DYNAMICS OF DOUBLE STANDARD MAPS

Universidade de Évora

Escola de Ciências e Tecnologia

Lecture summary

Provas de Agregação em Matemática

ANA MARGARIDA DA SILVA AFONSO RODRIGUES
Professora Auxiliar
Departamento de Matemática
Escola de Ciências e Tecnologia
Universidade de Évora

Contents

1. Introduction

We can in a way date back the idea of iterations of one-dimensional maps to the construction of a Babylonian calendar. There was then the need to consider a rotation of the circle, assigning the estimate of the angle of rotation to a piece of the orbit. Thus, in a certain sense, Babylonians considered already the rotation $f(x) = x + \alpha \pmod 1$ on the circle.

The theory of dynamical systems deals with the evolution of systems. It describes processes in motion, tries to predict the future of these systems or processes and also to understand the limitations of these predictions.

Poincaré at the end of the 19th century considered the dynamics of more general maps of the circle, giving rise to the programme of study of modern dynamical systems. Differential equations are not always easy to solve. Attempts to integrate analytically the three-body problem (the motion of the Earth around the Sun taking into account the gravitation of, say, Jupiter) had failed. An attempt to solve this problem led Poincareé to a radical change of viewpoint, instead of finding explicit analytic solutions of differential equations, one can try to describe qualitative behaviour of these solutions.

Continuous mappings of the circle onto itself were studied by Poincaré in connection with the qualitative study of trajectories on the torus [31]. The problem of Dirichlet for the equation of the string can be reduced to such mappings but the topological investigation turns to be insufficient [22]. While still a student, V.I.Arnold attempted an analytic refinement of Denjoy's [15] theorem completing the work of Poincaré [1].

V. I. Arnold considered analytic mappings of the circle onto itself depending on two-parameters a and ϵ, $x \mapsto x + a + \epsilon f(x)$ with $f(x + 2\pi) = f(x)$. He defined the *domains of ressonance* as follows. A point (a, ϵ) belongs to the domain of ressonance m/n if some point of the circle is shifted by m revolutions when the map is applied n times. That is, if the Poincaré rotation number of the map is equal to m/n. The study of such domains was carried out in connection to problems of Mathieu type using perturbation theory. At the same time, other authors became interested in this same problem [9, 27, 19].

In this work, we will study a particular family of maps that are double covers of the circle, to be introduced in our next section. We called this family *double standard maps.*

For some interval $I \subset \mathbb{R}$, suppose that $f : I \to I$ is a continuous function, x is a periodic point of period m for f if $f^{(m)}(x) = x$. We say x has least period m if, in addition, $f^{(k)}(x) \neq x$ for all $0 < k < m$. In 1964, Sharkovskii introduced what is nowadays known as the Sharkovskii order giving birth to the field of combinatorial and topological dynamics. Sharkovskii's theorem concerns the possible least periods of periodic points of f [40].

Much later, Milnor and Thurston analysed the iterates of piecewise monotone mappings of an interval into itself investigating the properties of the mapping that are invariant under topological conjugacy giving rise to kneading theory. Kneading theory provided an effective calculus for describing the qualitative behaviour of the iterates of a piecewise monotone mapping f of a closed interval I of the real line into itself [29].

In the eighties, Guckenheimer studied the iteration of maps of the interval which have negative Schwarzian derivative and one critical point. The maps in this class are classified up to topological equivalence. The equivalence classes of maps which display sensitivity to initial conditions for large sets of initial conditions are characterized [17].

We will investigate the kneading sequences for *double standard maps.*

Let f be a C^2 map of the circle or the interval and let $\Sigma(f)$ denote the basins of attraction of the attracting periodic orbits. Mañé proved that $\Sigma(f)$ is a hyperbolic expanding set if and only if every periodic point is hyperbolic and $\Sigma(f)$ does not contain the critical point [32]. Later, van Strien proved that piecewise continuous C^2 maps on $[0,1]$ or S^1 satisfying the so-called Misiurewicz condition are globally expanding and have absolute continuous invariant probability measures of positive entropy with the conditions that f is piecewise C^2, all critical points of f are non-flat and f has no periodic attractors [41].

A quadratic map $P_c : x \mapsto x^2 + c$ is called *regular* if it has an attracting cycle. It is called *stochastic* if it has a finite, absolutely continuous invariant measure (acim). It is known that the set of stochastic maps has positive measure [6, 21] and that the set of regular maps is open and dense [28].

For this family, it was recently proved that the parameter space $[-2, 1/4]$ can be written as a union of the set of parameters for which P_c is regular, nonregular at most finitely renormalizable and infinitely renormalizable [16].

We will finish this presentation by obtaining a similar result for *double standard maps.*

The goal of this lecture is to make a review of our investigation on the dynamics of a two-parameter family of maps in the interval which are double covers of the circle. In one-dimensional dynamics, a lot is known about the families of smooth maps with a critical point, like quadratic maps, and about the maps that have no critical points (local diffeomorphisms of the circle). Here we will investigate what happens at the interface of those two cases.

This work is organized as follows.

2. Double standard maps

When f is a homeomorphism of the circle S^1 to itself, the rotation number $\rho(f)$ of f is an invariant that measures the rate at which the orbit wraps around the circle. This concept was first investigated by Poincareé [38].

V.I. Arnold [1] was interested in the level sets of the function ρ in the (a, b) parameter plane of the family of *standard maps* of the circle, given by the formula

$$A_{a,b}(x) = x + a + \frac{b}{2\pi}\sin(2\pi x) \pmod 1 \tag{2.1}$$

(when we write "mod 1," we mean that both the arguments and the values are taken modulo 1), giving rise to the so-called *Arnold tongues*. This family has been investigated by various authors since then, see for instance [51, 45].

Families of branched covering maps in the plane have been studied, [42, 43, 44]. It motivated us to find similar one-dimensional maps and studying them. If we consider a branched covering map of the plane that has only one branching point and degree 2, a good choice is to study degree 2 circle maps.

We focused in a specific family of such maps. The most natural choice was the family similar to standard maps, but with the sinusoid added not to the identity but to the doubling map (we also rescaled the parameter b in order to keep its critical value at 1). In such a way we have the following family of *double standard maps*

$$f_{a,b}(x) = 2x + a + \frac{b}{\pi}\sin(2\pi x) \pmod 1. \tag{2.2}$$

There are also other reasons for studying this family. It is a hybrid between the family of standard maps and the family of expanding circle maps (see, e.g., [50]). Both families are of special interest, so it is an important problem to investigate what the result of the cross-breeding may be. Moreover, the circle maps with cubic critical points (this is what we get when we put $b = 1$ in (2.2)) already proved to be interesting (see, e.g., [47]).

A widely accepted method of investigating dynamical systems consists of some initial numerical investigation, formulating questions and conjectures based on it, and subsequent attempts to answer the questions and prove the conjectures. This was indeed the starting point for the work contained here.

The usual pictures of Arnold tongues for the family of standard maps (2.1) show the parameter values (a, b) for which there is an attracting periodic orbit (see Figure 1).

In this picture, the vertical axis is b, from 0 to 1. The horizontal axis is a, from 1/2 to 1. The tongues shown are all tongues of period 5 or less, and their order from left to right is $2, 5, 3, 4, 5, 1$. They correspond to the rotation numbers $1/2 < 3/5 < 2/3 < 3/4 < 4/5 < 1/1$.

Let us compare this picture to the analogous one for the double standard maps (see Figure 2). Here the vertical axis is b, from 1/2 to 1. The horizontal axis is a, from 1/2 to 1.

The tongues shown are all tongues of period 5 or less (in fact, almost all, because the last one is so small that it does not show on the picture), and their order from left to right is $1, 5, 5, 4, 5, 5, 4, 3, 5, 5, 4, 5, 5, 4, 3, 5, 5, 2, 5, 5, 4, 5, 3, 5, 4, 5$. As we will explain later, they correspond to the rational numbers $0/1 < 1/31 < 2/31 < 1/15 < 3/31 < 4/31 < 2/15 <$

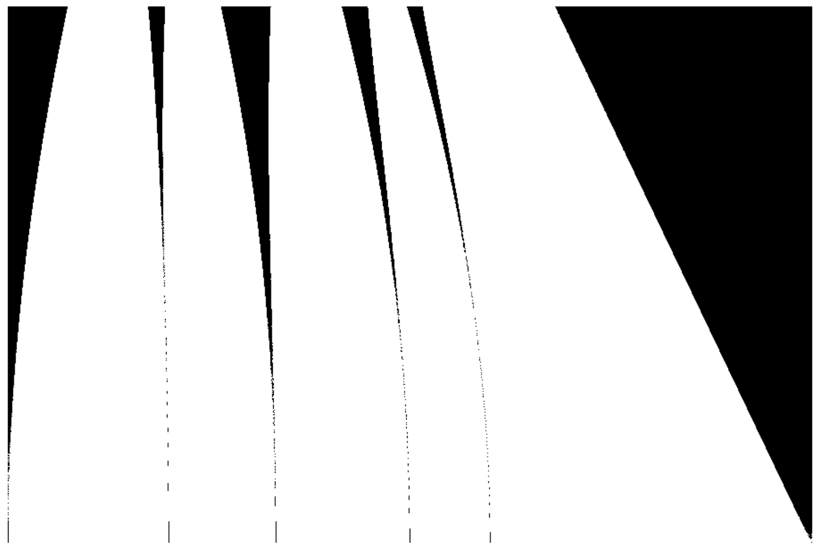

FIGURE 1. Arnold tongues for the family of (Arnold) standard maps

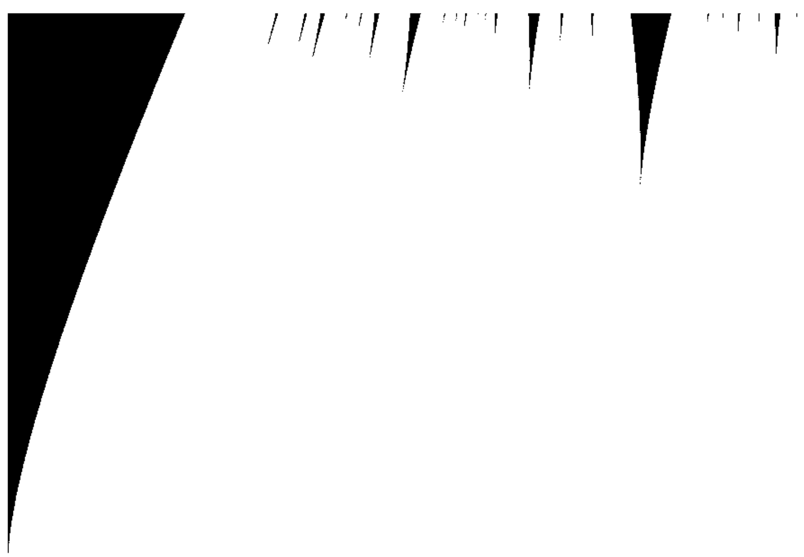

FIGURE 2. Arnold tongues for the family of double standard maps

$1/7 < 5/31 < 6/31 < 3/15 < 7/31 < 8/31 < 4/15 < 2/7 < 9/31 < 10/31 < 1/3 < 11/31 < 12/31 < 6/15 < 13/31 < 3/7 < 14/31 < 7/15 < 15/31$ (the denominators are of the form $2^n 1$, where n is the period). This order is completely different from that for the standard maps.

Another difference is that here the tongues begin not at the level $b = 0$, like for standard maps, but at much higher levels. The lowest tongue tip is at $b = 1/2$, for the period 1 tongue. There cannot be anything lower, because if $0 < b < 1/2$ then the map is expanding.

Our aim was to fully understand figure 2.

We start with some technical tools. Let $F_{a,b}$ be the lifting to the real line of $f_{a,b}$ (the map given by the same formula but not taken modulo 1).

In general, we can in fact assume that $F_{a,b}$ are maps from the real line to itself (not necessarily belonging to the double standard family of maps), satisfying the following properties:

1. Each $F_{a,b}$ is continuous increasing (as a function of x),
2. $F_{a,b}(x+k) = F_{a,b}(x) + 2k$ for every integer k,
3. $F_{a,b}(x)$ is increasing as a function of a and continuous jointly in x, a, b.

While the fact that local homeomorphisms of the circle of degree 2 are semiconjugate to the doubling map is well known, we need additionally monotonicity properties of the semiconjugacy as the function of a.

Lemma 2.1. [33] *Under the assumptions (1) and (2), the limit*

$$\Phi_{a,b}(x) = \lim_{n\to\infty} \frac{F^n_{a,b}(x)}{2^n} \tag{2.3}$$

exists uniformly in x. The limit $\Phi_{a,b}(x)$ is a continuous increasing function of x. Moreover, $\Phi_{a,b}(x+k) = \Phi_{a,b}(x)+k$ for every integer k and $\Phi_{a,b}(F_{a,b}(x)) = 2\Phi_{a,b}(x)$ for every x, so $\Phi_{a,b}$ semiconjugates $F_{a,b}$ with multiplication by 2.

Lemma 2.2. [33] *Under the assumptions of Lemma 2.1, the map $\Phi_{a,b}$ is a lifting of a monotone degree one map $\phi_{a,b}$ of the circle to itself, which semiconjugates $f_{a,b}$ with the doubling map $D : x \mapsto 2x \pmod 1$. Moreover, if p is a periodic point of $f_{a,b}$ of period n then $\phi_{a,b}(p)$ is a periodic point of D of period n.*

It is easy to prove monotonicity with respect to a. Under an additional assumption (3), $\Phi_{a,b}$ is increasing as a function of a and continuous as a function of x, a, b (jointly).

The next lemma adds monotonicity with respect to a.

Lemma 2.3. [33] *Under an additional assumption (3), $\Phi_{a,b}(x)$ is increasing as a function of a and continuous as a function of x, a, b (jointly).*

Let us now return to double standard maps. The way we think of the circle on which the maps $f_{a,b}$ act, this is the circle $\mathbb{R}/\mathbb{Z}$.

Theorem 2.4. [33] *If $0 \le b \le 1$ then the double standard map $f_{a,b}$, given by* (2.2)*, has at most one attracting or neutral periodic orbit.*

We can complexify $f_{a,b}$ by conjugating it via $e^{2\pi i x}$. Then we get the map

$$g_{a,b}(z) = e^{2\pi i a} z^2 e^{b\left(z-\frac{1}{z}\right)}, \tag{2.4}$$

of the unit circle to itself. This map is the restriction of the map of $\mathbb{C}\setminus\{0\}$ to itself given by the same formula.

By the results in the theory of iterations of complex maps it follows that for a map (2.4) any attracting periodic orbit of $g_{a,b}$ has to attract a critical point. A neutral periodic orbit on the unit circle is parabolic, so it is on a boundary of a periodic Leau domain. Therefore this result applies also to such orbit. If $b < 1$ then here is only one pair of critical points, symmetric (in the complex sense) with respect to the unit circle, and the map preserves this symmetry.

If $b = 1$, there is just one critical point, -1. Moreover, the map has then negative Schwarzian derivative. Therefore, there can be at most one attracting or neutral periodic orbit.

Let us now explain the order of tongues. Suppose that a double standard map $f_{a,b}$ has an attracting periodic orbit P of period n. By Theorem 2.4, the trajectories under $g_{a,b}^n$ of both critical points of $g_{a,b}$ (or of the critical point -1 if $b = 1$) converge to $e^{2\pi i p}$ for some $p \in P$. Let $\phi_{a,b}$ be the semiconjugacy from Lemma 2.2. Then by that lemma, $\phi_{a,b}(p)$ is a periodic point of period n of the doubling map D. We will denote this point by $T(P)$ and call the *type of the orbit* P.

For a periodic point T of D we define the *tongue of type* T as the set of parameter values (a, b) (where we think of a as taken modulo 1 and b is from $[0, 1]$) for which there exists an attracting periodic orbit of type T. If the period of T is n, we will say that the tongue of type T has period n.

Since $g_{a,b}$ and $\phi_{a,b}$ depend continuously of (a, b), each tongue is open.

We are interested in the order of the tongues as we vary a. While Lemma 2.3 gives us monotonicity of $\phi_{a,b}$ with respect to a, we cannot be sure where the point p from the definition of $T(P)$ is located. Fortunately, if $b = 1$, we know where on the circle the critical point of $f_{a,b}$ is located. Elementary computations show that this point is at $1/2$ and that $f_{a,1}$ has negative Schwarzian derivative. Therefore the whole interval joining p with $1/2$ is attracted to p under the iterates of $f_{a,1}^n$, where n is the period of P.

To simplify notation, we will write f_a for $f_{a,1}$ and ϕ_a for $\phi_{a,1}$.

Lemma 2.5. [33] *If f_a has an attracting periodic orbit P then $T(P) = \phi_a(1/2)$.*

The following theorem is a crucial point for what follows in this work.

Theorem 2.6. [33] *As a increases, the types of the tongues of f_a vary in the order of rational numbers.*

As a varies from $1/2$ to 1, the periodic points of D of period 5 or less are $0/1 < 1/31 < 2/31 < 1/15 < 3/31 < 4/31 < 2/15 < 1/7 < 5/31 < 6/31 < 3/15 < 7/31 < 8/31 < 4/15 < 2/7 < 9/31 < 10/31 < 1/3 < 11/31 < 12/31 < 6/15 < 13/31 < 3/7 < 14/31 < 7/15 < 15/31$, and they have periods $1, 5, 5, 4, 5, 5, 4, 3, 5, 5, 4, 5, 5, 4, 3, 5, 5, 2, 5, 5, 4, 5, 3, 5, 4, 5$, respectively.

3. Period 1 tongue

The boundary of the tongue corresponding to period 1 is given by the curves

$$a = \frac{1}{2} \pm \frac{\sqrt{4b^2 - 1} - \arctan\sqrt{4b^2 - 1}}{2\pi} \tag{3.1}$$

and the corresponding fixed point is

$$x = -\frac{1}{2} \pm \frac{\arctan\sqrt{4b^2 - 1}}{2\pi}.$$

Set $b = 1/2 + t$. Then (3.1) becomes

$$a = \frac{1}{2} \pm \frac{2\sqrt{t + t^2} - \arctan(2\sqrt{t + t^2})}{2\pi} \tag{3.2}$$

and the derivative of the right-hand side of (3.2) is

$$\frac{2\sqrt{t + t^2}}{\pi(1 + 2t)}.$$

At $t = 0$ this is of order $t^{1/2}$, so the tangency of the two lines bounding period 1 tongue is of order $t^{3/2}$. This tongue begins at the level $b = 1/2$.

For the double standard maps we have

$$f'_{a,b}(x) = 2 + 2b\cos(2\pi x). \tag{3.3}$$

Therefore, $f'_{a,b}$ has one minimum, at $x = 1/2 \pmod 1$, one maximum, at $x = 0 \pmod 1$, is decreasing on $(0, 1/2)$ and increasing on $(1/2, 1)$.

After obtaining some estimates on the derivatives of $F_{a,b}$ we proved the next theorem.

Theorem 3.1. [33] *If $0 \le b < 0.5$ then all periodic orbits of $f_{a,b}$ are repelling. Set $b_0 = 0.578$. If $0.5 \le b \le b_0$ then all periodic orbits of $f_{a,b}$, except perhaps one fixed point, are repelling.*

4. Mostly repelling attracting periodic orbits

We considered next a special class $\mathcal{P}$ of attracting periodic orbits. They are attracting periodic orbits for f_a of type $0.\overline{0001 * 1 * 1 \cdots * 1}$ (the line over a finite sequence means that it is repeated periodically), where each $*$ can be 0 or 1.

There are values of a, $a_l \approx -0.32221099$ and $a_r \approx -0.28609229$ for which $\Phi_{a_l}(1/2) = 1/16$ and $\Phi_{a_r}(1/2) = 1/8$. We have $1/16 = 0.0001\overline{0}$ and $1/8 = 0.000\overline{1}$. The numbers of the form $0.\overline{0001 * 1 * 1 \cdots * 1}$ are between those two, so any a for which f_a has a periodic orbit of such type is in (a_l, a_r).

Let us comment briefly on the structure of an orbit $P \in \mathcal{P}$. There is a point $p \in [1/3, 2/3] \cap P$, such that $1/2$ is in the basin of immediate attraction of p for f_a^n. The derivative of f_a at $f_a(p)$ is larger than c and at the points of P other than p and $f_a(p)$ is larger than λ.

We wanted to study the sizes in the directions of a and p of the region where the orbit $P \in \mathcal{P}$ is attracting (we called them the *P-windows* in the directions of a and p).

Denote those windows by $[a_1, a_2]$ and $[p_1, p_2]$ respectively. Since p depends on a, we write $p(a)$. Thus, we have $p_i = p(a_i)$ for $i = 1, 2$. We express those sizes in terms of the exponent of $P \setminus \{p(a)\}$

$$\alpha(a) = (f_a^{n-1})'(f_a(p(a)). \tag{4.1}$$

We had to choose some specific value of a, and the most natural such value is a_0 for which $p(a_0) = 1/2$.

Theorem 4.1. [33] *There exist positive constants K_1, K_2, K_3, K_4 such that if a periodic orbit P belongs to $\mathcal{P}$ then for the P-windows $[p_1, p_2]$ in the direction of p and $[a_1, a_2]$ in the direction of a we have*

$$K_1(\alpha(a_0))^{-1/2} \le p_2 - p_1 \le K_2(\alpha(a_0))^{-1/2} \tag{4.2}$$

and

$$K_3(\alpha(a_0))^{-3/2} \le a_2 - a_1 \le K_4(\alpha(a_0))^{-3/2}. \tag{4.3}$$

In particular, the size of the P-window in the direction of a is of order of the cube of the size of the P-window in the direction of p.

There are some features regarding Theorem 4.1 taht are worth commenting. The first one is that if instead of looking at the point p of the orbit P for which $1/2$ is in its immediate basin of attraction, we look at the next point along the orbit, $q = f_a(p)$ (the one that has a in its basin of attraction), then the scaling of the P-window in the direction of q will be the same as the scaling of the P-window in the direction of a. Indeed, if this window is $[q_1, q_2]$ then

$$q_2 - q_1 = F_{a_2}(p_2) - F_{a_1}(p_1) = (F_{a_1}(p_2) - F_{a_1}(p_1)) + (a_2 - a_1)$$

and since the map f_{a_1} in $[1/3, 2/3]$ is cubic up to a multiplicative constant (and in view of (4.3)), we get

$$K_5(\alpha(a_0))^{-3/2} \le q_2 - q_1 \le K_6(\alpha(a_0))^{-3/2} \tag{4.4}$$

for some positive constants K_5, K_6 independent of $P \in \mathcal{P}$.

The second comment is that since we expressed the sizes of the P-windows in terms of $\alpha(a_0)$, we have some information how those sizes behave as the period of $P \in \mathcal{P}$ goes to infinity. Then $\alpha(a_0)$ grows exponentially with the period n, in the sense that

$$c_1\lambda^{n-2} \le \alpha(a_0) \le c_2\Lambda^{n-2} \tag{4.5}$$

and $c_1, c_2 > 0$, $\Lambda \ge \lambda > 1$ (this follows immediately from the definition of α and our earlier estimates). However, whether $(1/n)\log\alpha(a_0)$ is closer to $\log\lambda$ or $\log\Lambda$, depends on a concrete orbit P.

The third comment is that although the orbits from $\mathcal{P}$ are kind of special, there are infinitely many of them. Moreover, the only properties of $\mathcal{P}$ that we used were that the growth of the derivatives along the pieces of the orbit $P \in \mathcal{P}$ not passing through p is exponential in the length of the piece, uniformly in $\mathcal{P}$. Thus, there are many other families similar to $\mathcal{P}$ for which the same properties can be proved.

Define *proper tongues* as those components of the tongues that have non-empty intersection with the line $b = 1$. The intersection of any tongue with the line $b = 1$ is connected and nonempty, and therefore there is exactly one proper tongue of each type.

For the types considered in this section, we were able to estimate the length of the proper tongues. We measure the length of a tongue in the direction of b.

Theorem 4.2. [33] *Let s, t be periodic points of D with $1/16 < s < t < 1/8$. Then there exist constants $\lambda > 1$, $N > 0$ and $K_5 > 0$ such that any proper tongue of a type between s and t, period $n \ge N$, and such that the orbit of this type for some f_a belongs to $\mathcal{P}$, has length smaller than $K_5\lambda^{-n}$.*

5. Intermittent periodic orbits

We consider here again the case $b = 1$. Set

$$a_I = \frac{\sqrt{3}}{2\pi} - \frac{2}{3} \approx -0.3910022190.$$

We have $F_{a_I}(2/3) = 2/3$ and $F'_{a_I}(2/3) = 1$. Thus, $2/3$ is a neutral fixed point and if a is slightly larger than a_I then we observe intermittency for f_a.

The trajectories of points in a rather large interval containing $1/2$ are increasing and spend a lot of time very close to $2/3$.

Denote by $\mathcal{R}$ the class of attracting periodic orbits for f_a such that if $p \in P \in \mathcal{R}$, $1/2$ is in the immediate basin of attraction of p and n is the period of P then $p < F_a(p) < F_a^2(p) < \cdots < F_a^{n-1}(p)$ and $p = F_a^n(p) - 1$. It follows that $\Phi_a(p) = D^n(\Phi_a(p)) - 1$, so $\Phi_a(p) = 1/(2^n - 1)$. Therefore the type of such an orbit is $1/(2^n - 1)$.

The general philosophy for intermittency is that as the period of the attracting periodic orbits increases, we have the same behaviour (even quantitatively) in the directions of the

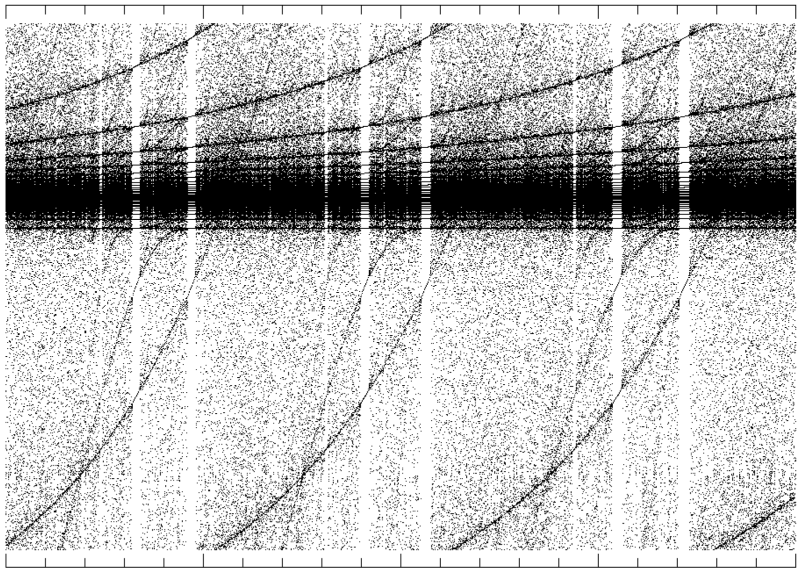

FIGURE 3. The (a, x)-plot, with a from 0.6107 to 0.6111, x from 0 to 1 and $b = 1$

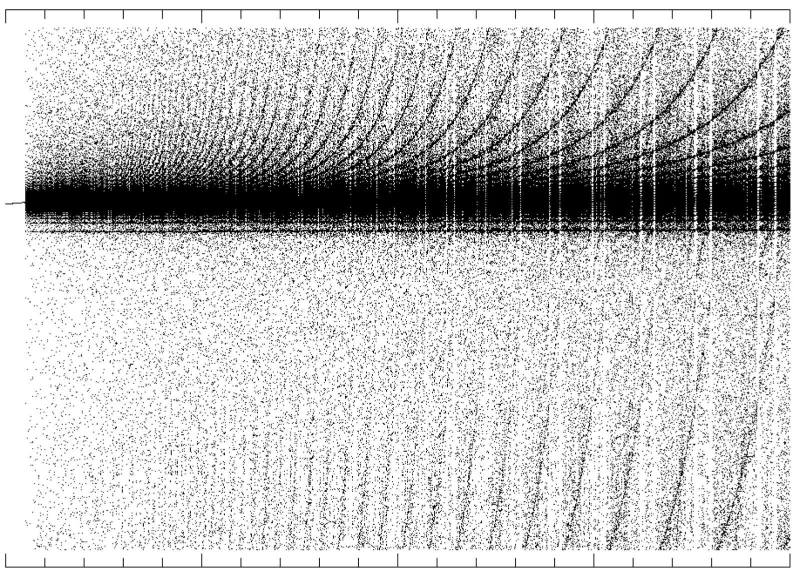

FIGURE 4. The (a, x)-plot, with a from 0.6089 to 0.6129, x from 0 to 1 and $b = 1$

variables x and b, while in the direction of a we have scaling depending on the order of tangency of the graph of F_{a_I} to the diagonal. We can also observe the repetition of the same behavior on Figures 3 and 4. Note that we see there wide windows coming in pairs. Such a pair consists of orbits of types $1/(2^n - 1)$ and $2/(2^{n+1} - 1)$.

We wanted to estimate sizes of the P-windows for $P \in \mathcal{R}$. We proved the next theorem.

Theorem 5.1. [33] *There exist positive constants M_1, M_2, M_3, M_4 such that if P_n is the periodic orbit of $\mathcal{R}$ of period n then for the P-windows $[p_1, p_2]$ in the direction of p and*

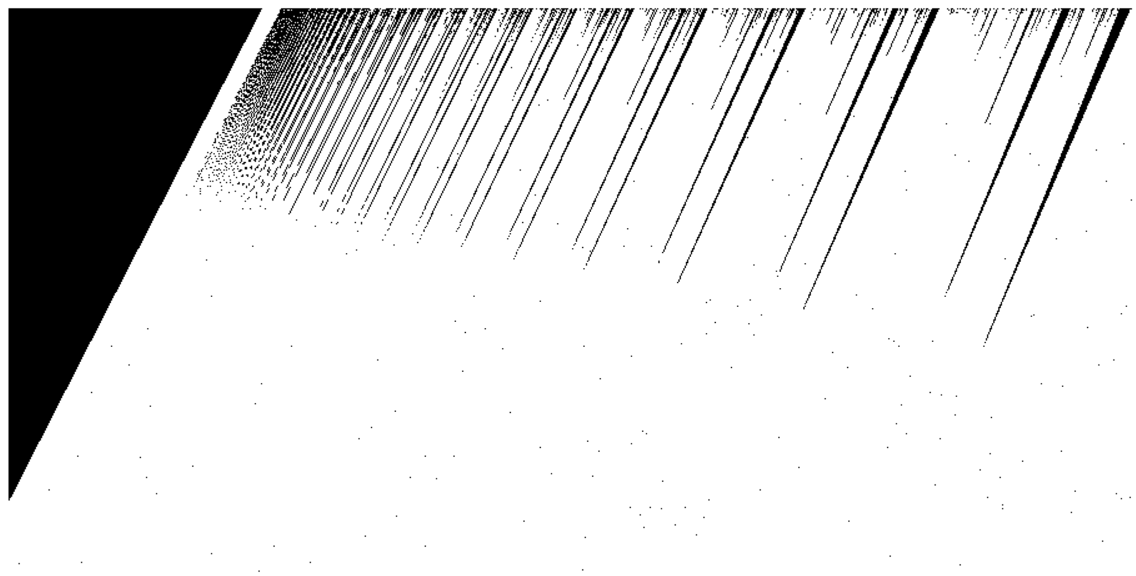

FIGURE 5. Tongues of period 50 or less in the intermittent region, $0.6 \le a \le 0.64$, $0.96 \le b \le 1$

$[a_1, a_2]$ in the direction of a, then

$$M_1 \le p_2 - p_1 \le M_2 \tag{5.1}$$

and

$$M_3 n^{-3} \le a_2 - a_1 \le M_4 n^{-3}. \tag{5.2}$$

Moreover, there exist positive constants M_5, M_6 such that if c_n is the value of the parameter a for which $1/2 \in P_n$, then

$$M_5 n^{-2} \le c_n - a_{_I} \le M_6 n^{-2}. \tag{5.3}$$

For orbits of this type, we proved that if $b < 1$ is sufficiently close to 1 then there are infinitely many tongues at that level (see Figure 5).

Theorem 5.2. [33] *There exists a constant $L > 0$ such that any proper tongue such that the orbit of this type for some f_a belongs to $\mathcal{R}$, has length larger than L.*

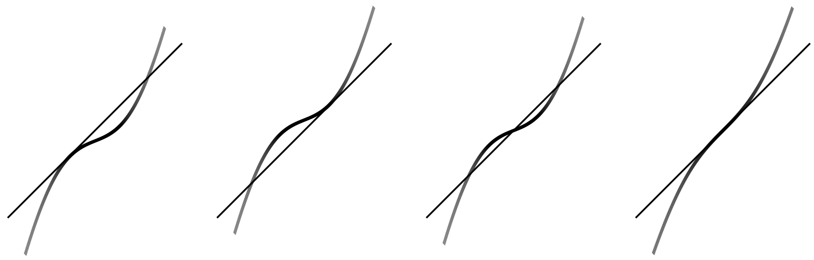

FIGURE 6. Four cases from Lemma 6.2

6. Boundary of the tongues

The next question was to describe the shape of the boundaries of the tongues, in particular close to their tips.

Let us call an orbit $(x, f(x), \dots, f^{n-1}(x))$, where $f^n(x) = x$, of a map f *attracting* if $(f^n)'(x) < 1$, *repelling* if $(f^n)'(x) > 1$, and *neutral* if $(f^n)'(x) = 1$. If a confusion is possible, we will call it *differentiably* attracting, repelling and neutral.

If there is $\varepsilon > 0$ such that $f^n(y) < y$ for $y \in (x, x+\varepsilon)$, then x will be called *topologically attracting from the right*. Similarly one defines periodic orbits *topologically attracting from the left*, and *topologically repelling from the right* (*left*). An orbit topologically attracting (repelling) from both sides is just *topologically attracting* (*repelling*).

Our map is analytic and not equal to the identity (we can assume this because double standard maps have degree 2), thus, each periodic orbit is either topologically attracting or topologically repelling from each side.

Since we were interested in the topological types of the periodic orbits more than in the differentiable ones, we added one more property to Theorem 2.4.

Lemma 6.1. [34] *A neutral periodic orbit mentioned in Theorem 2.4 cannot be topologically attracting from both sides.*

We have the following lemma (described in Figure 3).

Lemma 6.2. [34] *Assume that p is an attracting or neutral periodic point of $f_{a,b}$ of period n. Let J be the set of all points x for which $\phi_{a,b}(x) = \phi_{a,b}(p)$. Then J is either a closed interval (modulo 1) or a singleton and $f^n_{a,b}|_J$ is an orientation preserving homeomorphism of J onto itself. The endpoints of J are fixed points of $f^n_{a,b}$, and one of the following four possibilities holds. In the first three cases J is an interval.*

(a) *The left endpoint of J is neutral, topologically attracting from the right and topologically repelling from the left; the right endpoint of J is repelling; there are no other fixed points of $f^n_{a,b}$ in J.*
(b) *The right endpoint of J is neutral, topologically attracting from the left and topologically repelling from the right; the left endpoint of J is repelling; there are no other fixed points of $f^n_{a,b}$ in J.*

(c) *Both endpoints of J are repelling; there is an attracting fixed point of $f^n_{a,b}$ in the interior of J; there are no other fixed points of $f^n_{a,b}$ in J.*
(d) *The set J consists of one neutral fixed point of $f^n_{a,b}$, repelling from both sides.*

Our parameter space is $\mathbb{T} \times [0,1]$, and when we speak of the *boundary* of a tongue, we mean the boundary relative to this space. In particular, if a tongue contains a segment of the line $b = 1$, only the endpoints of this segment belong to the boundary of the tongue.

The following lemma provided a connection between the boundary of a tongue and the results presented earlier.

Lemma 6.3. [34] *If (a_0, b_0) belongs to the boundary of a tongue of period n then $f^n_{a_0,b_0}$ has a neutral fixed point.*

We say that a point (a, b) from a boundary of a tongue of period n belongs to the *left boundary of the tongue* if there is $\varepsilon > 0$ such that $(a+t, b)$ belongs to the tongue for $0 < t < \varepsilon$ and does not belong to the tongue for $-\varepsilon < t < 0$. Similarly, a point (a, b) from a boundary of a tongue of period n belongs to the *right boundary of the tongue* if there is $\varepsilon > 0$ such that $(a + t, b)$ belongs to the tongue for $-\varepsilon < t < 0$ and does not belong to the tongue for $0 < t < \varepsilon$. Moreover, a point (a, b) from a boundary of a tongue of period n is the *tip* of this tongue if there is $\varepsilon > 0$ such that $(a + t, b)$ does not belong to the tongue for $-\varepsilon < t < \varepsilon$.

In the classification of Lemma 6.2 if (c) is satisfied then the point (a, b) belongs to a tongue of period n. If (a, b) belongs to a boundary of a tongue of period n then, by Lemma 6.3, $f^n_{a,b}$ has a neutral fixed point. Thus, one of the Cases (a), (b) or (d) holds.

In the next lemma we explained Cases (a) and (b).

Lemma 6.4. [34] *In Case (a) of Lemma 6.2 (a, b) belongs to the left boundary of a tongue of period n. Similarly, in Case (b) (a, b) belongs to the right boundary of a tongue of period n.*

By considering a family of maps of an interval or a circle that looks locally more or less like the family of double standard maps, assuming that the maps are real analytic and that they depend on the parameters in a real analytic way we showed that if the dynamics of a map in this family locally looks like the one observed at the tip of a tongue and the parameters belong to the closure of a tongue, then it is really the tip. We considered only one parameter, whose change moves the graph of the map up or down.

Lemma 6.5. [34] *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $G : U \to \mathbb{R}$ be a real analytic function. Assume that*

$$G(0, x) < 0 \;\; \textit{for} \;\; x < 0, \quad G(0, 0) = 0, \quad \textit{and} \;\; G(0, x) > 0 \;\; \textit{for} \;\; x > 0; \tag{6.1}$$

and that

$$\frac{\partial G}{\partial t}(0, 0) \neq 0, \tag{6.2}$$

where t is the first variable. Then there are open intervals I, J containing 0 such that $I \times J \subset U$ and for every $t \in I$ is exactly one $x \in J$ such that $G(t, x) = 0$. Moreover, for those t and x, if $t \neq 0$ then $\frac{\partial G}{\partial x}(t, x) > 0$.

As a corollary we got the following result.

Theorem 6.6. [34] *Let U be a neighborhood of the origin in $\mathbb{R}^2$ and let $F : U \to \mathbb{R}$ be a real analytic function. Set $f_t(x) = F(t, x)$. Assume that f_0 has a topologically repelling fixed point at $x = 0$ and that*

$$\frac{\partial F}{\partial t}(0,0) \neq 0. \tag{6.3}$$

Then there are open intervals I, J containing 0 such that $I \times J \subset U$ and for every $t \in I$ the map f_t has exactly one fixed point $x \in J$. Moreover, if $t \neq 0$ then this fixed point is (differentiably) repelling.

The intuitive interpretation of the conclusion of this theorem is that the point $(0, 0)$ in the parameter space can be neither in the interior of a tongue nor in its left or right boundary. Thus, if it is in the closure of a tongue, it has to be its tip.

Applying Theorem 6.6 to the family of double standard maps, we get the following result.

Theorem 6.7. [34] *Consider a one-parameter subfamily $g_t = f_{a(t),b(t)}$ of the family of double standard maps, where $a(t)$ and $b(t)$ depend on t in an analytic way. Assume that $g^n_{t_0}$ has a topologically repelling neutral fixed point x_0 and $\partial G/\partial t(t_0, x_0) \neq 0$, where $G(t, x) = g^n_t(x)$. Then there is $\varepsilon > 0$ such that if $|t - t_0| < \varepsilon$ then g^n_t has no attracting or neutral fixed point.*

By Theorem 6.7 and Lemma 6.4 we get immediately the following theorem for the family of double standard maps.

Theorem 6.8. [34] *In Case (a) of Lemma 6.2 the point (a, b) belongs to the left boundary of a tongue of period n, in Case (b) to the right boundary of a tongue of period n, and in Case (d) it is a tip of a tongue of period n.*

This theorem has several consequences. Basically, it yields that the tongues have regular, tongue-like shapes.

Corollary 6.9. [34] *Every point on the boundary of a tongue either belongs to its left or right boundary or is its tip.*

In particular, degeneracies like a horizontal segment contained in the boundary of a tongue are ruled out.

Corollary 6.10. [34] *The intersection of every tongue with any horizontal line $b =$ constant is connected. In particular, every tongue is simply connected.*

The next corollary follows immediately from Theorem 6.8 and Proposition 4.6 of [33] (which states that whenever a piece of the boundary of a tongue consists of points for which the Case (a) or (b) of Lemma 6.2 holds, it has slope with the absolute value at least π).

Corollary 6.11. [34] *The left and right boundaries of a tongue have slope with the absolute value at least π.*

We can get even more information about the shape of a tongue at its tip.

Theorem 6.12. [34] *At a tip of a tongue the left and right boundaries are tangent to each other.*

7. Tongues and cusps

For our family (2.2), for the values of parameters at the tip of the tongue the system undergoes the cusp bifurcation [18]. The next natural question was to understand their shapes.

One of the basic characteristics of the shape is the order of contact of the left and the right boundaries of the cusp at its tip. If those boundaries are of the form $a = B_1(b)$ and $a = B_2(b)$ (here the a-axis is horizontal and the b-axis vertical) and the tip of the cusp is at (a_0, b_0), then this order is r if the limit

$$\lim_{b \to b_0} \frac{|B_1(b) - B_2(b)|}{|b - b_0|^{r+1}}$$

is positive and finite.

This is, the order of contact is r if the width of the cusp decreases to 0 as $|b - b_0|^{r+1}$ times a constant.

For Arnold tongues in the family of standard maps the order of contact depends on the rotation number of the tongue. If the rotation number is p/q (with p and q coprime), then the order is $q - 1$.

For the family of double standard maps the situation is different. By the results for the cusp bifurcation (see, e.g., [18]), the generic order of contact is 1/2. In [8] we checked that this is the order for the cusp corresponding to the attracting fixed point. However, we do not know whether the situation is generic for tips of cusps corresponding to the attracting periodic points of higher period. This motivated the study of the order of contact at the cusp bifurcation point for non-generic cases, shown below.

Let us now discuss the order of contact of the boundaries of the cusp for two-parameter families of vector fields on the real line or diffeomorphisms of the real line, in codimensions 1 and 2 (we also include the generic case for completeness).

What is perhaps more important, we created a machinery that can be used for the same problem in higher codimensions (although there will be more and more cases) and perhaps for other, similar problems.

As we said, we consider a two-parameter family of vector fields on the real line or diffeomorphisms of the real line. We look at them locally, so both parameters a, b and the variable x will be taken from some neighborhoods of 0. Thus, we work in a neighborhood U of $(0, 0, 0)$ in R^3. In order to have all continuous functions bounded automatically, we assume that U is compact. We will be analyzing the shape of the bifurcation set in the cusp bifurcation – generic and of small codimensions.

The bifurcation set in the cusp bifurcation is given as the projection to the (a, b)–plane of the set of all solutions to the system of equations:

$$F(a, b, x) = 0, \quad F_x(a, b, x) = 0 \quad (BS)$$

Here F is the vector field in the continuous case and is the map minus the identity in the discrete case. For simplicity, we will assume that F is real analytic. However, it is enough to assume that it is sufficiently smooth, that is, that all derivatives used in the formulas and proofs exist and are continuous. We will denote by T the set of solutions to (BS), that is,

$$T = \{(a, b, x) \in U : F(a, b, x) = 0, F_x(a, b, x) = 0\}.$$

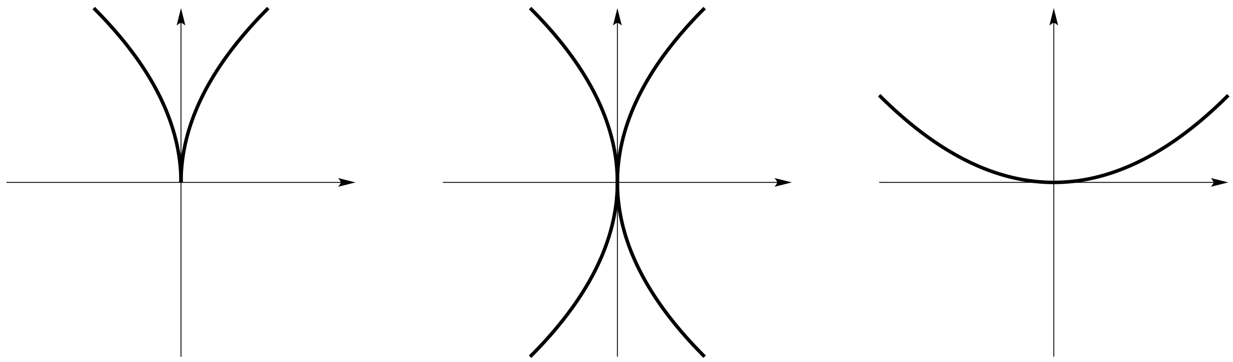

FIGURE 7. Simple, double and non-cusp.

In order to make (BS) relevant for the cusp bifurcation, we write $F(0,0,x) = f(x)$ and assume that there is an odd integer $k \geq 3$ such that

$$f(0) = f'(0) = f''(0) = \cdots = f^{(k)}(0) \neq 0.$$

If we write F in the form

$$F(a,b,x) = A_0(a,b) + A_1(a,b)x + A_2(a,b)x^2 + \ldots + A_k(a,b)x^k + \phi x^{k+1},$$

where ϕ is an analytic (or sufficiently smooth) function, we have

$$A_0(0,0) = A_1(0,0) = \cdots = A_k(0,0) \neq 0.$$

Consider the gradients V_0 of A_0 at the origin and V_1 of A_1 at the origin. Generically, for the cusp bifurcation, $k = 3$, and V_0 and V_1 are linearly independent. We increase the codimension by increasing the number of equalities in the assumptions. Observe that k in (1.4) has to be odd, and replacing $k = 3$ by $k = 5$ requires adding two extra equalities. Thus, the only possibility to get codimension 1 is to assume that V_0 and V_1 are non-zero, linearly dependent, and $k = 3$ (a gradient is a vector, so to make it zero we need two equalities). In the study of this case we come across certain scalar quantities, β and Δ, which we can assume to be non-zero without changing the codimension. In particular, the results will depend on the sign of Δ. In codimension 2 we have four possibilities. We can assume that $k = 5$, that V_0 is zero, that $\Delta = 0$, or that $\beta = 0$. Assuming that V_1 is zero does not give us anything new, since in the codimension 1 case we do not assume that it is non-zero.

In order to summarize the results, we produce a table, we indicate the codimension, the number k, information about V_0 and V_1, and about β and Δ (if relevant). Then we state what kind of cusp we get. It can be simple, consisting of two curves beginning at the origin and tangent there, or double, consisting of two curves passing through the origin and tangent there (see Figure 7). In those cases we also give the order of contact of those curves. However, we can also get a non-cusp, where the two curves begin at the origin, but the angle between them is (see also Figure 7), or we can even get the origin isolated (no curves nearby).

Codimension	Assumptions	Type of cusp	Order of contact
0	$k=3$, V_0 and V_1 are linearly independent	simple cusp	1/2
1	$k=3$, V_0 and V_1 are linearly dependent, V_0 non-zero, $\beta \neq 0$, $\Delta > 0$	double cusp	2
1	$k=3$, V_0 and V_1 are linearly dependent, V_0 non-zero, $\beta \neq 0$, $\Delta < 0$	isolated point	
2	$k=5$, V_0 and V_1 are linearly independent,	simple cusp	1/4
2	$k=3$, V_0 and V_1 are linearly dependent, V_0 non-zero, $\beta \neq 0$, $\Delta = 0$	simple cusp	7/2
2	$k=3$, $V_0=0$ and $V_1 \neq 0$	non-cusp	
2	$k=3$, V_0 and V_1 are linearly dependent, V_0 non-zero, $\beta = 0$, $\Delta \neq 0$	double cusp	2

8. Symbolic dynamics

In this section we present some results regarding the symbolic dynamics for the family of double standard maps (2.2) with $b = 1$. Let us fix the notation $f_{a,1} = f_a$.

In the classical theory of Milnor-Thurston the symbolic coding is associated to the two intervals where the restriction of the map to each of them is increasing or decreasing ([14]). Consider the *double standard family* (2.2) with $b = 1$ and $a \in [0, 1]$, then for each value of the parameter a the map is increasing for all values of $x \in [0, 1]$ except for the values of the parameter for which there is an attractive periodic orbit of period 1.

For $a_0 < a < 1$, f_a has a unique fixed point, which we denote by $p(a)$. This follows from the bifurcation behaviour of the fixed point(s). For $a = \frac{1}{2}$, f_a has three fixed points, one at $x = \frac{1}{2}$ and two repelling fixed points. According to the Implicit Function Theorem this behaviour persists for $a_0' < a < a_0$. For $a = a_0$, the two rightmost fixed points go through a saddle node bifurcation and disappear. Since the only bifurcation of fixed points for $1/2 < a < 1$ appears at $a = a_0$ it is clear that there is at most one fixed point in this interval, the continuation of the left fixed point that exists for $\frac{1}{2} < a < a_0$. We denote this fixed point by $p(a)$. We will also prove that $p(a)$ for $a_0 < a < 1$ has a unique preimage different from $p(a)$. We denote this preimage by $q(a)$. For $0 < a < a_0'$ the situation is completely symmetric.

We used a symbolic coding related to Yoccoz partitions of the interval [39], applied to the circle.

Let $J_0 = (p(a), q(a))$ for $a > \frac{1}{2}$, where the circle segments are chosen so that they have positive orientation on the circle, and let $J_1 = \mathrm{int}(\mathbb{T} \setminus J_0)$. In the case $0 < a < a_0'$, $J_0 = [0, q(a)) \cup (p(a), 1)$ where the circle is represented by the halfopen fundamental domain $[0, 1)$ and as before J_1 is the interior of its complement.

For a given initial point x on the circle such that its orbit does not land on p or q let

$$i_n(x) = \begin{cases} 0 \text{ if } f_a^n(x) \in J_0, \\ 1 \text{ if } f_a^n(x) \in J_1. \end{cases} \tag{8.1}$$

For a point that eventually hits p after possibly passing through q, we define the coding by either $i_0 \dots i_n\, 0\, \overline{1}$ or $i_0 \dots i_n\, 1\, \overline{0}$, and we identify these sequences. Note that this is exactly the same identification as is made of the binary expansions

$$0.i_0 \dots i_n\, 0\, \overline{1} \text{ and } 0.i_0 \dots i_n\, 1\, \overline{0},$$

when they are interpreted as real numbers.

Thus, we associate with each $x \in \mathbb{T}$ a finite or infinite sequence of the symbols $0, 1$ called its *itinerary*. We denote by $I(x)$ the sequence $\{i_n(x)\}_{n=0}^{\infty}$ and this sequence is also naturally identified with a real number in $[0, 1]$.

As usual the kneading sequence will be the itinerary of the critical value $f_a(1/2) = a$ and we denote it by $K(f_a)$. We will sometimes also use the notation $K(a) = K(f_a)$, in particular when we consider the function $a \mapsto K(a)$.

Let $I(x), I(y)$ be two sequences of the symbols $\{0, 1\}$ such that $I(x) \neq I(y)$. Suppose

$$I(x) = i_0^1 i_1^1 i_2^1 \cdots i_n^1, \quad I(y) = i_0^2 i_1^2 i_2^2 \cdots i_n^2, \tag{8.2}$$

where i_n is the smallest index for which $i_n^1 \neq i_n^2$. We order the sequences (8.2) lexicographically, if $i_n^1 = 0, i_n^2 = 1$ then $I(x) < I(y)$. This order coincides with the order of the real numbers corresponding to the symbol sequences interpreted as binary expansions. It is clear that the map $x \mapsto I(x)$ is continuous, where the topology on the kneading sequences is the topology of real numbers.

In fact, the kneading sequence corresponding to a periodic orbit is a periodic sequence that corresponds to the binary expansion of the rational number assigning the order of the given tongue as described in [33].

Theorem 8.1. [8] *If f_a belongs to the double standard family then*

(i) *f_a has a stable periodic orbit if and only if $K(f_a)$ is periodic.*

(ii) *If $K(f_{a_1})$ is periodic and $K(f_{a_1}) = K(f_{a_2})$ then f_{a_1} and f_{a_2} are topologically conjugate.*

This theorem implies that maps from our family of double standard maps with parameters inside of a given tongue are topologically conjugate. The maps with parameters corresponding to the boundary of the tongues are not conjugate to those with parameter values in the interior.

The order of the tongues for the family (2.2) correspond to the order of rational numbers with denominator given by $2^n - 1$. Consider $F_{a,b}$ the lift of $f_{a,b}$ to the real line. The limit

$$\Phi_{a,b}(x) = \lim_{j \to \infty} \frac{F^j_{a,b}(x)}{2^j}, \tag{8.3}$$

where each $F_{a,b}$ is continuous increasing (as a function of x) and $F_{a,b}(x + k) = F_{a,b}(x) + 2k$ for every integer k, exists uniformly in x (see Lemma 3.1 [33]).

The fact that the periodic part of the binary expansion of $\Phi_a(x)$ gives us a periodic coding in 0's and 1's suggests that there is a relation between a periodic kneading and the periodic part of this binary expansion.

Theorem 8.2. [8] *If a is such that f_a has a periodic orbit of period n ($n > 1$) then $K(f_a)$ corresponds to the shift of the binary expansion of $\Phi_a(\frac{1}{2}) = k/(2^n - 1)$ where k is a natural number and $k = 1, \ldots, 2^n - 2$.*

Let us now investigate aperiodic kneading sequences. The itinerary map is strictly increasing in this case.

Theorem 8.3. [8] *Assume that $K(f_{a_1})$ and $K(f_{a_2})$ are aperiodic kneading sequences. Then the two maps are topologically conjugate and both are conjugate to the doubling map of the circle*

$$x \mapsto 2x \pmod 1.$$

Further, we have that

Theorem 8.4. [8] *If $K(f_{a_1})$ and $K(f_{a_2})$ are aperiodic kneading sequences such that $K(f_{a_1}) = K(f_{a_2})$, then $a_1 = a_2$.*

9. Expanding properties

We finish this work on the dynamics of the family of double standard maps (2.2) by looking at the structure of the space of parameters for this family.

When $b = 1$, maps of this family have a critical point, and for a set of parameters E_1 of positive Lebesgue measure there is an absolutely continuous invariant measure for $f_{a,1}$. We denote $f_{a,1}$ by f_a. When $b \in [0, 1)$, there is a non-empty set E_b of parameters for which the map $f_{a,b}$ is expanding. We will see that as b goes to 1, the set E_b accumulates on many points of E_1 in a regular way (from the measure point of view).

Let us consider the family of *double standard maps* given by (2.2) with $b = 1$. Denote $f_a(x) = f_{a,1}(x)$, $\xi_j(a)$ the orbit of the critical point: $\xi_j(a) = f_a^j(c)$, and for a general $b \leq 1$, let $\xi_j(a, b) = f_{a,b}^j(c)$.

When $b < 1$, the point $c = \frac{1}{2}$ is an inflexion point. Sometimes we will also use the notations $f(x, a, b)$ for $f_{a,b}(x)$ and $f(x, a)$ for $f_a(x)$.

By $\partial_a f_{a,b}^j(x)$ we denote the partial derivative of $f_{a,b}^j(x)$ with respect to a and by $\partial_a f_a^j(x)$ the partial derivative of $f_a^j(x)$ with respect to a.

A parameter $a = a_0$ is an *MT parameter*, if there is an integer m and a period length ℓ such that $\xi_m(a_0)$ is a periodic point of f_{a_0} of period ℓ and the multiplier $\Lambda := (f_{a_0}^\ell)'(\xi_m(a_0))$ is larger than 1. This is, a parameter a_0 will be called an *MT parameter* if the trajectory of the critical point $c = 1/2$ is preperiodic (but not periodic).

When $b = 1$, the critical value $f_{a_0,1}(\frac{1}{2})$ satisfies the Collet-Eckmann condition, i.e. there is $C_{CE} > 0$ and $\kappa_1 > 0$ such that for $a = a_0$

$$(f_a^n)'(f_a(c)) \geq C_{CE} e^{\kappa_1 n},$$

for all $n \geq 0$ which implies the existence of an absolutely continuous invariant measure, [41].

Using the methods of [6] it is possible to prove that there is a set of positive Lebesgue measure $\tilde{E}_1$ such that for all $a \in \tilde{E}_1$ there is $n_0(a)$ so that

$$(f_a^n)'(f_a(c)) \geq e^{n^{2/3}},$$

for all $n \geq n_0(a)$, where $\frac{2}{3}$ can be replaced by any constant $\sigma < 1$.

A parameter exclusion requiring

$$\operatorname{dist}(f_a^j(c), c) \geq e^{-\sqrt{j}}, \quad j \geq 1,$$

is sufficient to prove the Collet-Eckmann condition in this case.

We also considered the non-critical case $0 < b < 1$ and used more elementary methods based on [6], which gave stretched exponential growth of the type

$$(f_{a,b}^n)'(f_{a,b}(c)) \geq e^{n^{2/3}}, \qquad n_0 \leq n \leq \hat{N}(a, b), \tag{9.1}$$

for all $a \in \tilde{E}_b$ for a set $\tilde{E}_b$, which is a finite union of intervals. To obtain this it was sufficient to do parameter exclusions of the type

$$\operatorname{dist}(f_{a,b}^j(c), c) \geq C_1 e^{-\sqrt{j}}, \quad j \geq 1, \tag{9.2}$$

and then prove exponential expansion.

By [11], if $f_a(c)$ satisfies the Collet-Eckmann condition, then f_a has an absolutely continuous invariant measure. This is the analogue of Jakobson's theorem [21] in this case.

As in [6] and [7], we need to define a partition $\mathcal{Q} = \{I_{r,l}\}$ of the *return interval* $I^* = (c-\delta, c+\delta)$, where $\delta = e^{-r_\delta}$. We first divide I^*

$$I^* = \bigcup_{r=r_\delta}^{\infty} I_r \cup \bigcup_{r=r_\delta}^{\infty} I_{-r},$$

where $I_r = (c + e^{-r-1}, c + e^{-r})$ for $0 < r \leq r_\delta$ and $I_{-r} = (c - e^{-r}, c - e^{-r-1})$.

We then subdivide I_r into r^2 intervals of equal length with disjoint interiors as follows

$$I_r = \bigcup_{\ell=0}^{r^2-1} I_{r,\ell}.$$

Let us also use the convention that $I_{r,r^2} = I_{r-1,0}$, $r > 0$, and the analogous convention for $r < 0$.

Note that $|I_{r,l}| = \frac{e^{-r}}{r^2}(1-e^{-1})$ and $|I_r| = e^{-r}(1-e^{-1})$. We will also need the extended interval

$$I^+_{r,\ell} = I_{r,\ell-1} \cup I_{r,\ell} \cup I_{r,\ell+1}.$$

Let us consider also a partition $\mathcal{Q}' = \{I_{r,l}\}$, $|r| \geq r^1_\delta$, of an interval

$$I^{**} = (c - \delta_1, c + \delta_1),$$

where $|r| \geq r^1_\delta$, for some $r^1_\delta < r_\delta$, i.e.

$$\delta_1 = e^{-r^1_\delta} > \delta.$$

A main tool is a sequence of partitions $\mathcal{P}_n$, $n = 0, 1, 2, \ldots$ of the parameter space which is induced by the phase space partition. We define

$$\mathcal{E}_n = \bigcup_{\omega \in \mathcal{P}_n} \omega.$$

Definition 9.1. [10] We call a time n a *free return* if there is a parameter interval ω belonging to a partition $\mathcal{P}_n$ such that

$$\xi_n(\omega) = I_{r,\ell}.$$

Similarly if we fix $b < 1$, we will have

$$\xi_n(\omega, b) = I_{r,\ell}.$$

(In some cases these two conditions for technical reasons were replaced by $I_{r,\ell} \subset \xi_n(\omega) \subset I^+_{r,\ell}$ or $I_{r,\ell} \subset \xi_n(\omega, b) \subset I^+_{r,\ell}$.)

We have exponential growth of the derivative for an orbit of a map $f_{a,b}$ that moves outside an open interval I containing c, when (a,b) is a small perturbation of an MT parameter $(a_0, 1)$.

We consider the parameter space $\mathbb{R}/\mathbb{Z} \times (0, 1]$, and when we speak of a neighbourhood of $(a_0, 1)$, we mean its neighbourhood in this space.

By $|x - y|$ we denote the distance between x and y on the circle. Since the points x and y will be usually close to each other, this makes perfect sense. Denote

$$\overline{d} = \min_{j \geq 1} |c - f^j_{a_0}(c)|. \tag{9.3}$$

By the definition of an MT parameter, we have $\overline{d} > 0$.

One of the main new features regarding the work in [6] and [7], is the startup of the induction. In our case this was done in the next lemma.

Lemma 9.2. [10] *Assume that δ_1 is sufficiently small and the neighbourhood $\mathcal{N}$ of $(a_0, 1)$ is sufficiently small. Then there are constants $C_1, C_2, \kappa_6 > 0$, so that for every $\varepsilon = 2^{-N_0}$ sufficiently small, there is a function $b_0(\varepsilon)$ so that for every $b_0(\varepsilon) \le b < 1$ one can partition $(a_0 - \varepsilon, a_0 - \varepsilon^2)$ into a partition $\mathcal{Q}$ of countable number of parameter intervals ω and an exceptional set $\mathcal{E}$ of measure $o(\epsilon)$, so that for all $\omega \in \mathcal{Q}$ there is an $n_0 = n_0(\omega)$ so that for some (r, ℓ), with $r \le n_0^2$, (or equivalently $e^{-r} \ge e^{-\sqrt{n_0}}$)*

$$I_{r,\ell} \subset \xi_{n_0}(\omega, b) \subset I^+_{r,\ell},$$

and such that for every $a \in \omega_b$

(a) $(f^j_{a,b})'(f_{a,b}(c)) \ge C_2 e^{\kappa_6 j}$ *for* $0 \le j \le n_0 - 1$*;*
(b) $\partial_a f^j_{a,b}(c) \ge C_2 e^{\kappa_6 (j-1)}$ *for* $1 \le j \le n_0$*;*
(c) $|\xi_j(a, b) - c| > C_1 e^{-\sqrt{j}}$ *for* $1 \le j < n_0$*;*
(d) $(f^{n_0-1}_{a,b})'(f_{a,b}(c)) \ge e^{(n_0-1)^{2/3}}$*;*
(e) $|\xi_{n_0}(a, b) - c| \ge e^{-\sqrt{n_0}}$

The corresponding statement holds also for the interval $(a_0 + \varepsilon^2, a_0 + \varepsilon)$.

Let us fix b, $0 < b < b_0(\varepsilon)$. Note that for every positive integer n we have a family $\mathcal{P}_n$ of subintervals of $(a_0 - \varepsilon, a_0 + \varepsilon)$ (as in Lemma 9.2) with pairwise disjoint interiors, such that each element of $\mathcal{P}_{n+1}$ is contained in some element of $\mathcal{P}_n$. In the set of pairs (n, ω) such that $\omega \in \mathcal{P}_n$ there is a natural structure of a combinatorial tree, that goes down with its branches. Pairs (n, ω) are vertices of this tree; n is the level on which the vertex lies; there is an edge from (n, ω) to $(n + 1, \omega')$ if and only if $\omega' \subset \omega$.

Definition 9.3. [10] We say a pair (n, ω) is a *free return pair* if

$$\xi_n(\omega, b) \subset I^*.$$

If (n, ω_n) is a free return pair, then we will call n a free return time.

The induction will be separate on every branch of the tree, fixing the branch results in considering a descending sequence of intervals $\omega_n \in \mathcal{P}_n$. An important feature of the construction is that if n is not a free return time then $\omega_n = \omega_{n-1}$. The main induction step will be from a free return time to the next free return time. The constants C_2 and C_1 are as in Lemma 9.2. In the whole induction they will stay the same.

The Induction Statement in our case is the following. If n is a free return time and $a \in \omega$, then:

(i) we have

$$(f^{n-1}_{a,b})'(f_{a,b}(c)) \ge e^{2(n-1)^{2/3}}, \tag{9.4}$$

(ii) for every $\nu \in [n_0, n)$

$$(f^{\nu}_{a,b})'(f_{a,b}(c)) \ge e^{\nu^{2/3}}, \tag{9.5}$$

(iii) for every $\nu \in [1, n)$

$$(f^{\nu}_{a,b})'(f_{a,b}(c)) \ge C_2 e^{\nu^{2/3}}, \tag{9.6}$$

(iv) if $\nu < n$ is also a free return time, then

$$(f_{a,b}^{n-\nu})'(f_{a,b}^{\nu}(c)) \geq C(\delta) >> 1, \tag{9.7}$$

(v) for every $\nu \in [n_0, n]$

$$|\xi_\nu(a,b) - c| \geq C_1 e^{-\sqrt{\nu}}, \tag{9.8}$$

(vi) for every $\nu \in [0, n]$

$$|\xi_\nu(a,b) - c| \geq C_1 e^{-\sqrt{\nu}}, \tag{9.9}$$

In [6] and [7] statements (v) and (vi) are called the basic assumption (BA).

We set $\mathcal{P}_n = \{\omega_b\}$ for $n = 1, 2, \ldots, N_0$. Thus, this is the beginning of every branch. Then we declare $n_0 = n_0(\omega)$ to be the first free return time according to the startup construction. Thus, for every branch we have to start induction by checking that the above conditions are satisfied for $n = n_0(\omega)$.

Let a' be the midpoint of the interval ω such that

$$\xi_n(\omega, b) \subset I_{r,l}^+ = I_{r,l-1} \cup I_{r,l} \cup I_{r,l+1}$$

for some n, r, l. We define the *bound period* as the maximal integer p so that for all $j \leq p$, $a \in \omega$, and $x \in \xi_n(\omega, b)$

$$|f_{a,b}^j(x) - f_{a',b}^j(c)| \leq e^{-4\sqrt{j}}. \tag{9.10}$$

Proposition 9.4. [10] *Let $a = a_0$ be an MT-parameter for f_a and let $\varepsilon > 0$ be given. There is a function $\eta(\varepsilon) \to 0$ and a function $b_0(\varepsilon) \to 1$ as $\varepsilon \to 0$ such that if $b_0(\varepsilon) < b < 1$, if ω_0 is a parameter interval such that*

$$\omega_0 \subset (a_0 - \varepsilon, a_0 - \varepsilon^2) \cup (a_0 + \varepsilon^2, a_0 + \varepsilon), \tag{9.11}$$

such that $I_{r,\ell} \subset \xi_{n_0}(\omega_0, b) \subset I_{r,\ell}^+$ and such that induction assumptions (i)–(vi) are satisfied for $n = n_0$. Then there is a set $\tilde{E}_b \subset \omega_0$ so that $|\tilde{E}_b| \geq (1 - \eta(\varepsilon))|\omega_0|$, $C = C(a_0)$ and $\hat{\kappa} = \hat{\kappa}(a_0) > 0$ so that

$$(f_{a,b}^n)'(f_{a,b}(c)) \geq C e^{\hat{\kappa} n}, \qquad \forall n \geq 0 \qquad \forall a \in \tilde{E}_b. \tag{9.12}$$

Note that the assumptions of Proposition 9.4 are satisfied by Lemma 9.2.

Corollary 9.5. [10] *The set E of parameters for which the double standard map is uniformly expanding accumulates on the MT points $(a_0, 1)$ in the parameter space.*

We now consider $b < 1$, and we construct a non-empty union of open intervals $\hat{E}_b \supset \tilde{E}_b$ so that for $(a, b) \in \hat{E}_b$ there is an integer N so that for $(a, b) \in \hat{E}_b$, $f_{a,b}^N$ is uniformly expanding. This is formulated in Proposition 9.6. The set $\hat{E}_b$ is obtained by stopping the construction of the parameter set $\tilde{E}_b$ of Proposition 9.4 at a finite stage.

Proposition 9.6. [10] *Let $a = a_0$ be a MT parameter. Then if $b_0 = b_0(a_0) < 1$ is sufficiently close to 1 then for all $b \in (b_0, 1)$ there is a set $\hat{E}_b$, which is a finite union of intervals $\{\omega_j\}_{j=0}^{J_0}$ so that for $a \in \omega_j$, there is an integer M_j so that for all $x \in \mathbb{T}$,*

$$(f_{a,b}^{M_j})'(x) \geq \lambda_j > 1. \tag{9.13}$$

Let us introduce some notations. For a fixed b, let us denote the sets of those parameters a for which $f_{a,b}$ has an attracting (resp. neutral) orbit T_b (resp. TN_b). Moreover, let E_b be the set of those parameters a for which $f_{a,b}$ is *expanding*, that is, there exist $C > 0$ and $\kappa > 0$ such that

$$(f_{a,b}^n)'(x) \geq Ce^{\kappa n}, \qquad \forall n \geq 0 \quad \forall x \in \mathbb{T}. \tag{9.14}$$

By the result of Mañé [32], if a does not belong to T_b or TN_b, then it belongs to E_b. Observe that by the definition, a small perturbation of an expanding map is also expanding, so E_b is open. In fact, the set $E = \{(a, b) : a \in E_b, 0 \leq b < 1\}$ is open in $[0, 1) \times [0, 1)$.

Note that our definition of E_1 or $\tilde{E}_1$ is quite different from the noncritical case, i.e. the case of of E_b for $b < 1$. Nevertheless, there are some common features of the noncritical case, because if $f_{a,b}$ is expanding, then by the results of Krzyżewski and Szlenk [24], or by the Lasota-Yorke Theorem [25], there exists an absolutely continuous invariant measure.

Let ω_0 be an interval as defined in Proposition 9.4 satisfying (9.11) and let $\tilde{E}_b$ be the set defined in this proposition. Let $\hat{E}_b = \hat{E}_b^{\hat{N}} \supset \tilde{E}_b$ be the set corresponding to the $\hat{N}$:th order construction of Proposition 9.6. $\hat{N}$ is here determined as the smallest integer satisfying $e^{-\hat{N}} \leq d$ as in the proof of Proposition 9.6. By (9.13) we have the next theorem.

Theorem 9.7. [10] *Let a_0 be a MT parameter for the family $\{f_a\}$. Denote $\omega(\varepsilon) = (a_0 - \varepsilon, a_0 + \varepsilon)$. Then for some $\varepsilon_0 > 0$ there is a function $b_0 : (0, \varepsilon_0) \to (0, 1)$ such that*

$$\lim_{\varepsilon \to 0} \frac{\inf\{|E_b \cap \omega(\varepsilon)| : b \in (b_0(\varepsilon), 1]\}}{|\omega(\varepsilon)|} = 1.$$

Here $|A|$ denotes the Lebesgue measure of the set A.

This theorem gives a quantitative relation between the behaviour of the system for $b < 1$, where the maps are local diffeomorphisms, and for $b = 1$, the critical case.

Finally, we prove a topological result, using very different methods.

Theorem 9.8. [10] *For each $b < 1$, the set E_b is dense in the complement of T_b. In particular, every interval of the parameters a either is contained in a closure of one tongue or intersects E_b.*

The above theorem in some sense complements Theorem A. Locally it says less about the set E_b, but it applies to all $b < 1$, not only to b sufficiently close to 1 (moreover, this closeness in Theorem A depends on a_0).

10. Future research topics

Double standard maps. Assume $K(f_{a_1})$ and $K(f_{a_2})$ are aperiodic kneading sequences. Then f_{a_1} and f_{a_2} are topologically conjugate and both are conjugate to the doubling map of the circle

$$x \mapsto 2x(mod 1).$$

One can look at this topological conjugacy as follows.

Let F_a be the lift of f_a to the real line and consider the limits

$$\phi_1 = \lim_{n\to\infty} \frac{F_{a_1}}{2^n},$$

and

$$\phi_2 = \lim_{n\to\infty} \frac{F_{a_2}}{2^n}.$$

Define a conjugacy function as

$$h = \phi_1 \circ \phi_2^{-1}.$$

We say a homeomorphism $h : [0,1] \to [0,1]$ is *quasi-symmetric* if there exists $k \leq 1$ such that for every $x \in [0,1]$ and $\epsilon > 0$ we have

$$\frac{1}{K} \leq \frac{|h(x+\epsilon) - h(x)|}{|h(x) - h(x-\epsilon)|} \leq K.$$

Sullivan initiated a programme to prove quasisymmetric rigidity in one-dimensional dynamics: interval or circle maps that are topologically conjugate are quasisymmetrically conjugate.

Open question 1: Can we prove using only real methods that

$$f_{a_1} = h \circ f_{a_2} \circ h^{-1}$$

where h is a quasi-symmetry?

Open question 2: Can we prove Theorem 2.4 using only real methods?

Piecewise isometries. *Piecewise Isometries* (PIWs) are higher order generalizations of *Interval Exchange Transformations* (IETs). Let X be a subset of $\mathbf{C}$ and $\mathcal{P} = \{X_\alpha\}_{\alpha\in\mathcal{A}}$ be a finite partition of X into convex sets (or *atoms*), that is $\bigcup_{\alpha\in\mathcal{A}} X_\alpha = X$ and $X_\alpha \cap X_\beta = \emptyset$ for $\alpha \neq \beta$. Given a *rotation vector* $\theta \in \mathbf{T}^{\mathcal{A}}$ (with $\mathbf{T}^{\mathcal{A}}$ denoting the torus $\mathbf{R}^{\mathcal{A}}/2\pi\mathbf{Z}^{\mathcal{A}}$) and a *translation vector* $\eta \in \mathbf{C}^{\mathcal{A}}$, we say (X,T) is a *piecewise isometry* if T is such that

$$T(z) := T_\alpha(z) = e^{i\theta_\alpha} z + \eta_\alpha, \text{ if } z \in X_\alpha.$$

Although it is commonly accepted that the phase space of typical Hamiltonian systems is divided into regions of regular and chaotic motion [13], there is still no general and rigorous study of this. Area preserving PWIs that have been studied as linear models for the standard map [3], can exhibit a similar phenomenon. Unlike IETs which are typically ergodic, there is numerical evidence [5], that Lebesgue measure on the exceptional set is typically not ergodic in some families of PWIs - there can be non-smooth invariant curves that prevent trajectories from spreading across the whole of the exceptional set. These curves were first observed in [2] for an isolated parameter and later found in [5] to be apparently abundant for a large family of PWIs.

In general, for a given PWI we may partition its domain X into a *regular* and an *exceptional set*. If we consider the zero measure set given by the union $\mathcal{E}$ of all preimages of the set of discontinuities D, then its closure $\overline{\mathcal{E}}$ (which may be of positive measure) is called the *exceptional set* for the map. The complement of the exceptional set is called the *regular set* for the map and consists of disjoint polygons or disks that, if X is compact, are periodically coded by their itinerary through the atoms of the PWI. There is numerical evidence that the exceptional set may have positive Lebesgue measure for typical PWIs [4].

Consider a family of dynamical systems $\mathcal{F} = \{f_\mu : X \to X\}$ parametrized by $\mu \in \mathcal{P}$, where $\mathcal{P}$ is called the parameter space of $\mathcal{F}$. A *renormalization scheme* for $\mathcal{F}$ is a decreasing chain of subsets of X, $X = Y_0(\mu) \supset Y_1(\mu) \supset Y_2(\mu) \supset \dots$, together with a renormalization operator $\mathcal{R} : \mathcal{P} \to \mathcal{P}$ such that the first return map of a point in $Y_{n+1}(\mu)$ under iteration by $f_{\mathcal{R}^n(\mu)} : Y_n(\mu) \to Y_n(\mu)$ is given by $f_{\mathcal{R}^{n+1}(\mu)} : Y_{n+1}(\mu) \to Y_{n+1}(\mu)$.

Open question 3 What is the measure of the exceptional set for a given PWI?

Open question 4 Can we renormalize a given PWI?

Open question 5 Can we prove the existence of invariant curves for PWIs?

Fair measures and fair entropy. Let X be a compact topological space. To each continuous transformation $f : X \to X$ we associate a non-negative real number or ∞, denoted by $h(f)$. If α is an open cover of X we define the *entropy of* α by $H(\alpha) = \log N(\alpha)$, where $N(\alpha)$ is the number of sets in a finite subcover of α with smallest cardinality, then the *entropy of f relative to* α is given by $h(f, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} f^{-i}\alpha\right)$. We define the *topological entropy* of f as $h(f) = \sup_\alpha h(f, \alpha)$, where α ranges over all open covers of X.

When working on topological entropy of one-dimensional Dynamical Systems, one would like to have some simple method of computing it and that is in general a very difficult task. For instance, for piecewise strictly monotone interval maps, various computational methods are known [12] but none of them is really general and simple.

One idea for computing topological entropy is to count preimages of a given point. For this we need some nice class of maps. We define the class PMM of continuous maps $f : [0, 1] \to [0, 1]$ piecewise monotone (finitely many pieces) and topologically mixing. For $f \in PMM$ and $x \in [0, 1]$ we have [36]

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log card(f^{-n}(x)).$$

Although this is a really nice theoretical result, it is useless if we want to compute $h(f)$ using a computer as the number of preimages of x under f grows exponentially with n, and keeping track of all of them requires a lot of memory, while computing their preimages (when we pass from n to $n + 1$) requires a lot of time.

In [36] we computed the entropy following backwards trajectories in a way that at each step every preimage can be chosen with equal probability introducing a new concept of entropy, the fair entropy. Fair entropy gives a lower bound for topological entropy and is simple to compute.

Let X be a compact metric space and let $f : X \to X$ be a continuous map (and a surjection). We consider the following partitions. A partition into laps (intervals of monotonicity), this is, partition $\mathcal{X}$ of X into Borel sets $X_i, i = 1, 2, \dots s$ such that $f(X_i)$ is a homeomorphism onto its image, and a partition $\mathcal{A}$ generated by the images of laps, this is, the common refinement of the partitions $\{f(X_i), X \setminus f(X_i)\}, i = 1, 2, \dots s$.

For each $A \in \mathcal{A}$ denote by $p(A)$ the set of those numbers $i \in \{1, 2, \dots s\}$ for which $A \subset f(X_i)$. Each $x \in A$ has a preimage in every X_i such that $i \in p(A)$ and no more preimages. The number $c(x)$ of preimages of x depends only on A (denoted $c(A)$). Since f is a surjection, this number is always positive.

Let $\mathcal{M}$ be the space of all probability Borel measures on X. We define an operator Φ from $\mathcal{M}$ to itself as follows. If $\mu \in \mathcal{M}$, then we chop μ into pieces $\mu|_A$, $A \in \mathcal{A}$, divide each piece by $c(A)$, and push via $(f|_{X_i})^{-1}$ to each X_i with $i \in p(A)$. That is, we set

$$\Phi(\mu) = \sum_{A \in \mathcal{A}} \sum_{i \in p(A)} (f|_{X_i})_*^{-1} \left(\frac{\mu|_A}{c(A)} \right), \tag{10.1}$$

and we define a *fair measure* as a measure $\mu \in \mathcal{M}$ for which $\Phi(\mu) = \mu$. The *fair entropy* is the entropy computed with respect to a fair measure.

Open question 6 Are there connections between the harmonic and the fair measure?

References

[1] V. I. Arnold, *Small denominators, I. Mappings of the circumference onto itself*, Amer. Math. Soc. Translations **46** (1965), 213–284.

[2] Ashwin, P. (1996). *Non-smooth invariant circles in digital overflow oscillations*. Proceedings of the 4th Int. Workshop on Nonlinear Dynamics of Electronic Systems, Sevilla, 417–422.

[3] Ashwin, P., (1997). *Elliptic behaviour in the sawtooth standard map*, Phys. Lett. A, 232, pp. 409-416.

[4] Ashwin, P., Fu, (2002). *On the Geometry of Orientation-Preserving Planar Piecewise Isometries*. Journal of Nonlinear Science, 12, no. 3, 207–240.

[5] Ashwin, P. , Goetz, A. (2005). *Invariant curves and explosion of periodic islands in systems of piecewise rotations*. SIAM J. Appl. Dyn. Syst., 4(2), 437-458.

[6] M. Benedicks and L. Carleson, On iterations of $1 - ax^2$ on $(- - 1; 1)$. Ann. of Math. (2) 122, no. 1, 1–25, (1985).

[7] M. Benedicks and L. Carleson, The dynamics of the Hénon map. Ann. of Math. (2) 133, no. 1, 73–169, (1991).

[8] M. Benedicks and A. Rodrigues, Kneading sequences for double standard maps. *Fund. Math.* **206**, 61–75, (2009).

[9] C. Bloch, *Sur la théorie des perturbations des états liés. Nuclear Physics* **6**, *329–347 (1958).*

[10] M. Benedicks, M. Misiurewicz and A. Rodrigues, Arxiv preprint (https://arxiv.org/abs/2002.02019)

[11] H. Bruin, J. Rivera-Letelier and S. van Strien. Large derivatives, backward contraction and invariant densities for interval maps. Invent. Math. ***172****, no. 3, 509–533, (2008).*

[12] L. Block et al J. Statist. Phys*, 55 (1989).*

[13] Chirikov B.V. (1983) Chaotic dynamics in Hamiltonian systems with divided phase space. In: Garrido L. (eds) Dynamical System and Chaos. Lecture Notes in Physics, vol 179. Springer, Berlin, Heidelberg.

[14] P. Collet, J.-P. Eckmann, Iterated maps on the interval as dynamical systems. In Progress in Physics. Birkhäuser Boston, Mass (1980).

[15] A. Denjoy, Sur les courbes definies par les equations differential à la surphace du tore, Journal de Math. , **11**, *333–375 (1932)*

[16] J Graczyk, G Swiatek. Generic hyperbolicity in the logistic family, Annals of mathematics 146 (1), 1–52 (1997)

[17] J. Guckenheimer, Sensitive dependence to initial conditions for one dimensional maps, Communications in Mathematical Physics, 70, 133–160 (1979)

[18] J.K.Hale and H. Koçak, Dynamics and bifurcations, Texts Appli. Math 3, Springer, New York (1991)

[19] H. Hochstadt, Instability intervals of Hill's equation, Comm. Pure Appl. Math, **17**, *251–255 (1964).*

[20] Hooper, W. Patrick (2013) Renormalization of polygon exchange maps arising from corner percolation*. Inventiones mathematicae, 191(2), 255-320.*

[21] M. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. Comm. Math. Phys.*,* ***81****, 39–88, (1981).*

[22] F. John, The Dirichlet problem for a hyperbolic equation, Amer. J. Math. **63** *141–154 (1941).*

[23] Keane, M. S., (1975) Interval exchange transformations*, Math. Z. 141, 25-31.*

[24] K. Krzyżewski and W. Szlenk, On invariant measures for expanding differentiable mappings. Studia Mathematica*,* ***33****, 83–92 (1969).*

[25] A. Lasota and J. A. Yorke, On the existence of invariant measures for piecewise monotonic transformations, Trans. Amer. Math. Soc. ***186****, 481–488 (1973).*

[26] G. Levin and G. Swiatek, Universality of critical circle covers. Comm. Math. Phys. 228 , 371–399 (2002).

[27] D.M.Levy and J.B. Keller, Instability intervals of Hill's equation, Comm. Pure Appl. Math, **16**, *469–476 (1963)*

[28] M. Lyubich, Acta Math. **178**, *185–297 (1997).*

[29] J. W. Milnor, W. Thurston, On iterated maps of the interval, Dynamical systems (College Park, MD, 198687), Lecture Notes in Mathematics, vol. 1342, Berlin: Springer, pp. 465–563 (1988).

[30] M. Misiurewicz, Absolutely continuous measures for certain maps of an interval. Inst. Hautes Etudes Sci. Publ. Math. No. 53, 17–51 (1981).

[31] H. Poincaré, On the curves defined by differential equations, Russian transl., Moscow (1947).

[32] R. Mañé, *Hyperbolicity, Sinks and Measure in One-Dimensional Dynamics.* Comm. Math. Phys. **100**, *495–524 (1985).*

[33] M. Misiurewicz and A. Rodrigues, *Double Standard Maps.* Comm. Math. Phys. ***273***, *37–65 (2007).*

[34] M. Misiurewicz and A. Rodrigues, *On the tip of the tongue.* J. of Fixed Point Theory Appl. **3**, *131–141 (2008).*

[35] M. Misiurewicz and A. Rodrigues, *Non-Generic Cusps.* Trans. Amer. Math. Soc. **363**, *3553–3572 (2011).*

[36] *M. Misiurewicz and A. Rodrigues, Counting preimages, Ergodic Theory and Dynamical Systems 38(05), 1–20 (2017).*

[37] M. Misiurewicz, *Maps of an interval,* Chaotic behaviour of deterministic systems, *North-Holland , pp. 565–590 (1983).*

[38] *Henri Poincare, Memoire sur les curbes definies par les equations differentielles I–IV, Oeuvres completes, Vol. 1, Gauthier–Villars, Paris (1952).*

[39] *J.-C. Yoccoz, manuscript.*

[40] *A. N. Sharkovskii, . "Co-existence of cycles of a continuous mapping of the line into itself". Ukrainian Math. J. 16: 61-71 (1964).*

[41] *S. van Strien, Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition.* Commun. Math Phys. ***128***, *437–496 (1990).*

[42] *A. Blokh, C. Cleveland and M. Misiurewicz, Expanding polymodials, in "Modern Dynamical Systems and Applications", eds. M. Brin, B. Hasselblatt and Ya. Pesin, Cambridge University Press, Cambridge, 2004, pp. 253–270.*

[43] *A. Blokh, C. Cleveland and M. Misiurewicz, Julia sets of expanding polymodials, Ergodic Theory Dynam. Systems **25** (2005), 1691–1718.*

[44] *A. Blokh and M. Misiurewicz, Branched derivatives, Nonlinearity **18** (2005), 703–715.*

[45] *A. Epstein, L. Keen and C. Tresser, The set of maps $F_{a,b} : x \mapsto x + a + \frac{b}{2\pi}\sin(2\pi x)$ with any given rotation interval is contractible, Comm. Math. Phys. **173** (1995), 313–333.*

[46] *L. B. Jonker, The scaling of Arnold's tongues for differentiable homeomorphisms of the circle, Commun. Math. Phys. **129** (1990), 1–25.*

[47] *G. Levin and G. Świątek, Universality of critical circle covers, Comm. Math. Phys. **228** (2002), 371–399.*

[48] *W. de Melo and S. van Strien, "One-dimensional dynamics", Springer-Verlag, Berlin, 1993*

[49] *S. E. Newhouse, J. Palis and F. Takens, Bifurcations and stability of families of diffeomorphisms, Publ. Math. IHES **57** (1983), 5–71.*

[50] *M. Shub and D. Sullivan, Expanding endomorphisms of the circle revisited, Ergodic Theory Dynam. Systems **5** (1985), 285–289.*

[51] *W. Wenzel, O. Biham and C. Jayaprakash, Periodic orbits in the dissipative standard map, Phys. Rev. A **43** (1991), 6550–6557.*