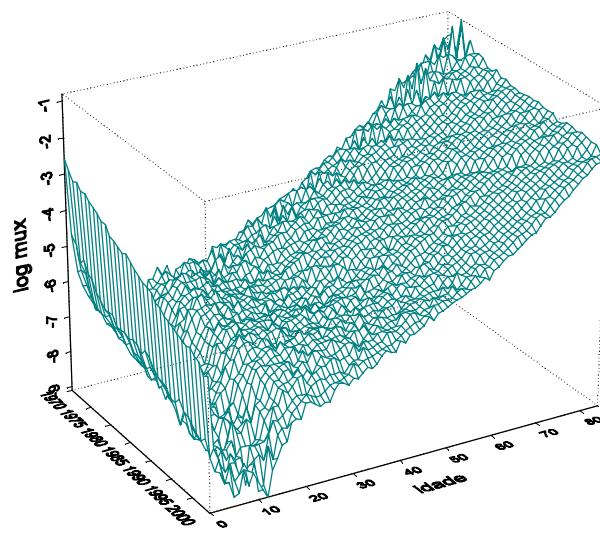


Figure 1: Crude mortality rates for the period 1970-2004, males

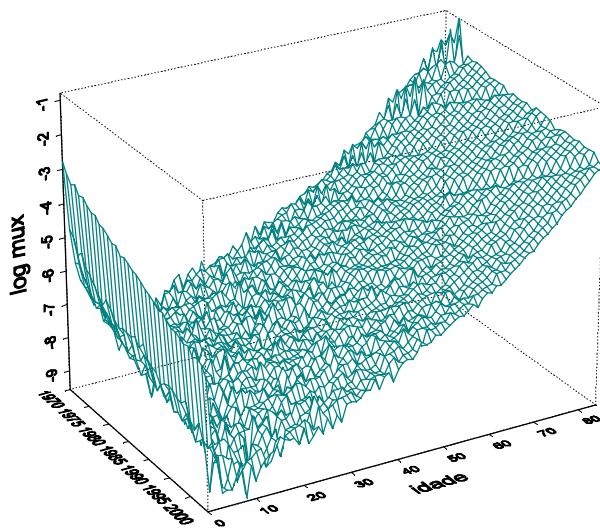


Figure 2: Crude mortality rates for the period 1970-2004, females

The database for this study comprises two elements: the observed number of death  $d_{x,t}$  given by age and year of death, and the observed population size  $l_{x,t}$  at December 31 of each year. We follow the INE definition of population at risk using the population counts at the beginning and at the end of a year and take migration into account. Figures 1 and 2 give us a first indication of mortality trends in Portugal during this period. Two trends dominated the global mortality decline: (i) a reduction in mortality due to infectious diseases, affecting mainly young ages, (ii) decreasing mortality at old ages.

### 3.5 Results

#### 3.5.1 Parameter estimates

We apply the Poisson modelling to the Portuguese data presented above. The Poisson parameters  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  implicated in (??) are estimated by maximum-likelihood methods using the iterative procedure described in Section ???. We started the updating scheme considering the following initial values  $\hat{\alpha}_x^{(0)} = 0$ ,  $\hat{\beta}_x^{(0)} = 1$ , and  $\hat{\kappa}_t^{(0)} = 0.1$ . The criterion to stop the iterative procedure is a very small increase of the log-likelihood function (in our case we used  $10^{-5}$ ). The routine was implemented within the SAS package. Figure 3 plots the estimated  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$ .

We note that the  $\hat{\alpha}_x$ 's represent the average of the  $\ln \hat{\mu}_x(t)$  across the time period. As expected, the average mortality rates are relatively high for newborn and childhood ages, then decrease rapidly towards their minimum (around age 12), increasing then in  $x$ , reflecting higher mortality at older ages. The only exception refers to the well known “mortality hump” around ages 20-25, more visible in the male population, a phenomena normally associated with accident or suicide mortality. We can see that young ages tend to be more affected by changes in the general time trends of mortality, probably due the evolution of medicine in reducing infantile and juvenile mortality. In effect, the  $\hat{\beta}_x$ 's decrease with age, except for the mortality hump phenomena, but remain positive for all ages. Note also that the sensitiveness of the male population to variations in parameter  $\kappa_t$  tends to be greater than that of the female population, which has a more stable pattern. Finally, we can see that the  $\hat{\kappa}_t$ 's exhibit a clear decreasing trend (approximately linear). This reveals the significant improvements of mortality at all ages both for men and women in the last 35 years.

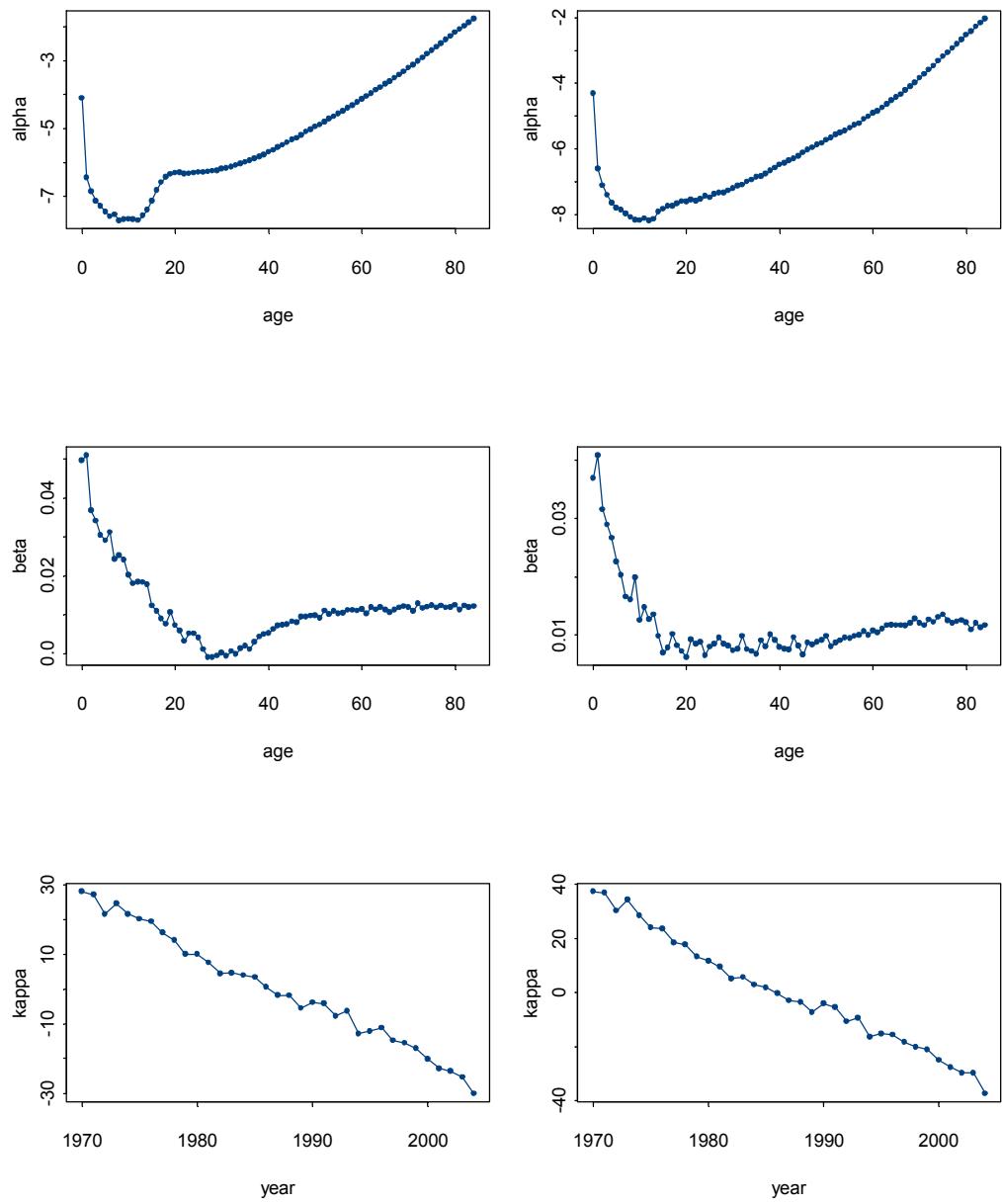


Figure 3: Estimations of  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  for men (left panels) and women (right panels).

### 3.5.2 Extrapolating time trends

Let  $\{\hat{\kappa}_t, t = t_{\min}, \dots, t_{\max}\}$  denote a realization of the finite chronologic time series  $\mathcal{K} = \{\kappa_t, t \in \mathbb{N}\}$ . Following the work of Lee and Carter (1992) and Brouhn et al. (2002a,b), we use standard Box-Jenkins methodology to identify, estimate and extrapolate the appropriate ARIMA  $(p, d, q)$  time series model for the male and female time indexes  $\kappa_t$ .

A good model for the male population is ARIMA(0, 1, 1), which is a moving average (MA(1)) model

$$(1 - B) \kappa_t^m = \rho^m + \theta^m \varepsilon_{t-1}^m + \varepsilon_t^m \quad (28)$$

whereas for women the ARIMA(1,1,0) autoregressive model was identified as a good candidate

$$(1 - B) \kappa_t^w = \rho^w + \phi^w \kappa_{t-1}^w + \varepsilon_t^w \quad (29)$$

where  $\varepsilon_t^m$  and  $\varepsilon_t^w$  are white noise error terms with variance  $\sigma_m^2$  and  $\sigma_w^2$ , respectively. The estimated parameters for the ARIMA  $(p, d, q)$  models (28) and (29) are given in Table 1. Note that all parameters are significant at a 5% significance level.

Sex	Parameter	Estimate	Std error	t-value	p-value
Men	$\rho^m$	-1.64623	0.11663	-14.11	<.0001
	$\theta^m$	0.64315	0.14831	4.34	0.0001
	$\sigma_m$	1.800992			
Women	$\rho^w$	-2.14802	0.23969	-8.96	<.0001
	$\phi^w$	-0.63606	0.15145	-4.20	0.0002
	$\sigma_w$	2.263249			

Table 1: Estimation of the parameters of the ARIMA(p,d,q) models

In Figure 4 we show the estimated values of  $\kappa_t$  together with the  $\kappa_t^*$  projected and the corresponding 95% confidence interval forecasts.

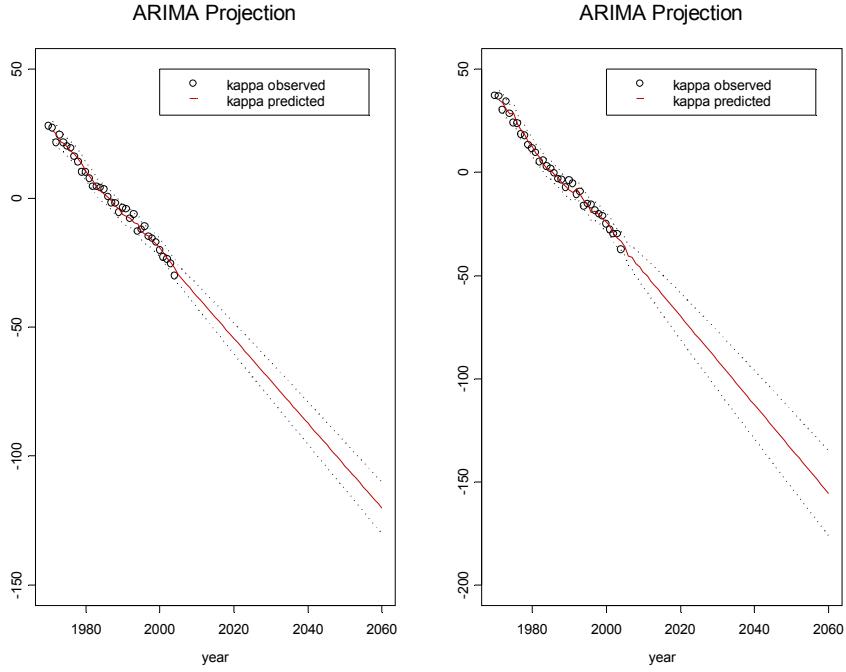


Figure 4: Estimated and projected values of  $\kappa_t$  with their 95% confidence intervals for males (left panel) and females (right panel)

Given the forecasted values of  $\kappa_t$   $\{\hat{\kappa}_{2004+s}^* : s = 1, 2, \dots\}$ , the reconstituted sex-specific forces of mortality are given by

$$\hat{\mu}_x(2004 + s) = \exp(\hat{\alpha}_x + \hat{\beta}_x \hat{\kappa}_{2004+s}^*), \quad s = 1, 2, \dots \quad (30)$$

and then used to generate sex-specific life expectancies and life annuities.

### 3.5.3 Projecting the mortality for the oldest-old: Closing Lifetables

According to the United Nations, it is estimated that in 2001 72 million of the 6.1 billion inhabitants of the world were 80 year or older. In the developing world, the population of the oldest-old (e.g., those 80 years and older) still represents a small fraction of the world's population but it is the fastest growing segment of the population. In addition, because life expectancy will continue to increase, not only we should expect to have an increasing number of people surviving to very old ages, but also anticipate that the deaths of the oldest-old will account for an increasing proportion of all deaths in a given population. In view of this, it is important to have detailed information about the age structure of the oldest-

old and about the behaviour of mortality at these ages. Unfortunately, in most countries reliable data on both the age distribution of population at risk and death counts of the oldest-old is not yet available. This is also our case since Portuguese statistics did not provide for this period an age breakdown for the group aged 85 and over. This poses a serious problem when it comes to complete lifetables.

Because of this, a number of research papers has addressed the issue of projecting mortality for the oldest-old (see, e.g., Buettner (2002)). In this paper we adopt the method proposed by Denuit and Goderniaux (2005) to extrapolate mortality rates at very old ages. The method is a two step method: first, a quadratic function is fitted to age-specific estimated mortality rates in a given age-band; second, the estimated function is used to extrapolated mortality rates up to a pre-determined maximum age. Formally, the following log-quadratic model is fitted by weighted least-squares

$$\ln \hat{q}_x(t) = a(t) + b(t)x + c(t)x^2 + \epsilon_x(t) \quad (31)$$

to age-specific mortality rates observed at older ages, where  $\epsilon_x(t) \sim \mathcal{N}(0, \sigma^2(t))$ , with additional constraints

$$q_{120} = 1 \quad (32)$$

$$q'_{120} = 0 \quad (33)$$

where  $q'_x$  denotes the first derivative of  $q_x$  with respect to age  $x$ . Constraints (32) and (33) impose a concave configuration to the curve of mortality rates at old ages and the existence of a horizontal tangent at  $x = 120$ . We then use this function to extrapolated mortality rates up to age 120. Figures 5 and 6 show the final result of this procedure.

### 3.5.4 Mortality Projections

**3.5.4.1 By chronologic year** Considering the prospective lifetables derived in the previous section, we can now analyse the evolution of mortality across time. Figure 7 represents the evolution of observed and estimated forces of mortality from 1970 to 2050 for both genders. In Figure 8 we can observe the evolution of observed and estimated mortality rates from 1970 to 2050 for both genders and some representative ages.

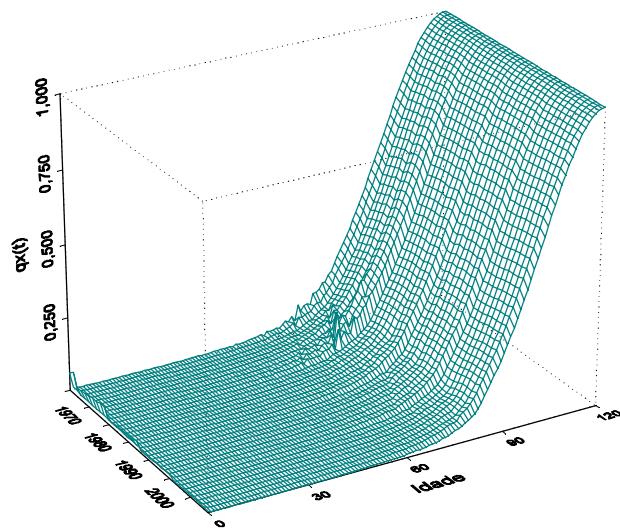


Figure 5: Mortality rates for closed lifetables, males

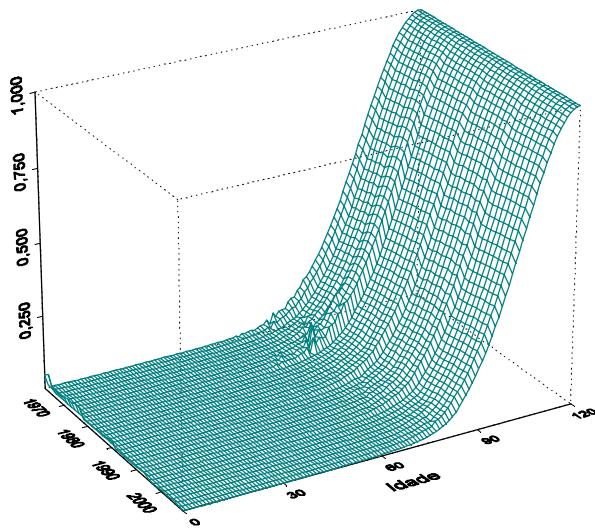


Figure 6: Mortality rates for closed lifetables, females

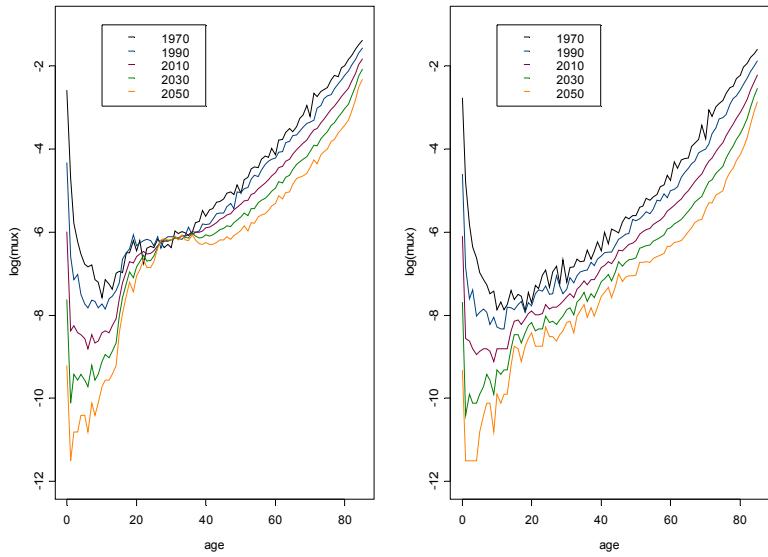


Figure 7: Evolution of  $\mu_x(t)$  for men (left panel) and women (right panel)

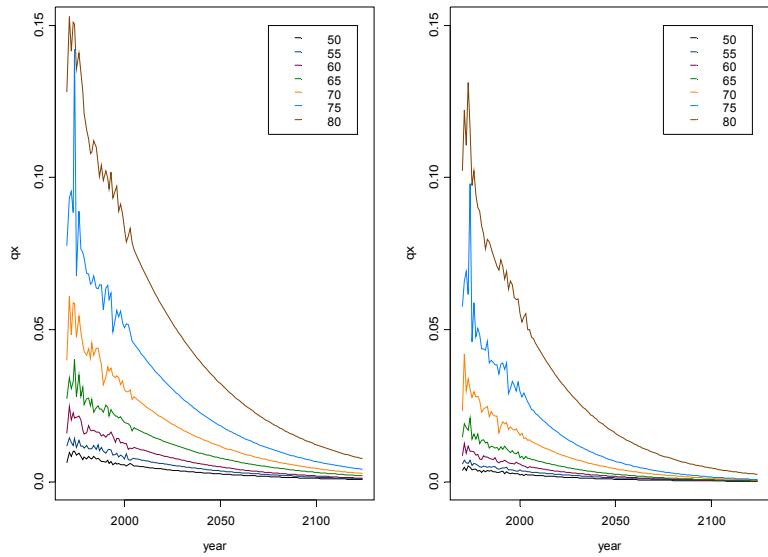


Figure 8: Evolution of  $q_x$  for some representative ages, from 1970 to 2124, for men (left) and women (right)

Overall, we can observe a clear and continuous decline in mortality throughout this period. It is also apparent that this mortality decline is more noticeable within younger ages. The mortality hump phenomenon is surprisingly persistent and tends to be more significant for the male population. In effect, we can observe a sort of mortality stagnation within this age-band. For older ages, we predict a decline in mortality rates.

**3.5.4.2 By Cohort** Prospective lifetables provide us with new tools for the analysis of mortality trends, namely the possibility to investigate the evolution of mortality not only in terms of calendar time but also in terms of year of birth or cohort. In brief, by using prospective lifetables we switch from a transversal approach to a longitudinal (or diagonal) approach to mortality.

In Figure 9 we can observe the evolution of the force of mortality for some representative generations born between 1970 and 2004.

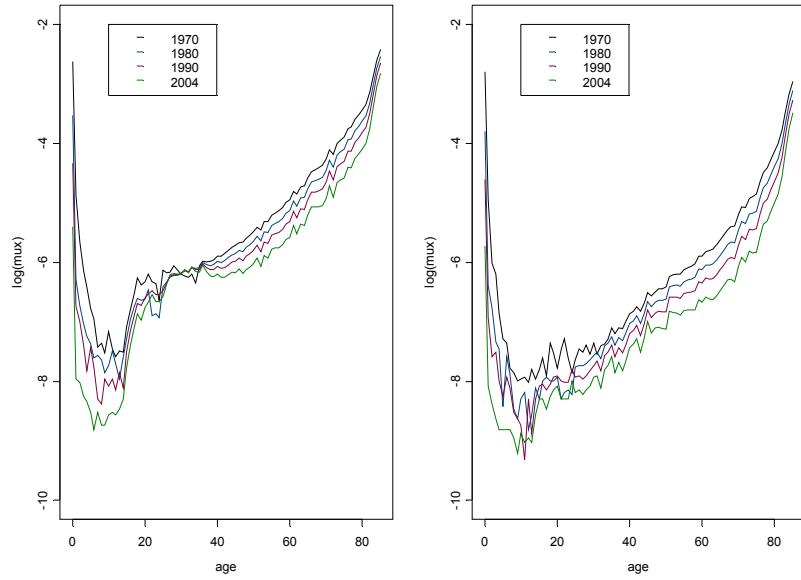


Figure 9: Evolution of the instantaneous force of mortality for some representative generations for men (left panel) and women (right panel)

We note that the main mortality features identified in the previous section within the transversal approach (decreasing mortality trends, mortality hump,...) are again easily recognized within the cohort approach. It should be mentioned,

however, that the evolution of mortality for successive generations seems to be more reliable and plausible when compared with that provided by the classic static approach.

In figure 10 we compare mortality rates obtained in both a transversal and diagonal approach for selected cohorts (and calendar years). We can observe that, in decreasing mortality environment, the predicted values within a diagonal approach are, as expected, lower than those estimated via a transversal approach. Note also that the differences in the projected values increase with the age of the individual and with the generation's year of birth. The only exception refers, once again, to the mortality hump phenomena, for which we project a stagnation (and even a slight increase) in mortality rates.

### 3.5.5 Life expectancy

In this section we analyse the evolution of life expectancy  $e_x(t)$  in terms of calendar year  $t = 1970, \dots, 2004$  for some representative ages  $x = 0$  and  $x = 65$ . In Section ?? we showed that within the transversal approach  $e_x(t)$  is calculated on the basis of mortality rates observed (or estimated) in year  $t$  (i.e., using probabilities  $q_{x+k}(t)$ ,  $k = 0, 1, 2, \dots$ ). For the contrary, within the diagonal approach  $e_x(t)$  represents the “true” remaining lifetime for individuals aged  $x$  in year  $t$ , and is calculated on the basis of mortality rates projected for that generation (i.e., using probabilities  $q_{x+k}(t+k)$ ,  $k = 0, 1, 2, \dots$ ). Table 2 summarizes the results obtained for the life expectancy calculated at birth and at age 65 for two selected calendar years. Column  $\bar{\Delta}_y$  indicates the average annual gain (measured in days) in the life expectancy registered between 1970 and 2004.

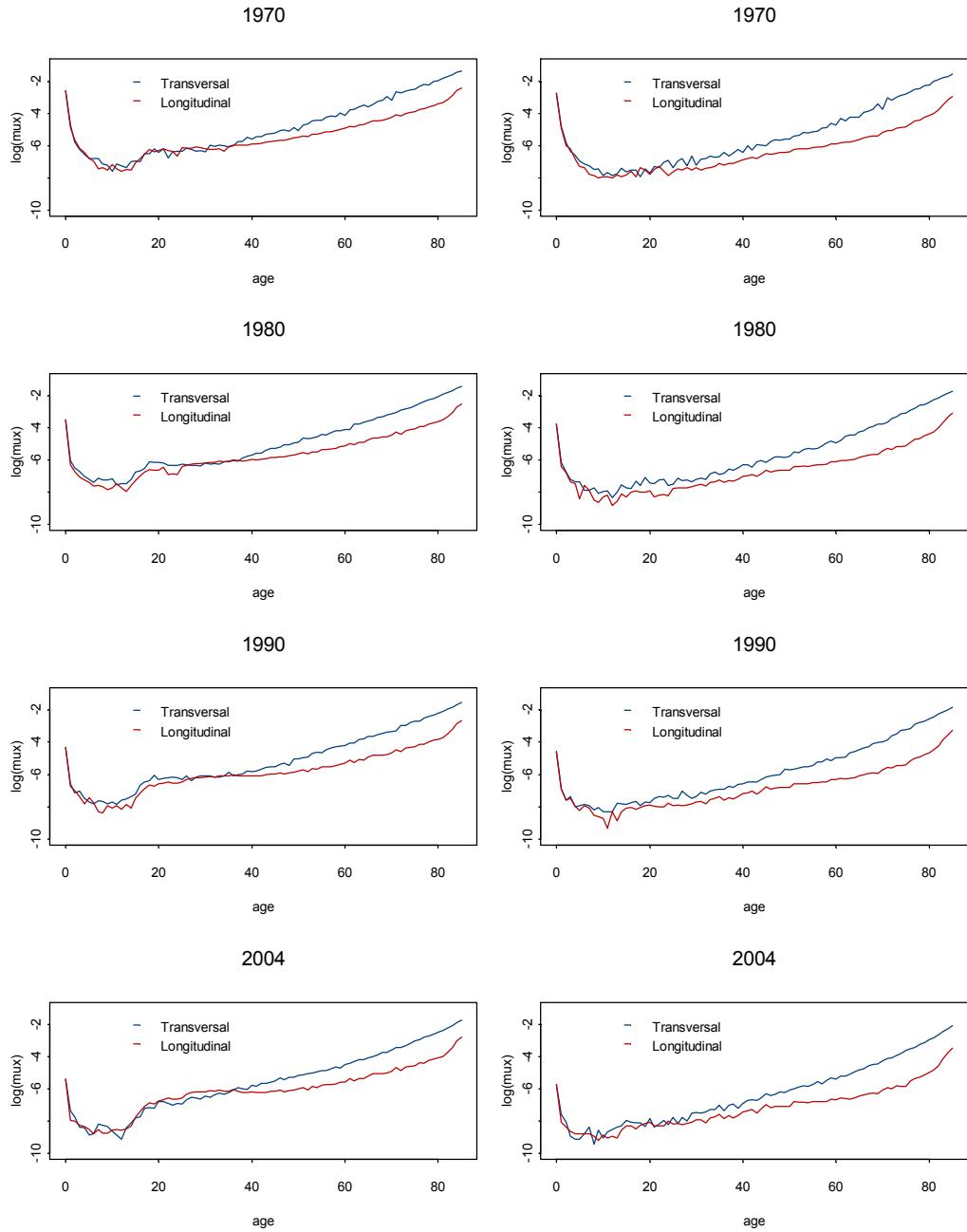


Figure 10: Transversal vs cohort approach, for selected calendar years, for men (left panel) and women (right panel)

Men	$e_0(t)$				$e_{65}(t)$			
$t$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$
1970	71.99		63.21		13.16		12.21	
2004	83.30	117.9	74.55	118.3	16.91	39.2	15.84	38.0
<hr/>								
Women	$e_0(t)$				$e_{65}(t)$			
$t$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$	Long	$\bar{\Delta}_y$	Trans	$\bar{\Delta}_y$
1970	80.37		69.32		16.24		14.53	
2004	90.63	107.0	81.05	122.4	20.87	48.3	19.22	49.0

Table 2: Evolution of life expectancy at birth and at age 65 calculated according to both a transversal and diagonal approach

The first noticeable aspect refers to the spectacular life expectancy gains observed during this period. In effect, when we can consider the transversal approach we observe that over this period life expectancy at birth increased, on an annual average, by approximately four months for both sexes (more precisely 118.3 and 122.4 days for men and women, respectively). These gains are slightly more moderate when considering the diagonal approach, particularly for the female population, with average annual gains amounting to 117.9 and 107.0 days for men and women, respectively. Similar conclusions may be stated when we examine the evolution of life expectancy at the age of 65.

The second main conclusion has to do with the significant difference between life expectancies estimated using the two approaches. In effect, when we use prospective lifetables we estimate that the “true” life expectancy at birth for an individual born in 2004 will be of 83.30 and 90.63 years for men and women, respectively, whereas the corresponding values estimated using the classic transversal approach are 74.55 and 81.05 years. In other words, when we project past trends observed in mortality to the future we conclude that adopting a transversal approach underestimates life expectancy at birth in 8.75 and 9.58 years for men and women, respectively. This apparently surprising conclusion highlights the importance of using prospective lifetables in life insurance and pension businesses. Actually, long-term calculations based on periodic lifetables are erroneous since they do not incorporate expected longevity improvements.

In Figure 11 we can see that the differentials between the values of  $e_0(t)$  and  $e_{65}(t)$  calculated according to the two methodologies considered are, for both sexes, relatively stable across the time period analysed.

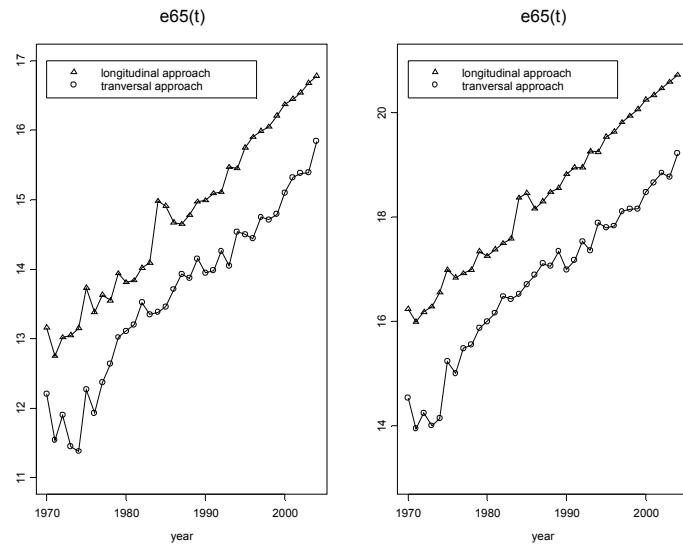


Figure 11: Life expectancy  $e_x(t)$  calculated at  $x = 0, 65$  for men (left panel) and women (right panel)

Finally, Figure 12 gives us a long term perspective of the evolution of  $e_0(t)$  and  $e_{65}(t)$  across the time period analysed.

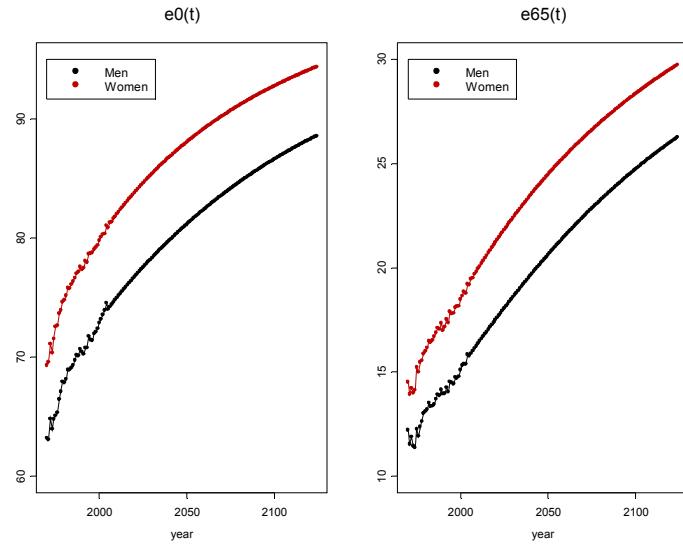


Figure 12: Projected life expectancy at birth and at age 65, calculated according a transversal approach

Our model estimates that life expectancy will continue to increase in the future in both sexes, although we expect longevity improvements to slow down.

### 3.5.6 Annuity prices

In this section we are interested in the evolution of the net single premium of an immediate life annuity sold to an  $x$ -aged individual in year  $t$ ,  $a_x(t)$  considered both a transversal and a diagonal approach. For simplicity of exposition, we assume a flat technical interest rate at 3%, i.e.  $i = 3\%$ . This means that we concentrate our analysis on the impact of longevity improvements on annuity prices. Given this, we examine the evolution of  $a_x(t)$  for  $x \in [0; 65]$  years.

Men				
$t$	$a_0(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_0(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	26.84		25.90	
2004	29.82	0.085	29.02	0.089

Women				
$t$	$a_0(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_0(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	28.18		27.06	
2004	30.72	0.073	29.89	0.081

Men				
$t$	$a_{65}(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_{65}(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	9.88		9.29	
2004	12.22	0.0668	11.64	0.0671

Women				
$t$	$a_{65}(t)$ Longitudinal	$\bar{\Delta}_y$ (annual)	$a_{65}(t)$ Transversal	$\bar{\Delta}_y$ (annual)
1970	11.86		10.85	
2004	14.56	0.077	13.70	0.081

Table 3: Evolution of  $a_x(t)$  for  $x = 0$  and  $x = 65$

In Table 3 we can appreciate the underestimation of annuity prices resulting from classic transversal lifetables. For example, the net single premium of an immediate life annuity sold to a female individual aged 65 in year 2004,  $a_{65}(2004)$ ,

will be 0.86\$ higher ( $14.56 - 13.70$ ) or 6.3% when compared with that calculated using classic static lifetables. Values in column  $\bar{\Delta}_y$ (annual) indicate the average annual gains in  $a_x(t)$  registered between 1970 and 2004.

## 4 Affine-Jump diffusion processes for mortality

Models following the approach of Lee and Carter typically adapt discrete-time time series models to capture the random element in the stochastic development of mortality rates. Given the unknown nature of future mortality, some authors have recently developed models in a continuous-time framework by modeling mortality intensity as a stochastic process (see, e.g., Milevsky and Promislow (2001), Dahl (2004), Biffis and Millossovich (2004, 2006), Biffis (2005), Dahl and Møller (2005), Miltersen and Persson (2005), Cairns *et al.* (2006a), Schrager (2006), Bravo (2007) and references therein).

Modelling the mortality intensity as a stochastic process allows us to capture two of its more significant features: time dependency and uncertainty of the future development. Additionally, this framework provides a more accurate description of both premiums and liabilities of life insurance companies and contributes to a proper quantification of systematic mortality risk faced by them. This framework and model application provides the theoretical foundation for financial pricing of longevity dependent financial claims and for the development of longevity risk hedging tools, namely mortality-linked contracts such as longevity bonds or other longevity-linked derivatives.

In this section we draw a parallel between insurance contracts and certain credit-sensitive securities and exploit some results of the intensity-based approach to credit risk modelling. Specifically, we use doubly stochastic processes (also known as *Cox processes*) in order to model the random evolution of the stochastic force of mortality of an individual aged  $x$  in a manner that is common in the credit risk literature. The model is then embedded into the well known affine-jump term structure framework, widely used in the term structure literature, in order to derive closed-form solutions for the survival probability, an key element when pricing life insurance contracts.

### 4.1 Mathematical framework

We are given a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  and concentrate on an individual aged  $x$  at time 0. Following the pioneering work of Artzner and Delbaen (1995) in the credit risk literature and the proposals by Dahl (2004) and Biffis

(2005) among others in the mortality area, we model his/her random lifetime as an  $\mathbb{F}$ -stopping time  $\tau_x$  admitting a random intensity  $\mu_x$ . Specifically, we consider  $\tau_x$  as the first jump-time of a nonexplosive  $\mathbb{F}$ -counting process  $N$  recording at each time  $t \geq 0$  whether the individual has died ( $N_t \neq 0$ ) or survived ( $N_t = 0$ ). The stopping time  $\tau_x$  is said to admit an intensity  $\mu_x$  if the compensator of  $N$  does, i.e., if  $\mu_x$  is a nonnegative predictable process such that  $\int_0^t \mu_x(s)ds < \infty$  for all  $t \geq 0$  and such that the compensated process  $M_t = \left\{ N_t - \int_0^t \mu_x(s)ds : t \geq 0 \right\}$  is a local  $\mathbb{F}$ -martingale. If the stronger condition  $\mathbb{E} \left( \int_0^t \mu_x(s)ds \right) < \infty$  is satisfied, then  $M_t$  is an  $\mathbb{F}$ -martingale.

From this, we derive

$$\mathbb{E} (N_{t+\Delta t} - N_t | \mathcal{F}_t) = \mathbb{E} \left( \int_t^{t+\Delta t} \mu_x(s)ds \middle| \mathcal{F}_t \right), \quad (34)$$

based on which we can write

$$E (N_{t+\Delta t} - N_t | \mathcal{F}_t) = \mu_x(t) \Delta t + o(\Delta t), \quad (35)$$

an expression comparable with that of the instantaneous probability of death  $\Delta t q_{x+t}$  derived in the traditional deterministic context.

By further assuming that  $N$  is a Cox (or doubly stochastic) process driven by a subfiltration  $\mathbb{G}$  of  $\mathbb{F}$ , with  $\mathbb{F}$ -predictable intensity  $\mu$  it can be shown, by using the law of iterated expectations, that the probability of an individual aged  $x + t$  at time  $t$  surviving up to time  $T \geq t$ , on the set  $\{\tau > t\}$ , is given by

$$\mathbb{P} (\tau > T | \mathcal{F}_t) = \mathbb{E} \left[ e^{- \int_t^T \mu_{x+s}(s)ds} \middle| \mathcal{F}_t \right]. \quad (36)$$

Readers who are familiar with mathematical finance and, in particular, with the interest rate literature, can without difficulty observe that the right-hand-side of equation (36) represents the price at time  $t$  of a unitary default-free zero coupon bond with maturity at time  $T > t$ , if the intensity  $\mu$  is to represent the short-term interest rate.

One of the main advantages of this mathematical framework is that we can approach the survival probability (36) by using well known affine-jump diffusion processes. In particular, an  $\mathbb{R}^n$ -valued affine-jump diffusion process  $X$  is an  $\mathbb{F}$ -Markov process whose dynamics is given by

$$dX_t = \delta(t, X_t)dt + \sigma(t, X_t)dW_t + \sum_{h=1}^m dJ_t^h, \quad (37)$$

where  $W$  is a  $\mathbb{F}$ -standard Brownian motion in  $\mathbb{R}^n$  and each component  $J^h$  is a pure-jump process in  $\mathbb{R}^n$  with jump-arrival intensity  $\{\eta^h(t, X_t) : t \geq 0\}$  and time-dependent jump distribution  $\nu_t^h$  on  $\mathbb{R}^n$ . An important requirement of affine processes is that the drift  $\delta : D \rightarrow \mathbb{R}^n$ , the instantaneous covariance matrix  $\sigma\sigma^T : D \rightarrow \mathbb{R}^{n \times n}$  and the jump-arrival intensity  $\eta^h : D \rightarrow \mathbb{R}_+$  must all have an affine dependency on  $X$ . The jump-size distribution is determined by its Laplace transform.

The convenience of adopting affine processes in modelling the mortality intensity comes from the fact that, for any  $a \in \mathbb{C}^n$ , for given  $T \geq t$  and an affine function  $R$  defined by  $R(t, X) = \rho_0(t) + \rho_1(t) \cdot X$ , under certain technical conditions we have

$$\phi^X(a, X_t, t, T) \doteq \mathbb{E} \left[ e^{-\int_t^T R(s, X_s) ds} e^{a \cdot X_T} \middle| \mathcal{F}_t \right] = e^{\alpha(t) + \beta(t) \cdot X_t}, \quad (38)$$

where  $\alpha(\cdot) \doteq \alpha(\cdot; a, T)$ ,  $\beta(\cdot) \doteq \beta(\cdot; a, T)$  satisfy generalized Riccati ordinary differential equations, that can be solved at least numerically and, in some cases, as we will see below, analytically.

#### 4.2 Mortality intensity as a stochastic process

To be useful for pricing purposes, the approach described above must specify an appropriate model for mortality dynamics. In Bravo (2007) the author tested a number of alternative specifications, considering mean-reverting and non-mean reverting stochastic processes, including or not jump components. Empirical results showed that one of best solutions is given by the classic Feller equation with jumps, an approach that we replicate in this paper. Formally, we assume that the mortality intensity  $\mu_{x+t}(t)$  solves the following stochastic differential equation

$$\begin{aligned} d\mu_{x+t}(t) &= a\mu_{x+t}(t)dt + \sigma\sqrt{\mu_{x+t}(t)}dW(t) + dJ(t) \\ \mu_{x+t}(0) &= \bar{\mu}_x, \end{aligned} \quad (39)$$

with

$$J(t) = \sum_{i=1}^{N_t} \varepsilon_i. \quad (40)$$

where  $\bar{\mu}_x > 0$ ,  $a > 0$ ,  $\sigma \geq 0$  and  $W(t)$  is a standard Brownian motion.

We assume that  $J(t)$  is a compound Poisson process, independent of  $W$ , with constant jump-arrival intensity  $\eta \geq 0$ , where  $\{\varepsilon_i : i = 1, \dots, \infty\}$  are i.i.d. variables. Following the results by Kou (2002), among others, we consider jump sizes that are random variables double asymmetric exponentially distributed with

density

$$f(z) = \pi_1 \left( \frac{1}{v_1} \right) e^{-\frac{z}{v_1}} \mathbb{I}_{\{z \geq 0\}} + \pi_2 \left( \frac{1}{v_2} \right) e^{\frac{z}{v_2}} \mathbb{I}_{\{z < 0\}} \quad (41)$$

where  $\pi_1, \pi_2 \geq 0$ ,  $\pi_1 + \pi_2 = 1$ , represent, respectively, the probabilities of a positive (with average size  $v_1 > 0$ ) and negative (with average size  $v_2 > 0$ ) jump. By setting  $\pi_1 = 0$  we are interested only on the importance of longevity risk (see, e.g., Biffis, 2005). By setting  $\eta = 0$  the model becomes deterministic. When  $v_1 = v_2$  and  $\pi_1 = \pi_2 = \frac{1}{2}$  we get the so-called “first Laplace law”. By adopting equation (41) we consider the significance of both positive mortality shocks (e.g., new medical breakthroughs) and negative mortality shifts (e.g., bird flu).

In the spirit of (38), let us now assume that the survival probability  $_{T-t}p_{x+t}(t)$  is represented by an exponentially affine function. By applying the framework described above, we have that

$$_{T-t}p_{x+t}(t) \equiv e^{\mathcal{A}(\tau) + \mathcal{B}(\tau) \cdot \mu_{x+t}(t)} \quad (42)$$

where  $\tau = T - t$ .

It can be shown that the solution to this problem admits the following Feynman-Kac representation

$$v(t, \mu_{x+t}(t)) \left\{ -\dot{\mathcal{A}}(\tau) - \dot{\mathcal{B}}(\tau) \mu_x(t) + a \mu_{x+t}(t) \mathcal{B}(\tau) + \frac{\sigma^2}{2} \mu_{x+t}(t) \mathcal{B}^2(\tau) + \eta \left( \frac{\pi_1}{1 - v_1 \mathcal{B}(\tau)} + \frac{\pi_2}{1 + v_2 \mathcal{B}(\tau)} - 1 \right) - \mu_{x+t}(t) \right\} = 0, \quad (43)$$

where  $v(t, \mu_{x+t}(t)) = _{T-t}p_{x+t}(t)$ .

Dividing both sides of this equation by  $v(t, \mu_{x+t}(t))$  we get

$$\left[ -\dot{\mathcal{B}}(\tau) + a \mathcal{B}(\tau) + \frac{\sigma^2}{2} \mathcal{B}^2(\tau) - 1 \right] \mu_{x+t}(t) + \left[ -\dot{\mathcal{A}}(\tau) + \eta \left( \frac{\pi_1}{1 - v_1 \mathcal{B}(\tau)} + \frac{\pi_2}{1 + v_2 \mathcal{B}(\tau)} - 1 \right) \right] = 0, \quad (44)$$

where  $A(\tau)$  and  $B(\tau)$  are solutions to the following system of ODEs'

$$\dot{\mathcal{B}}(\tau) = a \mathcal{B}(\tau) + \frac{1}{2} \sigma^2 \mathcal{B}^2(\tau) - 1 \quad (45)$$

$$\dot{\mathcal{A}}(\tau) = \eta \left( \frac{\pi_1}{1 - v_1 \mathcal{B}(\tau)} + \frac{\pi_2}{1 + v_2 \mathcal{B}(\tau)} - 1 \right) \quad (46)$$

with boundary conditions

$$\mathcal{B}(0) = 0, \quad \mathcal{A}(0) = 0. \quad (47)$$

where  $\dot{\mathcal{B}}(\tau) = \frac{\partial}{\partial \tau} \mathcal{B}(\tau)$ ,  $\dot{\mathcal{A}}(\tau) = \frac{\partial}{\partial \tau} \mathcal{A}(\tau)$ .

By solving the system (45)-(46)-(47), we get the following closed-form solutions for  $\mathcal{A}(\tau)$  and  $\mathcal{B}(\tau)$

$$\begin{aligned} \mathcal{A}(\tau) &= \eta\pi_1 \left\{ \frac{\alpha_0\tau}{(\alpha_0 - v_1)} + \frac{v_1(\alpha_0 + \alpha_1)[\ln(\alpha_0 + \alpha_1) - \ln(\alpha_0 - v_1 + (\alpha_1 + v_1)e^{\kappa\tau})]}{\kappa(\alpha_0 - v_1)(\alpha_1 + v_1)} \right\} \\ &\quad + \eta\pi_2 \left\{ \frac{\alpha_0\tau}{(\alpha_0 + v_2)} + \frac{v_2(\alpha_0 + \alpha_1)}{\kappa(\alpha_1 - v_2)(\alpha_0 + v_2)} [-\ln(\alpha_0 + \alpha_1) \right. \\ &\quad \left. + \ln(\alpha_0 + v_2 + (\alpha_1 - v_2)e^{\kappa\tau})] \} - \eta\tau \end{aligned} \quad (48)$$

$$\mathcal{B}(\tau) = \frac{1 - e^{\kappa\tau}}{\alpha_0 + \alpha_1 e^{\kappa\tau}} \quad (49)$$

with  $\kappa = \sqrt{a^2 + 2\sigma^2}$ ,  $\alpha_0 = \frac{(a+\kappa)}{2}$  and  $\alpha_1 = \frac{(\kappa-a)}{2}$ , defined for

$$-\frac{1}{v_2} < \mathcal{B}(\tau) < \frac{1}{v_1}. \quad (50)$$

We observe that the model stipulates an increasing (deterministic) trend for the mortality intensity, around which random fluctuations occur due to the stochastic component and due to the jump component. Additionally, the model offers a realistic process for the stochastic mortality rate since it ensures that the variable cannot take negative values. The model assumes that both negative and positive jumps can be registered in mortality, a feature that contrasts with similar models that are interested in sudden improvements in mortality (e.g., due to medical advances) only. We think this gives a more appropriate description of mortality, in which unexpected increases in mortality can occur (e.g., caused by natural catastrophes or epidemics). The model offers a nice analytical solution, easy to use in pricing and reserving applications within the life insurance industry.

### 4.3 Calibration to the Portuguese projected lifetables

We have calibrated model (39) to the Portuguese projected lifetables derived in Section 3. In fitting the model, we have adopted the ordinary least squares method, i.e., we minimize the quadratic deviations between the model survival probabilities,  ${}_{T-t}p_{65}^{\text{model}}(t)$ , and the prospective lifetable ones,  ${}_{T-t}p_{65}^{\text{TP}}(t)$  for an individual aged 65. Formally, parameter estimates  $\Theta$  solve the following optimization problem

$$\hat{\Theta} = \arg \min_{\Theta} \left\{ Q^2 = \sum_{T=t+1}^{t+(x_{\max}-65)+1} \left( {}_{T-t}p_{65}^{\text{model}}(t) - {}_{T-t}p_{65}^{\text{TP}}(t) \right)^2 \right\} \quad (51)$$

where  $x_{\max} = 120$  and  $t \in \{1970, 1980, 1990, 2004\}$ .

Table 4 reports the optimal values of the parameters, the calibration error and the initial value of  $\mu_{x+t}(t)$ ,  $\mu_{65}(t)$ , chosen to be equal to  $-\ln(p_{65}(t))$ , for both male and female populations. Figure 13 report, for the generations aged 65 in  $t \in [1970, 2004]$ , the survival function of the stochastic process analysed and of the prospective lifetable one.

Male				
	$t = 1970$	$t = 1980$	$t = 1990$	$t = 2004$
$\mu_{65}(t)$	0.02765901	0.02774125	0.02558451	0.01689187
$a$	0.09516212	0.09033169	0.08739382	0.09949474
$\sigma$	0.00001013	0.00001131	0.00000981	0.00000978
$\eta$	0.0117887	0.03936915	0.06544481	0.05226689
$v_1$	0.02654017	0.02876439	0.02726195	0.02757463
$v_2$	0.001128449	0.0001023	0.0001102	0.00009724921
$Q^2$	0.000483312	0.001135141	0.00423265	0.007431117
Female				
	$t = 1970$	$t = 1980$	$t = 1990$	$t = 2004$
$\mu_{65}(t)$	0.01472793	0.01375416	0.01163745	0.007780187
$a$	0.1119171	0.1096041	0.1101916	0.1199389
$\sigma$	0.00001044	0.00001033	0.00001082	0.00001049
$\eta$	0.01180289	0.03190069	0.05536174	0.05693019
$v_1$	0.0284391	0.02890383	0.02727089	0.02644525
$v_2$	0.0001189	0.0001098	0.0001072	0.0001066
$Q^2$	0.0003536312	0.0007131984	0.003482145	0.006155311

Table 4: Parameter estimates

The calibration error is quite small and the parameter estimates show that the value of  $\sigma$  is very low, particularly when compared with that of both positive and negative average size jumps. This is to some extent explained by the fact that the model is fitted to data that is partially smoothed by the closing procedure. We can also observe that the fit is very good, even when we consider the importance of the rectangularization phenomena, highly significant in the 2004 generation. The results also suggest that jumps seem to be an appropriate way to describe the random variations observed in mortality.

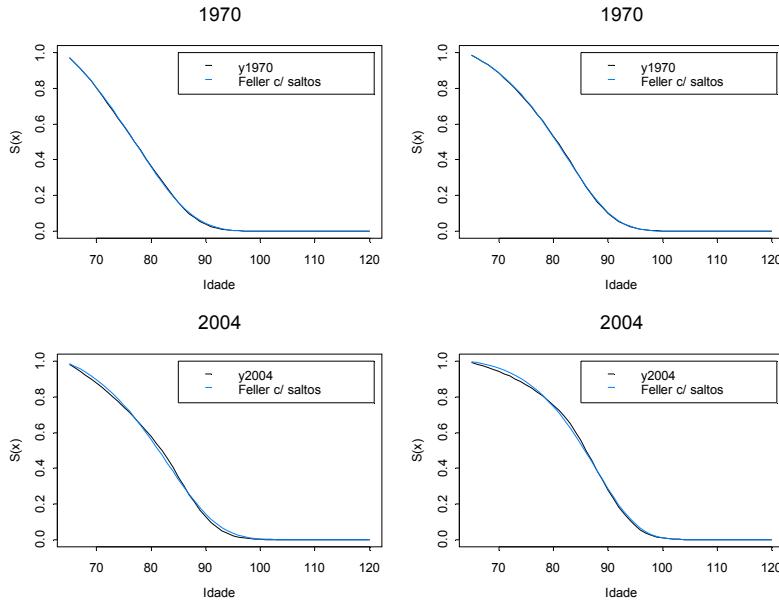


Figure 13: Survival probability  $T-t p_{65}(t)$  as a function of age  $x+T-t$  for  $t = 1970$  and  $t = 2004$  (the left panel corresponds to the male population)

## 5 Participating life annuity with longevity risk sharing mechanism

### 5.1 Structure of the contract

At least for developed markets, GSA doesn't seem to be a practical approach in trying to solve the problems posed to insurance companies by risk-averse individuals (who value annuities highly), since one of the main motivations to acquire an annuity is not fulfilled, because an annuitant can outlive his resources or, in older ages, to be receiving a very small amount, when compared with the face values of the initial annuities.

At the same time, companies are still very reluctant to keep doing business as usual in what concerns annuities, the main reason being the industry's perception that systematic risk, in the form of breakthrough life-prolonging technical innovation, may bankrupt an insurance company with a large life-annuity portfolio. In order to surpass this problem, companies usually use "very high" loadings, especially for small portfolios, where deviations in a few lives from the expected values

have a large impact in the final result of the business. All these factors produce a low voluntary demand for annuities, despite the fact that people, mainly at old age are risk averse and would be willing to pay the “true” price and share part of a “comprehensible” risk.

From the discussion in Section 2, the authors will now try to show that a partition of the risk is possible, with the advantage of having a - what we believe - marketable life-insurance product. Of course, we will use as the classical life annuity to compare results. In brief, a classical annuity does not provide the possibility for the provider to adjust benefits in any fashion since those will be fixed at the inception of the contract. Hence, the possible loss or profit will be dependent on actual mortality experience within the portfolio under investigation.

At this point we introduce a very simple model, that, we believe can be a starting point for the insurance and reinsurance companies to start taking a different approach to the annuities business. We consider that the risk bearers are in the presence of the adverse form of systematic risk, whenever the number of annuitants is above a defined threshold. In other words, assume that the limits of the confidence interval correspond to the value above which systematic deviations from projected survival rates will be observed, i.e., longevity risk is observed.

Whenever this happens, that is, the observed number of annuitants is larger than the defined boundary, we apply the same principle proposed in the GSA, but with a difference. In our model, we propose to reduce the annuity benefit payment, proportionally between the annuitants, but assume that benefits will be reduced only by an amount proportional to systematic risk, i.e., by an amount proportional to the difference between the observed number of annuitants and that of the pre-determined threshold.

When there is no violation of the threshold trigger, the future value of annuity benefits will remain constant, that is, we assume the reduction is not permanent, happening only when the number of observed annuitants systematically exceeded the expected one.

Using a simulation procedure, we show that even when we consider that the thresholds can be exceeded with a law that has a heavier tail than the (natural) binomial distribution, the price for the incorporation of this safeguard is considered acceptable, the underlying risk can be easily be explained and understood by the annuitants and, at the same time, by bringing together insurance and reinsurance companies we believe that the business is feasible.

For instance, let us suppose that the company started an annuity contract, with a single cohort of annuitants, all age  $x$ . At the inception of the contract,

the company will state what are the thresholds to be observed during the lifetime of the this contract, that is, for each instant (presumably the end of the year)  $t = 1, \dots, n$ . Let us now consider that, the the threshold is surpassed at time  $t = n$ , meaning that the number of annuitants alive  $l_{x+n}^o$  is greater than pre-determined threshold for that age,  $l_{x+n}^{(\alpha)}$ .

In this case, the underlying fund would only pay up to the limit defined by the threshold, so that annuity payments are reduced proportionally between all the survivors, meaning that everybody would suffer a reduction of the annuities payment. In this case the benefit for year  $n$  will be

$$B_{x+n} = B_0 \left( 1 - \frac{l_{x+n}^o - l_{x+n}^{(\alpha)}}{l_{x+n}^o} \right) = B_0 \left( \frac{l_{x+n}^{(\alpha)}}{l_{x+n}^o} \right). \quad (52)$$

In what follows, we develop only formulae for risk premiums, so that, all the usual loads should be applied (contingency, expenses, profit,...). For simplicity of exposition, we consider a single cohort that, namely a cohort aged  $x = 65$  at the inception of the contract. In this case the initial fund (risk premium) should be obtained by the following expression:

$$F_0 = B_0 l_x \ddot{a}_x - \mathbb{E} \left( B_0 \sum_{t=1}^{+\infty} v^t \left( L(x+t) - l_{x+n}^{(\alpha)} \right)^+ \right) \quad (53)$$

where  $L(x+t)$  is the number of annuitants alive, and  $(x-a)^+$  is a function that is equal to  $(x-a)$ , if  $x \geq a$ , and is equal to 0 if  $x < a$ .

We will simulate the evolution of this fund in the case where the distribution of  $L(x+t)$  is negative binomial, allowing thus for greater variance when compared to the traditional Binomial model, so producing a higher value for  $\mathbb{E} \left( \sum_{t=1}^{+\infty} \left( L(x+t) - l_{x+n}^{(\alpha)} \right)^+ \right)$  in the above expression.

## 5.2 Simulations

Since we are trying to show that the model is robust and adequate to analyze situations that, in time, start to deviate from the expected value, we start to notice that, our risk premium (7) is determined by the difference of 2 factors. The first is the risk premium as determined by the classical formulae, and the second factor, will be evaluated using a distribution with a heavier tail than what is used in the classical model and that was used to valuate the first term.

In this way, it is expected that a company that uses the formula (53), to determine the premium applied to the contract defined above will end up with a

loss, since the possibility of the heavy tail, in this case the negative binomial, was not incorporated in the determination of the first factor. We use this approach since, in this way, we can valuate the impact of a deviation caused by the increase of the longevity, that, in a systematic way, increases the cost of the contract and that was not predictable at the inception of the contract.

In this simulation we use the prospective lifetables derived in Section 3. Although this table considers already a relative high life expectancy in that it projects future mortality rates and “allows” people to live up to 120 years, we will see that it’s not enough to support the costs associated with deviations of observed survivors from their expected value, originated by a distribution with a heavier tail than the one used to valuate the annuity.

We will model the situation of a single cohort aged  $x = 65$  in 2004 and size  $l_{65}$ . We consider an initial benefit  $B_0 = 1$ . To evaluate  $\ddot{a}_x$  we will use a determinist interest rate of 3.5% per annum. As explained before, in order to incorporate larger deviations from the expected values and from the thresholds to be defined, we will consider that  $L(x + t)$  in formula (53) does not follows a binomial distribution with parameters  $l_{65}$ , the cohort size at the contract inception and  $tp_x$ . Instead, we consider a negative binomial with the same set of parameters, allowing in this way for larger deviations from the expected value and from the thresholds defined. We will define the thresholds equal to the 95%-quantiles for the binomial with parameters  $l_{65}$  and  $tp_x$ . Tables 5 and 6 exhibit the simulation results considering cohorts of different size.

Initial age	cohort size	$Premium_1$	$Premium_2$	$Premium$
65	1.000	12.155, 76	451, 93 (3, 72%)	11.703, 83

Simulations	Result	Mean Benefit	Load	
100	-1.534, 48	0, 9973538	3, 71%	
200	-1.528, 85	0, 9972818	3, 70%	

Table 5: Simulated premiums (cohort size 1.000)

In each table,  $Premium_1$  and  $Premium_2$  (the value in parenthesis corresponds to  $Premium_2$  in percentage of  $Premium_1$ ) denote, respectively, the first and second terms in expression (53). The column “Result” refers to the mean result of the contracts in the simulations conducted, “Mean Benefit” represents the mean benefit paid, and “Load” denotes the % of the premium necessary to have a null result for the contract.

Initial age	cohort size	$Premium_1$	$Premium_2$	$Premium$
65	10.000	121.557,6	4.776,47 (3,93%)	116.781,1

Simulations	Result	Mean Benefit	Load	
100	-17.989,98	0,9983938	3,98%	
200	-17.739,60	0,99900191	3,92%	

Table 6: Simulated premiums (cohort size 10.000)

We can understand from the simulations that, by establishing a threshold that is high enough, as the number of simulations increase,  $Premium_2$  converges to the value that would be necessary to add to the premium in order to obtain a risk premium according to the principle of equivalence. In other words, we show in exchange for a relatively small extra premium it is possible to have a contract that protects both annuitants and the insurance companies.

Annuitants will gain since in this way companies will be more willing to accept this type of risk, and this can be done by accepting an expected small penalty in the presence of the adverse form of systematic risk. The model seems to be robust, since, even in the case of a small cohort (1000 annuitants) and a small number of simulations (100), the value of Load converges to premium2 although a lot of work would still be necessary to fully confirm the simulations conducted.

## 6 Conclusion

Longevity risk, i.e., the risk that members of some reference population might live longer, on average, than anticipated, has recently emerged as one of the largest sources of risk faced by individuals, life insurance companies, pension funds and annuity providers. In order to measure the significance of longevity dynamics in Portugal, we derive in this paper the first prospective lifetables for the Portuguese general population. Contrary to classic static lifetables, the use of projected lifetables allows us to estimate the “true” (diagonal) expected residual lifetime of an individual. Comparing the results with that derived from classical static lifetables, we gave an indication of the longevity risk currently faced by insurance companies.

Using an innovative approach to mortality modelling, we argue that a better assessment of longevity risk would be one that consists of both a mean estimate and a measure of uncertainty. In this sense, we use affine-jump stochastic differential equations in order to derive closed-form solutions for the survival probability.

The development of generally agreed market models for longevity risk measurement is seen as one of the key conditions for the development of longevity-linked products and markets and for the hedging of longevity risk, a crucial element when developing annuity markets.

In this paper, we consider traditional pooled Group Self Annuitization life annuities and develop a new participating life annuity product in which the risk associated with systematic deviations from mortality rates derived using prospective life tables for the Portuguese population is shared between policyholders and life insurance companies. Contrary to traditional GSA's, in which surviving policyholders bear both systematic and unsystematic longevity risk, we propose a contract in which, in exchange for a relatively small premium, annuitants will bear only the part of longevity that exceeds pre-determined thresholds. Using a simulation procedure, we show that in exchange for an extra premium it is possible to have a contract that protects both annuitants and the insurance companies.

Future research will analyse the robustness of the simulation results derived in this paper and seek for alternative contract specifications.

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