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Higher order functional boundary value problems without monotone assumptions

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Abstract

In this paper, given $f:[a,b]\times(C([a,b]))^{n-2}\times\mathbb{R}^2\to\mathbb{R}$ a L^1-Carathéodory function, it is considered the functional higher order equation

$$u^{(n)}(x)=f(x,u,u',\ldots,u^{(n-2)}(x),u^{(n-1)}(x))$$

together with the nonlinear functional boundary conditions, for $i=0,\ldots,n-2$

$$L_i(u,u',\ldots,u^{(n-1)},u^{(i)}(a))=0,$$
$$L_{n-1}(u,u',\ldots,u^{(n-1)},u^{(n-2)}(b))=0.$$

Here, L_i, $i=0,\ldots,n-1$, are continuous functions. It will be proved an existence and location result in presence of not necessarily ordered lower and upper solutions, without assuming any monotone properties on the boundary conditions and on the nonlinearity f.

1 Introduction

In this paper, it is considered the functional higher order boundary value problem, for $n\geq 2$ composed by the equation

$$u^{(n)}(x)=f\bigl(x,u,\ldots,u^{(n-3)},u^{(n-2)}(x),u^{(n-1)}(x)\bigr) \tag{1}$$

for a.a. $x\in I:=[a,b]$, where $f:I\times(C(I))^{(n-2)}\times\mathbb{R}^2\to\mathbb{R}$ is a L^1-Carathéodory function, and the function boundary conditions

$$\begin{aligned}&L_i\bigl(u,u',\ldots,u^{(n-1)},u^{(i)}(a)\bigr)=0,\quad i=0,\ldots,n-2,\\&L_{n-1}\bigl(u,u',\ldots,u^{(n-1)},u^{(n-2)}(b)\bigr)=0,\end{aligned} \tag{2}$$

where L_i, $i=0,\ldots,n-1$, are continuous functions without assuming monotone conditions or another type of variation.

The functional differential equation (1) can be seen as a generalization of several types of full differential and integro-differential equations and allow to consider delays, maxima or minima arguments, or another kind of global variation on the unknown function or its derivatives until order $(n-3)$. On the other hand, the functional dependence in (2) makes possible its application to a huge variety of boundary conditions, such as Lidstone, separated, multipoint, nonlocal and impulsive conditions, among others. As example, we mention the problems contained in [1–15]. A detailed list about the potentialities of functional problems and some applications can be found in [16].

Recently, functional boundary value problems have been studied by several authors following several approaches, as it can be seen, for example, in [17–24]. In this work, the lower and upper solutions method is applied together with topological degree theory, according some arguments suggested in [25–27].

The novelty of this paper consists in the following items:

- There is no monotone assumptions on the boundary functions L_i, $i = 0, \dots, n-1$, by using adequate auxiliary functions and global arguments. This fact with the functional dependence on the unknown function and its derivatives till order $(n-1)$ will allow that problem (1)-(2) can include the periodic and antiperiodic cases, which were not covered by the existent literature on functional boundary value problems. In this sense, the results in this area, as for instance [28–32], are improved, even for $n = 2$, where equation (1) loses its functional part.
- No extra condition on the nonlinear part of (1) is considered, besides a Nagumo-type growth assumption. In fact, as far as we know, it is the first time where lower and upper solutions technique is used without such hypothesis on function f, by the use of stronger definitions for lower and upper solutions.
- No order between lower and upper solutions is assumed. Putting the 'well ordered' case on adequate auxiliary functions, it allows that lower and upper solutions could be well ordered, by reversed order or without a defined order.

The last section contains an example where the potentialities of the functional dependence on the equation and on the boundary conditions are explored.

2 Definitions and auxiliary functions

In this section, it will be introduced the notations and definitions needed forward together with some auxiliary functions useful to construct some ordered functions on the basis of the not necessarily ordered lower and upper solutions of the referred problem.

A Nagumo-type growth condition, assumed on the nonlinear part, will be an important tool to set an *a priori* bound for the $(n-1)$th derivative of the corresponding solutions.

In the following, $W^{m,1}(I)$ denotes the usual Sobolev Spaces in I, that is, the subset of $C^{m-1}(I)$ functions, whose $(m-1)$th derivative is absolutely continuous in I and the mth derivative belongs to $L^1(I)$ and the usual norms

$$\|u\|_p = \begin{cases} (\int_0^1 |u(x)|^p \, dx)^{1/p}, & 1 \le p < \infty, \\ \sup\{|u(x)| : x \in I\}, & p = \infty, \end{cases}$$

for spaces L^p, $1 \le p \le \infty$.

The function $f : I \times (C(I))^{(n-2)} \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, that is, $f(x, \cdot, \dots, \cdot, \cdot, \cdot)$ is a continuous function for a.e. $x \in I$; $f(\cdot, y_0, \dots, y_{n-2}, y_{n-1})$ is measurable for $(y_0, \dots, y_{n-2}, y_{n-1}) \in (C(I))^{(n-2)} \times \mathbb{R}^2$; and for every $M > 0$ there is a real-valued function $\psi_M \in L^1(I)$ such that

$$\bigl|f(x, y_0, y_1, y_2, y_3)\bigr| \le \psi_M(x), \quad \text{for a.e. } x \in [0,1]$$

and for every $(y_0, y_1, y_2, y_3) \in (C(I))^{(n-2)} \times \mathbb{R}^2$ with $|y_i| \le M$, for $i = 0, \dots, n-1$.

The main tool to obtain the location part is the upper and lower solutions method. However, in this case, they must be defined as a pair, which means that it is not possible to define them independently from each other. Moreover, it is pointed out that lower and upper functions, and the correspondent first derivatives, are not necessarily ordered.

To introduce 'some order', some auxiliary functions must be defined.

For any $\alpha, \beta \in W^{n-2,1}(I)$ define functions $\alpha_i, \beta_i : I \to \mathbb{R}$, $i = 0, \ldots, n-3$, as it follows:

$$\begin{aligned}
\alpha_{n-3}(x) &= \min\{\alpha^{(n-3)}(a), \beta^{(n-3)}(a)\} + \int_a^x \alpha^{(n-2)}(s)\,ds, \\
\beta_{n-3}(x) &= \max\{\alpha^{(n-3)}(a), \beta^{(n-3)}(a)\} + \int_a^x \beta^{(n-2)}(s)\,ds, \\
\alpha_i(x) &= \min\{\alpha^{(i)}(a), \beta^{(i)}(a)\} + \int_a^x \alpha_{i+1}(s)\,ds, \\
\beta_i(x) &= \max\{\alpha^{(i)}(a), \beta^{(i)}(a)\} + \int_a^x \beta_{i+1}(s)\,ds,
\end{aligned} \tag{3}$$

for $i = 0, \ldots, n-4$.

The Nagumo-type condition is given by next definition.

Definition 1 Consider $\Gamma_i, \gamma_i \in C(I)$, $i = 0, \ldots, n-2$, such that $\Gamma_i(x) \leq \gamma_i(x)$, $\forall x \in I$, and the set

$$E = \{(x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x), i = 0, \ldots, n-2\}.$$

A function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is said to verify a Nagumo-type condition in E if there exists $\varphi_E \in C([0, +\infty), (0, +\infty))$ such that

$$\left|f(x, y_0, \ldots, y_{n-1})\right| \leq \varphi_E(|y_{n-1}|), \tag{4}$$

for every $(x, y_0, \ldots, y_{n-1}) \in E$, and

$$\int_r^{+\infty} \frac{t}{\varphi_E(t)}\,dt > \max_{x\in I} \Gamma_{n-2}(x) - \min_{x\in I} \gamma_{n-2}(x), \tag{5}$$

where $r \geq 0$ is given by

$$r := \max\left\{\frac{\Gamma_{n-2}(b) - \gamma_{n-2}(a)}{b-a}, \frac{\Gamma_{n-2}(a) - \gamma_{n-2}(b)}{b-a}\right\}.$$

The next result gives an *a priori* estimate for the $(n-1)$th derivative of all possible solutions of (1).

Lemma 2 *There exists $K > 0$ such that for every L^1-Carathéodory function $f : I \times (C(I))^{n-2} \times \mathbb{R}^2 \to \mathbb{R}$ satisfying* (4) *and* (5) *and every solution u of* (1) *such that*

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \quad a.e.\ x \in I, \tag{6}$$

for $i = 0, \dots, n-2$, *we have*

$$\|u^{(n-1)}\|_\infty < R. \tag{7}$$

Moreover, the constant R *depends only on the functions* φ *and* γ_i, Γ_i $(i = 0, \dots, n-2)$ *and not on the boundary conditions.*

Proof The proof is similar to [19, Lemma 2.1]. □

The upper and lower solution definition is then given by the following.

Definition 3 The functions $\alpha, \beta \in W^{n,1}(I)$ are a pair of lower and upper solutions for problem (1)-(2) if $\alpha^{(n-2)}(x) \le \beta^{(n-2)}(x)$, on I, for all $(v_0, \dots, v_{n-3}) \in A := [\alpha_0, \beta_0] \times \cdots \times [\alpha_{n-3}, \beta_{n-3}]$, and for every $(w_1, w_2) \in B := [\alpha^{(n-2)}, \beta^{(n-2)}] \times [-K, K]$, for some $K > 0$, the following inequalities hold for a.e. $x \in [a, b]$,

$$\begin{aligned} &\alpha^{(n)}(x) \ge f\bigl(x, v_0, \dots, v_{n-3}, \alpha^{(n-2)}(x), \alpha^{(n-1)}(x)\bigr), \\ &\beta^{(n)}(x) \le f\bigl(x, v_0, \dots, v_{n-3}, \beta^{(n-2)}(x), \beta^{(n-1)}(x)\bigr), \end{aligned} \tag{8}$$

and for $j = 0, \dots, n-3$,

$$\begin{aligned} &L_j\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \alpha_j(a)\bigr) \ge 0, \\ &L_{n-2}\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \alpha^{(n-2)}(a)\bigr) \ge 0, \\ &L_{n-1}\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \alpha^{(n-2)}(b)\bigr) \ge 0, \\ &L_j\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \beta_j(a)\bigr) \le 0, \\ &L_{n-2}\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \beta^{(n-2)}(a)\bigr) \le 0, \\ &L_{n-1}\bigl(v_0, \dots, v_{n-3}, w_1, w_2, \beta^{(n-2)}(b)\bigr) \le 0. \end{aligned} \tag{9}$$

3 Existence and location result

In this section, it is provided an existence and location theorem for the problem (1)-(2). More precisely, sufficient conditions are given for, not only the existence of a solution u, but also to have information about the location of u, and all its derivatives up to the $(n-1)$ order.

The arguments of the proof require the following lemma, given on [29].

Lemma 4 *For* $v, w \in C(I)$ *such that* $v(x) \le w(x)$, *for every* $x \in I$, *define*

$$q(x, u) = \max\bigl\{v, \min\{u, w\}\bigr\}.$$

Then, for each $u \in C^1(I)$ *the next two properties hold*:

(a) $\frac{d}{dx} q(x, u(x))$ *exists for a.e.* $x \in I$.
(b) *If* $u, u_m \in C^1(I)$ *and* $u_m \to u$ *in* $C^1(I)$ *then*

$$\frac{d}{dx} q\bigl(x, u_m(x)\bigr) \to \frac{d}{dx} q\bigl(x, u(x)\bigr) \quad \text{for a.e. } x \in I.$$

Now, we are in a position to prove the main result of this paper.

Theorem 5 *Assume that there exists a pair of lower and upper solutions* (α,β) *of problem* (1)-(2).

If $f: I\times(C(I))^{n-2}\times\mathbb{R}^2\to\mathbb{R}$ *is a* L^1*-Carathéodory function, satisfying a Nagumo-type condition in*

$$E_* = \begin{cases}(x,y_0,\dots,y_{n-1})\in I\times\mathbb{R}^{n-1}:\alpha_i(x)\le y_i\le\beta_i(x), \quad i=0,\dots,n-3,\\ \alpha^{(n-2)}(x)\le y_{n-2}\le\beta^{(n-2)}(x),\end{cases}$$

then problem (1)-(2) *has at least one solution u such that*

$$\begin{aligned}&\alpha_i(x)\le u^{(i)}(x)\le\beta_i(x), \quad i=0,\dots,n-3,\\ &\alpha^{(n-2)}(x)\le u^{(n-2)}(x)\le\beta^{(n-2)}(x), \quad \forall x\in I\end{aligned}$$

for every $x\in I$, *and* $|u^{(n-1)}(x)|\le K$, $\forall x\in I$, *where*

$$K=\max\{R,|\alpha^{(n-1)}(x)|,|\beta^{(n-1)}(x)|\} \tag{10}$$

and $R>0$ *is given by* (7).

Proof Define the continuous functions, for $i=0,\dots,n-3$,

$$\begin{aligned}&\delta_i(x,y_i)=\max\{\alpha_i(x),\min\{y_i,\beta_i(x)\}\},\\ &\delta_{n-2}(x,y_{n-2})=\max\{\alpha^{(n-2)}(x),\min\{y_{n-2},\beta^{(n-2)}(x)\}\},\end{aligned} \tag{11}$$

and the truncation, not necessarily continuous,

$$\xi(z)=\max\{-K,\min\{z,K\}\},$$

with K given by (10).

Consider the modified problem composed by the equation

$$u^{(n)}(x)=f\begin{pmatrix}x,\delta_0(\cdot,u),\dots,\delta_{n-3}(\cdot,u^{(n-3)}),\\ \delta_{n-2}(x,u^{(n-2)}(x)),\xi(\frac{d}{dx}(\delta_{n-2}(x,u^{(n-2)}(x))))\end{pmatrix} \tag{12}$$

and the boundary conditions, for $i=0,\dots,n-2$,

$$\begin{aligned}&u^{(i)}(a)=\delta_i\left(a,u^{(i)}(a)+L_i\begin{pmatrix}\delta_0(\cdot,u),\dots,\delta_{n-2}(\cdot,u^{(n-2)}),\\ \xi(\frac{d}{dx}(\delta_{n-2}(\cdot,u^{(n-2)}))),u^{(i)}(a)\end{pmatrix}\right),\\ &u^{(n-2)}(b)=\delta_{n-2}\left(b,u^{(n-2)}(b)+L_{n-1}\begin{pmatrix}\delta_0(\cdot,u),\dots,\delta_{n-2}(\cdot,u^{(n-2)}),\\ \xi(\frac{d}{dx}(\delta_{n-2}(\cdot,u^{(n-2)}))),u^{(n-2)}(b)\end{pmatrix}\right).\end{aligned} \tag{13}$$

The proof will follow the next steps:

Step 1. Every solution u of problem (12)-(13), satisfies

$$\begin{aligned}
&\alpha^{(n-2)}(x) \le u^{(n-2)}(x) \le \beta^{(n-2)}(x),\\
&\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x), \quad i = 0, \dots, n-3,
\end{aligned}$$

and $|u^{(n-1)}(x)| < K$, for every $x \in I$, with $K > 0$ given in (10).

Let u be a solution of the modified problem (12)-(13). Assume, by contradiction, that there exists $x \in I$ such that $\alpha^{(n-2)}(x) > u^{(n-2)}(x)$ and let $x_0 \in I$ be such that

$$\min_{x\in I}(u-\alpha)^{(n-2)}(x) := (u-\alpha)^{(n-2)}(x_0) < 0.$$

As, by (13), $u^{(n-2)}(a) \ge \alpha^{(n-2)}(a)$ and $u^{(n-2)}(b) \ge \alpha^{(n-2)}(b)$, then $x_0 \in (a,b)$. So, there is $(x_1, x_2) \subset (a,b)$ such that

$$\begin{aligned}
&u^{(n-2)}(x) < \alpha^{(n-2)}(x), \quad \forall x \in (x_1, x_2),\\
&(u-\alpha)^{(n-2)}(x_1) = (u-\alpha)^{(n-2)}(x_2) = 0.
\end{aligned} \tag{14}$$

Therefore,

$$\delta_{n-2}\bigl(x, u^{(n-2)}(x)\bigr) = \alpha^{(n-2)}(x), \quad \forall x \in (x_1, x_2),$$

and

$$\frac{d}{dx}\delta_{n-2}\bigl(x, u^{(n-2)}(x)\bigr) = \alpha^{(n-1)}(x), \quad \text{a.e. } x \in (x_1, x_2).$$

Now, since for all $u \in C^{n-2}(I)$ it is satisfied that $(\delta_0(\cdot,u), \dots, \delta_{n-3}(\cdot,u')) \in A$, we deduce that

$$\begin{aligned}
u^{(n)}(x) &= f\begin{pmatrix} x, \delta_0(\cdot,u), \dots, \delta_{n-3}(\cdot,u^{(n-3)}), \delta_{n-2}(x,u^{(n-2)}(x)),\\ \xi\bigl(\frac{d}{dx}(\delta_{n-2}(x,u^{(n-2)}(x)))\bigr) \end{pmatrix}\\
&= f\bigl(x, \delta_0(\cdot,u), \dots, \delta_{n-3}\bigl(\cdot,u^{(n-3)}\bigr), \alpha^{(n-2)}(x), \alpha^{(n-1)}(x)\bigr)\\
&\le \alpha^{(n)}(x) \quad \text{for a.e. } x \in (x_1, x_2).
\end{aligned}$$

As $(u-\alpha)^{(n-1)}(x_0) = 0$ and $(u-\alpha)^{(n-1)}$ is nonincreasing in (x_1, x_2), this contradicts the definitions of x_0 and x_2.

The inequality $u^{(n-2)}(x) \le \beta^{(n-2)}(x)$, in I, can be proved in same way and so,

$$\alpha^{(n-2)}(x) \le u^{(n-2)}(x) \le \beta^{(n-2)}(x), \quad \forall x \in I. \tag{15}$$

By (13) and (3), the following inequalities hold for every $x \in I$:

$$\begin{aligned}
u^{(n-3)}(x) &= u^{(n-3)}(a) + \int_a^x u^{(n-2)}(s)\,ds \ge \alpha_{n-3}(a) + \int_a^x \alpha^{(n-2)}(s)\,ds\\
&\ge \min\bigl\{\alpha^{(n-3)}(a), \beta^{(n-3)}(a)\bigr\} + \int_a^x \alpha^{(n-2)}(s)\,ds = \alpha_{n-3}(x).
\end{aligned}$$

Analogously, it can be obtained $u^{(n-3)}(x) \le \beta_{n-3}(x)$, for $x \in I$.

The remaining inequalities are obtained by the same integration process.

Applying previous bounds in Lemma 2, and remarking that

$$\int_r^K \frac{s}{\varphi(s)}\,ds \geq \int_r^R \frac{s}{\varphi(s)}\,ds,$$

for K given by (10), it is obtained, by Lemma 2, the *a priori* bound $|u^{(n-1)}(x)| < K$, for $x \in I$. For details, see [33, Lemma 2].

Step 2. Problem (12)-(13) has at least one solution.

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$u^{(n)}(x) = \lambda f\begin{pmatrix} x, \delta_0(\cdot, u), \ldots, \delta_{n-3}(\cdot, u^{(n-3)}), \\ \delta_{n-2}(x, u^{(n-2)}(x)), \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))) \end{pmatrix} \tag{16}$$

and the boundary conditions, for $i = 0, \ldots, n-2$,

$$\begin{aligned} u^{(i)}(a) &= \lambda\delta_i\left(a, u^{(i)}(a) + L_i\begin{pmatrix} \delta_0(\cdot, u), \ldots, \delta_{n-2}(\cdot, u^{(n-2)}), \\ \xi(\frac{d}{dx}(\delta_{n-2}(\cdot, u^{(n-2)}))), u^{(i)}(a) \end{pmatrix}\right) := \lambda L_{A_i}, \\ u^{(n-2)}(b) &= \lambda\delta_{n-2}\left(b, u^{(n-2)}(b) + L_{n-1}\begin{pmatrix} \delta_0(\cdot, u), \ldots, \delta_{n-2}(\cdot, u^{(n-2)}), \\ \xi(\frac{d}{dx}(\delta_{n-2}(\cdot, u^{(n-2)}))), u^{(n-2)}(b) \end{pmatrix}\right) \\ &:= \lambda L_B. \end{aligned} \tag{17}$$

Let us consider the norms in $C^{n-1}(I)$ and in $L^1(I) \times \mathbb{R}^n$, respectively,

$$\|v\|_{C^{n-1}} = \max\bigl\{\|v\|_\infty, \ldots, \bigl\|v^{(n-1)}\bigr\|_\infty\bigr\}$$

and

$$\bigl|(h, h_1, \ldots, h_n)\bigr| = \max\bigl\{\|h\|_{L^1}, \max\bigl\{|h_1|, \ldots, |h_n|\bigr\}\bigr\}.$$

Define the operators $\mathcal{L}: W^{n,1}(I) \subset C^{n-1}(I) \to L^1(I) \times \mathbb{R}^n$ by $\mathcal{L}u = (u^{(n)}, u(a), \ldots, u^{(n-2)}(a), u^{(n-2)}(b))$ and, for $\lambda \in [0,1]$, $i = 0, \ldots, n-2$, $\mathcal{N}_\lambda : C^{n-1}(I) \to L^1(I) \times \mathbb{R}^n$ by

$$\mathcal{N}_\lambda u = \begin{pmatrix} \lambda f\begin{pmatrix} x, \delta_0(\cdot, u), \ldots, \delta_{n-3}(\cdot, u^{(n-3)}), \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))) \end{pmatrix}, \\ \lambda L_{A_1}, \ldots, \lambda L_{A_{n-2}}, \lambda L_B \end{pmatrix}.$$

Since $L_0, \ldots, L_{n-1}$ are continuous and f is a L^1-Carathéodory function, then, from Lemma 4, $\mathcal{N}_\lambda$ is continuous. Moreover, as $\mathcal{L}^{-1}$ is compact, it can be defined the completely continuous operator $\mathcal{T}_\lambda : C^{n-1}(I) \to C^{n-1}(I)$ by $\mathcal{T}_\lambda u = \mathcal{L}^{-1}\mathcal{N}_\lambda(u)$.

It is obvious that the fixed points of operator $\mathcal{T}_\lambda$ coincide with the solutions of problem (16)-(17).

As $\mathcal{N}_\lambda u$ is bounded in $L^1(I) \times \mathbb{R}^n$ and uniformly bounded in $C^{n-1}(I)$, we have that any solution of the problem (16)-(17), verifies the following *a priori* bound

$$\|u\|_{C^{n-1}} \leq \bigl\|\mathcal{L}^{-1}\bigr\|_{C^{n-1}}\bigl|\mathcal{N}_\lambda(u)\bigr| \leq \bar{K},$$

for some $\bar{K} > 0$ independent of λ.

In the set $\Omega = \{u \in C^{n-1}(I) : \|u\|_{C^{n-1}} < \bar{K} + 1\}$, the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by the invariance under homotopy, $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0)$.

As the equation $x = \mathcal{T}_0(x)$ is equivalent to the problem

$$\begin{cases} u^{(n)}(x) = 0, \\ u^{(i)}(a) = u^{(n-2)}(b) = 0, \quad i = 0, \dots, n-2, \end{cases}$$

which has only the trivial solution, then $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = \pm 1$. So, by degree theory, the equation $x = \mathcal{T}_1(x)$ has at least one solution, that is, the problem (12)-(13) has at least a solution in Ω.

Step 3. Every solution u of problem (12)-(13) is a solution of (1)-(2).

Let u be a solution of the modified problem (12)-(13). By previous steps, function u fulfills equation (1). So, it will be enough to prove the following inequalities, for $i = 0, \dots, n-3$:

$$\alpha_i(a) \le u^{(i)}(a) + L_i \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(i)}(a) \end{pmatrix} \le \beta_i(a),$$

$$\begin{aligned} \alpha^{(n-2)}(a) &\le u^{(n-2)}(a) + L_{n-1} \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(n-2)}(a) \end{pmatrix} \\ &\le \beta^{(n-2)}(a) \end{aligned}$$

and

$$\begin{aligned} \alpha^{(n-2)}(b) &\le u^{(n-2)}(b) + L_{n-1} \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(n-2)}(b) \end{pmatrix} \\ &\le \beta^{(n-2)}(b). \end{aligned}$$

Assume that

$$u(a) + L_0 \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u(a) \end{pmatrix} > \beta_0(a). \tag{18}$$

Then, by (13), $u(a) = \beta_0(a)$. By previous steps, it is obtained the following contradiction with (18):

$$\begin{aligned} & u(a) + L_0 \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u(a) \end{pmatrix} \\ & = \beta_0(a) + L_0 \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), \beta_0(a) \end{pmatrix} \\ & \le \beta_0(a). \end{aligned}$$

Applying similar arguments, it can be proved that

$$\alpha_0(a) \le u(a) + L_0 \begin{pmatrix} \delta_0(\cdot, u), \dots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u(a) \end{pmatrix}$$

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and analogously, for $j = 1, \ldots, n-3$,

$$\alpha_j(a) \le u^{(j)}(a) + L_j \begin{pmatrix} \delta_0(\cdot, u), \ldots, \delta_{n-2}(x\cdot, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(j)}(a) \end{pmatrix} \le \beta_j(a).$$

Also, using the same arguments and the same techniques, it can be proved that

$$\begin{aligned} \alpha^{(n-2)}(a) &\le u^{(n-2)}(a) + L_{n-1} \begin{pmatrix} \delta_0(\cdot, u), \ldots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(n-2)}(a) \end{pmatrix} \\ &\le \beta^{(n-2)}(a), \\ \alpha^{(n-2)}(b) &\le u^{(n-2)}(b) + L_{n-1} \begin{pmatrix} \delta_0(\cdot, u), \ldots, \delta_{n-2}(x, u^{(n-2)}(x)), \\ \xi(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)}(x)))), u^{(n-2)}(b) \end{pmatrix} \\ &\le \beta^{(n-2)}(b). \end{aligned}$$

□

4 Example

This section contains a problem composed by an integro-differential equation with some functional boundary conditions, whose solvability is proved in presence of nonordered lower and upper solutions. We remark that such fact was not possible with the results in the current literature. This example does not model any particular problem arising in real phenomena. Our purpose consists on emphasizing the powerful of the developed theory in this paper by showing what kind of problems we can deal with.

Consider, for $x \in [0,1]$, the fourth-order equation

$$u^{(iv)}(x) = \int_0^x u(s)\,ds + \max_{x\in[0,1]} \{u'(x)\} + \left(u''(x)\right)^3 - \left(u'''(x) + 1\right)^{\frac{2}{3}} \tag{19}$$

coupled with the boundary value conditions

$$\begin{aligned} &-\min_{x\in[0,1]} u''(x) - 26u(0) = 0, \\ &u(s) - \left(u(0)\right)^3 + 14 = 0, \\ &\max_{x\in[0,1]} u(x) - 2u''(0) = 0, \\ &\sqrt[3]{u''(1)} = 0. \end{aligned} \tag{20}$$

One can verify that functions

$$\alpha(x) = -\frac{x^3}{6} - 12x^2 + 20x - 1 \quad \text{and} \quad \beta(x) = \frac{x^3}{3} + 12x^2 + 1$$

are, respectively, lower and upper solutions for the problem (19)-(20). Moreover, we deduce that

$$\begin{aligned} \alpha_1(x) &= -\frac{x^2}{2} - 24x - \frac{9}{2}, \qquad \alpha_0(x) = -\frac{x^3}{6} - 12x^2 - \frac{9}{2}x - 1, \\ \beta_1(x) &= x^2 + 24x + 25, \qquad \beta_0(x) = \frac{x^3}{3} + 12x^2 + 125x + \frac{40}{3} \end{aligned}$$

and

$$\begin{aligned}
&f(x,y_0,y_1,y_2,y_3) = \int_0^x y_0(s)\,ds + \max_{x\in[0,1]}\{y_1(x)\} + \big(y_2(x)\big)^3 - \big(y_3(x)+1\big)^{\frac{2}{3}},\\
&L_0(z_1,z_2,z_3,z_4,z_5) = -\min_{x\in[0,1]} z_3 - 26z_5,\\
&L_1(z_1,z_2,z_3,z_4,z_5) = z_1 - (z_5)^3 + 14,\\
&L_2(z_1,z_2,z_3,z_4,z_5) = \max_{x\in[0,1]} z_1 - 2z_5,\\
&L_3(z_1,z_2,z_3,z_4,z_5) = -\sqrt[3]{z_5}.
\end{aligned}$$

As the continuous function f verifies (4) and (5) for $\varphi_{E_*}(y_3) = \frac{1{,}847}{12} + (y_3+1)^{\frac{2}{3}}$ in

$$E_* = \left\{\begin{array}{c}
(x,y_0,y_1,y_2,y_3)\in[0,1]\times\mathbb{R}^4:\\
-\frac{x}{6}^3 - 12x^2 - \frac{9}{2}x - 1 \le y_0 \le \frac{x}{3}^3 + 12x^2 + 125x + \frac{40}{3}\\
-\frac{x}{2}^2 - 24x - \frac{9}{2} \le y_1 \le x^2 + 24x + 25\\
-x - 24 \le y_2 \le 2x + 24
\end{array}\right\},$$

then, by Theorem 5, there is a nontrivial solution u for problem (19)-(20) such that

$$\begin{aligned}
&-\frac{x}{6}^3 - 12x^2 - \frac{9}{2}x - 1 \le u(x) \le \frac{x}{3}^3 + 12x^2 + 125x + \frac{40}{3},\\
&-\frac{x}{2}^2 - 24x - \frac{9}{2} \le u'(x) \le x^2 + 24x + 25,\\
&-x - 24 \le u''(x) \le 2x + 24,
\end{aligned}$$

for all $x\in[0,1]$.

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