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Assemblies as semigroups

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ABSTRACT

In this paper we give an algebraic characterization of assemblies in terms of bands of groups. We also consider substructures and homomorphisms of assemblies. We give many examples and counterexamples.

1. Introduction

The notion of assembly was introduced in [1], as a generalization of the notion of group. The main idea is that every element x of an assembly has its own "neutral" element denoted $e(x)$ which can be seen as a sort of error term, or the degree of flexibility of the element x. Some basic properties of assemblies can be found in [2–4]. In [1] it was also shown that besides groups (which are assemblies for which the function e is always constantly equal to the universal unique neutral element), the so-called *external numbers* also satisfy the assembly axioms.

In order to better understand this sort of algebraic structures and to compare them with existing structures based on semigroups, we consider a slightly modified definition in which commutativity is not required. An additional novelty lies in the last condition in Definition 2.1. Prior to this, the requirement was that for all x and y, $e(xy)$ would have to be equal to either $e(x)$ or $e(y)$ (we sometimes call this a *strong assembly*). We now assume a weaker condition which is indeed implied by the previous one: we require $e(xy) = e(x)e(y)$, for all x and y, which means that the function of local neutral elements is an homomorphism of semigroups. Thus, by changing the definition, we are proposing a more general notion that, for example, allows us to consider the not necessarily commutative assembly $\mathcal{A}(G)$ of all cosets gN where g and N range in a group G and in the lattice $n(G)$ of all normal subgroups of G, respectively (see Examples 2.2). Moreover, some expected results such as the fact that any Cartesian product of assemblies is an assembly now hold, while before this was not the case (see Examples 2.2).

In this paper we give a characterization of assemblies from a purely algebraic point of view by showing that *a semigroup is an assembly if and only if it is a band of groups and even a semilattice of groups if idempotent elements commute among themselves*. This provides perhaps a new way to approach the theory of bands of groups. For example, we can state that *an assembly is strong if and only if the set of idempotents (which are called magnitudes in the literature) is totally ordered* by the usual relation $x \leq y$ if and only if $xy = y$. We also briefly consider subassemblies, i.e. substructures which are also assemblies, and homomorphisms of assemblies, i.e. homomorphisms of semigroups which are assemblies.

For fundamental results and/or undefined notions about semigroups we refer to [5] and [6]. According to what is customary in semigroup theory we generally (but not always) use multiplicative notation (and consequently juxtaposition) for the binary operation.

2. Assemblies as bands of groups

We introduce the definition of assembly in multiplicative notation. By a semigroup we mean a nonempty set with a binary associative operation.

Definition 2.1. A nonempty semigroup $(S, \cdot)$ is called an *assembly* if the following hold

(A_1) $\forall x\, \exists e = e(x)\, (xe = ex = x \wedge \forall f\ (xf = fx = x \rightarrow ef = fe = e))$
(A_2) $\forall x\, \exists s = s(x)\, (xs = sx = e(x) \wedge e(s) = e(x))$
(A_3) $\forall x\, \forall y\, (e(xy) = e(x)e(y))$

To make explicit the functions that exist by conditions (A_1) and (A_2) we write $(S, \cdot, e, s)$ instead of $(S, \cdot)$.

The functional notation $e(x)$ and $s(x)$ used above is justified by the fact that the elements e and s are unique. Indeed,

- if e' satisfies condition (A_1), one has $e' = e'e = ee' = e$,
- if s' satisfies condition (A_2), one has $s' = s'e(s') = s'e(x) = s'xs = xs's = e(x)s = e(s)s = s$.

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So, we may write indiscriminately $e(x)$ or x^0 to denote the unique element e associated with x and $s(x)$ or x^{-1} to denote the unique element s such that $sx = xs = x^0$.

Examples 2.2.

1. *Every (possibly non-commutative) group G is an assembly with $e(x)$ constantly equal to the neutral element of G. Furthermore, the semigroup $G \cup \{0\}$, obtained by adding a zero to G (in the usual way, by postulating $0x = x0 = 0$), is also an assembly. In particular, the multiplicative semigroup of a (possibly skew) field is an assembly, while on the other hand the usual multiplicative semigroup of the integers is not.*
2. *An element e of a semigroup S is said idempotent if $e^2 = e$. A semigroup in which all elements are idempotent is called a band and is clearly an assembly. A group with more than 1 element is not a band. A commutative band is called a semilattice since it may be regarded as a lower semilattice with meet operation equal to the product. Clearly, for any set S, its powerset has two canonical semigroup structures, $(P(S), \cap)$ and $(P(S), \cup)$, which are both semilattices.*
3. *Any totally ordered set is a strong assembly with $xy = \inf\{x, y\}$, and $e(x) = x = s(x)$. In particular, $(B, \cup)$ where B is the set of all ordinals less than a given ordinal is a strong assembly.*
4. *The cartesian product of any family of assemblies is an assembly, with respect to the pointwise multiplication (the proof is a straightforward verification). However, the product of strong assemblies may be not strong as in the example $\{0, 1\} \times \{0, 1\}$.*
5. *The structures $(\mathbb{E}, +)$ and $(\mathbb{E} \setminus \mathcal{N}, \cdot)$ are strong assemblies, where $\mathbb{E}$ denotes the external set of external numbers and $\mathcal{N}$ the external set of all neutrices (see [1, Theorem 4.10]).*
6. *Let $\mathbb{F}$ be a non-archimedean ordered field. Let C be the set of all convex subgroups for addition of $\mathbb{F}$ and Q be the set of all cosets with respect to the elements of C. The set Q is called the quotient class of $\mathbb{F}$ with respect to C. In [2] it was show that Q is a strong assembly.*

By using a standard technique, let us rewrite the assembly axioms with details and proofs.

Lemma 2.3. *Condition (A_1) is equivalent to*

$$\forall x\, \exists! e = e(x)\, ((xe = ex = x) \wedge e^2 = e). \qquad (A'_1)$$

So, in particular the set $e(S) = \{e(a) \mid a \in S\}$ of all magnitudes of an assembly coincides with the set of idempotents of S, usually denoted by $E(S)$.

Proof. If (A_1) holds, $e = e(x)$ is unique by the above and $ee = e$; therefore (A'_1) holds. Conversely if (A'_1) holds and $xf = fx = x$, then we have $f = e$ by unicity in (A'_1). Hence $e^2 = ef = fe = e$ and thus (A_1) holds. □

Note that trivially, if all idempotents commute, i.e. if $\forall x, y \in E(S)\, (xy = yx)$, then $E(S)$ is a semigroup, even a semilattice, as $(xy)(xy) = x(yx)y = x(xy)y = x(xy)y = (xx)(yy) = xy$. However, this is not always the case.

Example 2.4. If $A = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)$, then $S = \{0, A, M, AM\}$ is a semigroup under the usual matrix multiplication, where $MA = 0$ and $E(S) = \{0, A, M\}$, but this set is not closed under multiplication.

Lemma 2.5. *Assume that in a semigroup S condition (A_1) holds. Then condition (A_2) is equivalent to*

$$\forall x\, \exists! s = s(x)\, (xs = sx = e(x)). \qquad (A'_2)$$

Proof. If condition (A_2) holds, then it is clear that the same s works in (A'_2) as well. Let us verify that it is unique. If s' is in the same condition w.r.t. x, then by (A_2) we have that $e(s) = e(x) = e(s')$, hence $s' = s'e(s') = s'e(x) = s'xs = e(s')s = e(s)s = s$, as desired. Conversely, from (A'_2) it follows immediately that $s(s(x)) = x$ and $e(x) = e(s)$ by the symmetric roles of x and s. □

The first two conditions in the definition of assembly may now be regarded from a different point of view by the next proposition which deals with associative *union of groups*. They are also known as *completely regular semigroups* and have been studied in many papers. For the fundamental results on the topic we refer to [5, ch. IV] and [6, ch. IV]. However, here we prefer to give a short direct proof which uses the Clifford decomposition argument to see a completely regular semigroup as a union of groups.

Proposition 2.6. *For a semigroup S the following are equivalent:*

1. *Conditions (A'_1) and (A'_2) hold;*
2. *$S = \bigcup_{e \in E(S)} S_e$, where $S_e = \{a \in S \mid e(a) = e\}$ is a group (called the Clifford component of S at e);*
3. *S is a union of (disjoint) groups.*

Proof. First of all recall that (A'_1) and (A_1) are equivalent and the same happens with (A'_2) and (A_2). If all of them hold, since $e^2 = e$ by (A'_1) we have $e = e(e)$ and then $e \in S_e$. Moreover, if $a \in S_e$ then $s(a) \in S_e$ by (A_2). Finally, if $a, b \in S_e$, then $abe = aeb = eab$ and so $ab \in S_e$. Thus each S_e is a group with neutral element e. Finally, it is clear that $a \in S_{e(a)}$ for each $a \in S$. Furthermore, all S_e are disjoint since such are distinct subgroups in any semigroup as groups have only one neutral element (compare also to (A'_1)).

Finally, it is clear that if S is a union of (disjoint) groups then (A'_1) and (A'_2) hold by considering, for each $a \in S$, the neutral element and the inverse (resp.) from the unique group in which a lies. □

If we denote by $s(x) = x^{-1}$ and $e(x) = x^0$, then we have the formulas

$$x^0 = x^{-1}x = xx^{-1}, \quad (x^{-1})^{-1} = x, \quad (xy)^{-1} = y^{-1}x^{-1}, \qquad (F)$$

which are consistent with the language of group theory and appear also in [1], but in a commutative context. Note that the formulas in (F) hold when the conditions (A_1) and (A_2) are satisfied.

Thus, it seems natural to ask if also the formula

$$(xy)^0 = x^0 y^0 \qquad (A'_3)$$

holds. In other words, we investigate if (A_1) and (A_2) imply (A_3), that is if the function $e(x)$ is an homomorphism. This is certainly *true when A is commutative*, as indeed we have $(xy)(x^0y^0) = x(x^0y)y^0 = x(yx^0)y^0 = xx^0yy^0 = xy$ and similarly $(x^0y^0)(xy) = xy$. Moreover, since here we have not used plain commutativity but just the fact that idempotents commute with all other elements, then, according to [7, Lemma 3.1], we know that the *formula (A'_3) always holds if idempotents commute.* So we are in a condition to state a consequence of the fundamental Clifford's Theorem [7, Theorem 3].

Theorem 2.7. *Semigroups which are a union of groups and in which idempotents commute are assemblies.*

Of course there exist elementary semigroups in which idempotents do not commute.

Example 2.8. Any left-zero band B, that is any semigroup in which the formula $xy = x$ holds, is an assembly in which distinct idempotents do not commute (even if they still are a semigroup).

In particular the multiplicative structure $B = \{a, b\}$, with $a^2 = ab = a \neq b = ba = b^2$ is a non-commutative band (and hence an assembly).

Let us see now a more complicated, but very natural, example of assembly. Before, let us recall that in the set $P(S)$ of all subsets of a semigroup S one can define a multiplication of $X, Y \in P(S)$ by the setwise product $XY = \{xy \mid x \in X, y \in Y\}$ and get a semigroup structure for $P(S)$, called the *power-semigroup* [6, I.7.5].

Example 2.9. The set $\mathcal{A}(G)$ of all cosets gN of all normal subgroups N of a group G is a subsemigroup of the power-semigroup $P(G)$ since $g_1N_1 = \{g_1\}N_1$ and for each $g_1, g_2 \in G$ we have

$$(g_1N_1)(g_2N_2) = (g_1g_2)N_1N_2,$$

where N_1, N_2 lie in the (semi)lattice $\mathfrak{n}(G)$ of all normal subgroups of G, which is a subsemigroup of $\mathcal{A}(G)$.

The functions $e(gN) = N$ and $s(gN) = g^{-1}N$ equip $\mathcal{A}(G)$ with a structure of assembly, as it can be easily checked, which is non-commutative if G is non-commutative. Then $E(G) = \mathfrak{n}(G)$ and any coset gN belongs only to the group G/N which is, of course, a subsemigroup of $\mathcal{A}(G)$. Thus, the semigroup $\mathcal{A}(G)$ is the union of the factor groups G/N (with their multiplicity):

$$\mathcal{A}(G) = \bigcup_{N \in \mathfrak{n}(G)} G/N$$

In such a structure idempotents do commute. Finally, when G is cyclic with order p^n, the assembly $\mathcal{A}(G)$ has order $1 + p + \cdots + p^n$ while the assembly $G \times \mathfrak{n}(G)$ has order $(n+1)p^n$.

Now a crucial example: there are unions of groups which are not assemblies.

Counterexample 2.10. *Let R be the semigroup $M(2, G, 2, P)$ of so-called Rees 2×2-matrices where $G = \{1, -1\}$ is a multiplicative group of order 2 and $P = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then R is a union of groups which is not an assembly.*

Proof. If we equip the set of 2×2 matrices over any field with characteristic $\neq 2$ with the multiplication $*$ defined by $A * B = APB$ (where juxtaposition on the right-hand side means the usual row-by-column product), which is clearly associative, then the set S consisting of the matrices: $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and their opposites $-A, -B, -C, -D$ is a semigroup (with respect to $*$) which is the union of the non-trivial groups $\{A, -A\}, \{B, -B\}, \{-C, C\}, \{D, -D\}$ with neutral elements $A, B, -C, D$ respectively. Thus, (A_1) and (A_2) hold. However (A_3) fails since, on the one hand, from $B * C = A$ one has $e(B * C) = e(A) = A = A^2$, while on the other hand $e(B) * e(C) = B * (-C) = -A$ is not even idempotent as $(-A)^2 = A^2 = A$. □

Definition 2.11. Let S be a semigroup. If there exists $\varphi : S \to B$ a semigroup homomorphism where B is a band (resp. semilattice), we say that S is a *band (resp. semilattice) of the subsemigroups* $S_e := \varphi^{-1}\{e\}$ with $e \in B$.

Note that in the above circumstances we have that

$$S = \bigcup_{e \in B} S_e$$

where S_e is a subsemigroup since by $x, y \in S_e$ it follows $xy \in S_e$ since $\varphi(xy) = \varphi(x)\varphi(y) = ee = e$.

We are now able to characterize assemblies in terms of bands of groups.

Theorem 2.12. *A semigroup is an assembly if and only if it is a band of groups.*

Proof. Let $(A, \cdot)$ be an assembly, then by Proposition 2.6 and (A_3) the map $x \mapsto e(x)$ is an homomorphism whose image is a subsemigroup which is the band $E(A)$.

Conversely, let A be a band of groups via the homomorphism $\varphi : A \to B$ an homomorphism of semigroups where B is a band. If φ is the canonical map $e(x)$ there is nothing left to show since we may apply Proposition 2.6 to get that (A_1) and (A_2) hold and note that (A_3) just means that the map e is an homomorphism.

To treat the general case, let us show that, up to an isomorphism, we can reduce to the case $\varphi = e$. Let us define $\psi : B \to E(A)$ where $\psi(b)$ is the identity element of the group $\varphi^{-1}\{b\}$. Let us show that ψ is the inverse map of the restriction $\varphi_1 : E(S) \to B$ of φ. In fact $\varphi_1(\psi(b)) = \varphi(\psi(b)) = b$ by definition of ψ. Moreover for each $\varepsilon \in E(S)$ we have that $\psi(\varphi_1(\varepsilon))$ is the identity element of the group $\varphi^{-1}(\varphi(\varepsilon))$. On the other hand, $\varepsilon \in \varphi^{-1}(\varphi(\varepsilon))$ and ε is idempotent. Thus ε is the identity element of $\varphi^{-1}(\varphi(\varepsilon))$, hence $\psi(\varphi_1(\varepsilon)) = \varepsilon$.

Thus $\psi = \varphi_1^{-1}$ is an homomorphism, as φ is by hypothesis. Then $e = \psi\varphi$ is an homomorphism as well, as wished. □

Corollary 2.13. *For a semigroup A the following are equivalent:*

1. *A is an assembly whose magnitudes commute.*
2. *A is an assembly whose magnitudes are central.*
3. *A is a semilattice of groups,*

Proof. Observe that (1) and (2) are equivalent by Clifford's Lemma [7, Lemma 3.1]. If they hold, then (3) holds since $e(x)$ is the wished homomorphism. Finally, (3) implies (1) via Theorem 2.12. □

Example 2.14. If G is a group and S a semilattice, then $S \times G$ is a possibly non-commutative semilattice of groups.

3. Subassemblies

In group theory it is possible to characterize the subgroups of a given group $(G, \cdot)$ as any nonempty subset of G which is closed under multiplication and inversion. A similar characterization of subassemblies of a given assembly holds.

Proposition 3.1. *If $(A, \cdot, s, e)$ is an assembly and B is a non-empty subset of A, then the following are equivalent.*

1. *$\forall x, y \in B \;\; x \cdot s(y) \in B$*
2. *$(B, \cdot_B, s_B, e_b)$ is an assembly, where $\cdot_B, s_B, e_b$ are the restrictions to B of $\cdot, s, e$, respectively.*

If these condition hold, we say that B is a subassembly of A. Moreover, if A_e is a Clifford component of A with $e \in B$, then $B_e = A_e \cap B$ is a Clifford component of B.

Proof. It is clear that (2) implies (1). Assuming (1), we have to prove that B is closed under the maps $\cdot, e, s$. If $b \in B$, then $e(b) = bs(b) \in B$, as wished. Moreover, $s(b) = e(s(b))s(b) = e(b)s(b) \in B$. Finally if $b_1 \in B$, then $b_1 b = b_1 s(s(b)) \in B$. Therefore (2) holds. □

To prove that a structure is a subassembly of a given assembly becomes quite simpler using the previous result. We illustrate this with some relevant examples.

Example 3.2. The following are subassemblies of $(\mathbb{E}, +)$.

1. $(\mathbb{R}, +)$, because $\mathbb{R} \subset \mathbb{E}$ and $(\mathbb{R}, +)$ is a group.
2. $B = \{x + A : x \in \mathbb{R}\}$, where A is a given neutrix. We have $A \in B$ and $B \subseteq \mathbb{E}$. If $\alpha = a + A$, $\beta = b + A \in B$ then $\alpha - \beta = (a + A) - (b + A) = (a - b) + A \in B$.
3. $(\mathcal{N}, +)$, where $\mathcal{N}$ is the class of all neutrices. Note that the class of all neutrices is nonempty because $0 \in \mathcal{N}$ and the difference of two neutrices is equal to the larger of the two.
4. $(\mathbb{E}\backslash\mathbb{R}, +)$. Clearly $\oslash \in \mathbb{E}\backslash\mathbb{R}$, hence $\mathbb{E}\backslash\mathbb{R}$ is nonempty. Let $x = a + A$, $y = b + B \in \mathbb{E}\backslash\mathbb{R}$. Then $x - y = (a - b) + \max(A, B) \in \mathbb{E}\backslash\mathbb{R}$.
5. $(A_\rho, +)$, where $\rho \in \mathbb{R}$ and $A_\rho = \{x \in \mathbb{E} : x \subseteq \bigcup_{\mathrm{st}(n)} [-\rho^n, \rho^n]\}$. Clearly $\emptyset \neq A_\rho \subseteq \mathbb{E}$. Let $x, y \in A_\rho$. Then there are standard m, n such that $x \subseteq [-\rho^m, \rho^m]$ and $y \subseteq [-\rho^n, \rho^n]$. Let $p = \max\{m, n\}$. Then $|x - y| \leq 2\max\{x, y\} \leq 2\rho^p \leq \rho^{p+1}$.
6. Let $(A, +)$ and $(B, \cdot)$ be assemblies. Let $(G, +)$ be a subassembly of $(A, +)$ and $(H, \cdot)$ be a subassembly of $(B, \cdot)$. Then $(G \times H, *)$ is a subassembly of $(A \times B, *)$.

7. If H is any subgroup of a group G, then $\mathcal{A}(H)$ is a subassembly of $\mathcal{A}(G)$.

An important difference between assemblies and groups is that subassemblies do not need to contain all a universal neutral element, allowing both $(\mathbb{E}\backslash\mathbb{R},+)$ and $(\mathbb{R},+)$ to be subassemblies of $(\mathbb{E},+)$. This fact shows that, unlike what happens with groups, *it is possible for the intersection of two subassemblies of a given assembly to be empty.* Moreover, for $(B,\cdot),(C,\cdot)$ subassemblies of an assembly $(A,\cdot)$ it may happen that $B\cup C$ is a subassembly of A and both $B\nsubseteq C$ and $C\nsubseteq B$. However, the following holds.

Proposition 3.3. *Let B,C be subassemblies of an assembly A. Then $B\cap C$ is either empty or a subassembly of A. Moreover, if A is commutative, the set $B\cdot C$ is a subassembly of A, where the product is meant to be defined in the power-semigroup $P(A)$.*

Proof. Suppose that $B\cap C$ is nonempty. Let $x,y\in B\cap C$. Then, because B and C are assemblies, $x\cdot y^{-1}\in B$ and $x\cdot y^{-1}\in C$ and then $x\cdot y^{-1}\in B\cap C$. Hence $(B\cap C,\cdot)$ is a subassembly of $(A,\cdot)$, by Proposition 3.1.

Suppose now that $x,y\in B\cdot C$. Then there are $u,v\in B$ and $r,t\in C$, such that $x=u\cdot r$ and $y=v\cdot t$. Because B and C are assemblies, $u\cdot v^{-1}\in B$ $r\cdot t^{-1}\in C$ and then $x\cdot y^{-1}=(u\cdot r)\cdot(v\cdot t)^{-1}=(u\cdot v^{-1})\cdot(r\cdot t^{-1})\in B\cdot C$, by formulas (F) and because A is commutative. Hence $(B\cdot C,\cdot)$ is a subassembly of $(A,\cdot)$, by Proposition 3.1. □

Example 3.4. By Proposition 3.3 we have that $A=\{x+N : x\in\mathbb{Z}, N\in\mathcal{N}\}$, $B=\{x+\oslash : x\in\mathbb{Q}\}$ and $C=\{x\in\mathbb{Z} : x \text{ is limited}\}$ are assemblies because $A=\mathbb{Z}+\mathcal{N}$, $B=\mathbb{Q}+\oslash$ and $C=\mathbb{Z}\cap\pounds$.

Proposition 3.5. *The subset $Z(A)$ of elements of an assembly A commuting with all elements of A is a subassembly of A that we call the center of A.*

Proof. If $z,z'\in Z(A)$ it is trivial that $zz'\in Z(A)$. Let us prove that $z^{-1}\in Z(A)$. For each $a\in A$, by formulas (F) we have $az^{-1}=((az^{-1})^{-1})^{-1}=(za^{-1})^{-1}=(a^{-1}z)^{-1}=z^{-1}a$ and we may apply Proposition 3.1. □

4. Homomorphisms

An homomorphism φ between two assemblies $(A,\cdot_A,s_A,e_A)$ and $(B,\cdot_B,$
$s_B,e_B)$ is expected to be a map $\varphi : A\to B$ which respects the 3 given operations. However, in the case under consideration, the request is so easily fulfilled that we can proceed even with a slight abuse of notation as in the next proposition.

Proposition 4.1. *Let A and B be assemblies.*
If $\varphi : A\to B$ is a semigroup homomorphism, i.e. if $\varphi(xy)=\varphi(x)\varphi(y)$, for all $x,y\in A$, then for each x in A

1. $\varphi(x^0)=\varphi(x)^0$,
2. $\varphi(x^{-1})=\varphi(x)^{-1}$.

Proof. We have $\varphi(x)\varphi(x^0)=\varphi(xx^0)=\varphi(x)=\varphi(x^0x)=\varphi(x^0)\varphi(x)$, hence by the uniqueness of $\varphi(x)^0$ we deduce $\varphi(x)^0=\varphi(x^0)$. The second part follows in a similar way since $\varphi(x)\varphi(x^{-1})=\varphi(xx^{-1})=\varphi(x^0)=\varphi(x)^0=\varphi(x^0)=\varphi(x^{-1}x)=\varphi(x^{-1})\varphi(x)$. □

Thus the homomorphic image of a magnitude is a magnitude and the homomorphic image of the inverse of a given element is the inverse of the homomorphic image of that same element. These properties generalize similar properties for group homomorphisms.

Example 4.2.
The following are assembly homomorphisms (sometimes in additive notation):

1. All group homomorphisms, because every group is an assembly.
2. The identity map $f(x)=x$ and the map $e(x)=x^0$ are assembly homomorphisms.
3. Let A be a neutrix. Then $f : (\mathbb{E},+)\to(\mathbb{E},+)$ such that $f(x)=x+A$ is an homomorphism. In fact, if $x,y\in\mathbb{E}$,
$$f(x+y)=(x+y)+A=x+y+A+A=(x+A)+(y+A)=f(x)+f(y).$$
4. The function $f : (\mathbb{E},+)\to(\mathbb{E},+)$ such that $f(x)=\omega x$ for some $\omega\simeq+\infty$ is an homomorphism. Let $x,y\in\mathbb{E}$. Then, using [1, Lemma 5.12],
$$f(x+y)=\omega(x+y)=\omega x+\omega y=f(x)+f(y).$$
5. The function $f : (\mathcal{N},+)\to(\mathcal{N},+)$, $f(x)=\oslash x$, where $\oslash$ is the external set of infinitesimal numbers. Let $x,y\in\mathcal{N}$. Using [1, Corollary 5.10],
$$f(x+y)=\oslash(x+y)=\oslash x+\oslash y=f(x)+f(y).$$
6. The function $f : (\mathcal{N},+)\to(\mathbb{E}\backslash\{0\},\cdot)$ such that $f(x)=\exp_S(x)\equiv[-e^x,e^x]$ (see [8, Def. 1.4.2]). Let $A,B\in\mathcal{N}$. Then
$$\begin{aligned}\exp_S(A+B) &= [\left(-e^A\right)e^B,\left(e^A\right)e^B]=[-e^A,e^A]e^B\\ &=[-e^A,e^A][-e^B,e^B]=\exp_S(A)\exp_S(B).\end{aligned}$$
7. If G is a group, the function
$$(g,N)\in G\times n(G)\mapsto gN\in\mathcal{A}(G)$$
is a possibly non-injective epimorphism of assemblies, where from $g_1N_1=g_2N_2$ it follows if $N_1=N_2$, by applying the function e.

Counterexample 4.3. *Obvious examples of functions which are not homomorphisms are nonlinear functions. Consider for instance the function $f : (\mathbb{E},+)\to(\mathbb{E},+)$ such that $f(x)=x^2$. In fact, if $x=-1+\oslash$ and $y=1+\oslash$ then*
$$f(x+y)=f(\oslash)=\oslash^2$$
and
$$f(x)+f(y)=(1+\oslash)^2+(-1+\oslash)^2=(1+\oslash)+(1+\oslash)=2+\oslash.$$

However there are also functions which may appear to be linear but are really not. As such one may not extend Example 4.2. 5 to the whole of $\mathbb{E}$:
$$\oslash(1-1)=0,$$
while
$$\oslash 1-\oslash 1=\oslash.$$

Proposition 4.4. *Let $\varphi : A\to B$ be an assembly homomorphism. Then $\varphi(A)$ is a subassembly of B.*

Proof. Apply Propositions 3.1 and 4.1. □

Thus, in studying assembly homomorphisms, there is not much loss of generality in assuming that these are onto. Furthermore, since – via the Clifford decomposition – any assembly A may be partitioned into disjoint groups and the homomorphic image of a group is likewise a group, one may regard any assembly homomorphism
$$A=\bigcup_{e\in E(A)}A_e\xrightarrow{\varphi}B=\bigcup_{\varepsilon\in E(B)}B_\varepsilon$$
as a disjoint union of group homomorphisms $A_e\xrightarrow{\varphi_e}B_{\varphi(e)}$. Then, for $e\in E(A)$, we define φ_e to be the eth component of φ and we have the following result.

Proposition 4.5. *An assembly homomorphism is into (resp. onto) if and only if all of its components are into (resp. onto).*

Therefore, if we denote by $\mathrm{Ker}(\varphi)$ the usual kernel of φ we have that it is the union of the kernels of its components φ_e. By the above this is a subassembly and we have:

$$\begin{aligned}\mathrm{Ker}(\varphi) &= \bigcup_{e\in E(A)} \{x \in A : \varphi(x) = \varphi(e)\} \\ &= \bigcup_{e\in E(A)} ker_{\varphi_e} = \varphi^{-1}(\varphi(e(A))) \supseteq E(A).\end{aligned}$$

Corollary 4.6. *An homomorphism of assemblies φ is injective if and only if $\mathrm{Ker}_\varphi = E(A)$, i.e. its kernel coincides with the set of idempotents of S.*

Example 4.7. If A is an assembly and its semilattice $E(A)$ of idempotents has maximum m, then there are no non-trivial homomorphisms $\varphi : A \to G$ to any group G. This holds in particular for the assembly of cosets $A = \mathcal{A}(G)$.

Proof. By Proposition 4.1, the element $\varphi(m)$ must be idempotent, but in G there is only one idempotent: its unique neutral element 1_G. Then $\varphi(a) = \varphi(a) \cdot 1_G = \varphi(a)\varphi(m) = \varphi(am) = \varphi(m) = 1_G$, for all $a \in A$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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