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Laws of non-symmetric optimal flow structures, from the macro to the micro scale

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Abstract. Many natural systems and engineering processes occur in which a fluid invades a territory from one entry point (invasion), or conversely is expelled from the territory through an outlet (drainage). In any such situation an evolutionary flow structure develops that bridges the gap between the micro-scale (diffusion dominant) and the macro-scale (convection dominant). The respiratory and circulatory systems of animals are clear examples of complex flow trees in which both the invasion and drainage processes occur. These flow trees display successive bifurcations (almost always non-symmetric) which allow them to cover and serve the entire territory to be bathed. Although they are complex, it is possible to understand its internal structuring in the light of Constructal Law.

A scaling law for optimal diameters of symmetric bifurcations was proposed by Murray (1926), while Bejan and co-workers (2000-2006) added a new scaling law for channel lengths, and based scaling laws of tree shaped structures on theoretical grounds. In this work we use the Constructal Law to study the internal structure and scaling laws of non-symmetric flow structures, and show how the results might help understand some flow patterns found in Nature. We show that the global flow resistances depend on the parameter $\xi=D_2/D_1=L_2/L_1$ defining the degree of asymmetry between branches 1 and 2 in a bifurcation. We also present a more accurate and general form, of Murray's law, as a result of the application of the Constructal law to branching flow structures. We end with a brief analysis of the use of these results in the analysis of flow structures of the human respiratory and circulatory systems.

Keywords: flow structures; branching, scaling laws, Constructal Law.
PACS: 89.75.Da, 89.75, 87.10.Ca

INTRODUCTION

The first attempts to study resistance to flow of the human arterial system date back to Young (1806) [1]. In his study Young assumed the ratio 4/5 between the diameters of daughter and parent vessels. However, a better founded study was carried out by C. D. Murray in 1926 [2], who derived a rule for evaluating that ratio based on the minimization of the "power to maintain a given flow". Then he arrived at the famous relation between the cubes of the ratios r_0 of radii of daughter $(r_1; r_2)$ and parent vessels (r_0), now known as the Murray's law: $r_0^3=r_1^3+r_2^3$.

Murray's law has been verified to hold, at least in an approximate way, for biologic flow structures (see [3] for more details). Interestingly, Sherman notes in his review [3] that "Murray's law will hold for any branching vascular system that, within a given volume, requires minimum flow resistance."

More recently Bejan and co-workers have derived Murray's law from the Constructal Law [4], by assuming that "the geometric structure of the flow system springs out of the principle of global performance maximization subject to global constraints."[5]. In their study, Bejan and co-workers not only derived the usual form of Murray's law, but also extended their analysis to turbulent flow and found a scaling law, akin of Murray's, in which the exponent is -3/7, instead of exponent -1/3 that holds for laminar flow. Based both on Murray's law and the Constructal Law, in 2004 Reis et al. [6,7] were able to anticipate the lung airway structure, namely its 23 levels of branching, a well-known anatomic feature of the human lung.

In this paper, we go beyond Murray's law to find the scaling laws of branching flows that are asymmetric both with respect to diameters and vessel lengths. We use Constructal Law as the principle that governs design of flow architectures. In doing this we are especially interested in the branching laws of the vessels in the circulatory system, in which branching asymmetry is very common.

GENERALIZATION OF MURRAY'S LAW

In its original form Murray's law: $r_0^3=r_1^3+r_2^3$, already accommodates branching with asymmetric vessel radii. However, in Murray's law the radii

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$(r_1; r_2)$ are not necessarily related to each other, neither they are related to the respective vessel lengths.

Here we use the Constructal Law formalism for optimization of flow structures that is akin of that of Equilibrium Thermodynamics for characterization of equilibrium states. Details of this Constructal formalism may be found in references [8, 9].

The optimal flow structure design is the one that leads to global optimal performance with respect to fluid flow. The performance is optimal when flow access is the easiest, or said another way when the global resistance to flow is the smallest under the existing constraints.

Due to the value of space the volume allocated to each function in living systems must be the smallest, and therefore any deviation of this rule reduces global performance. In this way, either global flow resistance R, or global volume V occupied by the flow structure might be considered as important parameters affecting global performance. Then, if $\{ X^j, j = 1,\ldots,n \}$ denotes the ensemble of the n design variables that are kept free to describe design evolution, the condition of easiest flow access may equivalently be formulated as the maximum of the global resistance to flow, under constant volume V:

$$\sum_i \left(\frac{\partial R}{\partial X^i} \right)_{V, X^k} dX^i = 0, X^i \neq V, i = 1,\ldots,n, \quad (1)$$

or alternatively the minimum of the global volume allocated to the flow structure, under constant flow resistance R:

$$\sum_j \left(\frac{\partial V}{\partial X^j} \right)_{R, X^k} dX^j = 0, X^j \neq R, j = 1,\ldots,n. \quad (2)$$

In normal conditions, biological flows are not turbulent. In fact, laminar flows dissipate less energy to move the fluids within the bodies and in this way match one of the basic requirements for the continued existence of living systems. Living systems are highly organized societies of cells that need permanent supply of water, blood and organic substances to keep functioning well. Then, in living systems the currents are fixed, and in case of altered resistance, the system adjusts the respective driving potential in order to keep currents constant. If ΔP is the potential that drives the current $I = \Delta P / R$, then the associated exergy dissipation rate reads: $\Delta E = R I^2$. Because most of the biological currents are fixed, it becomes clear that the minimization of the global resistance R is equivalent to minimization of the global exergy dissipation rate ΔE.

Let us consider the flow system represented in Fig. 1, where L and D stand for vessel length and diameter, respectively.

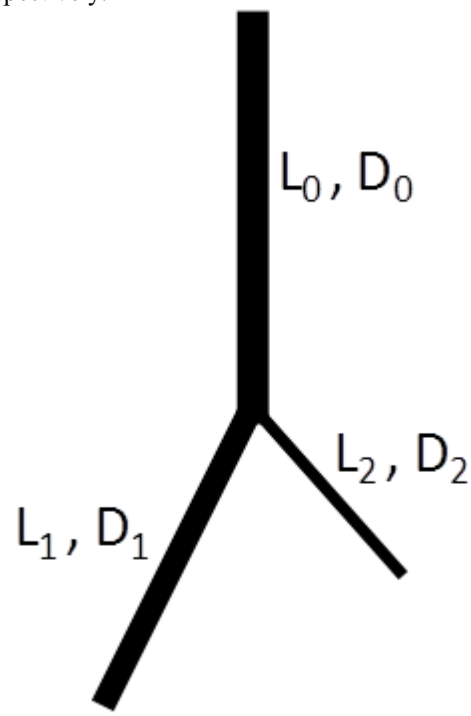

FIGURE 1. Branching flow structure. L and D represent length and diameter of vessels, respectively.

We will assume that the flow is laminar, and therefore resistance to flow reads:

$$R = \frac{128 \mu L}{\pi D^4}. \quad (3)$$

where μ stands for the dynamic viscosity of the fluid.

Then, by defining a reduced flow resistance as $\tilde{R} = R\pi / 128\mu$, the global resistance of the flow system in Fig 1 reads:

$$\tilde{R} = \frac{L_0}{D_0^4} + \left(\frac{D_1^4}{L_1} + \frac{D_2^4}{L_2} \right)^{-1}. \quad (4)$$

On the other hand, the total volume V occupied by the flow system expressed in the reduced form $\tilde{V} = V 4 / \pi$ is given by:

$$\tilde{V} = D_0^4 L_0 + D_1^4 L_1 + D_2^4 L_2. \quad (5)$$

Thought both the equations (4) and (5) are expressed in terms of geometric variables, the equation (4) describes the flow dynamics while the equation (5)

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expresses the constraints to the flow. As Murray's law relates the diameters of daughter and parent vessels, we will consider the ensemble of design variables $\{D_0, D_1, D_2,\}$. By using equation (1) we maximize the global resistance $\tilde{R}$ under constant volume, $\tilde{V}$. Then we obtain a system of 3 equations, which provide the following relationships:

$$2/D_0^6 = \lambda, \tag{6}$$

where λ is a constant (Lagrange multiplier used in the optimization). The remaining 2 equations read:

$$\frac{D_1}{D_0} = \left[1 + \left(\frac{L_2}{L_1}\right)^3\right]^{-1/3}, \tag{7}$$

and

$$\frac{D_2}{D_0} = \left[1 + \left(\frac{L_1}{L_2}\right)^3\right]^{-1/3}. \tag{8}$$

The equations (7) and (8) show that the ratio between the diameters of daughter and parent vessels depends on the ratio of the lengths of daughter vessels. By combining the equations (7) and (8) and defining $\xi = L_2/L_1$, we obtain:

$$D_0^3 = \frac{\left(1+\xi^3\right)D_1^3 + \left(1+\xi^{-3}\right)D_2^3}{2}, \tag{9}$$

which reduces to Murray's law when $\xi = 1$ (branching symmetry).

Additionally, from the equations (7) and (8) we obtain the following relationship:

$$\frac{D_2}{D_1} = \frac{L_2}{L_1} = \xi, \tag{10}$$

which means that in branching flow structures that perform optimally the ratio between the diameters of daughter vessels must be equal to the ratio between the respective lengths.

Therefore, not only the equations (7) and (8) are more general than Murray's law, but also modify the ratio between diameters of daughter and parent vessels.

By defining $\phi = (1+\xi^3)/2$ and $\psi = (1+\xi^{-3})/2$ we are able to assess the importance of the ratio ξ on the equation (9), as it is represented in Figure 2. In the case when daughter vessels are almost of same length the difference with respect to the ratios D_1/D_0 and

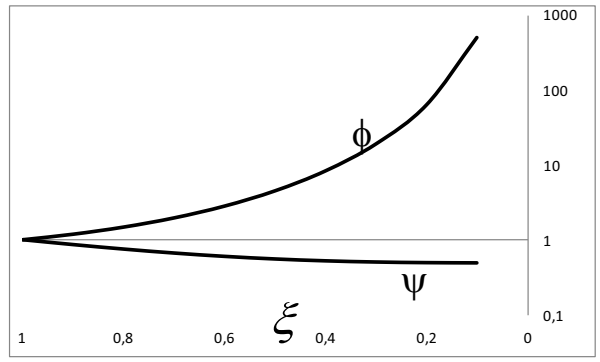

FIGURE 2. Variation of the weighting coefficients ϕ and ψ (see equation 9) with asymmetry parameter ξ.

D_2/D_0 provided by Murray's law is quite small. However, it becomes increasingly important as branching asymmetry gets higher. In fact, the equations (7) and (8) show that as ξ becomes smaller, the value for D_2 provided by the equation (8) is smaller than that provided by Murray's law, while the opposite occurs with D_1.

The equations (7), (8) and (10) enable us to evaluate the resistance (see equation 4) that corresponds to best performance in the form:

$$\tilde{R}_{opt} = \frac{L_0}{D_0^4}\left[1 + \frac{L_1}{L_0}\left(1+\xi^3\right)^{1/3}\right], \tag{11}$$

while the corresponding volume allocated to the flow system reads:

$$\tilde{V}_{opt} = D_0^4 L_0\left[1 + \frac{L_1}{L_0}\left(1+\xi^3\right)^{1/3}\right]. \tag{12}$$

In both the equations (11) and (12) the first term within the square brackets corresponds to the parent vessel while the second one corresponds to the daughter vessels.

EQUIPATITITION OF RESISTANCES AND FLOW VOLUMES AT EVERY SCALE

The previous optimization has not provided any relationship between the lengths of daughter and parent vessels. To find out such a relationship we will

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consider a second optimization procedure based on the dynamic variables of the flow.

The currents I_j in the flow structure of Figure 1 are driven by pressure differences ΔP_j such that the respective flow resistance reads: $R_j = \Delta P_j / I_j$. On the other hand, the assumption of current conservation entails $\sum_j (I_j - 2I_{j+1}) = 0$. Additionally, we assume that the overall pressure difference ΔP_0 in the flow structure is given by summing up the pressure differences corresponding to each branching level, i.e. $\Delta P_0 = \Delta P_j + \Delta P_{j+1}$. Then, in terms of the dynamic variables R_j, I_j and ΔP_j, minimization of the global resistance reads:

$$\sum_j \left[\left(\frac{\partial R}{\partial I_j} \right)_{\Delta P_0} dI_j + \left(\frac{\partial R}{\partial (\Delta P_j)} \right)_I d(\Delta P_j) \right] = 0 \quad (13)$$

subject to the constraints:

$$\sum_j (I_j - 2I_{j+1}) = 0 \ , \ \ \Delta P_0 = \sum_j \Delta P_j, \quad (14)$$

where the overall pressure difference ΔP_0 is a constant. As the results we obtain:

$$R_n = \lambda_1 / \lambda_2 \ , \ \ \Delta P_n = \lambda_2 \text{ with } n = 1,2 \quad (15)$$

where R_n stands for the total resistance of the vessels of rank n, and (λ_1, λ_2) are the Lagrange multipliers (constants) used with the purpose of taking into account the first and second of the constraints (14), respectively.

What the equations (15) tell us is that both the overall resistance and the pressure drop have the same values at every branching level. This means that in the best performing (optimal) flow system design both the overall resistance and pressure drop distribute in such a way that their respective values does not change from a branching level to the next one.

In equation (11) the first term in the square brackets correspond to the resistance of the first level (parent vessel), while the second one corresponds to the contribution of the second level (daughter vessels) to the global resistance. As resistance does not change from one level to the next one, we conclude that the second term must be equal to unity. Then it follows that:

$$\frac{L_1}{L_0} = \left(1 + \xi^3\right)^{-1/3}, \quad (16)$$

and

$$\frac{L_2}{L_0} = \left(1 + \xi^{-3}\right)^{-1/3}. \quad (17)$$

The equations (16) and (17) show that the ratio between the lengths of daughter and parent vessels follows the same rule as that of the ratio of the respective diameters (see equations 7 and 8). The coupling between those ratios is bridged by the asymmetry parameter ξ (see equation 10).

Now we are able to express the minimal (optimal) resistance of the flow structure represented in Figure 1 in the form:

$$\widetilde{R}_{opt} = n \frac{L_0}{D_0^4}, \ n = 2, \quad (18)$$

while the optimal volume allocated to the flow system reads:

$$\widetilde{V}_{opt} = n D_0^4 L_0, \ n = 2. \quad (19)$$

We note that also the global volume allocated to each level has the same value as that allocated to the next one. The n in both the equations (18) and (19) means that for the case of a flow structure of higher branching level we just have to multiply both the flow resistance and the volume of the first level by the number of levels of branching (bifurcation) to find out the respective global values for that flow structure.

Through summarizing the results (15) and (19) we are now able to put forward an "*equipartition rule for the geometric and the dynamic variables of flow trees*" in the form: *"In the best performing (optimal) flow tree design, resistance, volume allocated to the flow, and pressure drop, distribute equally for every level of branching"*.

This rule is akin of the rule of equipartion of energy in Equilibrium Thermodynamics. Bejan has long noted this similarity and called it "*the optimal distribution of imperfection*" (see [10] for details).

OPTIMAL DESIGN OF BRANCHING FLOW SYSTEMS

The design of optimal branching flow systems can now be guided by rules that result from principle. This is a characteristic of Constructal theory, in which design is not assumed in advance, but springs always from the result of the application of constructal rules (see also [11]).

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In the present case, the rules for optimal branching design hold at every scale in which flows are described by Hagen-Poiseuille law. We could also have determined the corresponding rules for turbulent flow, which would represent a generalization of those already presented by Bejan et al. [5]. However, the main purpose in this work was to find out the laws of asymmetric branching that is commonly found in the

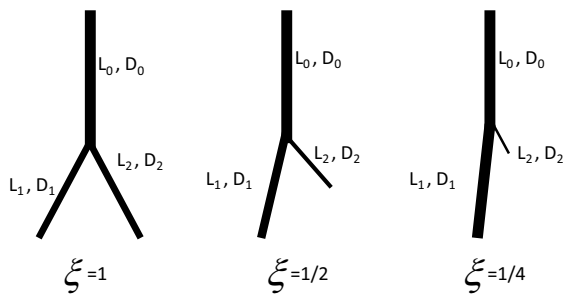

FIGURE 3. Design of optimal flow tree for symmetric branching (ξ=1), and for moderate (ξ=1/2), and high asymmetry (ξ=1/4).

circulatory systems of animals in which flows are laminar. Special care has to taken with respect to circulatory systems, because blood is not a perfect Newtonian fluid. However, as a first approach it can be viewed as having a quasi-Newtonian behavior.

The circulatory systems have evolved in time such as to meet best performing designs. We believe that actual designs of circulatory systems are close to optimal, or perform as near-optimal. In this way, at every scale in actual systems it will be possible to find structures that follow the rules of best performance. Some cases are depicted in Figure 3.

The first case corresponds to symmetric branching, ξ=1. In this case $L_1/L_0= D_1/D_0= 0.79$. Symmetric branching rarely occurs in the human circulatory system but is common in the lung airway tree. Reis et al. [6] used the rules (7) and (8), i.e. Murray's law to anticipate the structure of the human lung airway tree, namely the number of bifurcations – 23 – that match optimal performance.

The second and third cases are characteristic of the circulatory system. In fact that flow structure has the purpose of delivering blood to the organs and tissues that, differently to the lung alveoli, have very different flow requirements and are located in different parts of the body. In the second case (ξ=1/2) we have $L_1/L_0= D_1/D_0=0.96$, i.e. the lengths of the main daughter vessel and parent vessel are almost the same. In the third case (ξ=1/4), the main daughter vessel and the parent vessel have practically the same length and diameter ($L_1/L_0= D_1/D_0= 0.99$). In this case, the secondary daughter vessel is a small perturbation that almost does not affect the main vessel.

CONCLUSIONS

The design of optimal branching flow systems can be guided by principle. We used Constructal Law to find out the scaling rules of flow systems that sequentially branch to bridge the gap between the scales of different order. In this way we found relationships between diameters of daughter and parent vessels and between the respective lengths as well. We found that a coupling parameter exists $\xi = L_{i1}/L_{i2}= D_{i1}/D_{i2}$ that accounts for the degree of asymmetry of the daughter vessels and affects both the ratios of the diameters and lengths of daughter to parent vessels. This coupling parameter is relatively unimportant at very low asymmetry level (ξ~1) but becomes increasingly important as ξ deviates from unity. In this way, Murray's law appears as an approximation that is as accurate as ξ is close to unity.

We also found that in an optimally performing flow tree, resistance, pressure drop, and volume allocated to the flow system distribute equally for every level of branching. We noted that this law of equipartition is akin of the law of energy equipartition in Equilibrium Thermodynamics.

In a final and brief analysis we showed that these scaling rules may describe many aspects of living flow systems, namely the human respiratory and circulatory systems.

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