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Chapter 2

Parametric Interest Rate Risk Immunization

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Abstract

In this chapter we develop a new immunization model based on a parametric specification of the term structure of interest rates. The model extends traditional duration analysis to account for both parallel and non-parallel term structure shifts that have an economic meaning. Contrary to most interest rate risk models, we formally analyse both first-order and second-order conditions for bond portfolio immunization, emphasizing that the key to successful immunization will be to build up a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. We provide explicit formulae for new parametric interest rate risk measures and present alternative approaches to implement the immunization strategy. Additionally, we develop a more accurate approximation for the price sensitivity of a bond based upon new parametric interest rate risk measures and revise both classic and modern approaches to convexity in order to highlight the risks of convexity when changes other than parallel shifts in the term structure are considered. Furthermore, we provide useful expressions for the sensitivity of interest rate risk measures to changes in term structure shape parameters.

1. Introduction

Interest rate risk immunization, which may be defined as the protection of the nominal value of a portfolio (or the net value of a firm) against changes in the term structure of interest rates, is a well-known area of portfolio management. The term “immunization” describes the steps taken by a bond manager to build up and manage a bond portfolio in such a way that this portfolio reaches a predetermined goal. That goal can be either to guarantee a set

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of future payments, to obtain a certain rate of return for the investment or, in certain cases, to replicate the performance of a bond market index.

Immunization models (also known in the literature as interest-rate risk or duration models) control risk through duration and convexity measures. These measures capture the sensitiveness of bond-returns to changes in one or more interest rate risk factors. For a given change in the yield curve, the estimate of the change in bond price is typically approximated by multiplying the duration (and eventually the convexity) by the change in the yield curve factor.

The classical approach to immunization employs duration measures derived analytically from prior assumptions regarding specific changes in the term structure of interest rates. For instance, the duration measure developed by Fisher and Weil (1971) assumes that a parallel and instantaneous shift in the term structure of interest rates occurs immediately after the bond portfolio is build up. In this case, the recipe was basically to build up a portfolio such that its duration was equal to the investor's horizon. In order to take into account the fact that interest rates do not always move in a parallel way, a number of alternative models considering non-parallel shifts were proposed by Bierwag (1977), Khang (1979) and Babbel (1983) or, in an equilibrium setting, by Cox *et al.* (1979), Ingersoll *et al.* (1978), Brennan and Schwartz (1983), Nelson and Schaefer (1983) and Wu (2000), among many others.

This approach has several drawbacks. The earliest and most widespread refers to the fact that the investment is protected only against the particular type of interest rate change assumed. In this sense, the need to identify correctly the "true" stochastic process becomes obvious. If identified incorrectly, the effectiveness of the strategy is compromised and the investor is subject to a new type of risk - stochastic process (or immunization) risk. The second drawback concerns the nature of the interest rate uncertainty that can be described by a single factor model. In effect, in this case the changes in all interest rates along the term structure must be perfectly correlated, an assumption frequently rejected in empirical studies. Moreover, the existence of non-parallel movements in the yield curve limits the use of single factor models.

Fong and Vasicek (1983, 1984) developed the M-Squared model in order to minimize the immunization risk due to non-parallel (slope) shifts in the term structure of interest rates. The authors show in particular that by setting the duration of a bond portfolio equal to its planning horizon and by minimizing a quadratic cash flow dispersion measure, the immunization risk due to adverse term structure shifts can be reduced.[1] More recently, new immunization risk (dispersion) measures were proposed by Nawalkha and Chambers (1996), Balbás and Ibáñez (1998) and Balbás et al. (2002).

In recent years researchers have redirected their attention towards the development of alternative formulations which try to capture more effectively the interest rate risk faced by fixed-income portfolios, without relying on any particular assumptions as to the type of stochastic process which governs interest rate movements. A popular approach is to assume that interest rate changes can be accurately described by shifts in the level of a limited number of segments (vertices or yield curve drivers) into which the term structure is subdivided, generalizing then the concepts of duration and convexity to a multivariate context by considering the portfolio's joint exposure to these key rates. Specifically, we refer to the

[1] Nawalkha and Chambers (1997) and Nawalkha, Soto and Zhang (2003) derive a multiple-factor extension to the M-Squared model termed M-Vector Model.

directional duration and to the partial duration models of Reitano (1990, 1991a,b, 1992), to the key-rate duration model of Ho (1992) and to the reshaping duration model suggested by Klaffky et al.. (1992). In these models, the direction of interest rate shifts can be set on an a priori basis or can be based on real data. In the later case, the historical movements in the term structure of interest rates are used to identify a limited number of state variables, observable or not, which govern the yield curve.[2]

An alternative line of attack to the problem of immunization involves the use of parametric duration models. In this kind of formulation, which has its roots in the work of Cooper (1977), all that is assumed is that at each moment in time the term structure of interest rates adheres to a particular functional form, which expresses itself as a function of time and a limited number of shape parameters. In this line of thought, provided that the mathematical function fits accurately most yield curves all interest rate movements can be expressed in terms of changes in one or more shape parameters that characterize this function. In other words, it is apparent that in this kind of models the interest rate risk uncertainty is reflected by the unknown nature of future parameter values. Differentiating the bond price with respect to each shape parameter we obtain a vector of parametric interest rate risk measures. Choosing a particular functional form involves obviously some pricing errors. The difference is that in this case the errors can be quantified and controlled systematically, as long as we are able to choose the appropriate specification for the yield curve, where by appropriate we mean the one that minimizes immunization risk.

After the work of Cooper there has been little research in this area. Garbade (1985), Chambers et al. (1988) and Prisman and Shores (1988) assume that a polynomial may be used to fit the term structure of interest rates as a first step to derive a vector of interest rate risk measures - termed duration vector -, in which each element corresponds, basically, to the moment of order k of a bond[3]. Although simple, the use of polynomial functions to estimate the yield curve has been subject to great criticism since it can lead to forward curves that exhibit undesirable (and unrealistic) properties for long maturities, namely high instability. In Willner (1996) the actual yield curve risk exposure of a bond portfolio is decomposed using the Nelson and Siegel (1987) parametrization of the yield curve, a mathematical function that expresses interest rates in terms of four parameters and is compatible with standard increasing, decreasing, flat and inverted yield curve shapes.

Another major issue in the duration literature refers to the importance of portfolio design in immunization performance. In constructing a bond portfolio that immunizes the investment against changes in the term structure of interest rates, investors normally select the portfolio's composition so that its duration measures match the length of the planning period. When the number of bonds available is large enough, there are multiple solutions which satisfy the immunization constraints. Fong and Vasicek (1983, 1984) developed the M-Squared Model to minimize the stochastic process risk due to non-parallel shifts in the

[2]See, for example, Gultekin and Rogalski (1984), Elton et al. (1990), Garbade (1986), Litterman and Scheinkman (1991), Knez et al. (1994), D'Ecclesia and Zenios (1994), Barber and Copper (1996) and Bravo and Silva (2005).

[3]The moment of order k of a bond is defined as the weighted average of the kth power of its times of payments, the weights being the shares of the bond's cash flows in present value in the bond's present value. Chambers et al. (1988) perform immunization tests for the U.S. market over single and multiperiod horizons and conclude that the improvement in the immunization performance is considerable with the addition of at least four interest rate risk measures.

yield curve, providing at the same time a method to select the best duration-matching portfolio from the set of potential portfolios.

Fooladi and Roberts (1992) and Bierwag *et al.* (1993) extended the research into the importance of portfolio design by comparing the performance of duration-matching portfolios constrained to include a bond maturing near the end of the holding period, the so-called maturity bond, with that of traditional duration-matching portfolios, or with that of duration-matching portfolios which minimize or equate to zero the risk measure of Fong and Vasicek (1983, 1984). In their simulations for the Canadian market – using different term-structure estimation procedures, different investment horizons and different duration measures – Fooladi and Roberts (1992) report that "...a constraint forcing the duration-matching portfolio to include a bond with maturity equal to the time remaining in the horizon appears to add significantly to hedging performance". This result is often referred to as the "duration puzzle". Furthermore, contrary to Fong and Vasicek, their results suggest that forcing the duration-matching portfolio to include a maturity bond is a better design criterion than choosing a bullet portfolio, although the bullet portfolio has lower . By the same token, Bierwag et al. (1993) conclude that "...minimum portfolios fail to hedge as effectively as portfolios including a bond maturing on the horizon date", offering more evidence in favor of using the maturity bond.

More recently, Bravo and Silva (2006) and Soto (2001, 2004) investigated the immunization performance of alternative single- and multiple-factor duration-matching strategies and other models, using Portuguese and Spanish government bond data, , in order: (i) to evaluate whether the success of duration-matching strategies is primarily attributable to the particular model chosen to explain term structure movements, or to the number of interest rate risk factors considered and (ii) to confirm the importance of portfolio design in immunization performance. The results obtained by Bravo and Silva (2006) suggest that immunization models (single- and multi-factor) remove most of the interest rate risk underlying a more naïve maturity strategy, and that duration-matching portfolios constrained to include the maturity bond and formed using a single-factor model provide the best immunization performance overall, particularly in highly volatile term structure environments and shorter holding periods. Soto (2004) argues that for multiple-factor models, the number of risk factors considered in immunization strategies is definitely more important than the particular model chosen, but also warn that the addition of duration constraints to the immunization program beyond the third might impair the performance.

In this this chapter we develop a new immunization model based on the Svensson (1994) specification of the yield curve. The model is parametric by nature, i.e., the interest rate risk factors correspond to the parameters of the mathematical function used to represent the yield curve, and adopts a multivariate setting, being compatible with both parallel and non-parallel term structure shifts. Since we do not impose any previous assumptions about the way yield curve changes the model is applicable in virtually all yield curve environments. In addition, the model is intuitive and relatively easy to apply.

This chapter is related to Willner (1996), but there are some important differences. First, we adopt Svensson's parametrization instead of Nelson and Siegel's mathematical function. As shown by Svensson (1994) the extended form allows more flexibility in the yield curve estimation, in particular in the short-term end of the yield curve. In addition, the model assumes that every movement in the term structure of interest rates can be approximated by changes in a small number of factors and that these factors can be directly interpreted as representing parallel, slope and curvature shifts in the yield curve.

Previous research on duration models was not able to establish a link between first- and second order conditions for immunization.In this sense, contrary to Willner and most interest rate risk models we formally analyse both first-order and second-order conditions for bond portfolio immunization, emphasizing that the key to successful immunization will be to build up a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. In addition, we provide explicit formulae for new parametric interest rate risk measures and present alternative approaches to implement the immunization strategy.

Finally, we extend previous analysis on the sensitivity of a bond's duration to changes in the yield to maturity by developing useful expressions for the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters.

The outline of the remaining part of the chapter is as follows. In Section 2, we briefly characterize Svensson's specification of the yield curve, and theoretically justify its use in the context of the immunization problem. In Section 3 we introduce the concepts of parametric duration and parametric convexity and formally derive first-order and second-order conditions for immunization. We show that it is impossible to achieve immunization simply by meeting first-order conditions and that second-order conditions must be addressed conveniently. In Section 4 we develop a more accurate approximation for the price sensitivity of a bond based upon new definitions for parametric interest rate risk measures and revise both classic and modern approaches to convexity. In particular, we demonstrate that important negative effects of convexity are revealed when changes other than parallel shifts in the term structure are considered. In Section 5 we provide simple expressions for the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters. Section 6 summarizes the main conclusions of this chapter.

2. Term Structure Specification

Svensson (1994) proposed a mathematical characterization of the yield curve based on the following parametric specification of the instantaneous forward rate, $f(t, \mathbf{a})$:

$$f(t, \mathbf{a}) = a_0 + a_1 e^{-\frac{t}{a_4}} + a_2 \left(\frac{t}{a_4} e^{-\frac{t}{a_4}} \right) + a_3 \left(\frac{t}{a_5} e^{-\frac{t}{a_5}} \right), \tag{1}$$

where $f(t, \mathbf{a})$ is a function of both the time to maturity t and a (line) vector of parameters $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ to be estimated, with $(a_0, a_4, a_5) > 0$. To increase the flexibility of the curves and to improve the fit, Svensson extended the Nelson and Siegel's functional form by adding a potential extra hump in the forward curve. It is well known that the Nelson-Siegel method admits the existence of only one extremum and one point of inflection in the concavity. This means that when there are disturbances in the money market that lead to curves with two local extrema, the fit in the short segment of the yield curve turns out to be very poor. Given its higher adjustment capacity, the Svensson model has proven to be more adequate in estimating the term structure of interest rates and it is widely used by practitioners and major central banks.[4]

The parameters in the forward rate function are estimated by solving a non-linear optimization procedure to data observed on a trade day, which consists in minimizing the sum of squared yield (or price) deviations between observed and theoretical yields (or prices) as estimated with the model. The optimization problem can be solved using either a grid search procedure or a partial estimation technique[5]. In most practical applications fitting was found relatively insensitive to changes in parameters a_4 and a_5 (e.g. Barrett et al., 1995, Willner, 1996 and Diebold and Li, 2003). This means that, without loss of generality, we can follow standard practise and assume at any stage that these parameters are fixed at prespecified values. Note also that by setting a_3 equal to zero in (1) we obtain the Nelson and Siegel forward rate function.

Regardless of their popularity, the Nelson-Siegel-Svensson family of curves has been criticized because of two theoretical shortcomings. The first, pointed out by Björk and Christensen (1999) and Filipovic (1999, 2000), is that models fitted sequentially to cross-sectional data are not intertemporally consistent with the dynamics of a given interest rate model. Björk and Christensen (1999) prove, for instance, that the Nelson-Siegel family of curves is inconsistent with the Ho-Lee interest rate model and with the Hull-White extension of the Vasicek model. This feature weakens the validity of the model for applications that involve a time-series context. It can be shown, however, that a simple manifold expansion (i.e. the addition of appropriate functions of maturity) is sufficient to make the Nelson and Siegel model consistent with given interest rate models,, namely with the generalized Vasicek short rate model.[6] These adjustments impose, nonetheless, additional constraints on the estimation of the models to cross-sectional data leading thus to a non-trivial deterioration of the fitting performance when compared with that provided by the Nelson-Siegel-Svensson family of curves. On the other hand, it is not obvious to us that the use of arbitrage-free models is necessary or desirable for accomplishing good immunization performance. As a matter of fact, if the theoretical superiority of equilibrium term structure models is unquestionable, when compared to traditional immunizing duration models, the truth is that a number of papers, such as Ingersoll (1983), Nelson and Schaefer (1983) and Brennan and Schwartz (1983), have show that their immunization performance is rather similar. In addition, Brandt and Yaron (2003) prove that typical no-arbitrage models are actually time-inconsistent because their parameters are assumed constant for pricing purposes even though the parameters change each time the model is recalibrated to data observed on a given date. Moreover, recent studies (e.g. Duffie, 2002 and Dai and Singleton, 2002) have shown that affine no-arbitrage models can produce poor forecasts.

The second theoretical shortcoming is that these models apparently lack a fundamental economic foundation, which leaves researchers cautious about interpreting the parameters in conjunction with economic variables, and may explain why their use has been limited to cross-sectional applications, namely yield-curve fitting and interest rate risk manage-

[4]Bank of International Settlements (1999) notes that ten Central Banks (of twelve surveyed) routinely use either the Nelson and Siegel (1987) and/or the Svensson (1994) model as their primary method for analysing the yield curve. See Bravo (2001), Barrett et al. (1995), Diebold and Li (2003) for other uses of the NS model.

[5]For more details on the estimation process see, for example, Nelson and Siegel (1987), BIS (1999) and Bolder and Stréliski (1999).

[6]See Björk and Christensen (1999), Filipovic (2000) and Krippner (2005a).

ment. An exception is given by Diebold and Li (2003) who use variations on the Nelson-Siegel framework to model the entire yield curve on an intertemporally basis, as a three-dimensional parameter evolving dynamically.[7] The authors prove, first, that the model is consistent with standard stylized facts regarding the yield curve and, second, that the three-time varying parameters may be roughly interpreted as factors corresponding to level, slope and curvature, a result consistent with previous studies on this subject.[8]

From (1) the continuously compounded zero-coupon curve $r(t, \mathbf{a})$ can be derived noting that $r(t, \mathbf{a}) = \frac{1}{t} \int_0^t f(t, \mathbf{a}) dt$:

$$\begin{aligned} r(t, \mathbf{a}) \quad = \quad & a_0 + a_1 \frac{a_4}{t} \left(1 - e^{-\frac{t}{a_4}}\right) + a_2 \frac{a_4}{t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4}\right)\right] \\ & + a_3 \frac{a_5}{t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5}\right)\right], \end{aligned} \tag{2}$$

whereas the discount function $d(t, \mathbf{a})$ is defined as:

$$d(t, \mathbf{a}) = \exp\left[-r(t, \mathbf{a})t\right]. \tag{3}$$

Each parameter in (1) has a particular impact on the shape of the forward rate curve. Parameter a_0, which represents the asymptotic value of $f(t, \mathbf{a})$ (i.e., $\lim_{t\to\infty} f(t, \mathbf{a}) = a_0$), can actually be regarded as a long-term (consol) interest rate. Parameter a_1 defines the speed with which the curve tends towards its long-term value. The yield curve will be upward sloping if $a_1 < 0$ and downward-sloping if $a_1 > 0$. The higher the absolute value of a_1 the steeper the yield curve. Notice also that the sum of a_0 and a_1 corresponds to the instantaneous forward rate with an infinitesimal maturity ($\lim_{t\to 0} f(t, \mathbf{a}) = a_0 + a_1$), i.e., it defines the intercept of the curve. Parameters a_2 and a_3 have similar meaning and influence the shape of the yield curve. They determine the magnitude and the direction of the first and second humps, respectively. For example, if a_2 is positive, a hump will occur at a_4 whereas, if a_2 is negative, a U-shape value will emerge at a_4. Parameters a_4 and a_5, which are always positive, have similar roles and define the position of the first and second humps, respectively.

The Svensson model is very intuitive since parameters a_0, $a_{1,}$ a_2 and a_3 (the interest rate factors) can directly be linked to parallel displacements, slope changes and curvature shifts in the yield curve, given that scale coefficients are fixed. To perceive this behaviour, **Figure 1** displays the sensitivity $S_k = \frac{\partial f(t,\mathbf{a})}{\partial a_k}$ of forward rates to each parameter a_k, for $k = 0, ..., 3$.

As can be seen, the sensitivity of forward rates with respect to the consol rate is constant across the whole maturity spectrum, which means that it can actually be regarded as a level factor. In other words, the level factor S_0 fundamentally represents a parallel displacement in the term structure of interest rates. The sensitivity of interest rates to changes in parameter a_1 shows a descending shape, first larger for shorter maturities, then declining exponentially toward zero as maturity increases. In this sense, factor S_1 is a slope factor and represents changes in the steepness of the yield curve. Finally, factors S_2 and S_3 have

[7] See also Krippner (2005b).

[8] See, for instance, Litterman and Scheinkman (1991), Barber and Copper (1996), Knez et al. (1994), D'Ecclesia and Zenios (1994) and Bravo and Silva (2005).

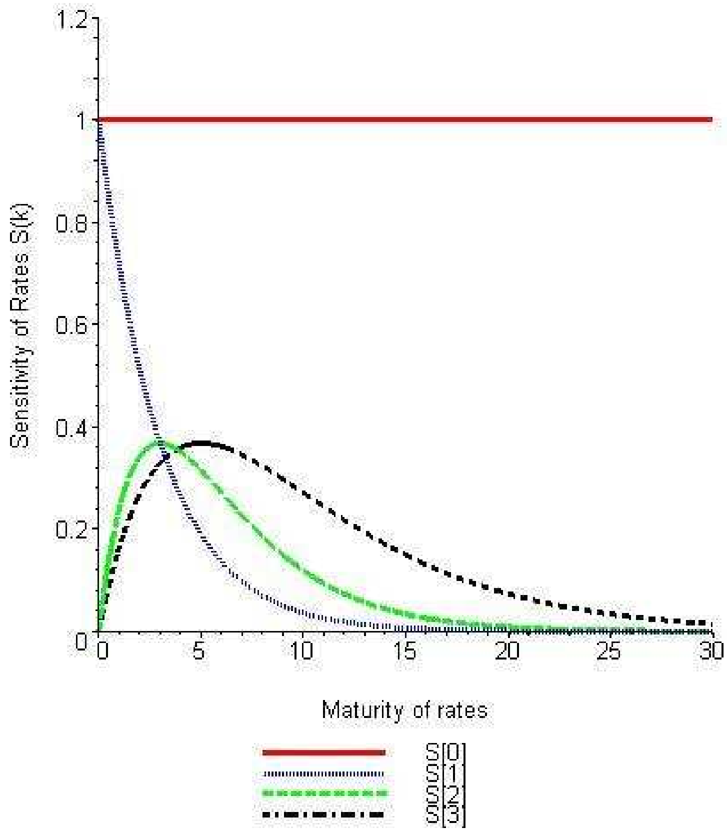

Figure 1. Sensitivity of forward rates to the parameters of the Svensson mathematical function; these sensitivities are obtained by fixing parameter values equal to $a_4 = 3$ and $a_5 = 5$.

different impacts on intermediate rates as opposed to extreme maturities (short and long), reaching a maximum on those points (a_4 and a_5, respectively) where the yield curve has humps. Hence, these factors may be interpreted as curvature factors. In brief, the Svensson model assumes that: (i) every movement in the term structure of interest rates can be approximated by changes in only four factors; (ii) these factors take familiar shapes, namely parallel shifts, changes in steepness, and changes in the curvature of the yield curve.

3. Constructing Immunized Portfolios

Consider an investor who has a position in a number L of default-free bonds. Let c_{lt} denote the nominal cash flow (in monetary units) received from bond l $(l = 1, ..., L)$ at time t $(t = 1, ..., N)$. Let $t = 0$ be the current date, and H a known, finite investment horizon, measured in years. Assuming that the initial term structure is known and described by the parametric function (2), which assigns a spot rate to each payment date t, the present value of bond l, $B_0^l(\mathbf{a})$, is given by:

$$\begin{aligned} B_0^l(\mathbf{a}) &= \sum_{t=1}^{N} c_{lt} \exp\left[-r(t,\mathbf{a})t\right] \\ &= \sum_{t=1}^{N} c_{lt} e^{-\left\{a_0+a_1\frac{a_4}{t}\left(1-e^{-\frac{t}{a_4}}\right)+a_2\frac{a_4}{t}\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]+a_3\frac{a_5}{t}\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]\right\}t}, \end{aligned} \tag{4}$$

where we have stressed the functional relationship between the bond price $B^l(\mathbf{a})$ and the initial vector $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)^T$ of parameters of the forward rate function. Let n_l represent the number of type l bonds in the portfolio. In this case, the present value (at time 0) of this bond portfolio, $P_0(\mathbf{a})$, is given by:

$$\begin{aligned} P_0(\mathbf{a}) &= \sum_{l=1}^{L} n_l B_0^l(\mathbf{a}) = \sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{a})t\right] \\ &= \sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} e^{-\left\{a_0+a_1\frac{a_4}{t}\left(1-e^{-\frac{t}{a_4}}\right)+a_2\frac{a_4}{t}\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]+a_3\frac{a_5}{t}\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]\right\}t} \end{aligned} \tag{5}$$

For simplicity of exposition, consider now that the investor is interested only in his wealth position at some future time H (where H might represent, for example, the due date on a single liability payment). The value of this portfolio at time H, under the expectations hypothesis of the term structure assuming no change in the yield curve, $P_H(\mathbf{a})$, will be:

$$\begin{aligned} P_H(\mathbf{a}) &= P_0(\mathbf{a}) \exp\left[r(H,\mathbf{a})H\right] \\ &= \left[\sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{a})t\right]\right] \exp\left[r(H,\mathbf{a})H\right] \end{aligned} \tag{6}$$

Suppose now that at time τ, immediately after the investor purchased the portfolio, the spot rate function has undergone a variation, which may be viewed here as a vector $d\mathbf{A}$ of multiple random shifts and represent both parallel and nonparallel shifts, such that the new term structure, represented again by Svensson's model, is $r_\tau(t,\mathbf{A}) = r(t, \mathbf{a} + d\mathbf{A})$:

$$\begin{aligned} r_\tau(t,\mathbf{A}) &= A_0 + A_1\frac{A_4}{t}\left(1-e^{-\frac{t}{A_4}}\right) + A_2\frac{A_4}{t}\left[1-e^{-\frac{t}{A_4}}\left(1+\frac{t}{A_4}\right)\right] \\ &\quad + A_3\frac{A_5}{t}\left[1-e^{-\frac{t}{A_5}}\left(1+\frac{t}{A_5}\right)\right], \end{aligned} \tag{7}$$

where $\mathbf{A} = (A_0, ..., A_5)^T$ denotes the new vector of coefficients of the spot rate function estimated at time τ. The new terminal value of the portfolio, $P_H(\mathbf{A})$, keeps the same form as above, except that vector $\mathbf{A}$ now replaces the initial vector of parameters $\mathbf{a}$:

$$P_H(\mathbf{A}) = \left[\sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{A})t\right]\right] \exp\left[r(H,\mathbf{A})H\right] \tag{8}$$

The traditional definition of immunization (e.g. Fisher and Weil, 1971) for the case of a single liability establishes that a portfolio of default-free bonds is said to be immunized against any type of interest rate shifts if its accumulated value at the end of the planning horizon is at least as great as the target value, where the target value is defined as the portfolio value at the horizon date under the scenario of no change in the spot (and forward) rates. Stated more formally, by immunization we mean selection of a bond portfolio such that the actual future value of the income stream $P_H(\mathbf{A})$ at time H will exceed the initially expected value $P_H(\mathbf{a})$, i.e., $P_H(\mathbf{A}) \geq P_H(\mathbf{a})$ (or equivalently, $\Delta P_H = P_H(\mathbf{A}) - P_H(\mathbf{a}) \geq 0$), if the interest rates $r(t, \mathbf{A})$ shift to their new value $r_\tau(t, \mathbf{A})$.

Under the assumption that interest rates only change by a parallel shift, the main conclusion of Fisher and Weil was that immunization is achieved when the duration of the portfolio is set equal to the length of the investment horizon. The assumption that interest rates can change only by a parallel shift is very restrictive and can carry serious risks. In this chapter we offer a more generalized approach to immunization by deriving the conditions under which the investment is protected against both parallel and non-parallel yield curve shifts.

3.1. First-Order Conditions

Let $P_H(\mathbf{A})$ be a multivariate price function, assumed to be twice continuously differentiable. The idea is to use a Taylor series expansion of $P_H(\mathbf{A})$ around the initial vector of parameters in order to evaluate the necessary and sufficient conditions for a local minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. For most practical applications, an expansion up to the second order is sufficient to obtain a reasonable approximation. The quadratic approximation for (8) is then given by:[9]

$$dP_H(\mathbf{A}) = P_H(\mathbf{A}) - P_H(\mathbf{a}) = \nabla P_H(\mathbf{a})^T \cdot d\mathbf{a} + \frac{1}{2} d\mathbf{a}^T \cdot \nabla^2 P_H(\mathbf{a}) \cdot d\mathbf{a} + R_2(\mathbf{a}, d\mathbf{A}), \quad (9)$$

where $d\mathbf{A} = (da_i)^T_{i=0,\ldots,5}$ denotes the (column) vector of variations of parameters $\mathbf{a}$, $\cdot$ denotes the inner product of two vectors and $R_2(\mathbf{a}, d\mathbf{A})$ represents the remaining terms of the series. Terms $\nabla P_H(\mathbf{A})$ and $\nabla^2 P_H(\mathbf{A})$ represent, respectively, the gradient vector and the Hessian matrix of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. Alternatively, if we divide (9) by $P_H(\mathbf{A})$ we obtain the percentage change in the terminal value of the bond portfolio

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{P_H(\mathbf{A})} \nabla P_H(\mathbf{a})^T \cdot d\mathbf{a} + \frac{1}{2} d\mathbf{a}^T \cdot \frac{1}{P_H(\mathbf{A})} \nabla^2 P_H(\mathbf{a}) \cdot d\mathbf{a} + R_2^*(\mathbf{a}, d\mathbf{A}), \quad (10)$$

where $R_2^*(\mathbf{a}, d\mathbf{A}) = R_2(\mathbf{a}, d\mathbf{A})/P_H(\mathbf{A})$. Let now $c_t = \sum_{l=1}^{L} n_l c_{lt}$ denote the total nominal cash flows received by the holder of the portfolio at time t. To determine the nature of the horizon value near the origin we compute the first-order partial derivative of (8) with respect to A_k $(k = 0, ..., 5)$. This yields the generic element of the gradient vector $\frac{\partial P_H(\mathbf{A})}{\partial A_k}$

$$\begin{aligned} \frac{\partial P_H(\mathbf{A})}{\partial A_k} &= \sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]} \left[H \frac{\partial r(H, \mathbf{A})}{\partial A_k} - t \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] \quad (11) \\ &= P_H(\mathbf{A}) \left[H \frac{\partial r(H, \mathbf{A})}{\partial A_k} \right] - \left[\sum_{t=1}^{N} t c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] \end{aligned}$$

[9]Note that the change in the portfolio value resulting from the passage of time is ignored here due to its deterministic nature.

which, after dividing by $P_H(\mathbf{A})$, can be written as

$$\frac{1}{P_H(\mathbf{A})}\frac{\partial P_H(\mathbf{A})}{\partial A_k} = H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - \frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right] \tag{12}$$

In anticipation of combining (12) and (10) we introduce new definitions for parametric interest rate risk measures.

Definition 1 *The parametric duration of a bond is a measure of first-order sensitivity of bond prices to changes in interest rates as given by modifications in parameters* A_k $(k = 0, ..., 5)$*. For bond* l*, the parametric duration is denoted* $D^{(l)}(k,\mathbf{A})$*, and is defined, for* $B_0^l(\mathbf{A}) \neq 0$*, as follows:*

$$\begin{aligned} D^{(l)}(k,\mathbf{A}) &= -\frac{1}{B_0^l(\mathbf{A})}\frac{\partial B_0^l(\mathbf{A})}{\partial A_k} \\ &= \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} tc_{lt} e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]. \end{aligned} \tag{13}$$

Definition 2 *Let* $w_l = \frac{n_l B_0^l(\mathbf{A})}{P_0(\mathbf{A})}$ *denote the percentage of portfolio invested in bond* l*, such that* $\sum_{l=1}^{L} w_l = 1$*. The parametric duration of a bond portfolio is a measure of first-order sensitivity of a bond portfolio to changes in interest rates as given by modifications in parameters* A_k $(k = 0, ..., 5)$*. It is calculated as the weighted average of the parametric durations of the bonds making up the portfolio, the weights being the shares of each bond in the portfolio. Denoted* $D^{(P)}(k,\mathbf{A})$*, it is defined, for* $P_0(\mathbf{A}) \neq 0$*, as follows:*

$$\begin{aligned} D^{(P)}(k,\mathbf{A}) &= -\frac{1}{P_0(\mathbf{A})}\frac{\partial P_0(\mathbf{A})}{\partial A_k} \\ &= \sum_{l=1}^{L} w_l D^{(l)}(k,\mathbf{A}). \end{aligned} \tag{14}$$

Each equation in (13) represents a bond's interest rate risk measure for a particular type of shift in the yield curve. For instance, the first element, $D^{(l)}(0,\mathbf{A})$, is defined as

$$D^{(l)}(0,\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} tc_{lt} e^{-r(t,\mathbf{A})t}\right] \tag{15}$$

and corresponds to the traditional Fisher-Weil duration measure. It is defined as the weighted average the times of payment of all the cashflows generated by the bond, the weights being the shares of the bond's cashflows in the bond's present value, and captures the sensitivity of bond returns to changes in the consol factor a_0, i.e., the responsiveness of bond returns to height shifts in the term structure of interest rates. The second element, $D^{(l)}(1,\mathbf{A})$, is defined as

$$D^{(l)}(1,\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}\left(1 - e^{-\frac{t}{a_4}}\right)a_4\right] \tag{16}$$

and captures the sensitivity of bond returns to changes in parameter a_1, that is, to changes in the slope of the yield curve. The third, $D^{(l)}(2, \mathbf{A})$, and fourth, $D^{(l)}(3, \mathbf{A})$, elements of the duration vector summarize the sensitivity of bond returns to changes in the curvature parameters a_2 and a_3, and are defined as

$$D^{(l)}(2, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] a_4 \right\} \tag{17}$$

and

$$D^{(l)}(3, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right] a_5 \right\} \tag{18}$$

respectively. Finally, The fourth, $D^{(l)}(4, \mathbf{A})$, and fifth, $D^{(l)}(5, \mathbf{A})$, elements of the duration vector summarize the sensitivity of bond returns to changes in the location parameters a_4 and a_5, and are defined as

$$D^{(l)}(4, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[\left(\frac{a_1}{t} + \frac{a_2}{t} \right) (1 - e^{-\frac{t}{a_4}}) - \left(\frac{a_1}{a_4} + \frac{a_2}{a_4} \right) e^{-\frac{t}{a_4}} - a_2 \frac{t}{a_4^2} e^{-\frac{t}{a_4}} \right] \right\} \tag{19}$$

and

$$D^{(l)}(5, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[\frac{a_3}{t} \left(1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} - \frac{t^2}{a_5^2} \right) \right) \right] \right\} \tag{20}$$

respectively. Taking this into account, the generic element of the gradient vector (12) can be simplified to

$$\frac{1}{P_H(\mathbf{A})} \frac{\partial P_H(\mathbf{A})}{\partial A_k} = H \frac{\partial r(H, \mathbf{A})}{\partial A_k} - D^{(P)}(k, \mathbf{A}) \tag{21}$$

Let us now address first-order conditions for bond portfolio immunization. For simplicity of exposition, we assume that parameters a_4 and a_5 are fixed at prespecified values.[10] We know from standard optimization theory that if a function partial differentiable has an extremum at an interior point then all first-order derivatives are required to be zero.[11] In other words, setting the gradient vector equal to zero is a necessary (but clearly not sufficient) condition for an interior local minimum. From (21) this is equivalent to a fourth-dimensional vector of the form

$$D^{(P)}(k, \mathbf{A}) = H \frac{\partial r(H, \mathbf{A})}{\partial A_k} \qquad (k = 0, \ldots, 3). \tag{22}$$

Each of the conditions in (22) defines an immunization condition for a different type of yield curve shift. For instance, selecting a bond portfolio such that its $D^{(P)}(0, \mathbf{A})$ is set equal to the planning horizon H protects the investment against a parallel shift in the yield curve. In other words, the traditional approach to immunization can be considered, to some extend, a particular case of the parametric model. Similarly, immunization against slope

[10] The approach can easily be expanded to admit changes in the location of the humps of the forward curve.
[11] See, for example, Apostol (1969).

shifts is attained if the condition $D^{(P)}(1, \mathbf{A}) = a_4 \left[1 - \exp(-\frac{H}{a_4})\right]$ is fulfilled. Finally, appropriate protection against changes in the curvature of the term structure is obtained by choosing a portfolio's composition such that

$$D^{(P)}(2, \mathbf{A}) = a_4 \left[1 - e^{-\frac{H}{a_4}} \left(1 + \frac{H}{a_4}\right)\right] \text{ and } D^{(P)}(3, \mathbf{A}) = a_5 \left[1 - e^{-\frac{H}{a_5}} \left(1 + \frac{H}{a_5}\right)\right].$$

To sum up, from equation (22) two implications follow immediately. First, the vector of parametric duration measures is determined only by the structure of the bond portfolio and, therefore, can be controlled by the portfolio manager. Second, given that convexity conditions are respected and a sufficient number of bonds are available (i.e. $L \geq 4$ or $L \geq 5$, if we include the initial self-financing constraint), complete immunization against interest rate changes (both parallel and non-parallel) can be achieved by selecting a bond portfolio such that all of the first-order immunization constraints are satisfied. Note that the investor can always adopt a more active role in the immunization strategy by choosing, deliberately, to satisfy only some of the conditions in (22). He can, for example, use the principal components analysis to select those interest rate shifts that are more likely or account most for the volatility of the yield curve and then engage in the appropriate immunization strategy. Alternatively, investors may try to obtain a yield pick-up and at the same time to be risk neutral against a change in the level and/or the yield curve by engaging in butterfly trades.

In those cases where there is more than one bond portfolio satisfying all of the immunization constraints, a particular objective function might be considered. For example, Chambers et al. (1988) argue that an acceptable portfolio construction criteria would be to minimize the sum of squared weights, i.e., $Min \sum_{l=1}^{L} w_l^2$. According to them, this will lead to a diversified portfolio that minimizes the impact of unsystematic risk caused by transitory pricing errors.

Finally, note that similar to Prisman and Shores (1988), except for the trivial case where a single zero coupon bond maturing on the planning horizon composes the portfolio[12], the solution to the immunization constraints given in equation (22) requires short positions in some bonds, i.e., any immunized portfolio must have both positive and negative cash flows. The non-monotone nature of the cash flow structure makes the existence of local minima at $\mathbf{A} = \mathbf{a}$ more problematic. In particular, we will see below that 'most' first-order immunized portfolios yield a horizon value which is not locally convex with respect to perturbations in the yield curve parameters.

3.2. Second-Order Conditions

We know from standard optimization theory that setting the gradient vector $\nabla P_H(\mathbf{A})$ equal to zero is a necessary but not sufficient condition for a minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. Let us now address second-order conditions and their implications for portfolio construction. For a local minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$, second-order conditions stipulate that to equations (22) we have to add those corresponding to a positive semidefinite Hessian matrix for $P_H(\mathbf{A})$. The generic element of the Hessian matrix, $\gamma_{km}(\mathbf{A}) = \frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m}$, is derived from (11) by

[12]Paradoxically, the existence of such a bond would mean that the immunization strategy is unnecessary.

taking the partial derivative with respect to A_m $(m = 0, ..., 3)$.[13]

$$\begin{aligned}\gamma_{km}(\mathbf{A}) &= \frac{\partial}{\partial A_m}\left\{\sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]}\left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - t\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\right\} \qquad (23)\\ &= \sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]}\left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - t\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\left[H\frac{\partial r(H,\mathbf{A})}{\partial A_m} - t\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right]\end{aligned}$$

To simplify notation, let

$$q_t = c_t \exp\left[r(H,\mathbf{A})H - r(t,\mathbf{A})t\right] \qquad (t = 1, ..., N) \qquad (24)$$

represent the cash flow received from portfolio at time t expressed in future value. From (23) $\gamma_{km}(\mathbf{A})$ is then

$$\begin{aligned}\gamma_{km}(\mathbf{A}) = \sum_{t=1}^{N} q_t \Bigg[&H^2\frac{\partial r(H,\mathbf{A})}{\partial A_k}\frac{\partial r(H,\mathbf{A})}{\partial A_m} - H\frac{\partial r(H,\mathbf{A})}{\partial A_k}t\frac{\partial r(t,\mathbf{A})}{\partial A_m}\\ &-H\frac{\partial r(H,\mathbf{A})}{\partial A_m}t\frac{\partial r(t,\mathbf{A})}{\partial A_k} + t^2\frac{\partial r(t,\mathbf{A})}{\partial A_k}\frac{\partial r(t,\mathbf{A})}{\partial A_m}\Bigg]\end{aligned}$$

with $(k, m = 0, ..., 3)$, or equivalently

$$\begin{aligned}\gamma_{km}(\mathbf{A}) = {}& H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right)\sum_{t=1}^{N} q_t - H\frac{\partial r(H,\mathbf{A})}{\partial A_k}\sum_{t=1}^{N} t q_t \frac{\partial r(t,\mathbf{A})}{\partial A_m}\\ &-H\frac{\partial r(H,\mathbf{A})}{\partial A_m}\sum_{t=1}^{N} t q_t\frac{\partial r(t,\mathbf{A})}{\partial A_k} + \sum_{t=1}^{N} t^2 q_t \frac{\partial r(t,\mathbf{A})}{\partial A_k}\frac{\partial r(t,\mathbf{A})}{\partial A_m} \qquad (25)\end{aligned}$$

Dividing both term in (23) by $P_H(\mathbf{A})$ we get

$$\begin{aligned}\frac{1}{P_H(\mathbf{A})}\frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} = {}& H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) \qquad (26)\\ &-H\frac{\partial r(H,\mathbf{A})}{\partial A_k}\left\{\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right]\right\}\\ &-H\frac{\partial r(H,\mathbf{A})}{\partial A_m}\left\{\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\right\}\\ &+\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right]\end{aligned}$$

where in (26) we have made use of the fact that $\sum_{t=1}^{N} q_t = P_H(\mathbf{A}) = P_0(\mathbf{A})e^{r(H,\mathbf{A})H}$. We are now in conditions to introduce the essential definitions of parametric convexity of a bond and of a bond portfolio.

[13] In Equation (23) we have made use of the fact that all second-order cross partial derivatives are zero, i.e., $\frac{\partial}{\partial A_m}\left(\frac{\partial^2 r(\cdot,\mathbf{A})}{\partial A_k}\right) = 0$, $m = 0, ..., 3$.

Definition 3 *The parametric convexity of a bond is a measure of second-order sensitivity of bond prices to changes in interest rates as given by modifications in parameters A_k and A_m $(k, m = 0, ..., 3)$. For bond l, the parametric convexity is denoted $C^{(l)}(k, m, \mathbf{A})$, and is equal, for $B_0^l(\mathbf{A}) \neq 0$, to:*

$$
\begin{aligned}
C^{(l)}(k, m, \mathbf{A}) &= \frac{1}{B_0^l(\mathbf{A})} \frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_k \partial A_m} \\
&= \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t} \left(\frac{\partial r(t, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(t, \mathbf{A})}{\partial A_m} \right) \right] \qquad (27)
\end{aligned}
$$

Definition 4 *Let $w_l = \frac{n_l B_0^l(\mathbf{A})}{P_0(\mathbf{A})}$ denote the percentage of portfolio invested in bond l, such that $\sum_{l=1}^{L} w_l = 1$. The parametric convexity of a bond portfolio is a measure of second-order sensitivity of a bond portfolio to changes in interest rates as given by modifications in parameters A_k and A_m $(k, m = 0, ..., 3)$. It is calculated as the weighted average of the parametric convexities of the bonds making up the portfolio, the weights being the shares of each bond in the portfolio. Denoted $C^{(P)}(k, m, \mathbf{A})$, is equal, for $P_0(\mathbf{A}) \neq 0$, to:*

$$
\begin{aligned}
C^{(P)}(k, m, \mathbf{A}) &= \frac{1}{P_0(\mathbf{A})} \frac{\partial^2 P_0(\mathbf{A})}{\partial A_k \partial A_m} \\
&= \sum_{l=1}^{L} w_l C^{(l)}(k, m, \mathbf{A}) \qquad (28)
\end{aligned}
$$

To simplify notation let $C_{k,m}^{(l)}(\mathbf{A}) = C^{(l)}(k, m, \mathbf{A})$. Each equation in (27) measures second-order effects for a particular type of shift in the term structure. For instance, the equation for $C_{0,0}^{(l)}(\mathbf{A})$ is defined as:

$$
C_{0,0}^{(l)}(\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_0^2} = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t} \right]. \qquad (29)
$$

Surprisingly, or not, the parametric model provides a second-order sensitivity measure of bond's price to changes in the level coefficient of the yield curve that is similar to the traditional (continuously compounded) definition of convexity.[14] We can then conclude, once again, that the traditional approach to immunization can be considered a particular case of the parametric model. Second-order effects caused by shifts in the slope parameter a_1 only can now be quantified by using:

$$
C_{1,1}^{(l)}(\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right)^2 a_4^2 \right], \qquad (30)
$$

[14]See, for example, Lacey and Nawalkha (1993).

and so on. The complete set of definitions can be found in the **Appendix**. Substituting (28) and (14) in (26) yields

$$\begin{aligned}\frac{1}{P_H(\mathbf{A})}\frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} &= H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) - H\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)D^{(P)}(k,\mathbf{A}) \\ &\quad -H\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right)D^{(P)}(m,\mathbf{A}) + C^{(P)}(k,m,\mathbf{A}) \qquad (31)\end{aligned}$$

From the first order conditions (22) for bond portfolio immunization we know that

$$D^{(P)}(k,\mathbf{A}) = H\frac{\partial r(H,\mathbf{A})}{\partial A_k} \qquad (k=0,...,3) \qquad (32)$$

which is also valid when k is replaced by m $(m=0,...,3)$. Substituting this expression in (31), the generic element of the Hessian matrix at $\mathbf{A}=\mathbf{a}$ becomes $(k,m=0,...,3)$:

$$\begin{aligned}\frac{1}{P_H(\mathbf{A})}\frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} &= H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) \\ &\quad -2H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(H\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) + C^{(P)}(k,m,\mathbf{A}) \\ &= C^{(P)}(k,m,\mathbf{A}) - H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) \qquad (33)\end{aligned}$$

Let ω_{km} denote the difference

$$\omega_{km}(\mathbf{A}) = C^{(P)}(k,m,\mathbf{A}) - H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) \qquad (34)$$

Each element in (34) has a clear interpretation since it defines the difference between the parametric convexity of a bond portfolio and the sensitivity of the perfect immunization asset (i.e. of a zero coupon maturing on the horizon date) to changes in the yield curve shape parameters.[15] That is, each element in (34) represents the extent to which second-order interest rate risk measures deviate from the target. This is not surprising since from equation (33) we observe that all elements $\gamma_{km}(\mathbf{A})$ of the Hessian matrix $\nabla^2 P_H(\mathbf{A})$ are those of the matrix $\Pi = [\omega_{km}]_{k,m=0}^{3}$. This means that the discussion of the positive semidefiniteness of $\nabla^2 P_H(\mathbf{A})$ reduces to that of the symmetric matrix Π. At least two alternative methodologies can be used to determine the sign definiteness of the Hessian matrix: The **determinantal test approach** and the **eigenvalue test approach**. We will show how can both be used in the context of immunization.

3.2.1. Determinantal Test Approach

Let us focus first on the use of the determinantal test approach. Let Π be a square $(n \times n)$ symmetric matrix of the form

$$\Pi = \begin{bmatrix} \omega_{00} & \omega_{01} & ... & \omega_{0k} \\ \omega_{10} & \omega_{11} & ... & \omega_{1k} \\ ... & ... & & ... \\ \omega_{k0} & \omega_{k1} & ... & \omega_{kk} \end{bmatrix},\ \omega_{ij} = \omega_{ji},\ i \neq j$$

[15]Note also that we can interpret $w_{km}(\mathbf{A})$ as a sort of "generalized variance" since its expression is analogous to the formula $Var(X) = E(X^2) - (E(X))^2$.

with $n = 4$. The jth order leading principal minors of the matrix Π, denoted D_j $(j = 1, ..., 4)$, are the determinants of the submatrices formed by deleting the entries in the last $n - j$ rows and columns of Π. Given Π, we may also define the jth order principal minors of Π, denoted $|D_j|$, as the determinants of the submatrices formed by deleting the entries in the $n - j$ rows and the corresponding $n - j$ columns of Π. Following these definitions, the criteria for semidefiniteness requires that for Π to be positive semidefinite, all of its principal minors of order j must be non-negative, i.e., $|D_j| \geq 0$.[16] Let us consider now the implications of this result for bond portfolio immunization. The first-order principal minors of Π, $\left|D_1^i\right|$ $(i = 1, ..., 4)$ are:

$$\left|D_1^1\right| = \omega_{00} \quad \text{and} \quad \left|D_1^2\right| = \omega_{11} \quad \text{and} \quad \left|D_1^3\right| = \omega_{22} \quad \text{and} \quad \left|D_1^4\right| = \omega_{33}, \tag{35}$$

which must be all positive or zero. From the definitions of ω_{00}, ω_{11}, ω_{22} and ω_{33} above we can observe that its sign is determined by the portfolio structure and cannot, unfortunately, be determined without ambiguity. The task is even more difficult when we recap that matching first-order conditions requires short positions in some bonds. Consequently, since the positive definiteness of $\nabla^2 P_H(\mathbf{A})$ cannot be guaranteed by first-order conditions, we are forced to conclude that setting the gradient vector $\nabla P_H(\mathbf{A})$ equal to zero is not sufficient to protect the investment against changes in the yield curve. This means that second-order conditions play an important role in the immunization problem and need to be addressed in a convenient way.

To ensure the positive semidefiniteness of Π we need then to impose certain restrictions on portfolio's composition. Assume, for instance, that $\omega_{00} = \omega_{11} = \omega_{22} = 0$. The second-order principal minors of Π, $\left|D_2^i\right|$ $(i = 1, ..., 6)$, are defined as:

$$\begin{aligned}
\left|D_2^1\right| &= \begin{vmatrix} \omega_{00} & \omega_{01} \\ \omega_{10} & \omega_{11} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{01} \\ \omega_{01} & 0 \end{vmatrix} \quad \text{and} \quad \left|D_2^2\right| = \begin{vmatrix} \omega_{00} & \omega_{02} \\ \omega_{20} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{02} \\ \omega_{02} & 0 \end{vmatrix} \\
\left|D_2^3\right| &= \begin{vmatrix} \omega_{00} & \omega_{03} \\ \omega_{30} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{03} \\ \omega_{03} & \omega_{33} \end{vmatrix} \quad \text{and} \quad \left|D_2^4\right| = \begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{12} \\ \omega_{12} & 0 \end{vmatrix} \\
\left|D_2^5\right| &= \begin{vmatrix} \omega_{11} & \omega_{13} \\ \omega_{31} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{13} \\ \omega_{13} & \omega_{33} \end{vmatrix} \quad \text{and} \quad \left|D_2^6\right| = \begin{vmatrix} \omega_{22} & \omega_{23} \\ \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{23} \\ \omega_{23} & \omega_{33} \end{vmatrix}.
\end{aligned} \tag{36}$$

From (36) we observe that the determinants $\left|D_2^1\right|$, $\left|D_2^2\right|$ and $\left|D_2^3\right|$ are equal to $-(\omega_{01})^2$, $-(\omega_{02})^2$ and $-(\omega_{03})^2$, respectively, which are all negative, violating thus the conditions for positive semidefiniteness. For these minors to be positive or zero ω_{01}, ω_{02} and ω_{03} must be all set equal to zero. Similarly, from (36) we note that the values of $\left|D_2^4\right|$, $\left|D_2^5\right|$ and $\left|D_2^6\right|$ are all negative and equal to $-(\omega_{12})^2$, $-(\omega_{13})^2$ and $-(\omega_{23})^2$, respectively. Using the same argument, to ensure the positive semidefiniteness of Π, we need to select a bond portfolio such that the entries ω_{12}, ω_{13} and ω_{23} are all equal to zero. Let us turn now to the third-order principal minors of Π, $\left|D_3^i\right|$ $(i = 1, ..., 4)$. They can be written as:

$$\left|D_3^1\right| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

[16] See Takayama (1990) and references therein for an extensive discussion of the determinantal test for second-order necessary conditions for a minimum.

$$\left|D_3^2\right| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{13} \\ \omega_{30} & \omega_{31} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}$$

$$\left|D_3^3\right| = \begin{vmatrix} \omega_{00} & \omega_{02} & \omega_{03} \\ \omega_{20} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}$$

$$\left|D_3^4\right| = \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}, \tag{37}$$

and, as can be seen above, their values are all equal to zero. Finally, by definition the fourth-order principal minor of Π, $|D_4|$, is equal to the determinant of Π. Therefore, we have

$$|D_4| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{33} \end{vmatrix}, \tag{38}$$

which is also equal to zero. To sum up, to guarantee the positive semidefiniteness of Π we need to select a bond portfolio such that all entries ω_{km} are equal to zero, except one, equal to $\omega_{33} = C^{(P)}(3,3,\mathbf{A}) - \left\{a_5\left[1 - e^{-\frac{H}{a_2}}\left(1 + \frac{H}{a_5}\right)\right]\right\}^2$, which must be set to an arbitrary positive value U. Accordingly, whereas first-order conditions for bond portfolio immunization imply the following $k+1$ $(k = 0, ..., 3)$ restrictions

$$D^{(P)}(k,\mathbf{A}) = H\frac{\partial r(H,\mathbf{A})}{\partial A_k},$$

second-order conditions entail the subsequent equations

$$C^{(P)}(k,m,\mathbf{A}) = \begin{cases} H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) + U, & k = m = 3 \\ H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right), & \text{other cases} \end{cases}, \tag{39}$$

to which the self-financing constraint may be added. The solution to the above immunization problem requires a considerable number of different bonds ($L \geq 14$ or $L \geq 15$, if we include the initial self-financing constraint) in the portfolio. Given that a sufficient number of bonds are available, it is theoretically possible to immunize a bond portfolio against both parallel and non-parallel interest rate shifts. Standard optimization techniques may be used to determine the immunizing portfolio. Let us now come back to the Hessian matrix. From (39) it reduces to:

$$\frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})} = \begin{bmatrix} 0 & ... & & 0 \\ ... & & & ... \\ & & 0 & 0 \\ 0 & ... & 0 & U \end{bmatrix}. \tag{40}$$

The associated quadratic form is then

$$d\mathbf{A}^T \begin{bmatrix} 0 & \dots & & 0 \\ \cdots & & & \cdots \\ & & 0 & 0 \\ 0 & \dots & 0 & U \end{bmatrix} d\mathbf{A} = U \cdot (dA_3)^2. \tag{41}$$

Taking into account both first-order and second-order conditions for immunization, the percentage change in the terminal value of the bond portfolio can be expressed in the following manner:

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{2}U \cdot (dA_3)^2 + R_2^*(\mathbf{a}, d\mathbf{A}), \tag{42}$$

where $R_2^*(\mathbf{a}, d\mathbf{A})$ represents again the remaining terms of the Taylor series. Given that by definition $R_2^*(\mathbf{a}, d\mathbf{A}) \to 0$ as $d\mathbf{A} \to 0$, we can always choose a value U such that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} > 0$, i.e., we can always choose a value U such that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ is convex at $\mathbf{a}$ whatever the magnitude of the displacement of $\mathbf{A}$, $d\mathbf{A}$.

3.2.2. Eigenvalue Test Approach

As we mentioned before, the solution to the immunization problem requires a considerable number of different bonds in the portfolio. If for large investment banks this is not a major problem, since they usually hold and manage many different bonds in several markets, for small investors based on emerging markets this may pose a serious obstacle when it comes to implement the strategy. In these cases, the investor may opt to select a bond portfolio that matches first-order conditions for immunization and then evaluate the sufficiency of these conditions on a particular basis using an alternative test to determine the sign-definiteness of the quadratic form: The Eigenvalue Test. Recap that at a stationary point we have $\frac{\nabla P_H(\mathbf{A})}{P_H(\mathbf{A})} = 0$, which means that the Taylor expansion in (10) reduces to:

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{2} d\mathbf{A}^T \cdot \frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})} \cdot d\mathbf{A} + R_2^*(\mathbf{a}, d\mathbf{A}) \tag{43}$$

Let $\mathbf{S} = \frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})}$ be a $n \times n$ symmetric matrix. From standard linear algebra we know that because $\mathbf{S}$ is symmetric, is has real eigenvalues, $\{\lambda_n\}$, and n independent unit eigenvectors, $\{\nu_n\}$, which are mutually orthogonal. Let $\mathbf{V}$ denote the a $n \times n$ matrix with $\{\nu_n\}$ as column vectors. By construction, $\mathbf{V}$ is an orthogonal matrix, $\mathbf{V}^T = \mathbf{V}^{-1}$. Changing coordinates to the $\{\nu_n\}$ basis, let $d\mathbf{A} = \mathbf{V}\mathbf{y}$. Substituting into (43) we obtain:

$$\begin{aligned} d\mathbf{A}^T \mathbf{S} d\mathbf{A} &= \mathbf{y}^T (\mathbf{V}^T \mathbf{S} \mathbf{V}) \mathbf{y} \\ &= \sum_{n=1}^{4} \lambda_n y_n^2, \end{aligned} \tag{44}$$

We also know that $\mathbf{S}$ is positive-definite (resp. negative definite) iff all its eigenvalues are positive (resp. negative). In other words, if $\mathbf{S}$ is positive-definite (resp. negative definite) we can conclude that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ has a minimum (resp. maximum) at the stationary point.

In Section 3.2.1 we were able to conclude that unless additional restrictions on portfolio structure are imposed we cannot guarantee that the hessian matrix is positive semidefinite. As a result, the possibility of getting both positive and negative eigenvalues from the spectral decomposition of matrix $\mathbf{S}$ cannot be disregarded. In other words, the possibility of obtaining negative eigenvalues means that for certain 'directions' (interest rate shifts) the portfolio's horizon value will not be convex at $\mathbf{A} = \mathbf{a}$ and the investor is, thus, exposed to interest rate risk.

Taking this into account, the solution to the immunization problem must be evaluated on a particular basis. For that, we now propose a three step procedure to find bond portfolios that satisfy both first-order and second-order immunization conditions.

Step 1 - Select a bond portfolio that matches the gradient conditions for immunization, as defined in (22);

Step 2 - Calculate the eigenvalues of $\mathbf{S}$ in order to assess if first-order conditions are sufficient to guarantee that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ has a minimum at the stationary point derived. First-order conditions will be sufficient iff all of the eigenvalues of the hessian matrix are positive;

Step 3 - If first-order conditions are not sufficient, i.e., if not all of the eigenvalues of the Hessian matrix are positive, we recommend a sort of "second-best" strategy. Since there is usually more than one bond portfolio satisfying first-order conditions, repeat Steps 1 and 2 for all of the candidate solutions and select the bond portfolio that most closely matches the conditions for a minimum. Since negative eigenvalues represent yield curve displacements for which the portfolio's horizon value is not convex, we think that a reasonable criteria for selecting an acceptable portfolio will be to minimize the impact of those yield curve directions. In this sense, we recommend to choose the candidate solution for which the sum of the absolute value of the negative eigenvalues is minimum, i.e., the one for which the quantity $\sum_{\lambda_n<0} |\lambda_n|$ is minimum. To implement the procedure standard optimization algorithms may be used.

4. Bond Price Sensitivity and the Risks of Convexity

In this section we develop a more accurate approximation for the price sensitivity of a bond based upon the new definitions for parametric interest rate risk measures given above. In addition, we revise both classic and modern approaches to convexity and demonstrate that important negative effects of convexity are revealed when changes other than parallel shifts in the term structure are considered.

Consider again the present value of bond at time $t = 0$, $B_0(\mathbf{a})$, as given by (4). If we ignore the effects of the passage of time, the price sensitivity around the initial vector of parameters can be approximated by the two first terms of a Taylor series expansion as follows:

$$\frac{dB_0(\mathbf{A})}{B_0(\mathbf{A})} \simeq \frac{1}{B_0(\mathbf{A})} \nabla B_0(\mathbf{a})^T \cdot d\mathbf{a} + \frac{1}{2} d\mathbf{a}^T \cdot \frac{1}{B_0(\mathbf{A})} \nabla^2 B_0(\mathbf{a}) \cdot d\mathbf{a} \tag{45}$$

where $\nabla B_0(\mathbf{A})$ and $\nabla^2 B_0(\mathbf{A})$ represent, respectively, the gradient vector and the Hessian matrix of $B_0(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$, and the remaining variables keep their previous meaning. If we substitute the definitions of parametric duration and parametric convexity as stated in

(13) and (27), the approximation for the bond price sensitivity can be written in terms of duration and convexity as follows:

$$\frac{dB_0(\mathbf{A})}{B_0(\mathbf{A})} \simeq -\mathbf{D}(\mathbf{A}) \cdot d\mathbf{a} + \frac{1}{2} d\mathbf{a}^T \cdot \mathbf{C}(\mathbf{A}) \cdot d\mathbf{a} \tag{46}$$

where

$$\begin{aligned} \mathbf{D}(\mathbf{A}) &= [D(0,\mathbf{A}),\ldots,D(k,\mathbf{A})] \\ \mathbf{C}(\mathbf{A}) &= \begin{bmatrix} C_{0,0}(\mathbf{A}) & \cdots & C_{0,m}(\mathbf{A}) \\ \vdots & \ddots & \vdots \\ C_{k,0}(\mathbf{A}) & \cdots & C_{k,m}(\mathbf{A}) \end{bmatrix} \end{aligned}$$

Assume now the term structure experiences only level shifts. A simple characterization of level shifts is given by assuming that the height coefficient a_0 experiences a non-infinitesimal, instantaneous change, and all other coefficients (i.e. a_1, a_2, etc.) in equation (1) remain constant. From (46), the total instantaneous change in bond price due this additive shift is given simply by:

$$\frac{dB_0(\mathbf{A})}{B_0(\mathbf{A})} \simeq -D(0,\mathbf{A}) \cdot \Delta a_0 + \frac{1}{2} C_{0,0}(\mathbf{A}) \cdot (\Delta a_0)^2 \tag{47}$$

The term $C_{0,0}(\mathbf{A})$ has been traditionally defined as the convexity of a bond. Convexity captures most of the change in the bond price not captured by traditional Fisher-Weil duration. Because $(da_0)^2$ is always positive, convexity is always beneficial for level shifts in the term structure.

A similar result holds when considering percentage changes in the reinvested terminal value of a bond at a given planning horizon H. Following steps similar to equations (45) and (46), we get the change in the terminal value of a bond caused by a change in a_0 at planning horizon H given as:

$$\frac{dB_H(\mathbf{A})}{B_H(\mathbf{A})} \simeq [H - D(0,\mathbf{A})] \cdot \Delta a_0 + \frac{1}{2}\left[C_{0,0}(\mathbf{A}) - 2HD(0,\mathbf{A}) + H^2\right] \cdot (\Delta a_0)^2 \tag{48}$$

Under additive term structure shifts a bond's reinvested value is immunized when the duration of the bond equals its planning horizon. Therefore the above equation can be simplified to:

$$\frac{dB_H(\mathbf{A})}{B_H(\mathbf{A})} \simeq \frac{1}{2}\left[C_{0,0}(\mathbf{A}) - H^2\right] \cdot (\Delta a_0)^2 \tag{49}$$

In the above equation, the expression $C_{0,0}(\mathbf{A}) - H^2$ is higher whenever $C_{0,0}(\mathbf{A})$ (or the convexity) of a bond is higher. Since a higher value of the expression $C_{0,0}(\mathbf{A}) - H^2$ implies higher return to the terminal value in equation (49), a higher convexity should always be preferred for additive term structure shifts. Consequently, for additive shifts, maximizing convexity is always an appropriate immunization objective. This conclusion corresponds basically to the traditional approach to convexity (see, e.g. Fabozzi (2000), Garbade (1985a), Milgrow (1985), Bierwag et al. (1988), Grantier (1988)).

The traditional approach to convexity assumes that interest rate shifts are additive and that convexity is a desirable feature in a bond portfolio. We argue that important negative aspects of convexity are revealed when the stochastic process underlying term structure is richer and allows for both parallel and non-parallel shifts.[17] In our case, by allowing a_1, a_2 and the other coefficients in equation (1) to change randomly and simultaneously with coefficient a_0, term structure movements will no longer be restricted to any specific stochastic process.

Consider a simple case of a simultaneous change in both a_0 and a_1. Allowing both the level coefficient a_0 and slope parameter a_1 to change implies a non-infinitesimal and non-parallel term structure shift. For this kind of shift, the change in the bond's terminal value at the planning horizon can be approximated by:

$$\begin{aligned} dB_H(\mathbf{A}) \simeq\ & \frac{\partial B_H(\mathbf{A})}{\partial a_0}\Delta a_0 + \frac{\partial B_H(\mathbf{A})}{\partial a_1}\Delta a_1 + \frac{1}{2}\frac{\partial^2 B_H(\mathbf{A})}{\partial a_0^2}\left(\Delta a_0\right)^2 \\ & + \frac{1}{2}\frac{\partial^2 B_H(\mathbf{A})}{\partial a_1^2}\left(\Delta a_1\right)^2 + \frac{\partial^2 B_H(\mathbf{A})}{\partial a_0 \partial a_1}\left(\Delta a_0\right)\left(\Delta a_1\right) \end{aligned} \tag{50}$$

The magnitudes of the last two terms are small when compared to the magnitude of the first three terms, and therefore can be ignored for simplicity. Dividing by $B_H(\mathbf{A})$ and expressing this equation in terms of duration and convexity yields:

$$\begin{aligned} \frac{dB_H(\mathbf{A})}{B_H(\mathbf{A})} \simeq\ & \left[H - D(0,\mathbf{A})\right]\cdot\Delta a_0 + \frac{1}{2}\left[C_{0,0}(\mathbf{A}) - 2HD(0,\mathbf{A}) + H^2\right]\cdot\left(\Delta a_0\right)^2 \\ & + \left[a_4\left(1 - e^{-\frac{H}{a_4}}\right) - D(1,\mathbf{A})\right]\cdot\Delta a_1 \end{aligned} \tag{51}$$

Since first order conditions for bond portfolio immunization against level shifts require $D(0,\mathbf{A})$ be equal to the planning horizon H, the above equation can be simplified to:

$$\frac{dB_H(\mathbf{A})}{B_H(\mathbf{A})} \simeq \frac{1}{2}\left[C_{0,0}(\mathbf{A}) - H^2\right]\cdot\left(\Delta a_0\right)^2 + \left[a_4\left(1 - e^{-\frac{H}{a_4}}\right) - D(1,\mathbf{A})\right]\cdot\Delta a_1 \tag{52}$$

The above equation redefines the meaning of convexity which is significantly different from its traditional usage. Traditionally, convexity has been associated with the bond value change caused by a non-infinitesimal shift in the level of the term structure. Though equation (52) is consistent with this view, it introduces an additional link between bond value change and slope shifts (Δa_1) in the term structure. Therefore, provided that first order conditions for bond portfolio immunization against slope shifts are not met, whenever a simultaneous shift in the level and the slope of the term structure occurs, the effect of traditional convexity (i.e. $C_{0,0}(\mathbf{A})$) on the terminal value of the bond at the planning horizon becomes uncertain. Lacey and Nawalkha (1993) call this additional link *risk effect*, contrasting with the *convexity effect* underlying the classical approach to convexity.

In other words, whenever we consider more realistic term structure shifts maximizing convexity may no longer be considered a suitable immunization objective. In addition, the modern approach to convexity is consistent with equilibrium non-arbitrage conditions in bond markets (see e.g. Lacey and Nawalkha (1993)).

[17]Similar conclusions can be found in Kahn and Lochoff (1990), Lacey and Nawalkha (1993) and Reitano (1993), among others.

5. Interest Rate Sensitivity of Bond Risk Measures

In this section we derive a simple expression for the sensitivity of parametric durations to changes in term structure shape parameters. Portfolio managers are often required to maintain target levels of interest rate risk exposure, both for assets and liabilities. From standard duration theory we know that the duration of a bond changes as time passes, not only because the bond approaches maturity but mainly due to changes in the yield curve. In volatile interest rate environments interest rate risk measures can change rapidly as a result of modifications in the shape of the term structure of interest rates. For portfolio managers this is a subject of major interest since maintaining the portfolio exposure up to a desired level requires frequent portfolio rebalancing. To do so, it is of great interest to understand how interest rate risk measures themselves change with modifications in the yield curve.

The sensitivity of a bond's duration to changes in the bond's yield to maturity has been extensively analysed in the literature (e.g. Bierwag, 1987). In spite of this, it is well know that the usefulness of this analysis is limited when yield curves are not flat and non-parallel term structure shifts may occur. In this section we extend previous research by investigation the sensitivity of parametric duration measures to a wider a range of yield curve movements.

Consider again the definition of parametric duration presented in (13):

$$D^{(l)}(k,\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})}\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k} \quad (k=0,...,3). \tag{53}$$

Differentiating with respect to A_m $(m=0,...,3)$ yields:

$$\begin{aligned}
\frac{\partial D(k,\mathbf{A})}{\partial A_m} &= \frac{\partial}{\partial A_m}\left\{\frac{1}{B_0^l(\mathbf{A})}\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right\} \\
&= \frac{1}{B_0^l(\mathbf{A})^2}\left\{\frac{\partial}{\partial A_m}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]B_0^l(\mathbf{A})\right. \\
&\quad \left. -\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\frac{\partial B_0^l(\mathbf{A})}{\partial A_m}\right\} \\
&= \frac{1}{B_0^l(\mathbf{A})^2}\left\{-\left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right]B_0^l(\mathbf{A})\right. \\
&\quad \left. -B_0^l(\mathbf{A})^2\left[\frac{1}{B_0^l(\mathbf{A})}\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\frac{1}{B_0^l(\mathbf{A})}\frac{\partial B_0^l(\mathbf{A})}{\partial A_m}\right\} \\
&= -\frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right] \\
&\quad +\frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\left(-\frac{1}{B_0^l(\mathbf{A})}\frac{\partial B_0^l(\mathbf{A})}{\partial A_m}\right)
\end{aligned} \tag{54}$$

Substituting the definitions of parametric duration and parametric convexity in equation

(54) yields:

$$\frac{\partial D(k,\mathbf{A})}{\partial A_m} = D(k,\mathbf{A})D(m,\mathbf{A}) - C(k,m,\mathbf{A}) \qquad (k,m=0,...,3) \tag{55}$$

Equation (55) provides a general expression for the sensitivity of interest rate risk measures (parametric duration measures) to changes in interest rates as given by modifications in yield curve parameters. For any combination of term structure shifts the sensitivity of parametric duration is computed as a product of two duration measures minus the corresponding parametric convexity. Two have a broader understanding of the significance of equation (55) consider the following cases of interest.

Case 1: Let $k = 0$ and $m = 0$. From (55) we have

$$\frac{\partial D(0,\mathbf{A})}{\partial A_0} = [D(0,\mathbf{A})]^2 - C_{0,0}(\mathbf{A}) \tag{56}$$

Therefore, the sensitivity of traditional Fisher-Weil duration to changes in the level of the yield curve is equal to duration squared minus the traditional convexity measure. Note also that if gradient conditions for immunization against shifts in A_0 are satisfied (i.e., if $D(0,\mathbf{A}) = H$), the sensitivity $\frac{\partial D(0,\mathbf{A})}{\partial A_0}$ can be written as the negative of the popular M-squared dispersion measure (M^2) proposed by Fisher and Weil (1983, 1984), i.e.,

$$\frac{\partial D(0,\mathbf{A})}{\partial A_0} = -\left[C_{0,0}(\mathbf{A}) - (D(0,\mathbf{A}))^2\right] = -M^2 \tag{57}$$

Case 2: Let $k = 0$ and $m = 1$. Then,

$$\frac{\partial D(0,\mathbf{A})}{\partial A_1} = D(0,\mathbf{A})D(1,\mathbf{A}) - C_{0,1}(\mathbf{A}) \tag{58}$$

Hence, the sensitivity of traditional Fisher-Weil duration to changes in the slope parameter of the yield curve is equal to the product of duration and $D(1,\mathbf{A})$ minus $C_{0,1}(\mathbf{A})$. Generalising the above examples we can estimate the combined effects produced by changes in the term-structure level, slope and curvature on interest rate risk measures using the the concept of total differential

$$\begin{aligned} \Delta D(k,\mathbf{A}) &\approx \sum_{m=0}^{3} \frac{\partial D(k,\mathbf{A})}{\partial A_m}\Delta A_m \\ &\approx \sum_{m=0}^{3} \left[D(k,\mathbf{A})D(m,\mathbf{A}) - C(k,m,\mathbf{A})\right]\Delta A_m \end{aligned} \tag{59}$$

6. Conclusion

Traditionally, the study of the interest-rate sensitivity of the price of a portfolio of assets or liabilities has been performed using single factor models from which simple expressions for duration and convexity have been derived. In general, the ability of such models to predict price sensitivity or to achieve immunization is dependent on the validity of yield curve assumptions. In this sense, the classical duration analysis can greatly understate price sensitivity when non-parallel term structure shifts occur.

In this chapter, we have developed a general multivariate duration and convexity analysis that does not depend on previous statements about the way in which the yield curve moves. Differently, the model links interest rate risk factors to the parameters of the Svensson specification of the yield curve and is valid in virtually all yield curve environments. The model extends classical duration and convexity analysis to include yield curve shifts that are not parallel. The concepts of parametric duration and parametric convexity provide, in this context, natural first-order and second-order sensitivity measures of bond or bond portfolio prices to changes in interest rates. Moreover, the interest rate risk measures derived quantify the sensitivity of the portfolio to yield curve shifts that have an economic meaning, namely changes in the level, slope and curvature of the yield curve.

Contrary to most interest rate risk models we emphasize the importance of second-order conditions for bond portfolio immunization. In concrete, we show that it is impossible to achieve immunization simply by meeting first-order conditions and that the key to successful immunization will be to build up a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. We present two alternative methods to determine the sign definiteness of the Hessian matrix: the determinantal test and the eigenvalue test, emphasizing the advantages and shortcomings of both methods.

We have developed a more accurate approximation for the price sensitivity of a bond based upon new definitions for parametric interest rate risk measures. In addition, we examine the advantages and disadvantages of traditional convexity under realistic term structure shifts and prove that whenever we consider more realistic yield curve shifts, other than simply parallel shifts, maximizing convexity may no longer be considered a suitable immunization objective.

Finally, we analyse the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters, offering fixed-income portfolio managers a new powerful tool to assess the combined effects of changes in the term-structure level, slope and curvature on interest rate risk measures.

Future research should investigate the empirical performance of the parametric model when compared with that obtained with alternative single- and multiple-factor duration matching strategies.

Appendix: Formulae for Parametric Convexity

First recall that

$$r(t,\mathbf{A}) = a_0 + a_1\frac{a_4}{t}\left(1-e^{-\frac{t}{a_4}}\right) + a_2\frac{a_4}{t}\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right] + a_3\frac{a_5}{t}\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]. \tag{60}$$

The general expression for the parametric convexity of a bond, $C^{(l)}(k,m,\mathbf{A})$, is given by

$$\begin{aligned} C^{(l)}(k,m,\mathbf{A}) &= \frac{1}{B_0^l(\mathbf{A})}\frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_k A_m} \\ &= \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right],\ k,m=0,...,3 \end{aligned} \tag{61}$$

Differentiating equation (60) with respect to A_k $(k=0,...,3)$ and substituting in (61) we obtain the following complete set of formulas for parametric convexity:

Table 1. Formulae for Parametric Convexity

k	m	$C^{(l)}(k,m,\mathbf{A})$
0	0	$C_{0,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t}\right]$
0	1	$C_{0,1}^{(l)} = C_{1,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t}\left(1-e^{-\frac{t}{a_4}}\right)a_4\right]$
0	2	$C_{0,2}^{(l)} = C_{2,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t}\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]a_4\right\}$
0	3	$C_{0,3}^{(l)} = C_{3,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t}\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]a_5\right\}$
1	1	$C_{1,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}\left(1-e^{-\frac{t}{a_4}}\right)^2 a_4^2\right]$
1	2	$C_{1,2}^{(l)} = C_{2,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}a_4^2\left(1-e^{-\frac{t}{a_4}}\right)\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]\right\}$
1	3	$C_{1,3}^{(l)} = C_{3,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}\left(1-e^{-\frac{t}{a_4}}\right)a_4\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]a_5\right\}$
2	2	$C_{2,2}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}a_4^2\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]^2\right\}$
2	3	$C_{2,3}^{(l)} = C_{3,2}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}\left[1-e^{-\frac{t}{a_4}}\left(1+\frac{t}{a_4}\right)\right]a_4\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]a_5\right\}$
3	3	$C_{3,3}^{(l)} = \frac{1}{B_0^l(\mathbf{A})}\left\{\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t}\left[1-e^{-\frac{t}{a_5}}\left(1+\frac{t}{a_5}\right)\right]^2 a_5^2\right\}$

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