ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 26 (2026) #P1.09
https://doi.org/10.26493/1855-3974.2838.8be
(Also available at http://amc-journal.eu)

Numerical semigroups with non-admissible distances between gaps greater than its multiplicity*

J. C. Rosales †
Departamento de Álgebra, Universidad de Granada, E-18071 Granada, Spain

Manuel B. Branco ‡
Departamento de Matemática, Universidade de Évora, 7000 Évora, Portugal

Márcio A. Traesel
Departamento de Matemática, Instituto Federal de São Paulo, Caraguatatuba, SP, Brazil

Received 3 March 2022, accepted 29 August 2024, published online 17 December 2025

Abstract

Let A be a nonempty subset of positive integers. In this paper we study the set of numerical semigroups that fulfill: if $\{x, y\} \subseteq \mathbb{N} \backslash S$ and $x > y > \min(S \backslash \{0\})$, then $x - y \notin A$.

Keywords: Frobenius pseudo-varieties, genus number, numerical semigroups, $\mathsf{PD}(A)$*-semigroup and tree (associated to a* $\mathsf{PD}(A)$*-semigroup).*

Math. Subj. Class. (2020): 20M14, 11D07

1 Introduction

Let $\mathbb{Z}$ be the set of integers and let $\mathbb{N} = \{z \in \mathbb{Z} \mid z \geq 0\}$. A submonoid of $\mathbb{N}$ is a subset of $\mathbb{N}$ closed under addition, containing the zero element. A submonoid with finite complement in $\mathbb{N}$ is a numerical semigroup.

*The authors would also like to thank the referee for the constructive and helpful comments and corrections.
†The first author was partially supported by MTM-2017-84890-P and by Junta de Andalucia group FQM-343.
‡Corresponding author. The second author is supported by the project FCT PTDC/MAT/73544/2006).
E-mail addresses: jrosales@ugr.es (J. C. Rosales), mbb@uevora.pt (Manuel B. Branco), marciotraesel@ifsp.edu.br (Márcio A. Traesel)

If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{F}(S) = \max(\mathbb{Z}\backslash S)$ and $\mathrm{g}(S) = \#(\mathbb{N}\backslash S)$ (the cardinalty of $\mathbb{N}\backslash S$) are three importants invariants of S known as *multiplicity*, *Frobenius number* and *genus* of S, respectively (see [1] and [4]). We denote by $H(S) = \{x \in \mathbb{N}\backslash S \mid x > \mathrm{m}(S)\}$.

For m a positive integer, the semigroup $\{0, m, \rightarrow\}$ is denoted here by $\triangle(m)$ and it is called half-line or ordinary (the symbol $\rightarrow$ means that every integer greater than m belongs to the set).

Let A be a nonempty set of $\mathbb{N}\backslash\{0\}$. A $\mathsf{PD}(A)$-*semigroup* is a numerical semigroup S such that $H(S) + A \subseteq S$. Our main purpose in this work is to study this class of numerical semigroups. In particular, we will study the sets $\mathcal{P}(A) = \{S \mid S \text{ is a } \mathsf{PD}(A)\text{-semigroup}\}$ and $\mathcal{P}\big(A, m\big) = \{S \in \mathcal{P}(A) \mid \mathrm{m}(S) = m\}$. Its study is clearly motivated by generalizing to other classes of semigroups studied before, such as:

(1) We say that a numerical semigroup S is elementary if $\mathrm{F}(S) < 2\mathrm{m}(S)$. The concept of elementary numerical semigroup is studied in [2] and comprehensively in [7]. If we denote by $\mathcal{E}(m)$ the set of elementary numerical semigroups with multiplicity m, then we have that $\mathcal{E}(m) = \mathcal{P}\big(\{m\}, m\big)$.

(2) Given a numerical semigroup S and $s \in S$, we denote the concentration of a numerical semigroup S by $\mathsf{C}(S) = \max\{\mathrm{next}_S(s) - s \mid s \in S\backslash\{0\}\}$ wherein $\mathrm{next}_S(s) = \min\{x \in S \mid s < x\}$. Recently, the authors studied the class of numerical semigroups with concentration two [8]. It is easy to see that this class coincides with the set $\mathcal{P}\big(\{1\}, m\big)\backslash\triangle(m)$.

This work is organized as follows. In Section 2, we will order the elements of $\mathcal{P}(A)$ to construct a tree with root $\mathbb{N}$. This ordering will provide us an algorithmic procedure that allows us to recursively build the elements of $\mathcal{P}(A)$.

In Section 3, we will show that $\mathcal{P}\big(A, m\big)$ has infinite cardinality if and only if m is a even number and all elements in A are odd numbers.

If S is a numerical semigroup, we denote by $P(S) = \{x \in S \mid \mathrm{m}(S) < x < 2\mathrm{m}(S)\}$. A subset, X, of $\{m+1, m+2, \ldots, 2m-1\}$ is a $\mathsf{PD}(A, m)$-set if $X = P(S)$ for some a $\mathsf{PD}(A, m)$-semigroup. If X is a $\mathsf{PD}(A, m)$-set we denote by $\gamma(X) = \{S \in \mathsf{PD}(A, m) \mid P(S) = X\}$. In Section 4, we will see that the set $\{\gamma(X) \mid X \text{ is a } \mathsf{PD}(A, m)\text{-set}\}$ is a partition of the set $\mathcal{P}\big(A, m\big)$. Furthermore, following the notation introduced in [5], we will prove that $\gamma(X)$ is a Frobenius pseudo-variety. We will show that the elements of the set $\{\gamma(X) \mid X \text{ is a } \mathsf{PD}(A, m)\text{-set}\}$ can be ordered in a finite tree. From this, we will see that the set $\mathcal{P}\big(A, m\big)$ is a finite tree wherein its vertices are Frobenius pseudo-varieties.

Finally, in Section 5 we will provide algorithms to produce all elements in $\mathcal{P}\big(A, m\big)$ with fixed genus or fixed Frobenius number.

2 The tree of $\mathsf{PD}(A)$-semigroups

Throughout this paper A will be a nonempty set of $\mathbb{N}\backslash\{0\}$. The next result is easy to prove.

Lemma 2.1. *If S is a $\mathsf{PD}(A)$-semigroup and $S \neq \mathbb{N}$, then $S \cup \{F(S)\}$ is a $\mathsf{PD}(A)$-semigroup.*

The above result enable us, given a $\mathsf{PD}(A)$-semigroup, S, to define recursively the following sequence of a $\mathsf{PD}(A)$-semigroups, as:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \mathbb{N} \\ \mathbb{N} & \text{otherwise.} \end{cases}$

The next result is trivial.

Proposition 2.2. *If S is a $\mathsf{PD}(A)$-semigroup and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence of numerical semigroups, then there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$.*

A graph $G = (V, E)$ consists of nonempty set V and a collection E of ordered pairs (v, w) of distinct elements from V. Elements of V are called a vertices and elements of E are called edges. A path of length n connecting the vertices u and v of G is a sequence of n distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ with $v_0 = u$ and $v_n = v$.

A graph G is a tree if there exists a vertex r (known as the root of G) such that for every other vertex v of G, there exists a unique path connecting v and r. If (u, v) is a edge of the tree then we say that u is a child of v. If there exists a path connecting the vertices u and v, then we say that u is a descendant of v.

We define the graph $G\big(\mathcal{P}(A)\big)$ as the graph whose vertices are elements of $\mathcal{P}\big(A\big)$ and $(S, T) \in \mathcal{P}\big(A\big) \times \mathcal{P}\big(A\big)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. As a consequence of Lemma 2.2, we have the following.

Proposition 2.3. *The graph $G\big(\mathcal{P}(A)\big)$ is a tree with root equal to $\mathbb{N}$.*

Clearly, we can recursively construct the elements of the set $\mathcal{P}(A)$, starting in $\mathbb{N}$, we connect each vertex with its children. Therefore, we need to characterize the children of an arbitrary vertex of this tree, for that we need the following results.

If $\mathcal{X}$ is a nonempty subset of $\mathbb{N}$, we denote by $\langle \mathcal{X} \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by $\mathcal{X}$, that is,

$$\langle \mathcal{X} \rangle = \left\{ \sum_{i=1}^{n} \lambda_i\, x_i \mid n \in \mathbb{N}\backslash \{0\}\,,\ x_i \in \mathcal{X},\ \lambda_i \in \mathbb{N} \text{ and } i \in \{1, \ldots n\} \right\}.$$

Lemma 2.4 ([9, Lemma 2.1])**.** *Let $\mathcal{X}$ be a nonempty subset of $\mathbb{N}$. Then $\langle \mathcal{X} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{X}) = 1$.*

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle \mathcal{X} \rangle$ then we say that $\mathcal{X}$ is a system of generators of M. Moreover, if $M \neq \langle \mathcal{Y} \rangle$ for all $\mathcal{Y} \subsetneq \mathcal{X}$, then we say that $\mathcal{X}$ is a minimal system of generators of M.

Lemma 2.5 ([9, Corollary 2.8])**.** *Every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which is finite.*

Given a submonoid of $(\mathbb{N}, +)$, M, we denote by $\mathrm{msg}(M)$ the minimal system of generators of M, its cardinality is called the embedding dimension of M and it is denoted by $\mathrm{e}(M)$.

Lemma 2.6 ([6, Lemma 1.7])**.** *Let S be a numerical semigroup and $x \in S$. Then $S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

We already have conditions to characterize the children of a vertex S in the tree $G\big(\mathcal{P}(A)\big)$.

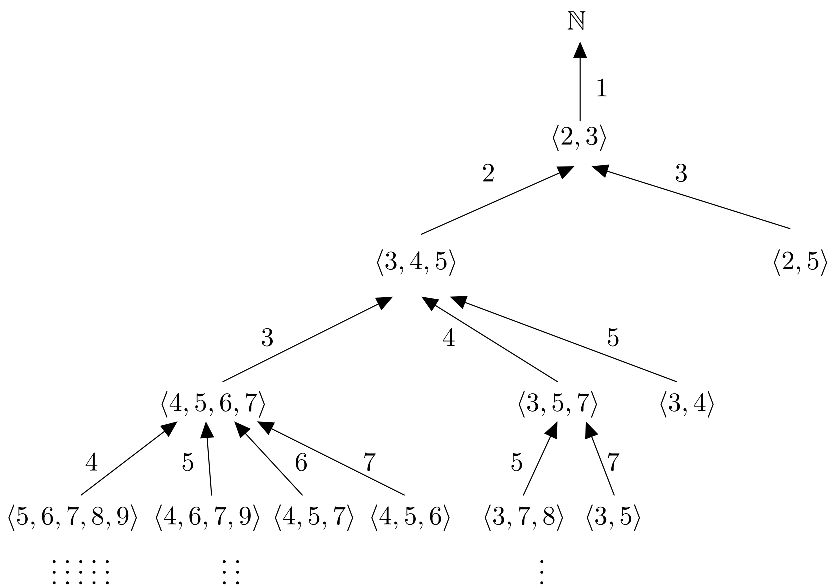
Proposition 2.7. *If $S \in \mathcal{P}(A)$, then the set of children of S in the tree $G(\mathcal{P}(A))$ is equal to $\{S\backslash\{x\} \mid x \in \operatorname{msg}(S),\ x > \mathrm{F}(S)$ and $x - a \notin H(S\backslash\{x\})$, for all $a \in A\}$.*

Proof. If T is a child of S, then $T \in \mathcal{P}(A)$ and $S = T \cup \{F(T)\}$. Hence, we can deduce that $T = S\backslash\{F(T)\}$. Using Lemma 2.6, we have that $F(T) \in \operatorname{msg}(S)$ and as $S = T \cup \{F(T)\}$ then $F(S) < F(T)$. Moreover, since $T \in \mathcal{P}(A)$ and $\mathrm{F}(T) \notin T$, then $\mathrm{F}(T) - a \notin H(T)$ for all $a \in A$.

Conversely, suppose that $x \in \operatorname{msg}(S)$, $x > \mathrm{F}(S)$ and $x - a \notin H(S\backslash\{x\})$ for all $a \in A$. Then, by Lemma 2.6, $S\backslash\{x\}$ is a numerical semigroup with $\mathrm{F}(S\backslash\{x\}) = x$. By applying $x - a \notin H(S\backslash\{x\})$ for all $a \in A$ and $S \in \mathcal{P}(A)$, we deduce that $S\backslash\{x\} \in \mathcal{P}(A)$. Finally, as $S = (S\backslash\{x\}) \cup \mathrm{F}(S\backslash\{x\})$ then $S\backslash\{x\}$ is a child of S in the tree $G(\mathcal{P}(A))$. □

Note that, since half-line $\triangle(m) \in \mathcal{P}(A)$ for all $m \in \mathbb{N}$, then we get that the set $\mathcal{P}(A)$ has infinite cardinality.

Example 2.8. Let us construct the tree $G(\mathcal{P}(\{2\}))$.



An edge $S \to T$ is labelled x whenever S is obtain from T by removing x, that is, $S = T\backslash\{x\}$.

The number that appears on each side of the edges is the element that we remove from the semigroup to obtain its corresponding new children. This number coincide with the Frobenius number of the corresponding child.

3 PD(A)-semigroups with a given multiplicity

From now on m denotes a positive integer greater than or equal to 2. Our first aim in this section is to see which conditions must m and A fulfill so that $\mathcal{P}(A, m)$ has infinite cardinality.

If S is a numerical semigroup, then $\mathbb{N}\backslash S$ is a finite set and so we get the following result.

Lemma 3.1. *Let S be a numerical semigroup, the set $\{T \mid T$ is a numerical semigroup and $S \subseteq T\}$ is a finite set.*

Lemma 3.2. *Let the hypothesis be as above. Then the following conditions hold:*

(1) $\{S \in \mathcal{P}(A, m) \mid m+1 \in S\}$ *is a finite set.*

(2) $\{S \in \mathcal{P}(A, m) \mid 2m-1 \in S\}$ *is a finite set.*

(3) *If $x \in \{m+1, m+2, \ldots, 2m-2\}$, then* $\{S \in \mathcal{P}(A, m) \mid \{x, x+1\} \subseteq H(S)\}$ *is a finite set.*

(4) *If $x \in \{m+1, m+2, \ldots, 2m-2\}$, then* $\{S \in \mathcal{P}(A, m) \mid \{x, x+1\} \subseteq S\}$ *is a finite set.*

Proof. (1) Since $\gcd\{m, m+1\} = 1$, then by Lemma 2.4, we get that $\langle m, m+1\rangle$ is a numerical semigroup. It is clear that $\{S \in \mathcal{P}(A, m) \mid m+1 \in S\} \subseteq \{T \mid T$ is a numerical semigroup and $\langle m, m+1\rangle \subseteq T\}$ and, by Lemma 3.1, the last set is finite.
(2) The proof is similar to (1) using $2m-1$ in place of $m+1$.
(3) As $\{x, x+1\} \subseteq H(S)$, if $a \in A$, then $\{x+a, x+a+1\} \subseteq S$ and $\gcd\{x+a, x+a+1\} = 1$. To conclude the proof it is enough to note that $\{S \in \mathcal{P}(A, m) \mid \{x, x+1\} \subseteq H(S)\} \subseteq \{T \mid T$ is a numerical semigroup and $\langle x+a, x+a+1\rangle \subseteq T\}$ and, by Lemma 3.1, the last set is finite.
(4) Clearly $\{S \in \mathcal{P}(A, m) \mid \{x, x+1\} \subseteq S\} \subseteq \{T \mid T$ is a numerical semigroup and $\langle x, x+1\rangle \subseteq T\}$ and, again by Lemma 3.1, the last set is finite. □

Lemma 3.3. *With above notation. If $\mathcal{P}(A, m)$ has infinite cardinality, then m is a even number.*

Proof. By using (1) and (3) of Lemma 3.2, we deduce that if the set $\mathcal{P}(A, m)$ has infinite cardinality, then $\{S \in \mathcal{P}(A, m) \mid m+1 \notin S$ and $m+2 \in S\}$ is also an infinite set. Since $\{S \in \mathcal{P}(A, m) \mid m+1 \notin S$ and $m+2 \in S\} \subseteq \{T \mid T$ is a numerical semigroup and $\langle m, m+2\rangle \subseteq T\}$, by applying Lemmas 2.4 and 3.1, we obtain that $\gcd\{m, m+2\} \neq 1$ and, consequently, m is an even number. □

Lemma 3.4. *With above notation. If $\mathcal{P}(A, m)$ has infinite cardinality, then all elements in A are odd numbers.*

Proof. For the same reasons as previously, we have that $\{S \in \mathcal{P}(A, m) \mid m+1 \notin S$ and $m+2 \in S\}$ is an infinite set. If $a \in A$, then we get that $\{m, m+2, m+1+a\} \subseteq S$ and $\{S \in \mathcal{P}(A, m) \mid m+1 \notin S$ and $m+2 \in S\} \subseteq \{T \mid T$ is a numerical semigroup and $\langle m, m+2, m+1+a\rangle \subseteq T\}$. By using Lemma 3.1, we can deduce that $\gcd\{m, m+2, m+1+a\} = \gcd\{m, 2, 1+a\} \neq 1$. Hence $\gcd\{m, 2, 1+a\} = 2$ and so a is an odd number. □

We are ready to show the above announced result.

Theorem 3.5. *With above notation. The set* $\mathcal{P}\big(A, m\big)$ *has infinite cardinality if and only if* m *is an even number and all elements in* A *are odd numbers.*

Proof. $Necessity$. This is an immediate consequence of Lemmas 3.3 and 3.4.

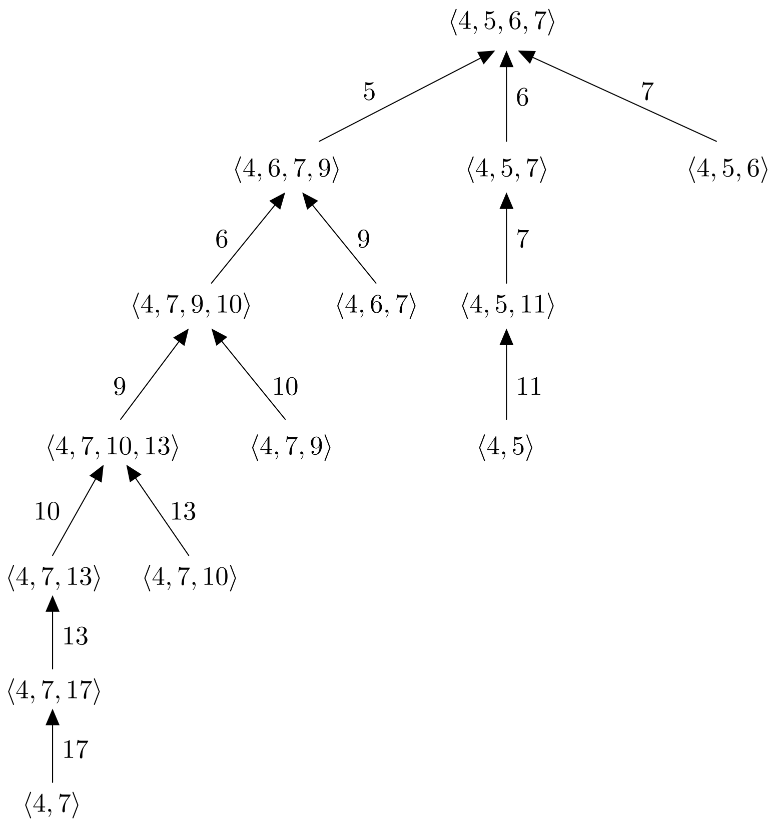
$Sufficiency$. For each $n \in \{m, \rightarrow\}$ we denote by $S(n) = \big\{2k \mid k \in \big\{0, \frac{m}{2}, \rightarrow\big\}\,\big\} \cup \{n, \rightarrow\}$. It is easy to see that $S(n) \in \mathcal{P}\big(A, m\big)$ for all $n \in \{m, \rightarrow\}$ and thus $\mathcal{P}\big(A, m\big)$ has infinite cardinality. □

We define the graph $G\big(\mathcal{P}(A, m)\big)$ as the graph whose vertices are elements of $\mathcal{P}\big(A, m\big)$ and $(S, T) \in \mathcal{P}\big(A, m\big) \times \mathcal{P}\big(A, m\big)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. In the same way as in Section 2, we have the following result.

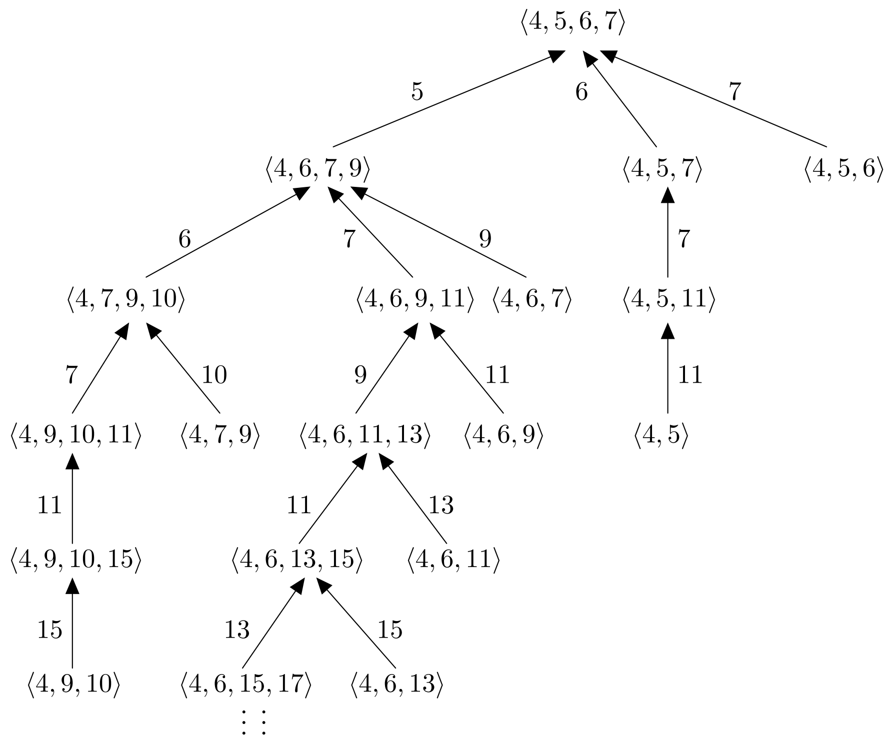
Proposition 3.6. *The graph* $G\big(\mathcal{P}(A, m)\big)$ *is a tree with root equal to* $\triangle(m)$. *Furthermore, the set of children of* S *in the tree* $G\big(\mathcal{P}(A, m)\big)$ *is equal to* $\{S \backslash \{x\} \mid x \in \mathrm{msg}(S),\ x \neq m,\ x > \mathrm{F}(S)$ *and* $x - a \notin H(S \backslash \{x\})$ *for all* $a \in A\}$.

In the next examples we are going to build the trees $G\big(\mathcal{P}(\{2\}, 4)\big)$ and $G\big(\mathcal{P}(\{3\}, 4)\big)$. Observe that by Theorem 3.5, we get that the first is finite and the second is infinite.

Example 3.7. We are going to build the tree $G\big(\mathcal{P}(\{2\}, 4)\big)$.



Example 3.8. We are going to construct the tree $G\big(\mathcal{P}(\{3\},4)\big)$.



4 Partition of the set $\mathcal{P}\big(A,m\big)$

Given a numerical semigroup S, we denote by

$$P(S) = \{x \in S \mid \mathrm{m}(S)+1 \leq x \leq 2\mathrm{m}(S)-1\} \text{ and by}$$
$$\overline{P(S)} = \{\mathrm{m}(S)+1,\ldots,2\mathrm{m}(S)-1\} \setminus P(S).$$

Proposition 4.1. *Let S be a numerical semigroup. Then S is an $\mathsf{PD}(A)$-semigroup if and only if $\overline{P(S)} + A \subseteq S$.*

Proof. As $\overline{P(S)} \subseteq H(S)$, if $x \in \overline{P(S)}$ then $x \in H(S)$ and so $\{x\}+A \subseteq S$. Conversely, if $h \in H(S)$ we have that $i = h \bmod m \in \{1,\ldots,m-1\}$ and $m+i \in \overline{P(S)}$. Moreover, there exists $q \in \mathbb{N}$ such that $h = m+i+q\cdot m$. Hence, $\{h\}+A = \{m+i\}+A+\{q\cdot m\} \subseteq S$. □

Let R be the equivalence relation defined on $\mathcal{P}\big(A,m\big)$ by

$$S\ R\ T \text{ if and only if } P(S) = P(T).$$

Let $[S]$ be denote the class of $S \in \mathcal{P}\big(A,m\big)$ modulo R, that is,

$$[S] = \big\{T \in \mathcal{P}\big(A,m\big) \mid S\ R\ T\big\}\,.$$

Hence, the quotient set of $\mathcal{P}(A, m)$ induce by R is the set

$$\mathcal{P}(A, m)/R = \{[S] \mid S \in \mathcal{P}(A, m)\}.$$

The power set of a set X is the set of all subsets of X, denote by $\mathbb{P}(X) = \{Y \mid Y \subseteq X\}$.

Proposition 4.2. *The correspondence*

$$\varphi\colon \mathcal{P}(A, m)/R \to \mathbb{P}(\{m+1, \dots, 2m-1\})$$

such that $\varphi([S]) = P(S)$ *is an injective map.*

Proof. Clearly, φ is a map, because if $[S] = [T]$ then $S\ R\ T$ and so $P(T) = P(S)$. Since $\varphi([S]) = \varphi([T])$ implies that $P(T) = P(S)$, and thus $[S] = [T]$, we get that φ is injective. □

A $\mathsf{PD}(A, m)$-set is a subset, X, of $\{m+1, \dots, 2m-1\}$ that verifies: if $a \in A$, $b \in \{m+1, \dots, 2m-1\} \setminus X$ and $m+1 \leq a+b \leq 2m-1$, then $a+b \in X$.

Proposition 4.3. *If* φ *is the map defined in* Proposition 4.2*, then* $Im(\varphi) = \{X \mid X$ *is a* $\mathsf{PD}(A, m)$*-set*$\}$.

Proof. If $X \in Im(\varphi)$, then there exists $S \in \mathcal{P}(A, m)$ such that $P(S) = X$ and thus $X \subseteq \{m+1, \dots, 2m-1\}$. Still, if $a \in A$, $b \in \{m+1, \dots, 2m-1\} \setminus X$ then $b \in H(S)$ and so $a+b \in S$. Consequently, if $m+1 \leq a+b \leq 2m-1$, then $a+b \in X$. Hence, we obtain that X is a $\mathsf{PD}(A, m)$-set.

Conversely, if X is a $\mathsf{PD}(A, m)$-set, then we deduce that $S_X = \{0, m\} \cup X \cup \{2m, \rightarrow\} \in \mathcal{P}(A, m)$ and $P(S_X) = X$. Wherefore, $X \in Im(\varphi)$. □

Given X a $\mathsf{PD}(A, m)$-set, we denote by $\gamma(X) = \{S \in \mathcal{P}(A, m) \mid P(S) = X\}$. As a consequence of Propositions 4.2 and 4.3, we establish the following result.

Theorem 4.4. *With above notation, the set* $\{\gamma(X) \mid X$*is a* $\mathsf{PD}(A, m)$*-set*$\}$ *defines a (disjoint) partition of* $\mathcal{P}(A, m)$.

Following the notation introduced in [5], a Frobenius pseudo-variety is a non-empty family $\mathfrak{P}$ of numerical semigroups that fulfils the following conditions:

(1) $\mathfrak{P}$ has a maximum element (with respect to the inclusion order).

(2) If $S, T \in \mathfrak{P}$, then $S \cap T \in \mathfrak{P}$.

(3) If $S \in \mathfrak{P}$ and $S \neq \max \mathfrak{P}$, then $S \cup \{\mathrm{F}(S)\} \in \mathfrak{P}$.

Our next aim in this section is to prove that if X is a $\mathsf{PD}(A, m)$-set, then $\gamma(X)$ is a Frobenius pseudo-variety.

Proposition 4.5. *If* X *is a* $\mathsf{PD}(A, m)$*-set, then* $\gamma(X)$ *is a Frobenius pseudo-variety.*

Proof. Clearly, $S_X = \{0, m\} \cup X \cup \{2m, \rightarrow\}$ is the maximum element in the set $\gamma(X)$. If $\{S, T\} \subseteq \gamma(X)$, then $P(S) = X$ and $P(T) = X$. Hence, we can conclude that $P(S \cap T) = X$ and so $\overline{P(S \cap T)} = \overline{P(S)} = \overline{P(T)}$. By using Proposition 4.1, we have that $\overline{P(S)} + A = \overline{P(S \cap T)} + A \subseteq S \cap T$ and thus $S \cap T \in \mathcal{P}(A, m)$. Consequently, $S \cap T$ is an element of $\gamma(X)$.

If $S \in \gamma(X)$ and $S \neq S_X$, then we obtain that $\mathrm{F}(S) > 2m$ and thus $S \cup \{\mathrm{F}(S)\} \in \gamma(X)$. □

Following the notation introduced in [2], a numerical semigroup S is elementary if $\mathrm{F}(S) < 2\mathrm{m}(S)$. In [7], a broad study of these semigroups is carried out which were also studied in [3] and [10].

Proposition 4.6. *The following conditions are equivalent.*

(1) *S is an elementary numerical semigroup and $S \in \mathcal{P}(A,m)$.*

(2) *$S = S_X$ for some $\mathsf{PD}(A,m)$-set X.*

Proof. (1) *implies* (2). By Proposition 4.3, we obtain that $P(S)$ is a $\mathsf{PD}(A,m)$-set. If S is elementary, then $\{2m, \rightarrow\} \subseteq S$ and thus $S = \{0,m\} \cup P(S) \cup \{2m, \rightarrow\} = S_{P(S)}$.

(2) *implies* (1). From the proof of Proposition 4.5, we have that $S_X \in \mathcal{P}(A,m)$ and so S_X is an elementary numerical semigroup. □

Let $\mathcal{E}(A,m) = \{S \in \mathcal{P}(A,m) \mid S \text{ is elementary}\} = \{S_X \mid X \text{ is a } \mathsf{PD}(A,m)\text{-set}\}$. The maximum element in the set $\mathcal{E}(A,m)$ is the half-line $\triangle(m)$. From Lemma 2.1, we can deduce the following result.

Lemma 4.7. *If $S \in \mathcal{E}(A,m)$ and $S \neq \triangle(m)$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{E}(A,m)$.*

The previous result allows us, given $S \in \mathcal{E}(A,m)$ to define recursively the following sequence of elements in $\mathcal{E}(A,m)$, as:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } S_n \neq \triangle(m) \\ \triangle(m) & \text{otherwise.} \end{cases}$

Now, we define the graph $G(\mathcal{E}(A,m))$ as the graph whose vertices are elements of $\mathcal{E}(A,m)$ and $(S,T) \in \mathcal{E}(A,m) \times \mathcal{E}(A,m)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. It is easy to prove the following result.

Proposition 4.8. *The graph $G(\mathcal{E}(A,m))$ is a finite tree with root equal to the half-line $\triangle(m)$.*

It is clear that, if X is a $\mathsf{PD}(A,m)$-set, then $\gamma(X) = [S_X]$. As a consequence of Theorem 4.4 and Proposition 4.6, we obtain the next result.

Proposition 4.9. *With above notation, the set $\{[S] \mid S \in \mathcal{E}(A,m)\}$ defines a (disjoint) partition of $\mathcal{P}(A,m)$.*

By using Propositions 4.5, 4.8 and 4.9 we can formulate the following result.

Corollary 4.10. *The set $\mathcal{P}(A,m)$ is a finite tree in which each vertex is a Frobenius pseudo-variety .*

5 Algorithms for computing all elements in $\mathcal{P}(A, m)$

From Theorem 4.4, we deduce that $\{\gamma(X) \mid X \text{ is a } \mathsf{PD}(A, m)\text{-set}\}$ is a partition of the set $\mathcal{P}(A, m)$. Hence, in order to determine explicitly the elements in $\mathcal{P}(A, m)$ we will need:

(1) an algorithm to compute the set of all $\mathsf{PD}(A, m)$-set.

(2) an algorithm to compute the set $\gamma(X)$, given X a $\mathsf{PD}(A, m)$-set.

In the literature, there are many algorithms devoted to computing the power set of the set $\{m+1, \dots, 2m-1\}$, that is, $\mathbb{P}(\{m+1, \dots, 2m-1\})$. Moreover, it is easy to check whether an element of this set is a $\mathsf{PD}(A, m)$-set. Since we have problem (1) solved, we give the following example.

Example 5.1. Let us fully compute the set $\mathsf{PD}(\{2\}, 5)$-set. We need the set

$$\mathbb{P}(\{6,7,8,9\}) = \{\emptyset, \{6\}, \{7\}, \{8\}, \{9\}, \{6,7\}, \{6,8\}, \{6,9\}, \{7,8\}, \{7,9\}, \{8,9\}, \{6,7,8\}, \{6,7,9\}, \{6,8,9\}, \{7,8,9\}, \{6,7,8,9\}\}.$$

Note that $X \subseteq \{6,7,8,9\}$ is a $\mathsf{PD}(\{2\}, 5)$-set that fulfills the following: if $6 \notin X$ then $8 \in X$, and if $7 \notin X$ then $9 \in X$. Hence, the $\mathsf{PD}(\{2\}, 5)$-sets are

$$\{6,7\}, \{6,9\}, \{7,8\}, \{8,9\}, \{6,7,8\}, \{6,7,9\}, \{6,8,9\}, \{7,8,9\}, \{6,7,8,9\}.$$

Our next goal in this section is to solve the issue (2). By Proposition 4.5, we know that $S_X = \{0, m\} \cup X \cup \{2m, \rightarrow\}$ is the maximum element in $\gamma(X)$ and if $S \in \gamma(X)$ such that $S \neq S_X$, then $S \cup \{\mathrm{F}(S)\} \in \gamma(X)$. Moreover, S_X is the unique element in $\gamma(X)$ such that $\mathrm{F}(S_X) < 2m$.

If X is a $\mathsf{PD}(A, m)$-set and $S \in \gamma(X)$, then we can define recursively the following sequence of elements in $\gamma(X)$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{F}(S_n)\} & \text{if } \mathrm{F}(S_n) > 2m \\ S_n & \text{otherwise.} \end{cases}$

The next result has immediate proof.

Lemma 5.2. *If X is a $\mathsf{PD}(A, m)$-set, $S \in \gamma(X)$ and $\{S_n \mid n \in \mathbb{N}\}$ is the previous sequence of numerical semigroups in $\gamma(X)$, then there exists $k \in \mathbb{N}$ such that $S_k = S_X$.*

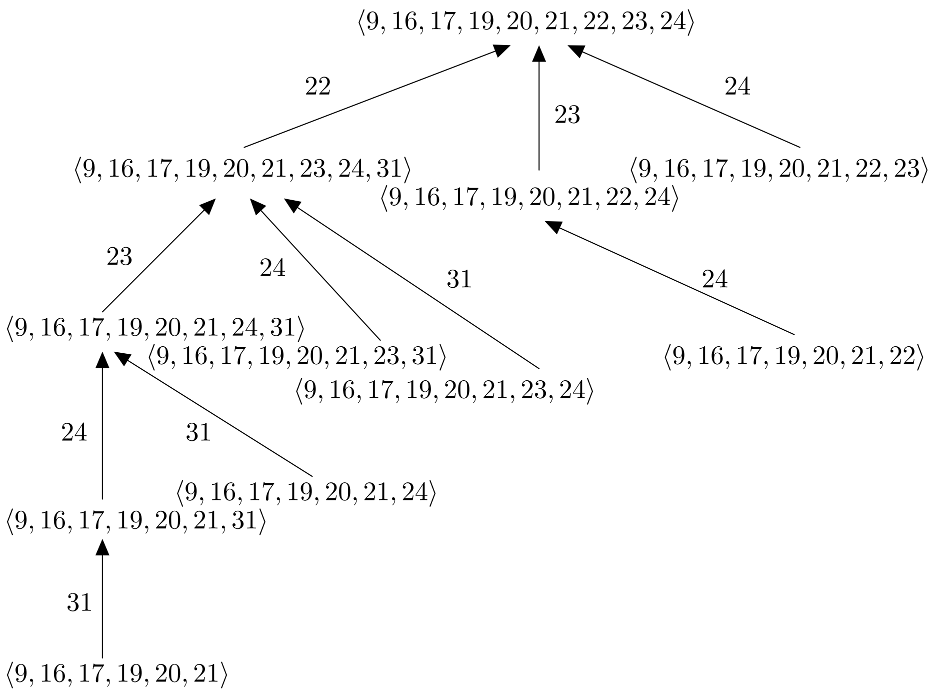
We define the graph $G(\gamma(X))$ as the graph whose vertices are elements of $\gamma(X)$ and $(S, T) \in \gamma(X) \times \gamma(X)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$. It is not hard to prove the following result.

Proposition 5.3. *If X is a $\mathsf{PD}(A, m)$-set, then the graph $G(\gamma(X))$ is a tree with root equal to S_X. Furthermore, the set of children of S in the tree $G(\gamma(X))$ is equal to $\{S\backslash\{b\} \mid b \in \mathrm{msg}(S),\ b > \max\{\mathrm{F}(S), 2m\}, \text{and } b - a \notin H(S\backslash\{b\}), \text{for all } a \in A\}$.*

We illustrate the above results with the following example.

Example 5.4. Clearly the set $\{16, 17\}$ is a $\mathsf{PD}(\{6\}, 9)$-set. Let us compute the graph $G(\gamma(\{16, 17\}))$. By using Proposition 5.3, the graph $G(\gamma(\{16, 17\}))$ is a tree with root equal to $S_{\{16,17\}} = \{0, 9\} \cup \{16, 17\} \cup \{18, \rightarrow\} = \langle 9, 16, 17, 19, 20, 21, 22, 23, 24\rangle$.

Using again Proposition 5.3, we compute the children of each vertex.



Now, we aim to give an algorithm to compute all elements in $\mathcal{P}(A, m)$ with a given genus. For this purpose, we need to introduce some concepts and results.

Let $G = (V, E)$ be a tree with root and $v \in V$. We define the depth of the vertex v as the length of the path that connects v to the root of G, denoted by d_v. If $k \in \mathbb{N}$, we denote by

$$N(G, k) = \{v \in V \mid d_v = k\}\,.$$

We define the height of the tree G by $h(G) = \ max\,\{d_v \mid v \in V\}$.

The next result is easy to prove.

Proposition 5.5. *Let $m \in \mathbb{N}\backslash\{0, 1\}$ and $k \in \mathbb{N}$. Then the following conditions hold.*

(1) $N\big(G(\mathcal{P}(A, m)), k\big) = \big\{S \in \mathcal{P}\big(A, m\big) \mid \mathrm{g}(S) = m - 1 + k\big\}$.

(2) $N\big(G(\mathcal{P}(A, m)), k + 1\big) = \big\{S \mid S$ *is a child of an element in* $N\big(G(\mathcal{P}(A, m)), k\big)\big\}$.

(3) *If* $\mathcal{P}\big(A, m\big)$ *is an infinite set, then* $\big\{\mathrm{g}(S) \mid S \in \mathcal{P}\big(A, m\big)\big\} = \{m - 1, \rightarrow\}$.

(4) *If* $\mathcal{P}\big(A, m\big)$ *is a finite set, then* $\big\{\mathrm{g}(S) \mid S \in \mathcal{P}\big(A, m\big)\big\} = \big\{m - 1, m, \ldots, m - 1 + h\big(G(\mathcal{P}(A, m))\big)\big\}$.

We are already to present the advertised algorithm.

Algorithm 5.6. Input: g a positive integer greater than or equal to $m-1$.
Output: The set $\{S \in \mathcal{P}(A,m) \mid \mathrm{g}(S)=g\}$.

1) Start with $i=m-1$ and $X=\{\langle m, m+1, \cdots, 2m-1\rangle\}$.

2) If $i=g$, then return X.

3) For each $S \in X$ compute $B_S = \{T \mid T \text{ is a child of } S \in G(\mathcal{P}(A,m))\}$.

4) If the set $\bigcup_{S\in X} B_S = \emptyset$, return $\emptyset$.

5) $X=\bigcup_{S\in X} B_S$, $i=i+1$ and go to step 2).

Let us see in an example how our algorithm works.

Example 5.7. Let us compute the set $\{S \in \mathcal{P}(\{3\},4) \mid \mathrm{g}(S)=6\}$.

(1) Start with $i=3$ and $X=\{\langle 4,5,6,7\rangle\}$.

(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$ and then $X = \{\langle 4,6,7,9\rangle, \langle 4,5,7\rangle, \langle 4,5,6\rangle\}$, $i=4$.

(3) The second loop constructs $B_{\langle 4,6,7,9\rangle} = \{\langle 4,7,9,10\rangle, \langle 4,6,9,11\rangle, \langle 4,6,7\rangle\}$, $B_{\langle 4,5,7\rangle} = \{\langle 4,5,11\rangle\}$ and $B_{\langle 4,5,6\rangle} = \emptyset$ and then $X = \{\langle 4,7,9,10\rangle, \langle 4,6,9,11\rangle, \langle 4,6,7\rangle, \langle 4,5,11\rangle\}$, $i=5$.

(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4,9,10,11\rangle, \langle 4,7,9\rangle\}$, $B_{\langle 4,6,9,11\rangle} = \{\langle 4,6,11,13\rangle, \langle 4,6,9\rangle\}$ and $B_{\langle 4,6,7\rangle} = \emptyset$ and $B_{\langle 4,5,11\rangle} = \{\langle 4,5\rangle\}$ then $X = \{\langle 4,9,10,11\rangle, \langle 4,7,9\rangle, \langle 4,6,11,13\rangle, \langle 4,6,9\rangle, \langle 4,5\rangle\}$, $i=6$.

(5) Return $\{S \in \mathcal{P}(\{3\},4) \mid \mathrm{g}(S)=6\} =$
$= \{\langle 4,9,10,11\rangle, \langle 4,7,9\rangle, \langle 4,6,11,13\rangle, \langle 4,6,9\rangle, \langle 4,5\rangle\}$.

We finish this section showing an algorithm which allows us to compute all elements in $\mathcal{P}(A,m)$, with a given Frobenius number. The operation of this algorithm is based on the fact that if S is a vertex of the tree $G(\mathcal{P}(A,m))$ then every descendant of S has a Frobenius number greater than $\mathrm{F}(S)$.

Algorithm 5.8. Input: F a positive integer greater than or equal to $m-1$ and $m \nmid F$.
Output: The set $\{S \in \mathcal{P}(A,m) \mid \mathrm{F}(S)=F\}$.

1) Start with $C=\emptyset$ and $X=\{\langle m, m+1, \ldots, 2m-1\rangle\}$.

2) For each $S \in X$ compute $B_S = \{T \mid T \text{ is a child of } S \in G(\mathcal{P}(A,m))\}$, $C_S = \{T \in B_S \mid \mathrm{F}(T)=F\}$ and $D_S = \{T \in B_S \mid \mathrm{F}(T)<F\}$.

3) Do $C = C \cup \{\bigcup_{S\in X} C_S\}$.

4) If the set $\bigcup_{S\in X} D_S = \emptyset$, return C.

5) Do $X=\bigcup_{S\in X} D_S$ and go to step 2).

Example 5.9. Let us compute the set $\{S \in \mathcal{P}(\{3\}, 4) \mid \mathrm{F}(S) = 7\}$.

(1) Start with with $C = \emptyset$ and $X = \{\langle 4, 5, 6, 7\rangle\}$.

(2) The first loop constructs $B_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle, \langle 4, 5, 6\rangle\}$, $C_{\langle 4,5,6,7\rangle} = \{\langle 4, 5, 6\rangle\}$ and $D_{\langle 4,5,6,7\rangle} = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\}$ then $C = \{\langle 4, 5, 6\rangle\}$ and $X = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle\}$.

(3) The second loop constructs $B_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 6, 7\rangle\}$, $C_{\langle 4,6,7,9\rangle} = \{\langle 4, 6, 9, 11\rangle\}$ and $D_{\langle 4,6,7,9\rangle} = \{\langle 4, 7, 9, 10\rangle\}$,
and it constructs $B_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$, $C_{\langle 4,5,7\rangle} = \{\langle 4, 5, 11\rangle\}$ and $D_{\langle 4,5,7\rangle} = \emptyset$ then $C = \{\langle 4, 5, 6\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 5, 11\rangle\}$ and $X = \{\langle 4, 7, 9, 10\rangle\}$.

(4) The third loop constructs $B_{\langle 4,7,9,10\rangle} = \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 9\rangle\}$, $C_{\langle 4,7,9,10\rangle} = \{\langle 4, 9, 10, 11\rangle\}$ and $D_{\langle 4,7,9,10\rangle} = \emptyset$ then $C = \{\langle 4, 5, 6\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 5, 11\rangle, \langle 4, 9, 10, 11\rangle\}$

(5) Return $\{S \in \mathcal{P}(\{3\}, 4) \mid \mathrm{F}(S) = 7\} =$
$= \{\langle 4, 5, 6\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 5, 11\rangle, \langle 4, 9, 10, 11\rangle\}$.

ORCID iDs

J. C. Rosales https://orcid.org/0000-0003-3353-4335
Manuel B. Branco https://orcid.org/0000-0003-0397-7602
Márcio A. Traesel https://orcid.org/0000-0002-7586-4861

References

[1] V. Barucci, D. E. Dobbs and M. Fontana, Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains, *Mem. Am. Math. Soc.* **125** (1997), x+78, doi:10.1090/memo/0598, https://doi.org/10.1090/memo/0598.

[2] V. Blanco and J. C. Rosales, The set of numerical semigroups of a given genus, *Semigroup Forum* **85** (2012), 255–267, doi:10.1007/s00233-012-9378-2, https://doi.org/10.1007/s00233-012-9378-2.

[3] N. Kaplan and L. Ye, The proportion of Weierstrass semigroups, *J. Algebra* **373** (2013), 377–391, doi:10.1016/j.jalgebra.2012.09.041, https://doi.org/10.1016/j.jalgebra.2012.09.041.

[4] J. L. Ramírez Alfonsín, *The Diophantine Frobenius Problem*, Oxford University Press, 2005, doi:10.1093/acprof:oso/9780198568209.001.0001, https://doi.org/10.1093/acprof:oso/9780198568209.001.0001.

[5] A. M. Robles-Pérez and J. C. Rosales, Frobenius pseudo-varieties in numerical semigroups, *Ann. Mat. Pura Appl. (4)* **194** (2015), 275–287, doi:10.1007/s10231-013-0375-1, https://doi.org/10.1007/s10231-013-0375-1.

[6] J. C. Rosales, Numerical semigroups that differ from a symmetric numerical semigroups in an element, *Algebra Colloq.* **15** (2008), 23–32, doi:10.1142/S1005386708000035, https://doi.org/10.1142/S1005386708000035.

[7] J. C. Rosales and M. B. Branco, On the enumeration of the set of elementary numerical semigroups with fixed multiplicity, Frobenius number or genus, *Kragujevac J. Math.* **46** (2022), 433–442.

[8] J. C. Rosales, M. B. Branco and M. A. Traesel, Numerical semigroups with concentration two, *Indag. Math. (N.S.)* **33** (2022), 303–313, doi:10.1016/j.indag.2021.07.004, https://doi.org/10.1016/j.indag.2021.07.004.

[9] J. C. Rosales and P. A. García-Sánchez, *Numerical Semigroups*, volume 20 of *Developments in Mathematics*, Springer, New York, 2009, doi:10.1007/978-1-4419-0160-6, https://doi.org/10.1007/978-1-4419-0160-6.

[10] Y. Zhao, Constructing numerical semigroups of a given genus, *Semigroup Forum* **80** (2009), 242–254, doi:10.1007/s00233-009-9190-9, https://doi.org/10.1007/s00233-009-9190-9.