Convergence: what's logic got to do with it?

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Amuse-bouche

- A convergence statement (for sequences) is a Π_3-statement,

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N (|x_n - \ell| \leq \varepsilon)$$

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  and thus a realizer for it (a rate of convergence) is not guaranteed to exist.
- In fact, there exist explicit examples ("Specker sequences") of sequences of computable reals with no computable limit and thus with no computable rate of convergence.

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Metastability

$$\forall \varepsilon > 0 \, \forall f : \mathbb{N} \to \mathbb{N} \, \exists N \, \forall i, j \in [N, N + f(N)] (|x_i - x_j| \leq \varepsilon)$$

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  which is a Herbrandization of the Cauchy property of a sequence.

Proof mining

Georg Kreisel asked the following question:

What more do we know given the proof of a statement than simply knowing that the statement is true?

Proof mining

Proof mining program

Analysis of mathematical proofs with the help of proof theoretic techniques, including functional interpretations, in search of concrete new information: effective bounds, algorithms, weakening of premisses, ...

Functional interpretations

A functional interpretation is a mapping $f : T \to T'$ such that a formula A (in classical logic) is mapped to a formula $A^f \equiv \forall x \exists y\, A_f(x, y)$ such that theorems of T are mapped to theorems of T', i.e.

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Moreover, f provides a witness for the existential quantifier (term).

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Functional interpretations allow for the extraction of the (hidden) computational content (captured by t) in the proof of the theorem.

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- D. and Pinto: Applied convergence results using the BFI

Proof mining with the BFI

We use Ferreira and Oliva's Bounded Functional Interpretation (BFI) and its characteristic principles plus a new base type for elements of the space and the (universal) axioms for the Hilbert space:

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- Completely new translation of formulas.
- Unlike the Monotone interpretation, with the BFI the independence on bounded parameters is made explicit.

Majorizability

Let PA^ω be Peano Arithmetic in all finite types. Types are defined inductively as follows

Definition

- 0 is a type.
- If σ, τ are types, then $\sigma \to \tau$ is also a type.

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- The Howard-Bezem strong majorizability $\leq^*_\sigma$ is defined by:
  - $s \leq^*_0 t :\equiv s \leq_0 t$;
  - $s \leq^*_{\rho\to\sigma} t :\tilde{\equiv} \forall v\, \forall u \leq^*_\rho v\, (su \leq^*_\sigma tv \wedge tu \leq^*_\sigma tv)$.
- We say that x^σ is monotone if and only if $x \leq^*_\sigma x$.

Majorizability

Proposition

1. $\mathsf{PA}^{\omega}_{\leq^*} \vdash x \leq^*_\sigma y \to y \leq^*_\sigma y$;
2. $\mathsf{PA}^{\omega}_{\leq^*} \vdash x \leq^*_\sigma y \wedge y \leq^*_\sigma z \to x \leq^*_\sigma z$.

Theorem (Howard's majorizability theorem)

For all closed terms t^σ *of* $\mathsf{PA}^{\omega}_{\leq^*}$*, there is a closed term* s^σ *of* $\mathsf{PA}^{\omega}_{\leq^*}$ *such that* $\mathsf{PA}^{\omega}_{\leq^*} \vdash t \leq^*_\sigma s$.

Quantifiers

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Formulas that don't contain unbounded quantifiers are called bounded formulas.

Bounded functional interpretation (Ferreira and Oliva)

Assign to each formula A of $\mathrm{PA}^{\omega}_{\leq^*}$ the formulas A^f and $A_f(a; b)$ of $\mathrm{PA}^{\omega}_{\leq^*}$ such that $A^f \equiv \tilde{\forall} a \, \tilde{\exists} b \, A_f(a; b)$ according to the following clauses.

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5. $(\forall x \, A(x))^f :\equiv \tilde{\forall} e \, \tilde{\forall} a \, \tilde{\exists} b \, \forall x \leq^* e \, A_f(x, a; b)$;

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6. $(\forall x \leq^* t \, A(x))^f :\equiv \tilde{\forall} a \, \tilde{\exists} b \, \forall x \leq^* t \, A_f(x, a; b)$.

Caracteristic Principles

Definition

1. $(\mathsf{mAC}^{\omega}_{\mathrm{bd}}) \equiv \tilde{\forall} x \, \tilde{\exists} y \; A_{\mathrm{bd}}(x, y) \to \tilde{\exists} f \, \tilde{\forall} x \, \tilde{\exists} y \leq^* fx \; A_{\mathrm{bd}}(x, y)$;

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2. $(\mathsf{Coll}^{\omega}_{\mathrm{bd}}) \equiv$
$$\forall x \leq^* t \exists y\, A_{\mathrm{bd}}(x,y) \to \tilde{\exists} Y\, \forall x \leq^* t \exists y \leq^* Y\, A_{\mathrm{bd}}(x,y);$$

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3. $(\mathrm{MAJ}^{\omega}) \equiv \forall x \exists y (x \leq^{*} y)$.

Abbreviation

$\mathrm{P} := \mathrm{mAC}^{\omega}_{\mathrm{bd}} + \mathrm{Coll}^{\omega}_{\mathrm{bd}} + \mathrm{MAJ}^{\omega}$.

Soundness

Theorem (soundness theorem of f)

For all formulas A *of* $\mathrm{PA}^{\omega}_{\leq^*}$, *if*

$$\mathrm{PA}^{\omega}_{\leq^*} + \mathrm{P} \vdash A,$$

then there are closed monotone terms t *of appropriate types such that*

$$\mathrm{PA}^{\omega}_{\leq^*} \vdash \tilde{\forall} a \, \tilde{\exists} b \; \leq^* ta \, A_f(a; b).$$

Characterization

Theorem (characterization theorem of f)

For all formulas A of $\mathrm{PA}^{\omega}_{\leq^*}$, we have

$$\mathrm{PA}^{\omega}_{\leq^*} + \mathrm{P} \vdash A \leftrightarrow A^f.$$

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Proposition

Let (x_n) be a non-increasing sequence of real numbers and let $D \in \mathbb{N}$ be such that

$$\forall n \in \mathbb{N} \, (0 \leq x_n \leq D) \, .$$

Then

$$\forall k \in \mathbb{N} \, \tilde{\forall} f : \mathbb{N} \to \mathbb{N} \, \exists N \leq f^{(D(k+1))}(0) \, \forall i, j \in [N, f(N)] \left(|x_i - x_j| \leq \frac{1}{k+1} \right)$$

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Then $f^{(D(k+1))}(0)$ is a rate of metastability for (x_n), where

$$f^{(r)} := \begin{cases} f^{(0)}(n) = n \\ f^{(r+1)}(n) = f(f^{(r)}(n)) \end{cases}$$

- In the previous Proposition, if one considers functions which are not monotone, the bound becomes

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- Observe that the bound is very uniform. It depends only on k and f, but not on the sequence (x_n).
- The analysis also extends to bounded sequences which are eventually non-increasing. In this case, the bound becomes

$$\max\{M, f^{D(k+1)}(M)\},$$

where M is the order after which (x_n) is non-increasing.

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- a new base type H for objects in an abstract Hilbert space and extend the notion of majorizability in an appropriate way.
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- modulus (of convergence, of Cauchyness, of asymptotic regularity, of metastability, etc.) witnessing problematic existential quantifiers.

As long as the new constants are majorizable and the new axioms are universal the proof of the Soundness theorem can be extended to this new theory.

Browder's theorem

Theorem (Browder 1967)

Let H be an Hilbert space and $U : H \to H$ a non-expansive map. Suppose that C is a convex, closed and bounded subset of H, $0 \in C$ and that U maps C into C. For every $n \in \mathbb{N}$, let $U_n : H \to H$ the strict contraction $U_n(x) = (1 - \frac{1}{n+1})U(x)$ and let u_n the unique fixed point of U_n. Then the sequence (u_n) strongly converges for a fixed point $u \in C$ of U

A quantitative version of Browder's theorem

Theorem (Kohlenbach 2011; Ferreira, Leustean, Pinto 2019)

For all $k \in \mathbb{N}$ *and function* $f : \mathbb{N} \to \mathbb{N}$,

$$\exists n \leq \phi(k, f) \forall i, j \in [n, n + fn] \left(\|u_i - u_j\| \leq \frac{1}{2^k} \right).$$

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For f non-decreasing one obtains the following rate

$$\phi(k, f) := 2^{2g_k^{(r)}(0)+4+2d},$$

where

- d is an upper bound of the diameter of C.
- $g_k(n) := 2k + d + 5 + \lceil \log_2(2^{2n+4+2d}) + f(2^{2n+4+2d}) + 1) \rceil$.
- $r := 2^{2k+4d+9}$.

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Thank you!