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Alfred Stadler

A Relativistic Theory of Few-Nucleon Systems

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Abstract This talk provides an overview of recent results for two- and three-nucleon systems obtained within the framework of the covariant spectator theory (CST). The main features of two recently published models for the neutron–proton interaction, that fit the 2007 world data base containing several thousands of neutron-proton scattering data below 350 MeV with $\chi^2/N_{\text{data}} \approx 1$, are presented. These one-boson-exchange models, called WJC-1 and WJC-2, have a considerably smaller number of adjustable parameters than are present in realistic nonrelativistic potentials. When applied to the three-nucleon bound state, the correct binding energy is obtained without additional three-body forces. First calculations of the electromagnetic form factors of helium-3 and the triton in complete impulse approximation also give very reasonable results. One can conclude that the CST yields a very efficient description of few-nucleon systems, in which the relativistic formulation of the dynamics is an essential element.

1 The Fundamental Ideas of the Covariant Spectator Theory

In low-energy few-body nuclear physics, relativity is often considered an unwelcome complication that gives rise to only small effects, which can be either neglected or included perturbatively. The argument is usually based on the observation that typical kinetic energies of nucleons in light nuclear systems are small compared to their rest mass. However, a relativistic description of such systems becomes unavoidable when they are investigated with hadronic or electromagnetic probes at high momentum transfer and the final nuclear states reach relativistic velocities.

This talk intends to show that relativity can also lead to a significant simplification of the description of few-nucleon systems. This will be demonstrated for the case of the covariant spectator theory (CST), where simple one-boson-exchange (OBE) models of the nucleon–nucleon (NN) interaction can be derived that provide a more efficient description of the NN observables than nonrelativistic models. This efficiency applies also to the $3N$ bound state, which can be well described without $3N$ forces. The obtained simplification depends crucially on relativity.

The basic idea of the CST is to reorganize the manifestly covariant Bethe–Salpeter (BS) equation with its complete kernel to another equivalent form, in which a different propagator and an accordingly modified kernel is used. The specific propagator of the CST places all particles but one in intermediate states on their mass shells [1–3]. This reduces the dimension of the integration over intermediate momenta from four to

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A. Stadler (✉)
Departamento de Física da Universidade de Évora, 7000-671 Évora, Portugal
E-mail: stadler@uevora.pt

A. Stadler
Centro de Física Nuclear da Universidade de Lisboa, 1649-003 Lisboa, Portugal

three, while maintaining the manifest covariance of the equation. This choice was originally motivated by an observed cancellation between two-body ladder and crossed-ladder diagrams in scalar theories of ϕ^3-type, where two heavier particles with unequal masses exchange a third lighter one. The CST two-body equation has the correct one-body limit: when one particle becomes infinitely massive, the two-body equation reduces to a relativistic one-body equation for the light particle moving in an effective potential created by the massive particle. The CST three-body equation satisfies the property of cluster-separability, without which a two-body CST amplitude could not be used consistently in the kernel of a three-body equation.

Note that when the kernel is truncated at the OBE level, the exact equivalence to the full BS equation is lost. However, this would only be an issue were one to attempt to use the CST to find an exact solution of the full BS equation. Instead, the equations of the CST can be taken as the starting point for a description of few-body systems that is in part phenomenological, through the way loop integrations are regularized, and in the determination of some parameters from fits to experimental data. Ultimately, the practical value of the CST will be judged from its efficiency in describing observables and its predictive power.

2 Covariant Spectator Theory of Two- and Three-Nucleon Systems

A good understanding of the interaction between two nucleons is essential for the study of nuclear structure and nuclear reactions. The construction of a CST kernel for NN scattering which reproduces the deuteron properties and the NN scattering observables is therefore of high importance.

The specific form of the CST equation for the NN scattering amplitude M, with particle 1 on-shell in both the initial and final state, is [4]

$$M_{12}(p, p'; P) = \overline{V}_{12}(p, p'; P) - \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_k} \overline{V}_{12}(p, k; P) G_2(k, P) M_{12}(k, p'; P), \tag{1}$$

where P is the conserved total four-momentum, and p, p', and k are relative four-momenta related to the momenta of particles 1 and 2 by $p_1 = \frac{1}{2}P + p$, $p_2 = \frac{1}{2}P - p$, and M_{12} is the matrix element of the Feynman scattering amplitude between positive energy Dirac spinors of particle 1. The covariant kernel $\overline{V}_{12}$ (which is also referred to as the "potential") is explicitly antisymmetrized, ensuring that the amplitudes M_{12} satisfy the generalized Pauli principle (Fig. 1). The propagator for the off-shell particle 2 is

$$G_2(k, P) \equiv G_{\beta\beta'}(k_2) = \frac{(m + \not{k}_2)_{\beta\beta'}}{m^2 - k_2^2 - i\varepsilon} h^2(k_2), \tag{2}$$

with $k_2 = P - k_1$, $k_1^2 = m^2$. It is dressed by the off-shell nucleon form factor $h(k_2)$, which can be related to the self-energy of the off-shell nucleon, and which is normalized to unity when $k_2^2 = m^2$.

The propagator of an off-shell particle can be decomposed into positive- and negative- energy contributions. Accordingly, the CST equations can be separated into positive- and negative-energy (also called "ρ-spin"-) channels, which is useful for their numerical solution. Negative-energy states are related to the "Z-graphs" of time-ordered perturbation theory. In this sense, the solutions of (1) automatically include Z-graphs to all orders. This is useful to keep in mind when CST is compared to other theories in which relativistic corrections are added perturbatively to nonrelativistic calculations.

The CST of $3N$ scattering was formulated for the first time in Ref. [3]. Assuming only two-body interactions, the main idea is to place spectators always on mass shell. If this is done consistently, in any intermediate

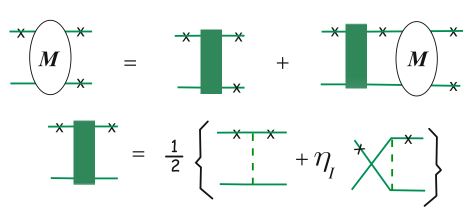

Fig. 1 *Top line* diagrammatic representation of the covariant spectator equation (1) with particle 1 on-shell (the on-shell particle is indicated with a *cross*). *Second line* diagrammatic representation of the definition of the antisymmetrized kernel $\overline{V}_{12}$, with $\eta_I = (-)^I$, where I is the NN isospin

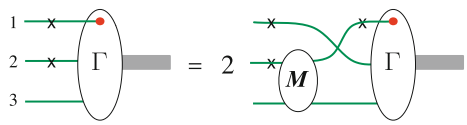

Fig. 2 Diagramatic representation of the covariant spectator equation for the three-body bound state vertex function Γ with particles *1* and *2* on-shell (labeled with a *cross*). Here particle *1* is the spectator to the last two-body interaction between particles *2* and *3*, described by the scattering amplitude *M* with particle *3* off-shell

state there are always two nucleons on mass shell, and one off mass shell, such that all loop integrations are three-dimensional.

The CST equations for the $3N$ bound state were formulated in a way suitable for a practical solution in Ref. [5]. One obtains a homogeneous equation for the vertex function Γ of the $3N$ bound state, shown graphically in Fig. 2. All relativistic effects can be calculated *exactly* in CST, and the full Dirac structure of the nucleons is also taken into account.

The CST $3N$ equation was solved numerically for the first time in Ref. [6], for a family of older OBE potentials. Since then, much progress has been made in the development of more accurate CST NN interaction models, which will be described in the following section.

3 The Two-Nucleon System with New High-Precision *np* Kernels

The first covariant NN OBE kernels in CST, based on the exchange of either four or six mesons, were published in Ref. [4]. After a long process of improvements of the applied numerical techniques, the structure of the kernels, and the enlargement of the np data base, two new models, called WJC-1 and WJC-2, were developed [7] whose precision in representing the most recent world np data is on par with all commonly used "realistic" potentials.

The kernels are sums of OBE contributions. In the notation of Fig. 3, the individual boson contributions are of the form

$$V_{12}^{b}(p,k;P)=\varepsilon_b\delta\frac{\Lambda_1^b(p_1,k_1)\otimes\Lambda_2^b(p_2,k_2)}{m_b^2+|q^2|}f(\Lambda_b,q). \tag{3}$$

Here, $b=\{s,p,v,a\}$ denotes the boson type (scalar, pseudoscalar, vector, axial vector), $q=p_1-k_1=k_2-p_2=p-k$ the momentum transfer, m_b the boson mass, ε_b a phase factor, $\delta=1$ for isoscalar bosons and $\delta=\tau_1\cdot\tau_2=-1-2(-)^I$ for isovector bosons, and $f(\Lambda_b,q)$ a boson form factor depending on a form factor mass Λ_b. The axial vector bosons are treated as contact interactions, with a structure as in (3), but with the propagator replaced by a constant. The explicit forms of the numerator functions $\Lambda_1^b\otimes\Lambda_2^b$ can be inferred from Table 1.

The use of the absolute value $|q^2|$ instead of $-q^2$ in the propagators and form factors amounts to a covariant redefinition in the region $q^2>0$. It is a significant new theoretical improvement that removes all singularities and can be justified by a detailed study of the structure of the exchange diagrams [7].

Note that terms proportional to the parameters ν_s for scalar and ν_v for vector meson exchanges contribute only if the nucleon is off mass shell on at least one side of the vertex. These terms are therefore called "off-shell couplings." In the case of pseudoscalar exchange, $\lambda_p\equiv 1-\nu_p$ parametrizes a mixing between pseudoscalar and pseudovector coupling, with $\lambda_p=0$ corresponding to pure pseudovector, and $\lambda_p=1$ to pure pseudoscalar coupling.

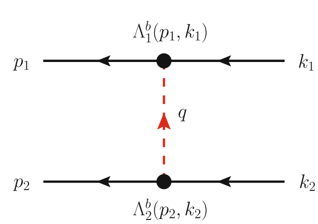

Fig. 3 The structure of a OBE kernel

Table 1 Mathematical forms of the bNN vertex functions, with $\Theta(p) \equiv (m - \not{p})/2m$

$J^P(b)$	ε_b	$\Lambda_1 \otimes \Lambda_2$	$\Lambda(p,k)$ or $\Lambda^\mu(p,k)$
$0^+(s)$	$-$	$\Lambda_1\Lambda_2$	$g_s - \nu_s\,[\Theta(p) + \Theta(k)]$
$0^-(p)$	$+$	$\Lambda_1\Lambda_2$	$g_p\gamma^5$
			$-g_p(1-\lambda_p)\left[\Theta(p)\gamma^5 + \gamma^5\Theta(k)\right]$
$1^-(v)$	$+$	$\Lambda_1^\mu\Lambda_2^\nu\Delta_{\mu\nu}$	$g_v\left[\gamma^\mu + \frac{\kappa_v}{2M}i\sigma^{\mu\nu}(p-k)_\nu\right]$
			$+g_v\nu_v\left[\Theta(p)\gamma^\mu + \gamma^\mu\Theta(k)\right]$
$1^+(a)$	$+$	$\Lambda_1^\mu\Lambda_2^\nu g_{\mu\nu}$	$g_a\gamma^5\gamma^\nu$

The vector propagator is $\Delta_{\mu\nu} = g_{\mu\nu} - q_\mu q_\nu/m_v^2$ with the boson momentum $q = p_1 - k_1 = k_2 - p_2$

Table 2 Comparison of precision np models and the 1993 Nijmegen phase shift analysis

Model			$\chi^2/N_{\text{data}}(N_{\text{data}})$		
Reference	N_{pars}	Year	1993	2000	2007
PWA93	39	1993	0.99 (2,514)	–	–
			1.09 (3,011)	**1.12** (3,336)	**1.13** (3,788)
Nijm I	41	1993	1.03 (2,514)	–	–
AV18	40	1995	1.06 (2,526)	–	–
CD-Bonn	43	2000	–	1.02 (3,058)	–
WJC-1	27	2007	**1.03** (3,011)	**1.05** (3,336)	**1.06** (3,788)
WJC-2	15	2007	**1.09** (3,011)	**1.11** (3,336)	**1.12** (3,788)

The first column specifies the model, the second the number of adjustable parameters (in the case of the first four models for both np and pp data), and the third the year of the data base (data prior to this year are included). Columns four to six are the obtained χ^2/N_{data} for various data bases (identified by their year), where the number of included data is given in parentheses. Our calculations are in bold face

Model WJC-1 was constructed with the goal to obtain the best possible fit, while the objective of WJC-2 was to use the smallest number of parameters without deteriorating the quality of the fit too much. Table 2 shows that we achieved excellent fits for the most complete data base of np scattering, and with a considerably smaller number of adjustable parameters than other realistic potential models. In fact, in view of the $\chi^2/N_{\text{data}} = 1.06$ of model WJC-1, the corresponding phase shifts can be considered a new phase shift analysis which includes many more data than the "standard" Nijmegen 93 analysis [8] to which all realistic potential models were fitted. Note that our phase shifts, shown in Fig. 4, can be used outside the framework of CST, just like any other phase shift analysis.

The deuteron binding energy was used as a constraint during the fitting of the CST NN kernels, and therefore they reproduce the experimental binding energy of $E_d = 2.2246$ MeV automatically. The deuteron vertex functions can be related to the well-known nonrelativistic S- and D-state deuteron wave functions $u(p)$ and $w(p)$, respectively. In addition one obtains spin singlet and triplet P-waves, $v_s(p)$ and $v_t(p)$, which are of relativistic origin. Tables with the numerical values, and convenient parameterizations using analytic functions, both in momentum and coordinate space, are given in Ref. [9].

4 Results for the Three-Nucleon Bound State

One of the few persistent problems of low-energy few-nucleon physics is the apparent inability of realistic NN potentials to explain the experimental triton binding energy $E_t = 8.48$ MeV. The potentials with the best fit to the NN data yield binding energies roughly between 7.6 and 8 MeV. The view commonly adopted in order to deal with this discrepancy is that $3N$ forces must be an essential part of $3N$ dynamics. Models for $3N$ forces introduce additional parameters, which are usually adjusted to reproduce the triton binding energy.

The CST calculations of Ref. [6] showed that scalar off-shell coupling terms in a relativistic NN kernel not only improve the fit to the NN data, but the model with the best fit, called W16, also predicts the correct triton binding energy without $3N$ forces. These terms, proportional to ν_s in the vertices for the coupling of scalar mesons to nucleons of Table 1, contribute only when the incoming or outgoing nucleon is off mass shell and are therefore not present in nonrelativistic theories.

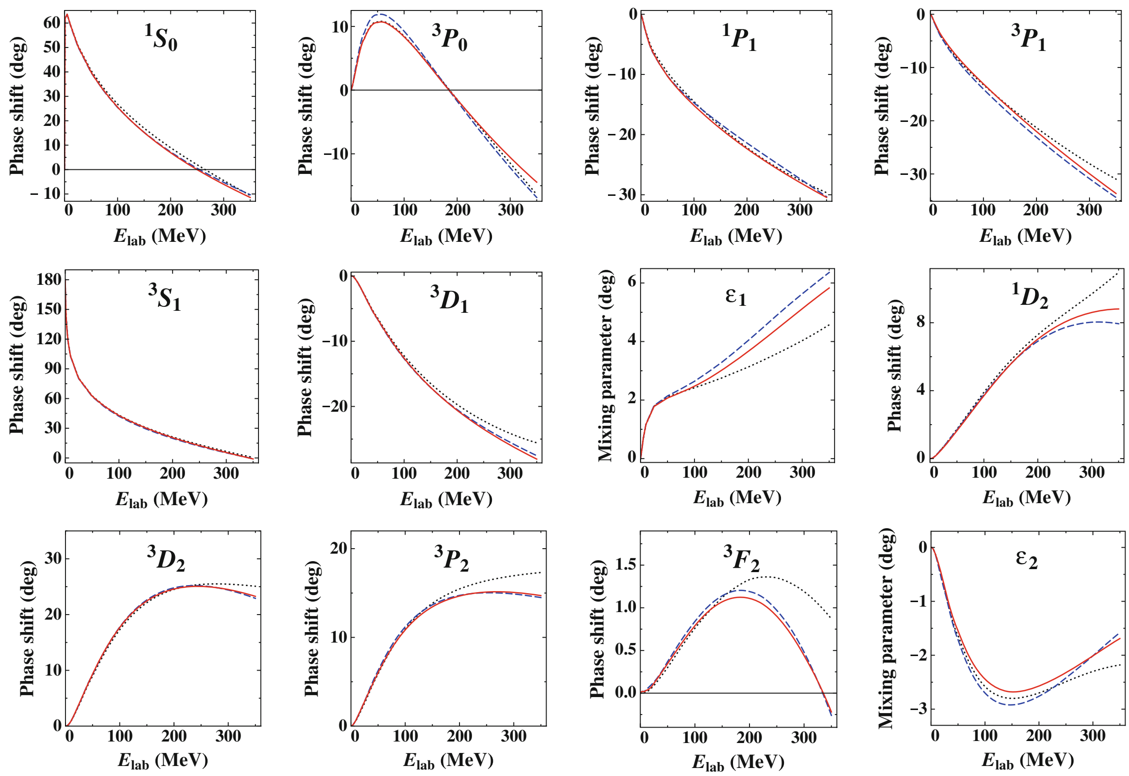

Fig. 4 Phase shifts of *np* scattering for partial waves with $J \leq 2$. The *solid* and *dashed lines* respresent the results of models WJC-1 and WJC-2, respectively. The *dotted line* shows the Nijmegen multienergy phase shift analysis of 1993 [8]

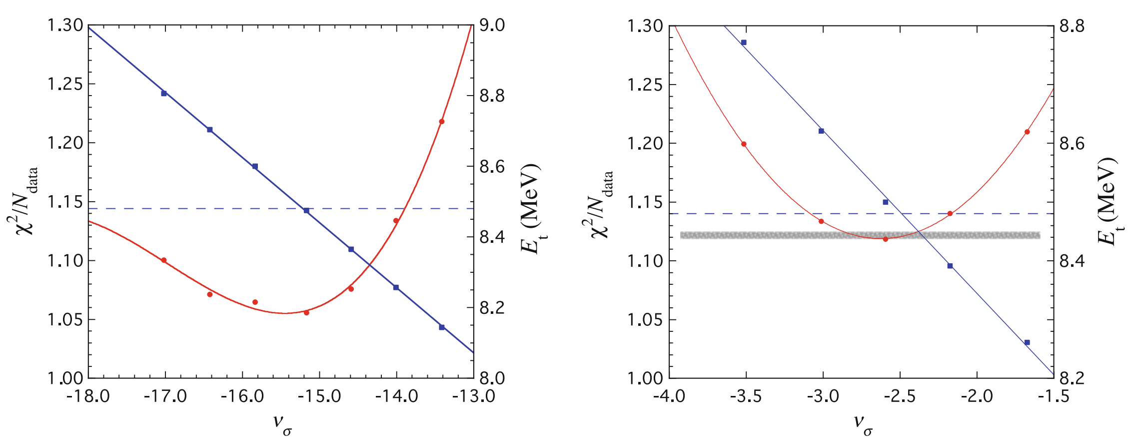

Fig. 5 Results for calculations of χ^2/N_{data} (*solid circles* on *curved line* and *left scale*) to a 2007 *np* data base, and triton binding energy E_t (*solid squares* on *straight line* and *right scale*) for WJC-1 family (*left panel*) and for WJC-2 family of models (*right panel*). The points with the lowest χ^2/N_{data} are models WJC-1 and WJC-2, respectively. The other models of the two families were obtained by fixing ν_σ at different values and refitting all other parameters. The curves are fits through the actually calculated points

After the fits of the new high-precision models WJC-1 and WJC-2 were completed, it came as a surprise that both models—with $E_t = 8.48$ MeV and $E_t = 8.50$ MeV, respectively—again predict the experimental binding energy very closely, even though their detailed structure and their parameters differ quite significantly from each other and from the old model W16. The result is robust, and it becomes difficult to believe in a mere coincidence! Fig. 5 shows the changes in χ^2/N_{data} and E_t when ν_σ is held fixed at certain values while all other potential parameters are refitted, confirming the importance of this mechanism in our NN kernels.

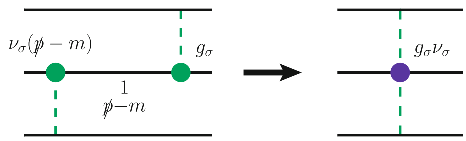

Fig. 6 Boson-nucleon vertices with off-shell coupling can generate effective $3N$ forces. In this example, an off-shell nucleon consecutively exchanges a scalar σ meson with two different nucleons. When a scalar off-shell vertex is multiplied with the nucleon propagator, the two separate boson-nucleon vertices shrink to a single contact vertex, and the whole diagram takes on the form of a $3N$ force

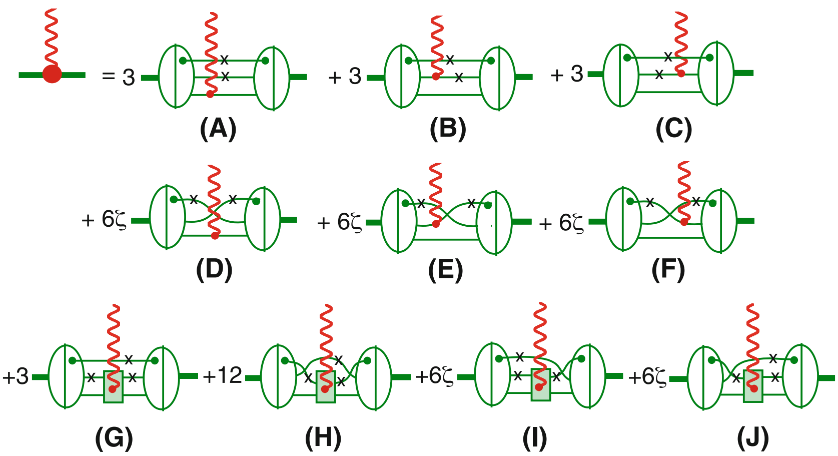

Fig. 7 The electromagnetic $3N$ current in CST for elastic electron scattering from the $3N$ bound state. A *cross* on a nucleon line indicates that the particle is on mass shell. Diagrams (*A*)–(*F*) define the complete impulse approximation (CIA), and diagrams (*G*)–(*J*) describe processes in which the photon couples to two-body currents associated with the two-nucleon kernel

One may wonder whether $3N$ forces still need to be included, which might spoil the nice agreement with the experimental value for E_t. However, this is not the case: in a true relativistic OBE theory for the interaction between nucleons no additional irreducible $3N$ forces can be derived from the basic vertices of the theory!

When discussing $3N$ forces it is important to keep in mind that it is a framework-dependent concept. Figure 6 illustrates that vertices with off-shell terms together with off-shell propagators can transform into contact vertices and take the form of $3N$ forces. Note that these "$3N$ forces," which are reducible in the framework of the CST and automatically included in a OBE model, contain no new parameters and are *completely determined from the NN interaction*. Our calculations provide true predictions of the triton binding energy and of the structure of the $3N$ bound state in the form of the $3N$ vertex function.

To test these vertex functions, we calculated the electromagnetic form factors of the $3N$ bound states. The CST $3N$ current displayed in Fig. 7 was was derived in [10]. The first six terms (diagrams A–F) are referred to as the "complete impulse approximation" (CIA), and the remaining diagrams (G–J) represent interaction currents. Note that the term "impulse approximation" can be misleading because the CIA in CST includes contributions that in nonrelativistic frameworks appear as interaction currents (pair terms related to Z-graphs). Both the complete current and CIA by itself are conserved.

The $3N$ form factors were calculated in the CIA approximation for the first time in [11]. This calculation explored the model-dependence of the CST predictions using the family of older ν-dependent NN models of [6]. Interaction currents are known to give important contributions to the $3N$ form factors. Therefore, a good description of the data over a large range of Q cannot be expected in CIA. Instead, we compare to calculations by the Pisa-Jlab collaboration, described in Ref. [12] and labeled "IARC" below. The IARC calculations use a nonrelativistic impulse approximation with a one-nucleon current and wave functions obtained from the Argonne AV18 NN and Urbana IX $3N$ potentials, and also include first-order relativistic corrections. The Coulomb interaction is not included in the IARC and CST calculations presented here.

Figure 8 shows the isoscalar and isovector charge and magnetic $3N$ form factors for models WJC-1 and WJC-2 in CIA-0 [13] (an approximation to CIA in which the $3N$ vertex function with two off-shell nucleons is replaced by a vertex function with only one nucleon off mass shell), W16 both in CIA and CIA-0, and IARC for the AV18/UIX interaction. Clearly, CIA-0 is an excellent approximation to CIA for W16.

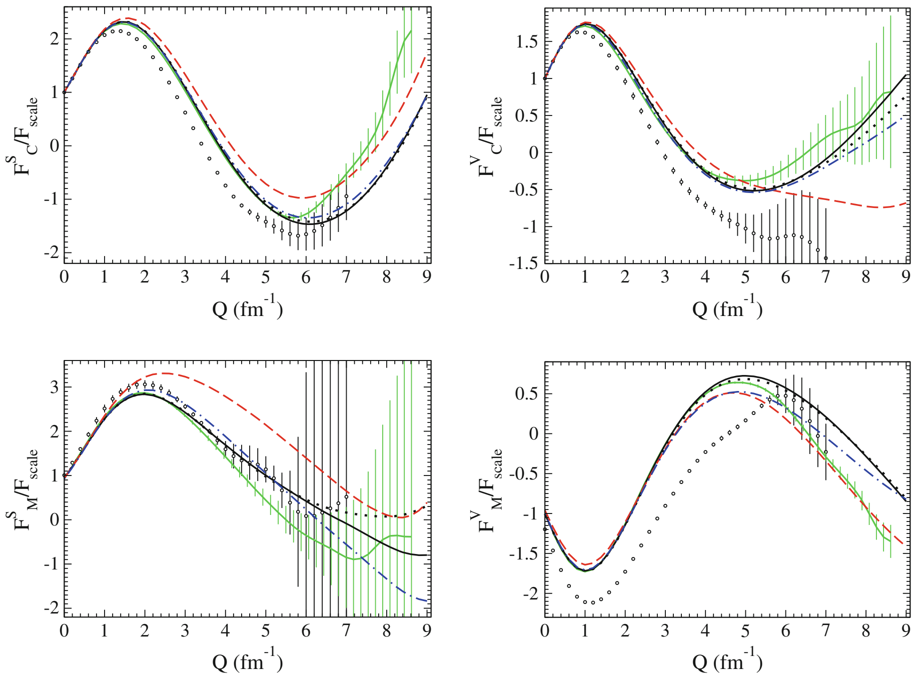

Fig. 8 Isoscalar (*first row*) and isovector (*second row*) charge form factors of the $3N$ bound states. In each case, the form factor is divided by a common scaling function $F_{\mathrm{scale}}(Q)$ [13]. The *solid line* is the result for NN model W16 in CIA, the *dotted line* is the approximation CIA-0 for the same model. The *dashed line* is model WJC-1, and the *dash-dotted line* is model WJC-2, both in CIA-0. For comparison, the *solid line* with theoretical error bars is the result of an IARC calculation by Marcucci based on the AV18/UIX potential. All calculations employ the on-shell single-nucleon current. The *full circles* represent the experimental data [14]

All models reproduce the correct $3N$ binding energy, and the form factors remain close to each other. The only exception is WJC-1, for which some deviations are observed already at relatively small Q. The reason for this behavior is instructive: WJC-1 is the only model with a mixed pseudoscalar–pseudovector πNN coupling. Its pseudoscalar part induces strong Z-graph-type currents, which are not present in the other cases.

To summarize, the $3N$ electromagnetic form factors obtained so far in CST present a very coherent picture, from which one can conclude that CST provides a sound description of the structure of the $3N$ bound states. For more detailed studies the interaction currents have to be calculated as well.

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