Computational Statistics manuscript No.
(will be inserted by the editor)

Moments and probability density of threshold crossing times for populations in random environments under sustainable harvesting policies

Nuno M. Brites · Carlos A. Braumann

Received: date / Accepted: date

Abstract Stochastic differential equations are used to model the dynamics of harvested populations in random environments. The main goal of this work is to compute, for a particular fish population under constant effort harvesting, the mean and standard deviation of first passage times by several lower and upper thresholds values. We apply logistic or logistic-like with Allee effects average growth dynamics. In addition, we present a method to obtain the probability density function of the first passage time by a threshold through the numerical inversion of its Laplace transform.

Keywords Allee effects · Constant effort harvesting · Laplace transform · Logistic growth · Stochastic differential equations · First passage times

1 Introduction

In random varying environments, we can describe the size evolution of a population under constant effort harvesting using stochastic differential equations as follows:

$$dX(t) = f(X(t))X(t)dt - qEX(t)dt + \sigma X(t)dW(t), \; X(0) = x. \qquad (1)$$

Nuno M. Brites
ISEG – School of Economics and Management, Universidade de Lisboa; REM - Research in Economics and Mathematics, CEMAPRE, Portugal
Rua do Quelhas, 6
Tel.: +351 213 925 800
E-mail: nbrites@iseg.ulisboa.pt

Carlos A. Braumann
Centro de Investigação em Matemática e Aplicações, Instituto de Investigação e Formação Avançada, Universidade de Évora and Departamento de Matemática, Escola de Ciências e Tecnologia, Universidade de Évora, Portugal
E-mail: braumann@uevora.pt

Following [1–7, 9, 10], we denote by $X(t)$ the population size at time t, $f(X)$ is the *per capita* natural growth rate, $q > 0$ is the catchability coefficient, $E > 0$ is the constant harvesting effort, $H(t) = qEX(t)$ represents the yield from harvesting, $\sigma > 0$ measures the intensity of environmental fluctuations, $W(t)$ is a standard Wiener process and $X(0) = x > 0$ is the population known size at time 0.

A particular case of SDE (1) is the logistic growth model (as in [12, 13]), i.e., $f(X(t)) = r\left(1 - \frac{X(t)}{K}\right)$, where $r > 0$ represents the intrinsic growth rate and $K > 0$ is the environment's carrying capacity. This formulation represents an evolution where the population grows almost exponentially followed by a period with less growth intensity. Sometimes, however, the population behaves differently: to low values of the population size, we observe *per capita* growth rates lower than the high rates one would expect considering the higher availability of resources per individual. When such behavior occurs we say that the population is influenced by Allee effects. Difficulty in finding mating partners or in setting up an effective pack-hunting size or, in the case of prey species, in constructing a strong enough group defence against predators can be some of the causes for the presence of Allee effects (see, for instance, [5, 11, 15, 21]). In this case, the *per capita* natural growth rate can be defined as $f(X(t)) = r\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)$, where A is the Allee parameter representing the strength of Allee effects. This functional form for f assumes that the natural growth rate follows a logistic-like model inspired by a similar deterministic model (see, for instance, [15]). However, without changing the logistic-like model for the average natural growth rate dynamics, we use a different parametrization of that model to allow easier comparisons with the logistic model without Allee effects (as in [11]). In particular, the logistic model and the logistic-like model here considered have in common the same carrying capacity K and the same slope of the natural growth rate at $X = K$. According to the value of A, one can refer to strong ($A \in (0, K)$) or weak ($A \in (-K, 0)$) Allee effects. Strong Allee effects will not be considered here since they drive the population to extinction even in the absence of harvesting (see, for instance, [8, 11]). Therefore, we will consider only weak Allee effects. The closer A is to 0, the more intense is the Allee effect. On the contrary, the closer A is to $-K$, the less intense is the Allee effect. Taking $A \to -\infty$ leads to the well-known logistic model.

For the models here considered, if the harvesting effort E is not too high, mathematical extinction (population size $X(t)$ converging to zero) has, as we will discuss later, a zero probability of occurring. However, since we work with ergodic processes, all states in the interior of the state space are attainable with probability one in finite time. In particular, we can consider a threshold $y > 0$ and study how long it takes for the process $X(t)$ to reach y for the first time. This threshold can be a low biological reference point $y < X(0)$, i.e., a minimum biomass value below which the population self-renewable capacity is endangered. It can also be some high biomass level $y > X(0)$ that is important for the fishery, such as a warning level of danger to the survival of another species or possible deviations from the optimal fishing effort.

Based on general expressions for the mean and standard deviation of first passage times by lower and upper thresholds, we compute such values for the particular cases of the logistic and the logistic-like with weak Allee effects model and several lower and upper threshold values y. For a fixed threshold value, we also present a way to obtain, by numerical inversion of its Laplace transform, the probability density function of the time to reach the threshold. To check the quality of this numerical approximation, we compare the mean and standard deviation of the first passage time obtained by using this approximated probability density function with the mean and standard deviation obtained directly.

This paper is organized as follows. In Section 2 we review some results on the optimal sustainable policy. Section 3 presents the expressions to compute the mean and standard deviation of first passage times, and respective application to the logistic model and the logistic-like model with weak Allee effects. In Section 4 we compute the probability density function of the first passage time using the inversion of the Laplace transform. Finally, some concluding remarks are presented in Section 5.

2 Optimal sustainable policy

In [9,18], for the logistic growth model, and in [5] for the logistic-like growth model with Allee effects, one can find conditions to avoid population extinction, to have a unique solution and to grant a stationary density for the population size. For the logistic model, it is sufficient to have $0 \leq E < \frac{r}{q}\left(1 - \frac{\sigma^2}{2r}\right)$ and, for the logistic-like growth model with weak Allee effects, it is sufficient that $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K} - \frac{\sigma^2}{2r}\right)$.

The state space of X is $(0, +\infty)$ for both models. If the above conditions on E hold, then the boundaries $X = 0$ and $X = +\infty$ are non-attractive. The non-attractiveness of $X = 0$ ensures that there is a zero probability of mathematical extinction. The non-attractiveness of $X = +\infty$ ensures non-explosion and the existence and uniqueness of the solution for all $t > 0$. Thus, the transient distribution of $X(t)$ can stabilize and converge to a stationary density as $t \to +\infty$. This is indeed the case when the above conditions on E are met. Denoting by X_∞ the random variable with such stationary density, a good approximation of the expected size of the population $\mathbb{E}[X_t]$, for large t, is the expected value of X_∞.

For the growth models here considered, the expected value and the second moment of X_∞ can be found in [5] and [9]. Their expressions are, respectively, $\mathbb{E}[X_\infty] = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)$ and $\mathbb{E}[X_\infty^2] = K^2\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE}{r}\right)$ for the logistic model, and $\mathbb{E}[X_\infty] = \dfrac{I_1(E)}{I_0(E)}$ and $\mathbb{E}[X_\infty^2] = \dfrac{I_2(E)}{I_0(E)}$ for the logistic-like model with Allee affects, where

$$I_j(E) = \int_0^{+\infty} z^{\alpha-\beta E+j-1} \exp\left\{-\gamma(z-(K+A))^2\right\} dz,$$

$\alpha = \dfrac{2rA}{\sigma^2(A-K)} - 1$, $\beta = \dfrac{2q}{\sigma^2}$, and $\gamma = \dfrac{r}{\sigma^2 K(K-A)}$.

According to [4–7,9,10], the sustainable profit per unit time is defined as the difference between sales revenues and harvesting costs at the steady sate, i.e., $\Pi_\infty = R_\infty - C$, where $R_\infty = (p_1 - p_2 qEX_\infty)qEX_\infty$ are the sales revenues per unit time ($p_1 > 0,\ p_2 \geq 0$) and $C = (c_1 + c_2 E)E$ represent the fishing costs per unit time ($c_1 > 0,\ c_2 > 0$). Hence, $\Pi_\infty = (p_1 q X_\infty - c_1)E - (p_2 q^2 X_\infty^2 + c_2)E^2$ and so, $\mathbb{E}[\Pi_\infty] = (p_1 q \mathbb{E}[X_\infty] - c_1)E - (p_2 q^2 \mathbb{E}[X_\infty^2] + c_2)E^2$. To determine the harvesting effort E that maximizes $\mathbb{E}[\Pi_\infty]$ we use numerical methods and denote this optimal sustainable effort by E^{opt}.

For each of the models here considered, the optimal expected sustainable profit per unit time is, respectively,

$$\begin{aligned}\mathbb{E}[\Pi_\infty^{opt}] = &\left(p_1 qK\left(1 - \frac{qE^{opt}}{r} - \frac{\sigma^2}{2r}\right) - c_1\right)E^{opt} \\ &- \left(p_2 q^2 K^2 \left(1 - \frac{qE^{opt}}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE^{opt}}{r}\right) + c_2\right)E^{opt2} \qquad (2)\end{aligned}$$

and

$$\mathbb{E}[\Pi_\infty^{opt}] = \left(p_1 q \frac{I_1(E^{opt})}{I_0(E^{opt})} - c_1\right)E^{opt} - \left(p_2 q^2 \frac{I_2(E^{opt})}{I_0(E^{opt})} + c_2\right)E^{opt2}. \qquad (3)$$

Note that, in [4–7,9,10], the authors refer to E^{opt} as E^{**} due to the presence of the alternative optimal variable effort $E^*(t)$.

3 Moments of the first passage times

In this section, we will consider, for comparison purposes, a logistic growth model and a logistic-like model with weak Allee effects, given by

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qEX(t)dt + \sigma X(t)dW(t),\ X(0) = x \qquad (4)$$

and

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t)-A}{K-A}\right)dt - qEX(t)dt + \sigma X(t)dW(t),\ X(0) = x, \qquad (5)$$

respectively.

For model (4), the scale and speed densities (see, for instance, [8] and [19]) are given, respectively, by

$$s(X) = C_1 X^{-\rho-1} \exp\{\theta X\},$$
$$m(X) = C_2 X^{\rho-1} \exp\{-\theta X\},$$

with $\rho = \dfrac{2(r-qE)}{\sigma^2} - 1$, $\theta = \dfrac{2r}{K\sigma^2}$ and $C_1,\ C_2$ are positive constants. For model (5), the scale and speed densities, obtained in [5], are given, respectively, by

$$s(X) = C_3 X^{-(\alpha+1)+\beta E} \exp\left\{\gamma(X-(K+A))^2\right\},$$
$$m(X) = C_4 X^{\alpha-\beta E-1} \exp\left\{-\gamma(X-(K+A))^2\right\},$$

with $\alpha = \dfrac{2rA}{(A-K)\sigma^2} - 1$, $\beta = \dfrac{2q}{\sigma^2}$, $\gamma = \dfrac{r}{K(K-A)\sigma^2}$ and $C_3,\ C_4$ are positive constants.

The definitions of the first passage time by a threshold are as follows:

- the first passage time of $X(t)$ by a lower threshold L $(0 < L < x < +\infty)$ is
$$T_L := \inf\{t \geq 0 : X(t) = L\};$$
- the first passage time of $X(t)$ by an upper threshold U $(0 < x < U < +\infty)$ is
$$T_U := \inf\{t \geq 0 : X(t) = U\},$$

where x is the initial population size. Our main interest is to study the mean of the first passage by lower and upper thresholds. Such values represent, on average, the amount of time that the process needs to attain L or U. In [11] one can see, for a general class of stochastic processes (where our models can be included with minor adaptations), the expressions for the mean and variance of T_L and T_U. They are given by

- Mean of T_L:
$$\mathbb{E}\left[T_L \middle| X(0) = x\right] = 2\int_L^x s(y) \int_y^{+\infty} m(z)dzdy. \qquad (6)$$
- Variance of T_L:
$$Var\left[T_L|X(0) = x\right] = 8\int_L^x s(y) \int_y^{+\infty} s(z) \left(\int_z^{+\infty} m(\theta)d\theta\right)^2 dzdy. \qquad (7)$$
- Mean of T_U:
$$\mathbb{E}\left[T_U \middle| X(0) = x\right] = 2\int_x^U s(y) \int_0^y m(z)dzdy. \qquad (8)$$

Table 1 Values used in the computations. Adapted from [5].

Item	Description	Value	Unit[c]
r	Intrinsic growth rate	0.71	y^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
A	Allee parameter	$-0.75K$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	$SFU^{-1}y^{-1}$
σ	Strength of environmental fluctuations	0.2	$y^{-1/2}$
x	Initial population size	$0.5K$	kg
E^{opt}	Optimal sustainable effort (logistic model)[a]	104540	SFU
E^{opt}	Opt. sust. effort (logistic-like model w/ Allee effects)[b]	60546	SFU

[a]Value taken from [9]. [b]Value taken from [5]. [c]SFU represents the Standardized Fishing Unit. The definition can be found in [17]. y stands for year.

– Variance of T_U:

$$Var\left[T_U|X(0)=x\right] = 8\int_x^U s(y)\int_0^y s(z)\left(\int_0^z m(\theta)d\theta\right)^2 dzdy. \quad (9)$$

In these expressions, we have used upper limits of integration $+\infty$ in (6) and (7), and lower limits of integration 0 in (8) and (9) because $+\infty$ and 0 are, respectively, the upper and lower boundaries of the state space. If they have different values, one should use such values instead.

So, we compute the values (6) to (9) by numerical integration methods that are very precise. We have done that for the data presented in Table 1. For both growth models, the effort E was set to the optimal effort E^{opt} obtained by maximizing (2) and (3), respectively. The mean and variance of the first passage time by T_L and T_U were computed at the following values for L and U:

$$L = (0.05, 0.10, 0.15, \ldots, 0.90, 0.95, 1.00) \times x, \quad (10)$$

and

$$U = (1.00, 1.05, 1.10, \ldots, 3.90, 3.95, 4.00) \times x, \quad (11)$$

where x is, in both cases, the initial population size.

Figures 1 and 2, for the logistic model, and Figures 3 and 4, for the logistic-like model with weak Allee effects ($A = -0.75K$), show, for the list of L and U values presented in (10) and (11), the mean value and the standard deviation (both in logarithmic scale) of T_L and T_U using expressions (6) to (9).

From Figures 1 and 3 one can see that, as L increases towards the initial population level x (the end of the X-axis), the mean and standard deviation of T_L are decreasing, taking obviously the value zero at $L = x$ (not depicted since $\log(0) = -\infty$). From these Figures it is also clear that the mean and the standard deviation of T_L have the same order of magnitude, except for values of L very close to x, where the standard deviation is slightly greater, being this difference more pronounced in the case of the logistic-like model with weak Allee effects. We also conclude that the results for the mean and standard deviation have qualitatively similar behavior for both models but,

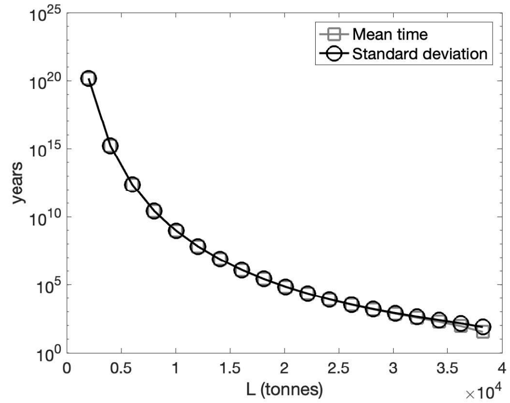


Fig. 1 Logistic model without Allee effects: mean (gray line with square points) and standard deviation (black line with circular points) of the first passage time by several values of L, when the initial population size is $x = 4.03 \times 10^4$ tonnes. We use a log scale on the vertical axis.

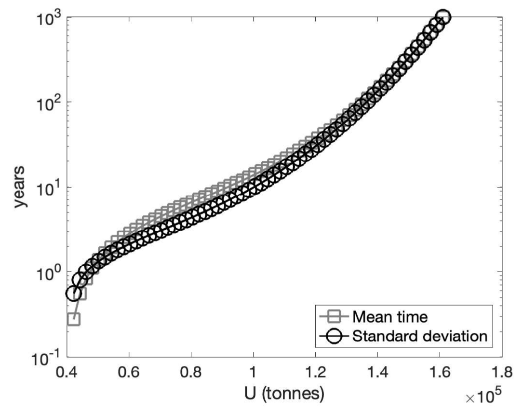


Fig. 2 Logistic model without Allee effects: mean (gray line with square points) and standard deviation (black line with circular points) of the first passage time by several values of U, when the initial population size is $x = 4.03 \times 10^4$ tonnes. We use a log scale on the vertical axis.

when there are weak Allee effects, it takes much less time to reach the lower thresholds, thus increasing the risk of the population reaching low dangerous levels (notice that, for both models, the values of K and x are the same).

Figures 2 and 4 show that the upper threshold U has a behavior different from the lower threshold L. One can see that the mean time and standard deviation are increasing as the threshold U increases. Of course, when $U = x$, the mean and standard deviation are zero. Again, the standard deviation has the same order of magnitude as the mean. In the presence of weak Allee effects,

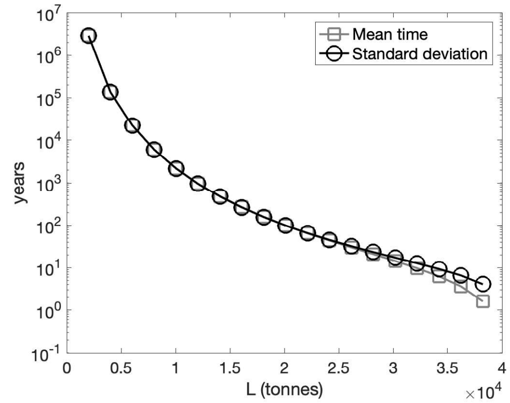


Fig. 3 Logistic-like model with weak Allee effects: mean (gray line with square points) and standard deviation (black line with circular points) of the first passage time by several values of L, when the initial population size is $x = 4.03 \times 10^4$ tonnes. We use a log scale on the vertical axis.

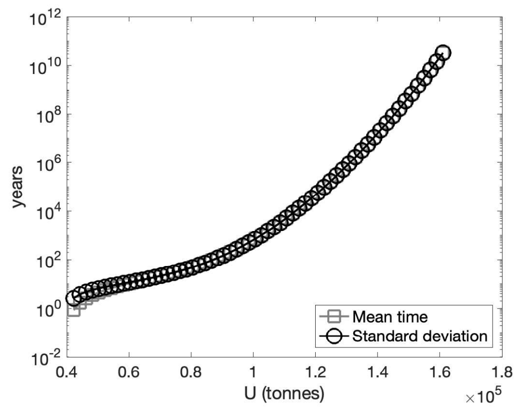


Fig. 4 Logistic-like model with weak Allee effects mean (gray line with square points) and standard deviation (black line with circular points) of the first passage time by several values of U, when the initial population size is $x = 4.03 \times 10^4$ tonnes. We use a log scale on the vertical axis.

the mean time (and the respective standard deviation) needed to reach any value of U is quite larger than for the logistic model, thus increasing the time to recovery.

Table 2 presents, for the logistic model, a list of scenarios with variations in thresholds L and U, in initial population size x and in effort E. For the first 3 scenarios (S_{L_1}, S_{L_2} and S_{L_3}) the mean and standard variation values of the first passage time by L are computed when the initial population size is x, and the lower threshold to reach is 10% of x, 50% of x and 75% of x. These 3

Table 2 Logistic model without Allee effects: alternative scenarios with approximate values for the mean and standard deviation of T_L and T_U when varying the parameters L, U, x and E. L, U, x are measured in tonnes $\times 10^4$, $\mathbb{E}[\cdot]$ and $\text{sd}[\cdot]$ are in years and harvesting efforts are measured in SFU.

Scenario	L	x	E	$\mathbb{E}[T_L]$	$\text{sd}[T_L]$
S_{L_1}	$0.10x = 0.40$	4.03	E^{opt}	$1.57 \cdot 10^{15}$	$1.57 \cdot 10^{15}$
S_{L_2}	$0.50x = 2.01$	4.03	E^{opt}	$7.02 \cdot 10^{4}$	$7.02 \cdot 10^{4}$
S_{L_3}	$0.75x = 3.02$	4.03	E^{opt}	734.86	800.66
S_{L_4}	$\mathbb{E}[X_\infty^{opt}] = 3.91$	4.03	E^{opt}	16.78	38.11
S_{L_5}	0.40	2.01	E^{opt}	$2.67 \cdot 10^{10}$	$2.67 \cdot 10^{15}$
S_{L_6}	$0.10x = 0.40$	4.03	$1.10E^{opt}$	$5.42 \cdot 10^{13}$	$5.42 \cdot 10^{13}$
S_{L_7}	$0.10x = 0.40$	4.03	$1.50E^{opt}$	$1.01 \cdot 10^{8}$	$1.00 \cdot 10^{8}$
S_{L_8}	$0.10x = 0.40$	4.03	$2.00E^{opt}$	$1.95 \cdot 10^{2}$	$1.72 \cdot 10^{2}$
Scenario	U	x	E	$\mathbb{E}[T_U]$	$\text{sd}[T_U]$
S_{U_1}	$\mathbb{E}[X_\infty^{opt}] = 3.91$	2.01	E^{opt}	8.78	4.07

cases are directly observed from Figure 1. Scenario S_{L_4} shows the case where the lower threshold is the value of the expected sustainable population size $\mathbb{E}[X_\infty^{opt}]$, i.e., the value given by $\mathbb{E}[X_\infty] = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)$when $E = E^{opt}$. Since this value is very close to $K/2 = x$, it is not a surprise to see that the mean time to reach the sustainable average threshold is only about 17 years.

Scenario S_{L_5} considers a low initial population size ($x = 0.25K$). Comparing this Scenario with S_{L_1}, which has a higher initial population size ($x = 0.5K$), we expect to have lower values for the mean and standard deviation of the first passage time. Indeed that is what one can observe from Table 2.

Scenarios S_{L_6}, S_{L_7} and S_{L_8} consider the cases where the effort is greater than E^{opt}. In the first one, the applied effort is 10% greater than E^{opt}. This will produce a higher catching rate, resulting in a stronger population decrease and, consequently, reducing the time required on average to reach L. Hence, the mean time to reach L will be lower than in scenario S_{L_1}. Scenario S_{L_7} is very similar to scenario S_{L_6} but with a higher effort, although still sustainable in the sense that the population will have a stationary density. Scenario S_{L_8} is a clear case of heavy over-fishing and the effort used almost reaches the value $\frac{r}{q}\left(1 - \frac{\sigma^2}{2r}\right)$ beyond which mathematical extinction occurs with probability one and there is no stationary density, as referred at the beginning of Section 2. In this case, reaching L will happen much faster.

Until now all the thresholds were smaller than the initial value. Scenario S_{U_1} considers the opposite case, i.e., we estimate the time that the population takes, on average, to reach the steady-state expected size but starting with an initial population of $0.25K$. Since the population tends to become close to $\mathbb{E}[X_\infty^{opt}]$, it seems natural that it took only about 9 years, on average, to reach the steady-state average value. So, even if we, by over-fishing or other reasons, have depleted the population to a low size of $0.25K$, applying the optimal effort E^{opt} from then on, the recovery will take on average only a few years.

4 Density probability functions of T_L and T_U

The Laplace transforms of the first passage times T_L and T_U, when the initial population size is x, are given respectively by

$$\mathbb{E}_x[\exp(-\lambda T_L)] \qquad \text{and} \qquad \mathbb{E}_x[\exp(-\lambda T_U)]. \tag{12}$$

In [14], for the stochastic logistic model (without harvesting and without Allee effects),

$$dX(t) = r_1 X(t)\left(1 - \frac{X(t)}{K_1}\right)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{13}$$

one can find the following expressions for (12):

$$\mathbb{E}_x[\exp(-\lambda T_L)] = \left(\frac{x}{L}\right)^{\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u} \cdot \frac{\Psi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vx\right)}{\Psi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vL\right)} \tag{14}$$

and

$$\mathbb{E}_x[\exp(-\lambda T_U)] = \left(\frac{x}{U}\right)^{\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u} \cdot \frac{\Phi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vx\right)}{\Phi\left(\sqrt{\frac{2\lambda}{\sigma^2}+u^2}+u, 1+2\sqrt{\frac{2\lambda}{\sigma^2}+u^2}, vU\right)}, \tag{15}$$

where $u = \frac{1}{2}\left(1 - \frac{2r_1}{\sigma^2}\right)$, $v = \frac{2r_1}{K_1\sigma^2}$, and Ψ and Φ are the hypergeometric confluent functions. Note that Ψ and Φ are also denoted by U and M (as in [20]).

The stochastic logistic model with harvesting based on constant effort, given by Equation (4), can be written as Equation (13) for $r_1 = (r - qE)$ and $K_1 = K\left(1 - \frac{qE}{r}\right)$. So, the Laplace transforms (14) and (15) are valid for the model with harvesting based on constant effort if one uses $r_1 = (r - qE)$ and $K_1 = K\left(1 - \frac{qE}{r}\right)$.

The inversion of the Laplace transforms (14) and (15) returns the probability density functions of T_L and T_U. Due to the non-linearity of (14) and (15), it is not possible to obtain explicitly those densities. However, in [16], one can find an algorithm to implement numerically the Laplace transform inversion for very simple functions. After some algorithm adaptations, we have applied it to the logistic model with harvesting and without Allee effects with parameter values of $r,\ K,\ q,\ \sigma$ taken from Table 1, with $E = E^{opt}$ and considering $L = 0.90x$ and $U = 1.10x$. The output returns the p.d.f. of T_L for $L = 0.90x$

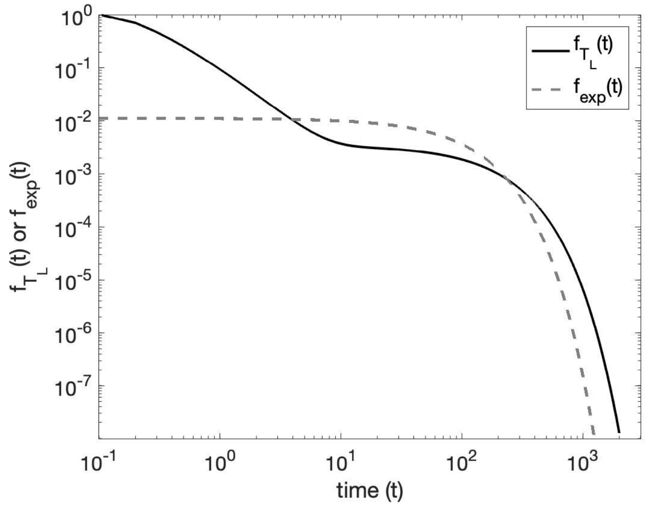
and the p.d.f. of T_U for $U = 1.10x$, whose graphics are depicted in Figure 5. In both graphics, the area under the thicker lines is approximately 1 (more specifically, 0.99), which indicates that they are a very good approximation of the p.d.f.. We also have computed the mean value of T_L for $L = 0.90x$ and of T_U for $U = 1.10x$ from the graphics and from the exact expressions (6) and (8). The differences are less than 0.1%, which reinforces the idea that the algorithm is credible and, therefore, the approximations of the p.d.f. are very accurate.

For visibility reasons, the densities are displayed on a log-log scale, as the depicted lines are too close to the axes on a natural scale. Actually, in the graphics on a natural scale, the depicted lines are too close to the axes. One can observe that the mean and standard deviation of the first passage times are very close, suggesting that the distribution of first passage times might be approximately exponential. To examine that suggestion more closely, Figure 5 also depicts the lines (dashed lines) of the p.d.f. of the exponential distributions with the same expected values as T_L (top panel) and T_U (bottom panel). Even though the top image appears to support that idea, namely that the p.d.f. of T_L is close to the p.d.f. of the exponential distribution with the same mean, the suggestion is invalid for small values of time t. Furthermore, even when the influence of the log scale is considered, the bottom graphic reveals a less than satisfactory agreement between the p.d.f. of T_U and the p.d.f. of the exponential distribution with the same mean.

5 Conclusions

In this work we have presented, for the logistic model and for the logistic-like model with weak Allee effects, the expressions for the mean and standard deviation of the first passage times by lower and by upper thresholds. For several lower and upper threshold values, we have computed the mean and standard deviation of the first passage time by such values. For both models, the results are qualitatively similar but, in the case of the logistic model, the population needs more time to reach lower thresholds and less time to reach upper thresholds. We have also seen that the mean and the standard deviation have the same order of magnitude, which might suggest first passage times distributions not far from being exponential, but we saw that this is not the case.

For the logistic model, we have set up 9 scenarios with parameter variations, namely the lower and upper thresholds, the initial population, and the effort. For these scenarios, the mean and standard deviation of the first passage time by lower and by upper thresholds were computed. The conclusions were very similar to the ones based on the Figures. The general idea is that there exists a decrease of the mean and the standard deviation values of the first passage time by the threshold when the threshold values approach the initial population value.



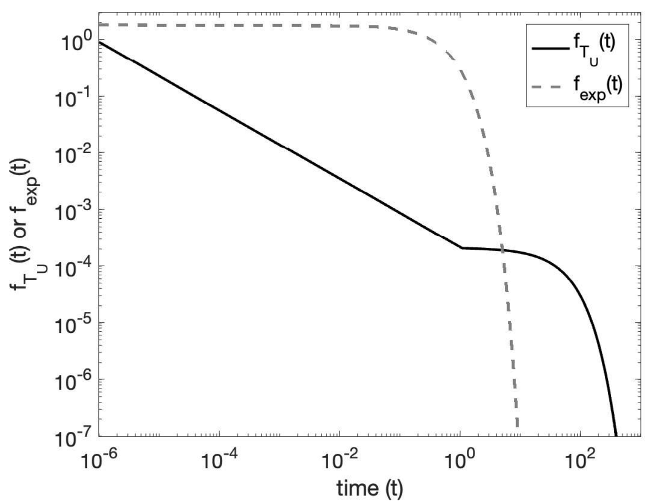


Fig. 5 P.d.f. of T_L and T_U for the logistic model when $x = 4.03 \times 10^4$ tonnes, obtained by numerical inversion of their Laplace transforms. In log-log scale, one can see on the top the p.d.f for $L = 0.90x$ and on the bottom the p.d.f. for $U = 1.10x$. The dashed lines refer to the p.d.f. of an exponential distribution with parameter $E[T_L]$ (top) and $E[T_U]$ (bottom).

For the logistic model without harvesting, we have found in the bibliography the expressions of the Laplace transform of the first passage time by the lower and by the upper thresholds. With some mild adaptations, we deduced the expressions for the logistic model with harvesting. The inversion of the Laplace transform gives the probability density functions of the first passage time by the lower and by the upper thresholds. Such densities cannot be obtained explicitly, so we resort to numerical methods to compute them. Using Matlab, we were able to plot an approximation of the probability densities functions for first passage time by the lower and by the upper thresholds.

Acknowledgements The helpful comments from anonymous Referees and from the Editor are gratefully acknowledged.
Nuno M. Brites was partially financed by Fundação para a Ciência e a Tecnologia (FCT), through national funds within the Project CEMAPRE/REM - UIDB/05069/2020.
Carlos A. Braumann is member of Centro de Investigação em Matemática e Aplicações, Universidade de Évora, a research center supported by FCT, project UID/04674/2020.

References

1. Alvarez L. H. R., Sheep L.A., Optimal harvesting of stochastically fluctuating populations. J. Math. Biol. 37, 155–177 (1998).
2. Alvarez L. H. R., On the option interpretation of rational harvesting planning. J. Math. Biol. 40, 383–405 (2000).
3. Alvarez L. H. R., Singular stochastic control in the presence of a state-dependent yield structure. Stochastic Processes and their applications. 86, 323–343 (2000).
4. Brites N.M., Braumann C.A., Harvesting policies with stepwise effort and logistic growth in a random environment. In: Ventorino E., Aguiar M.A.F., Stollenwek N., Braumann C.A., Kooi B., Pugliese A. (eds). Dynamical Systems in Biology and Natural Sciences. SEMA SIMAI Springer Series, Berlin (2020).
5. Brites N.M., Braumann C.A., Stochastic differential equations harvesting policies: Allee effects, logistic-like growth and profit optimization. Appl. Stochastic. Models Bus. Ind. 36, 825–835 (2020).
6. Brites N.M., Braumann C.A., Harvesting in a random varying environment: optimal, stepwise and sustainable policies for the Gompertz model. Statistics Opt. Inform. Comput. 7, 533–544 (2019).
7. Brites N.M., Braumann C.A., Fisheries management in randomly varying environments: Comparison of constant, variable and penalized efforts policies for the Gompertz model. Fisheries Research 216, 196–203 (2019).
8. Braumann C.A., Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance. John Wiley & Sons, Inc., New York (2019).
9. Brites N.M., Braumann C.A., Fisheries management in random environments: Comparison of harvesting policies for the logistic model. Fisheries Research 195, 238–246 (2017).
10. Brites N.M., Stochastic differential equation harvesting models: sustainable policies and profit optimization. PhD thesis, Universidade de Évora (2017).
11. Carlos C., Braumann C.A., General population growth models with Allee effects in a random environment. Ecological Complexity 30, 26–33 (2017).
12. Alvarez L. H. R., Hening A., Optimal sustainable harvesting of populations in random environments. Stochastic Processes and their Applications (2019). https://doi.org/10.1016/j.spa.2019.02.008.
13. Hening A., Tran K.Q, Harvesting and seeding of stochastic populations: analysis and numerical approximation. J. Math. Biol. 81, 65–112. (2020).
14. Giet J.S., Vallois P., Wantz-Mézières S., The logistic S.D.E.. Theory of Stochastic Processes 20(36), 28–62 (2015).
15. Dennis B., Allee effects in stochastic populations. Oikos 96(3), 389–401 (2002).
16. Valsa J., Brancik L., Approximate formulae for numerical inversion of Laplace transforms. Int. J. Numer. Model 11, 153–166 (1998).
17. Hanson F.B., Ryan D., Optimal harvesting with both population and price dynamics. Math. Biosci. 148(2), 129–146 (1998).

18. Braumann C.A., Stochastic differential equation models of fisheries in an uncertain world: extinction probabilities, optimal fishing effort, and parameter estimation. In: Capasso V., Grosso E., Paveri-Fontana S.L. (eds.) Mathematics in Biology and Medicine.; pp. 201–206. Springer, Berlin (1985).
19. Karlin S., Taylor H.M., A Second Course in Stochastic Processes. Academic Press, New York (1981).
20. Abramowitz M., Stegun I.A.: Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. National Bureau of Standards Applied Mathematics Series, volume 5. Washington DC (1964).
21. Allee W.C.: Principles of Animal Ecology, 837. Saunders Co., Philadelphia (1949).