DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2011

Website: www.aimSciences.org

pp. **209–218**

NON ORDERED LOWER AND UPPER SOLUTIONS TO FOURTH ORDER PROBLEMS WITH FUNCTIONAL BOUNDARY CONDITIONS

Alberto Cabada

Departamento de Análise Matemática. Facultade de Matemáticas
Universidade de Santiago de Compostela
15782 Santiago de Compostela, Galicia, Spain

João Fialho and Feliz Minhós

Centro de Investigação em Matemática e Aplicações da Universidade de Évora (CIMA-UE)
Rua Romão Ramalho, 59
7000-671 Évora. Portugal

Abstract. In this paper, given $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ a L^1-Carathéodory function, it is considered the functional fourth order equation

$$u^{(iv)}(x) = f(x, u, u', u''(x), u'''(x))$$

together with the nonlinear functional boundary conditions

$$\begin{array}{rcl} L_0(u, u', u'', u(a)) = & 0 & = L_1(u, u', u'', u'(a)) \\ L_2(u, u', u'', u''(a), u'''(a)) = & 0 & = L_3(u, u', u'', u''(b), u'''(b)). \end{array}$$

Here L_i, $i = 0, 1, 2, 3$, are continuous functions satisfying some adequate monotonicity assumptions.

It will be proved an existence and location result in presence of non ordered lower and upper solutions and without monotone assumptions on the right hand side of the equation.

1. **Introduction.** Let us consider the problem composed by the functional equation

$$u^{(iv)}(x) = f(x, u, u', u''(x), u'''(x)) \tag{1}$$

with $x \in I \equiv [a, b]$, $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ a L^1- Carathéodory function and the nonlinear functional boundary conditions

$$\begin{array}{rcl} L_0(u, u', u'', u(a)) & = & 0, \\ L_1(u, u', u'', u'(a)) & = & 0, \\ L_2(u, u', u'', u''(a), u'''(a)) & = & 0, \\ L_3(u, u', u'', u''(b), u'''(b)) & = & 0, \end{array} \tag{2}$$

where L_i, $i = 0, 1, 2, 3$, are continuous functions satisfying some monotonicity assumptions to be defined later.

2000 *Mathematics Subject Classification.* Primary: 34B15, 34K10; Secondary: 34B10, 34K45.

Key words and phrases. Higher order problems, functional boundary value problems, Nagumo condition, lower and upper solutions.

The first author is partially supported by FEDER and Ministerio de Educación y Ciencia, Spain, project MTM2010-15314. The third author is partially supported by Fundação para a Ciência e Tecnologia, SFRH/BSAB/849/2008.

This type of higher order functional boundary value problems has been recently studied in several works such that [3, 4], for third order and [2, 7], to fourth order problems. However, to the best of our knowledge, this is the first time where the functional dependence on the unknown function and its first derivative, is allowed, not only in the nonlinearity f, but also in the boundary functions L_0, L_1, L_2 and L_3. In this sense, this paper improves the results existent in the literature related to fourth order functional problems and it generalizes, also, boundary value problems with several types of differential equations, such as, delay equations, integro-differential or equations with maxima arguments, and many different boundary conditions, like Lidstone, separated, multipoint or nonlocal conditions, among others. As example, we refer the papers [1, 8, 11, 13, 14, 16, 17, 18, 19]. A detailed list about the potentialities and some applications of functional boundary value problems can be seen in [7].

The method used in this paper follows standard arguments in lower and upper solutions technique, as it was suggested, for instance, in [5, 9, 12]. It is also pointed out that this work makes use of a different technique of lower and upper solutions, which allows two features, not covered by the existent results:

- lower and upper functions can be considered with second order derivatives well ordered, but with the first derivative and/or the functions not ordered (see Definition 2.1). In this case, the main theorem (Theorem 3.2), that coincides with the classical theory if lower and upper solutions, and the corresponding derivatives, are "well ordered", gives the existence and the location of the solution u, and of u', in regions limited by a pair of functions that are obtained by translations of the initial lower and upper solutions. Therefore the set of admissible functions to be considered as lower and upper is hugely generalized.
- no monotone-type conditions are assumed on the nonlinearity f.

The last section contains, as example, a functional boundary value problem where differential equation and boundary conditions have both functional dependence, which could not be solved by the existent theory. In fact it includes an integro-differential equation and existence and location results are obtained in presence of non-ordered lower and upper solutions and the corresponding first derivatives.

2. **Definitions and auxiliary results.**

In this section it will be introduced the notations and definitions needed forward together with some auxiliary functions useful to construct some ordered functions on the basis of the not necessarily ordered lower and upper solutions of the referred problem.

A Nagumo-type growth condition, assumed on the nonlinear part, will be an important tool to set an *a priori* bound for the third derivative of the corresponding solutions.

In the following, $W^{4,1}(I)$ denotes the usual Sobolev space in I, that is, the subset of $C^3(I)$ functions, whose third derivative is absolutely continuous in I and the fourth derivative belongs to $L^1(I)$.

The nonlinear part f will be a locally L^1-bounded Carathéodory function, in the following standard sense:

$f(x, \cdot, \cdot, \cdot, \cdot)$ is continuous in $(C(I))^2 \times \mathbb{R}^2$ for a.e. $x \in I$; $f(\cdot, \eta, \xi, y_0, y_1)$ is measurable for all $(\eta, \xi, y_0, y_1) \in (C(I))^2 \times \mathbb{R}^2$; and for every $R > 0$ there exists $\psi \in L^1(I)$ and a null measure set $N \subset I$ such that $|f(x, \eta, \xi, y_0, y_1)| \leq \psi(x)$ for all $(x, \eta, \xi, y_0, y_1) \in (I \setminus N) \times (C(I))^2 \times \mathbb{R}^2$ with $\|(\eta, \xi, y_0, y_1)\|_\infty \leq R$.

The functions L_i, $i = 0, 1, 2, 3$, considered in boundary conditions, must verify the following monotonicity properties:

(H_0) $L_0,\ L_1 : (C(I))^3 \times \mathbb{R} \to \mathbb{R}$ are continuous functions, nondecreasing in first, second and third variables;

(H_1) $L_2 : (C(I))^3 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in first, second, third and fifth variables;

(H_2) $L_3 : (C(I))^3 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, nondecreasing in first, second and third variables and nonincreasing in the fifth one.

The main tool to obtain the location part is the upper and lower solutions method. However, in this case, they must be defined as a pair, which means that it is not possible to define them independently from each other. Moreover, it is pointed out that lower and upper functions, and the correspondent first derivatives, are not necessarily ordered.

To introduce "some order", it must be defined the following auxiliary functions: For any $\alpha,\ \beta \in W^{4,1}(I)$ define functions $\alpha_i,\ \beta_i : I \to \mathbb{R}$, $i = 0, 1$, as it follows:

$$\alpha_1(x) = \min\ \{\alpha'(a), \beta'(a)\} + \int_a^x \alpha''(s)\, ds, \tag{3}$$

$$\beta_1(x) = \max\ \{\alpha'(a), \beta'(a)\} + \int_a^x \beta''(s)\, ds, \tag{4}$$

$$\alpha_0(x) = \min\ \{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\, ds, \tag{5}$$

$$\beta_0(x) = \max\ \{\alpha(a), \beta(a)\} + \int_a^x \beta_1(s)\, ds. \tag{6}$$

Definition 2.1. The functions $\alpha, \beta \in W^{4,1}(I)$ are a pair of lower and upper solutions for problem (1) – (2) if $\alpha'' \leq \beta''$, on I, and the following conditions are satisfied: For all $(v, w) \in A := [\alpha_0, \beta_0] \times [\alpha_1, \beta_1]$, the following inequalities hold:

$$\alpha^{(iv)}(x) \geq f(x, v, w, \alpha'', \alpha'''(x)), \text{ for a. e. } x \in I, \tag{7}$$

$$\beta^{(iv)}(x) \leq f(x, v, w, \beta'', \beta'''(x)), \text{ for a. e. } x \in I, \tag{8}$$

$$\begin{array}{rcccl} L_0(\alpha_0, \alpha_1, \alpha'', \alpha_0(a)) & \geq & 0 & \geq & L_0(\beta_0, \beta_1, \beta'', \beta_0(a)) \\ L_1(\alpha_0, \alpha_1, \alpha'', \alpha_1(a)) & \geq & 0 & \geq & L_1(\beta_0, \beta_1, \beta'', \beta_1(a)) \\ L_2(\alpha_0, \alpha_1, \alpha'', \alpha''(a), \alpha'''(a)) & \geq & 0 & \geq & L_2(\beta_0, \beta_1, \beta'', \beta''(a), \beta'''(a)) \\ L_3(\alpha_0, \alpha_1, \alpha'', \alpha''(b), \alpha'''(b)) & \geq & 0 & \geq & L_3(\beta_0, \beta_1, \beta'', \beta''(b), \beta'''(b)). \end{array} \tag{9}$$

The Nagumo-type condition is given by next definition:

Definition 2.2. Consider $\Gamma_i,\ \gamma_i \in L^1(I)$, $i = 0, 1, 2$, such that $\gamma_i(x) \leq \Gamma_i(x)$, $\forall\, x \in I$, and the set

$$E = \left\{ \begin{array}{c} (x, z_0, z_1, y_2, y_3) \in\ I \times (C(I))^2 \times \mathbb{R}^2 : \gamma_0(x) \leq z_0(x) \leq \Gamma_0(x), \\ \gamma_1(x) \leq z_1(x) \leq \Gamma_1(x),\ \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\}.$$

A function $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is said to verify a Nagumo-type condition in E if there exists $\varphi_E \in C([0, +\infty), (0, +\infty))$ such that

$$|f(x, y_0, y_1, y_2, y_3)| \leq \varphi_E(|y_3|), \tag{10}$$

for every $(x, y_0, y_1, y_2, y_3) \in E$, and

$$\int_r^{+\infty} \frac{t}{\varphi_E(t)} dt > \max_{x \in I} \Gamma_2(x) - \min_{x \in I} \gamma_2(x), \tag{11}$$

where $r \geq 0$ is given by

$$r := \max\left\{\frac{\Gamma_2(b) - \gamma_2(a)}{b-a}, \frac{\Gamma_2(a) - \gamma_2(b)}{b-a}\right\}.$$

Next result gives an *a priori* estimate for the third derivative of all possible solutions of (1).

Lemma 2.3. *There exists $R > 0$ such that for every L^1-Carathéodory function $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ satisfying (10) and (11) and every solution u of (1) such that*

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \ \forall \, x \in I, \tag{12}$$

for $i = 0, 1, 2,$ we have $\|u'''\| < R$. Moreover the constant R depends only on the functions φ and γ_i, Γ_i $(i = 0,\ 1,\ 2)$ and not on the boundary conditions.

Proof. The proof is similar to [3, Lemma 2.1]. □

3. **Existence and location result.** In this section it is provided an existence and location theorem for the problem (1) – (2). More precisely, sufficient conditions are given for, not only the existence of a solution u, but also to have information about the location of u, u', u'' and u'''.

The arguments of the proof require the following lemma, given on [15, Lemma 2]:

Lemma 3.1. *For $z, w \in C^1(I)$ such that $z(x) \leq w(x)$, for every $x \in I$, define*

$$q(x, u) = max\{z(x), min\{u, w(x)\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\dfrac{d}{dx}q(x, u(x))$ *exists for a.e.* $x \in I$.

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}q(x, u_m(x)) \to \frac{d}{dx}q(x, u(x)) \text{ for a.e. } x \in I.$$

Now, we are in a position to prove the main result of this paper.

Theorem 3.2. *Assume that there exists a pair (α, β) of lower and upper solutions of problem (1) – (2), such that conditions (H_0), (H_1) and (H_2) hold.*
If $f : I \times (C(I))^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a L^1-Carathéodory function, satisfying a Nagumo-type condition in

$$E_* = \left\{ \begin{array}{c} (x, z_0, z_1, y_2, y_3) \in \ I \times (C(I))^2 \times \mathbb{R}^2 : \alpha_0(x) \leq z_0(x) \leq \beta_0(x), \\ \alpha_1(x) \leq z_1(x) \leq \beta_1(x), \ \alpha''(x) \leq y_2 \leq \beta''(x) \end{array} \right\},$$

then problem (1) – (2) has at least one solution u such that

$$\alpha_0(x) \leq u(x) \leq \beta_0(x), \quad \alpha_1(x) \leq u'(x) \leq \beta_1(x), \quad \alpha''(x) \leq u''(x) \leq \beta''(x),$$

for every $x \in I$, and $|u'''(x)| \leq K$, $\forall \, x \in I$, where

$$K = \max_{x \in I}\{R, |\alpha'''(x)|, |\beta'''(x)|\} \tag{13}$$

and $R > 0$ is given by Lemma 2.3 referred to the set E_*.

Proof. Define the continuous functions

$$\begin{aligned} \delta_i\left(x,y_i\right) &= \max\left\{\alpha_i\left(x\right),\min\left\{y_i,\beta_i\left(x\right)\right\}\right\}, \text{ for } i=0,1, \\ \delta_2\left(x,y_2\right) &= \max\left\{\alpha''\left(x\right),\min\left\{y_2,\beta''\left(x\right)\right\}\right\} \end{aligned} \tag{14}$$

and

$$q\left(z\right)=\max\left\{-K,\min\left\{z,K\right\}\right\} \text{ for all } z\in\mathbb{R}.$$

Consider the modified problem composed by the equation

$$u^{(iv)}\left(x\right)=f\left(x,\delta_0\left(\cdot,u\right),\delta_1\left(\cdot,u'\right),\delta_2\left(x,u''(x)\right),q\left(\frac{d}{dx}\left(\delta_2\left(x,u''(x)\right)\right)\right)\right) \tag{15}$$

and the boundary conditions

$$\begin{aligned} u\left(a\right) &= \delta_0\left(a,u\left(a\right)+L_0\left(u,u',u'',u\left(a\right)\right)\right), \\ u'\left(a\right) &= \delta_1\left(a,u'\left(a\right)+L_1\left(u,u',u'',u'\left(a\right)\right)\right), \\ u''\left(a\right) &= \delta_2\left(a,u''\left(a\right)+L_2\left(u,u',u'',u''\left(a\right),u'''\left(a\right)\right)\right), \\ u''\left(b\right) &= \delta_2\left(b,u''\left(b\right)+L_3\left(u,u',u'',u''\left(b\right),u'''\left(b\right)\right)\right). \end{aligned} \tag{16}$$

The proof will be proved by following several steps:

Step 1 - *Every solution u of problem (15) – (16), satisfies $\alpha''(x)\le u''(x)\le \beta''(x)$, $\alpha_1(x)\le u'(x)\le\beta_1(x)$, $\alpha_0(x)\le u(x)\le\beta_0(x)$* and $|u'''(x)|<K$, for every $x\in I$, with $K>0$ given in (13).

Let u be a solution of the modified problem (15) – (16). Assume, by contradiction, that there exists $x\in I$ such that $\alpha''(x)>u''(x)$ and let $x_0\in I$ be such that

$$\min_{x\in I}\left(u-\alpha\right)''\left(x\right)=\left(u-\alpha\right)''\left(x_0\right)<0.$$

As, by (16), $u''(a)\ge\alpha''(a)$ and $u''(b)\ge\alpha''(b)$, then $x_0\in(a,b)$. So, there is $(x_1,x_2)\subset(a,b)$ such that $x_0\in(x_1,x_2)$ and

$$u''\left(x\right)<\alpha''\left(x\right),\forall x\in(x_1,x_2), \quad \left(u-\alpha\right)''\left(x_1\right)=\left(u-\alpha\right)''\left(x_2\right)=0. \tag{17}$$

Therefore, for all $x\in(x_1,x_2)$ it is satisfied that $\delta_2(x,u''(x))=\alpha''(x)$ and $\frac{d}{dx}\delta_2(x,u''(x))=\alpha'''(x)$. Now, since for all $u\in C^1(I)$ we have that $(\delta_0(\cdot,u),\delta_1(\cdot,u'))\in A$, we deduce

$$\begin{aligned} u^{(iv)}\left(x\right) &= f\left(x,\delta_0\left(\cdot,u\right),\delta_1\left(\cdot,u'\right),\delta_2\left(x,u''(x)\right),q\left(\frac{d}{dx}\left(\delta_2\left(x,u''(x)\right)\right)\right)\right) \\ &= f\left(x,\delta_0\left(\cdot,u\right),\delta_1\left(\cdot,u'\right),\alpha''\left(x\right),\alpha'''\left(x\right)\right)\le\alpha^{(iv)}\left(x\right), \text{ for a.e. } x\in (x_1,x_2). \end{aligned}$$

In consequence we deduce that function $(u-\alpha)'''$ is monotone nonincreasing on the interval (x_1,x_2). From the fact that $(u-\alpha)'''(x_0)=0$, we know that $(u-\alpha)''$ is monotone nonincreasing too on (x_0,x_2), which contradicts the definitions of x_0 and x_2.

The inequality $u''(x)\le\beta''(x)$ in I, can be proved in same way and, so,

$$\alpha''\left(x\right)\le u''\left(x\right)\le\beta''\left(x\right), \ \forall\, x\in I. \tag{18}$$

By (16) and (3), the following inequalities hold for every $x\in I$,

$$\begin{aligned} u'\left(x\right) &= u'\left(a\right)+\int_a^x u''\left(s\right)ds \\ &\ge \alpha_1\left(a\right)+\int_a^x\alpha''\left(s\right)ds=\min\ \left\{\alpha'\left(a\right),\beta'\left(a\right)\right\}+\int_a^x\alpha''\left(s\right)ds=\alpha_1\left(x\right). \end{aligned}$$

Analogously, it can be obtained $u'(x)\le\beta_1(x)$, for all $x\in I$.

On the other hand, by using (16), (5) and (6), the following inequalities are fulfilled:

$$u(x) \quad \geq \quad \alpha_0(a) + \int_a^x \alpha_1(s)\,ds = \min \ \{\alpha(a), \beta(a)\} + \int_a^x \alpha_1(s)\,ds = \alpha_0(x).$$

The inequality $u(x) \leq \beta_0(x)$ for every $x \in I$ is deduced in the same way.

Applying previous bounds in Lemma 2.3, for K given by (13), it is obtained the *a priori* bound $|u'''(x)| < K$, for $x \in I$. For details, see [3, Lemma 2.1].

Step 2 - Problem (15) – (16) has at least one solution.

For $\lambda \in [0,1]$ let us consider the homotopic problem given by

$$u^{(iv)}(x) = \lambda f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right) \tag{19}$$

and the boundary conditions

$$\begin{array}{rcl} u(a) = & \lambda\delta_0(a, u(a) + L_0(u, u', u'', u(a))) & \equiv \lambda L_A, \\ u'(a) = & \lambda\delta_1(a, u'(a) + L_1(u, u', u'', u'(a))) & \equiv \lambda L_B, \\ u''(a) = & \lambda\delta_2(a, u''(a) + L_2(u, u', u'', u''(a), u'''(a))) & \equiv \lambda L_C, \\ u''(b) = & \lambda\delta_2(b, u''(b) + L_3(u, u', u'', u''(b), u'''(b))) & \equiv \lambda L_D. \end{array} \tag{20}$$

Let us consider the norms in $C^3(I)$ and in $L^1(I) \times \mathbb{R}^4$, respectively,

$$\|v\|_{C^3} = \max\{\|v\|_\infty, \|v'\|_\infty, \|v''\|_\infty, \|v'''\|_\infty\}$$

and

$$|(h, h_1, h_2, h_3, h_4)| = \max\{\|h\|_{L^1}, \max\{|h_1|, |h_2|, |h_3|, |h_4|\}\}.$$

Define the operators $\mathcal{L} : W^{4,1}(I) \subset C^3(I) \to L^1(I) \times \mathbb{R}^4$ by

$$\mathcal{L}u(x) = \left(u^{(iv)}(x), u(a), u'(a), u''(a), u''(b)\right), \quad x \in I,$$

and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3(I) \to L^1(I) \times \mathbb{R}^4$ by

$$\mathcal{N}_\lambda u(x) = \left(\begin{array}{c} \lambda f\left(x, \delta_0(\cdot, u), \delta_1(\cdot, u'), \delta_2(x, u''(x)), q\left(\frac{d}{dx}(\delta_2(x, u''(x)))\right)\right), \\ L_A, L_B, L_C, L_D \end{array} \right)$$

Since L_0, L_1, L_2 and L_3 are continuous and f is a L^1- Carathéodory function, then, from Lemma 3.1, $\mathcal{N}_\lambda$ is continuous (see [6, Theorem 3.5] for details). Moreover, as $\mathcal{L}^{-1}$ is compact, it can be defined the completely continuous operator $\mathcal{T}_\lambda : C^3(I) \to C^3(I)$ by $\mathcal{T}_\lambda u = \mathcal{L}^{-1}\mathcal{N}_\lambda(u)$. It is obvious that the fixed points of operator $\mathcal{T}_\lambda$ coincide with the solutions of problem (19) – (20). As $\mathcal{N}_\lambda u$ is bounded in $L^1(I) \times \mathbb{R}^4$ and uniformly bounded in $C^3(I)$, we have that any solution of the problem *(19) – (20)*, verifies the following *a priori* bound $\|u\|_{C^3} \leq \left\|\mathcal{L}^{-1}\right\| |\mathcal{N}_\lambda(u)| \leq \bar{K}$, for some $\bar{K} > 0$ independent of λ.

In the set $\Omega = \left\{u \in C^3(I) : \|u\|_{C^3} < \bar{K} + 1\right\}$ the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0,1]$ and, by the invariance under homotopy, $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0)$.

As the equation $x = \mathcal{T}_0(x)$ is equivalent to the problem

$$u^{(iv)}(x) = 0, \quad x \in I, \quad u(a) = u'(a) = u''(a) = u''(b) = 0,$$

which has only the trivial solution, then $d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = \pm 1$. So by degree theory, the equation $x = \ \mathcal{T}_1(x)$ has at least one solution, that is, the problem *(15) – (16)* has at least one solution in Ω.

Step 3 - Every solution u of problem (15) – (16) is a solution of (1) – (2).

Let u be a solution of the modified problem (15) − (16). By previous steps, function u fulfills equation (1). So, it will be enough to prove the following four inequalities:

$$\begin{array}{rcl}
\alpha_0\,(a) \leq & u\,(a) + L_0\,(u, u', u'', u\,(a)) & \leq \beta_0\,(a)\,, \\
\alpha_1\,(a) \leq & u'\,(a) + L_1\,(u, u', u'', u'\,(a)) & \leq \beta_1\,(a)\,, \\
\alpha''\,(a) \leq & u''\,(a) + L_2\,(u, u', u'', u''\,(a)\,, u'''\,(a)) & \leq \beta''\,(a)\,, \\
\alpha''\,(b) \leq & u''\,(b) + L_3\,(u, u', u'', u''\,(b)\,, u'''\,(b)) & \leq \beta''\,(b)\,.
\end{array}$$

Assume that

$$u\,(a) + L_0\,(u, u', u'', u\,(a)) > \beta_0\,(a)\,. \tag{21}$$

Then, by (16), $u\,(a) = \beta_0\,(a)$ and, by (H_0) and previous steps, it is obtained the following contradiction with (21):

$$u\,(a) + L_0\,(u, u', u'', u\,(a)) \leq \beta_0\,(a) + L_0\,(\beta_0, \beta_1, \beta'', \beta_0\,(a)) \leq \beta_0\,(a)\,.$$

Applying similar arguments it can be proved that $\alpha_0(a) \leq u\,(a) + L_0\,(u, u', u'', u\,(a))$ and $\alpha_1\,(a) \leq u'\,(a) + L_1\,(u, u', u'', u'\,(a)) \leq \beta_1\,(a)$. For the third case assume, again by contradiction, that

$$u''\,(a) + L_2\,(u, u', u'', u''\,(a)\,, u'''\,(a)) > \beta''\,(a)\,. \tag{22}$$

By (16), $u''\,(a) = \beta''\,(a)$ and, as $u'' \leq \beta''$ in I, then $u'''\,(a) \leq \beta'''\,(a)$ and, by (H_1) and (9), it is achieved this contradiction with (22):

$$\begin{array}{rcl}
u''\,(a) + L_2\,(u, u', u'', u''\,(a)\,, u'''\,(a)) & \leq & \beta''\,(a) + L_2\,(\beta_0, \beta_1, \beta'', \beta''\,(a)\,, \beta'''\,(a)) \\
& \leq & \beta''\,(a)\,.
\end{array}$$

The same technique yields the two last inequalities. $\square$

As a corollary of the previous existence and location theorem, we deduce the following result for multipoint boundary value problems.

Corollary 1. *Assume that there exist* $\alpha,\ \beta \in W^{4,1}(I)$ *satisfying the following inequalities:*

$$\alpha''(x) \leq \beta''(x), \quad \textit{for all } x \in I,$$
$$\alpha^{(iv)}\,(x) - f\,(x, v, w, \alpha''(x), \alpha'''\,(x)) \geq 0 \geq \beta^{(iv)}\,(x) - f\,(x, v, w, \beta''(x), \beta'''\,(x))$$

for a. e. $x \in I$ *and all* $(v, w) \in A$,

$$\begin{array}{rcl}
\alpha_0\,(a) & \leq & \displaystyle\sum_{i=1}^{m_1^0} a_i^0\,\alpha_0(\xi_i^0) + \sum_{i=1}^{m_2^0} b_i^0\,\alpha_1(\rho_i^0) + \sum_{i=1}^{m_3^0} c_i^0\,\alpha''(\zeta_i^0), \\
\alpha_1\,(a) & \leq & \displaystyle\sum_{i=1}^{m_1^1} a_i^1\,\alpha_0(\xi_i^1) + \sum_{i=1}^{m_2^1} b_i^1\,\alpha_1(\rho_i^1) + \sum_{i=1}^{m_3^1} c_i^1\,\alpha''(\zeta_i^1),
\end{array}$$

$$
\begin{aligned}
\alpha''(a) &\leq \sum_{i=1}^{m_1^2} a_i^2\,\alpha_0(\xi_i^2) + \sum_{i=1}^{m_2^2} b_i^2\,\alpha_1(\rho_i^2) + \sum_{i=1}^{m_3^2} c_i^2\,\alpha''(\zeta_i^2) + c\,\alpha'''(a),\\
\alpha''(b) &\leq \sum_{i=1}^{m_1^3} a_i^3\,\alpha_0(\xi_i^3) + \sum_{i=1}^{m_2^3} b_i^3\,\alpha_1(\rho_i^3) + \sum_{i=1}^{m_3^3} c_i^3\,\alpha''(\zeta_i^3) - d\,\alpha'''(b),\\
\beta_0(a) &\geq \sum_{i=1}^{m_1^0} a_i^0\,\beta_0(\xi_i^0) + \sum_{i=1}^{m_2^0} b_i^0\,\beta_1(\rho_i^0) + \sum_{i=1}^{m_3^0} c_i^0\,\beta''(\zeta_i^0),\\
\beta_1(a) &\geq \sum_{i=1}^{m_1^1} a_i^1\,\beta_0(\xi_i^1) + \sum_{i=1}^{m_2^1} b_i^1\,\beta_1(\rho_i^1) + \sum_{i=1}^{m_3^1} c_i^1\,\beta''(\zeta_i^1),\\
\beta''(a) &\geq \sum_{i=1}^{m_1^2} a_i^2\,\beta_0(\xi_i^2) + \sum_{i=1}^{m_2^2} b_i^2\,\beta_1(\rho_i^2) + \sum_{i=1}^{m_3^2} c_i^2\,\beta''(\zeta_i^2) + c\,\beta'''(a),\\
\beta''(b) &\geq \sum_{i=1}^{m_1^3} a_i^3\,\beta_0(\xi_i^3) + \sum_{i=1}^{m_2^3} b_i^3\,\beta_1(\rho_i^3) + \sum_{i=1}^{m_3^3} c_i^3\,\beta''(\zeta_i^3) - d\,\beta'''(b),
\end{aligned}
$$

with $m_k^j \in \mathbb{N}$ *for* $k = 1,2,3$ *and* $j = 0,1,2,3$, $a \leq \xi_1^j < \xi_2^j < \ldots < \xi_{m_k^j}^j \leq b$, $a \leq \rho_1^j < \rho_2^j < \ldots < \rho_{m_k^j}^j \leq b$, $a \leq \zeta_1^j < \zeta_2^j < \ldots < \zeta_{m_k^j}^j \leq b$, *and* c, d, a_i^j, b_i^j *and* c_i^j *non-negative constants.*

If f *is a* L^1-*Carathéodory function, satisfying a Nagumo-type condition in* E_*, *then problem*

$$
\begin{aligned}
u^{(iv)}(x) &= f(x,u,u',u''(x),u'''(x)) \quad \text{for a. e. } x \in I,\\
u(a) &= \sum_{i=1}^{m_1^0} a_i^0\,u(\xi_i^0) + \sum_{i=1}^{m_2^0} b_i^0\,u'(\rho_i^0) + \sum_{i=1}^{m_3^0} c_i^0\,u''(\zeta_i^0),\\
u'(a) &= \sum_{i=1}^{m_1^1} a_i^1\,u(\xi_i^1) + \sum_{i=1}^{m_2^1} b_i^1\,u'(\rho_i^1) + \sum_{i=1}^{m_3^1} c_i^1\,u''(\zeta_i^1),\\
u''(a) &= \sum_{i=1}^{m_1^2} a_i^2\,u(\xi_i^2) + \sum_{i=1}^{m_2^2} b_i^2\,u'(\rho_i^2) + \sum_{i=1}^{m_3^2} c_i^2\,u''(\zeta_i^2) + c\,u'''(a),\\
u''(b) &= \sum_{i=1}^{m_1^3} a_i^3\,u(\xi_i^3) + \sum_{i=1}^{m_2^3} b_i^3\,u'(\rho_i^3) + \sum_{i=1}^{m_3^3} c_i^3\,u''(\zeta_i^3) - d\,u'''(b).
\end{aligned}
$$

has at least one solution u *such that* $\alpha_0(x) \leq u(x) \leq \beta_0(x)$, $\alpha_1(x) \leq u'(x) \leq \beta_1(x)$, $\alpha''(x) \leq u''(x) \leq \beta''(x)$, *for every* $x \in I$.

Proof. The proof is a direct consequence of Theorem 3.2. In this case it is enough to define the following functions:

$$
L_0(u,v,w,z) = -z + \sum_{i=1}^{m_1^0} a_i^0\,u(\xi_i^0) + \sum_{i=1}^{m_2^0} b_i^0\,v(\rho_i^0) + \sum_{i=1}^{m_3^0} c_i^0\,w(\zeta_i^0),
$$

$$\begin{aligned}
L_1(u,v,w,z) &= -z+\sum_{i=1}^{m_1^1} a_i^1\, u(\xi_i^1)+\sum_{i=1}^{m_2^1} b_i^1\, v(\rho_i^1)+\sum_{i=1}^{m_3^1} c_i^1\, w(\zeta_i^1),\\
L_2(u,v,w,z,p) &= -z+\sum_{i=1}^{m_1^2} a_i^2\, u(\xi_i^2)+\sum_{i=1}^{m_2^2} b_i^2\, v(\rho_i^2)+\sum_{i=1}^{m_3^2} c_i^2\, w(\zeta_i^2)+c\,p,\\
L_3(u,v,w,z,p) &= -z+\sum_{i=1}^{m_1^3} a_i^3\, u(\xi_i^3)+\sum_{i=1}^{m_2^3} b_i^3\, v(\rho_i^3)+\sum_{i=1}^{m_3^3} c_i^3\, w(\zeta_i^3)-d\,p.
\end{aligned}$$

□

4. **Example.** This section contains a functional problem composed by an integro-differential equation with functional boundary conditions, which solvability is proved in presence of non-ordered lower and upper solutions. We remark that such fact was not possible with the results in the current literature. This example does not model any particular problem arising in real phenomena. Our purpose consists on emphasize the powerful of the developed theory in this paper by showing what kind of problems we can deal with.

Consider, for $x \in [0,1]$, the fourth order equation

$$u^{(iv)}(x)=\int_0^x u(s)\,ds+\max_{x\in[0,1]}\{u'(x)\}+2u''(x)-(u'''(x)+1)^{\frac{2}{3}}, \tag{23}$$

coupled with the boundary value conditions

$$\begin{gathered}
\sum_{i=1}^{\infty} a_i\, u(\xi_i)+\sum_{i=1}^{\infty} b_i\, u'(\varsigma_i)+\eta u(0)=0,\\
\max_{x\in[0,1]} u(x)+\min_{x\in[0,1]} u'(x)+\max_{x\in[0,1]} u''(x)-11u'(0)=0, \qquad (24)\\
\textstyle\int_0^1 u(s)\,ds-3u''(0)=0, \quad \max_{x\in[0,1]} u(x)-2u''(1)=0,
\end{gathered}$$

with $0\le \xi_i,\varsigma_i\le 1$, $i\in\mathbb{N}$, $\sum_{i=1}^{\infty} a_i$, $\sum_{i=1}^{\infty} b_i$ are nonnegative series convergent to a and b, respectively, and $\eta \le -3a-\frac{10}{3}b\le 0$. Considering as auxiliary functions

$$\begin{aligned}
\alpha_1(x) &= -2x-\frac{1}{2}, & \alpha_0(x) &= -x^2-\frac{x}{2}-\frac{3}{4},\\
\beta_1(x) &= 2x+\frac{1}{2}, & \beta_0(x) &= x^2+\frac{x}{2}+\frac{3}{4},
\end{aligned}$$

one can verify that functions $\alpha(x)=-x^2+\frac{x}{2}+\frac{3}{4}$ and $\beta(x)=x^2-\frac{x}{2}-\frac{3}{4}$ are, respectively, lower and upper solutions for the problem (23) – (24), with

$$\begin{aligned}
f(x,y_0,y_1,y_2,y_3) &= \int_0^x y_0(s)ds+\max_{x\in[0,1]}\{y_1(x)\}+2\,y_2(x)-(y_3(x)+1)^{\frac{2}{3}},\\
L_0(z_1,z_2,z_3,z_4) &= \sum_{i=1}^{\infty} a_i\, z_1(\xi_i)+\sum_{i=1}^{\infty} b_i\, z_2(\varsigma_i)+\eta\, z_4,\\
L_1(z_1,z_2,z_3,z_4) &= \max_{x\in[0,1]} z_1+\min_{x\in[0,1]} z_2+\max_{x\in[0,1]} z_3-11\, z_4,\\
L_2(z_1,z_2,z_3,z_4,z_5) &= \int_0^1 z_1(s)\,ds-3\, z_4, \quad L_3(z_1,z_2,z_3,z_4,z_5)=\max_{x\in[0,1]} z_1-2\, z_4.
\end{aligned}$$

As f is continuous and satisfies (10) and (11) for $\varphi_{E_*}(y_3) = \frac{47}{6} + (y_3+1)^{\frac{2}{3}}$ in

$$E_* = \left\{ (x, y_0, y_1, y_2, y_3) \in [0,1] \times \mathbb{R}^4 : \begin{array}{c} -x^2 - \frac{x}{2} - \frac{3}{4} \leq y_0 \leq x^2 + \frac{x}{2} + \frac{3}{4} \\ -2x - \frac{1}{2} \leq y_1 \leq 2x + \frac{1}{2} \\ -2 \leq y_2 \leq 2 \end{array} \right\},$$

by Theorem 3.2, there is a nontrivial solution u for problem (23) – (24) such that

$$-x^2 - \frac{x}{2} - \frac{3}{4} \leq u(x) \leq x^2 + \frac{x}{2} + \frac{3}{4}, \quad -2x - \frac{1}{2} \leq u'(x) \leq 2x + \frac{1}{2}, \quad -2 \leq u''(x) \leq 2,$$

for all $x \in [0,1]$.

REFERENCES

[1] Z. Bai, *The upper and lower solution method for some fourth-order boundary value problems,* Nonlinear Anal. **67** (2007), 1704–1709.

[2] A. Cabada, F. Minhós, *Fully nonlinear fourth order equations with functional boundary conditions*, J. Math. Anal. Appl., **340/1** (2008) 239–251.

[3] A. Cabada, F. Minhós, A. I. Santos, *Solvability for a third order discontinuous fully equation with functional boundary conditions* J. Math. Anal. Appl., **322** (2006) 735–748.

[4] A. Cabada, M. R. Grossinho, F. Minhós, *On the Solvability of some Discontinuous Third Order Nonlinear Differential Equations with Two Point Boundary Conditions*, J. Math. Anal. Appl. **285** (2003) 174–190.

[5] A. Cabada, M. R. Grossinho, F. Minhós, *Extremal solutions for third-order nonlinear problems with upper and lower solutions in reversed order*, Nonlinear Anal. **62** (2005) 1109–1121.

[6] A. Cabada, D. O'Regan, R. L. Pouso, *Second order problems with functional conditions including Sturm Liouville and multipoint conditions*, Math. Nachr. **281**, (2008) 1254–1263.

[7] A. Cabada, R. Pouso, F. Minhós, *Extremal solutions to fourth-order functional boundary value problems including multipoint condition*, Nonlinear Anal.: Real World Appl., **10** (2009) 2157–2170.

[8] H. Feng, D. Ji, W. Ge, *Existence and uniqueness of solutions for a fourth-order boundary value problem*, Nonlinear Anal., **70** (2009) 3761–3566.

[9] D. Franco, D. O'Regan, J. Perán, *Fourth-order problems with nonlinear boundary conditions*, J. Comput. Appl. Math. **174** (2005) 315–327.

[10] J. R. Graef, L. Kong, B. Yang, *Existence of solutions for a higher order multi-point boundary value problems*, Result. Math. **53** (2009), 77–101.

[11] D. Ma, X. Yang, *Upper and lower solution method for fourth-order four-point boundary value problems,* J. Comput. Appl. Math., **223** (2009), 543–551.

[12] F. Minhós, T. Gyulov, A. I. Santos, *Lower and upper solutions for a fully nonlinear beam equations,* Nonlinear Anal., **71** (2009) 281–292.

[13] H. Pang, W. Ge, *Existence results for some fourth order multi-point boundary value problem*, Math. Comput. Model., **49** (2009) 1319-1325.

[14] C. V. Pao, Y. M. Wang, *Fourth-order boundary value problems with multi-point boundary conditions*, Comm. Appl. Nonlin. Anal., **16** (2009), 1–22.

[15] M. X. Wang, A. Cabada, J. J. Nieto, *Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions,* Ann. Polon. Math. **58** (1993), 221–235.

[16] W. Wang, J. Shen, Z. Luo, *Multi-point boundary value problems for second-order functional differential equations,* Comput. Math. Appl. **56** (2008), 2065–2072.

[17] S. Weng, H. Gao, D. Jiang, X. Hou, *Upper and lower solutions method for fourth-order periodic boundary value problems.* J. Appl. Anal. **14** (2008), 53–61.

[18] X. Zhang, L. Liu, *Positive solutions of fourth-order multi-point boundary value problems with bending term*, Appl. Math. Comp., **194** (2007) 321-332.

[19] H. Zhao, *A note on upper and lower solutions method for fourth-order boundary value problems,* Ann. Differential Equations **24** (2008), 117–120.

Received July 2010; revised April 2011.

E-mail address: `alberto.cabada@usc.es`
E-mail address: `jfzero@gmail.com`
E-mail address: `fminhos@uevora.pt`