Estimation of the Weibull Tail Coefficient Through the Power Mean-of-Order-p

Frederico Caeiro, M. Ivette Gomes, and Lígia Henriques-Rodrigues

Abstract The *Weibull tail coefficient* (WTC) is the parameter θ in a right-tail function of the type $\overline{F} := 1 - F$, such that $H := -\ln \overline{F}$ is a *regularly varying* function at infinity with an index of regular variation equal to $\theta \in \mathbb{R}^+$. In a context of extreme value theory for maxima, it is possible to prove that we have an *extreme value index* (EVI) $\xi = 0$, but usually a very slow rate of convergence. Most of the recent WTC-estimators are proportional to the class of Hill EVI-estimators, the average of the log-excesses associated with the k upper order statistics, $1 \leq k < n$. The interesting performance of EVI-estimators based on generalized means leads us to base the WTC-estimation on the power *mean-of-order-p* (MO_p) EVI-estimators. Consistency of the WTC-estimators is discussed and their performance, for finite samples, is illustrated through a small-scale Monte Carlo simulation study.

Keywords Power mean-of-order-p · Semi-parametric estimation · Statistics of extremes · Weibull tail coefficient

F. Caeiro (✉)
NOVA School of Science and Technology (FCT NOVA) and CMA, NOVA University of Lisbon, Campus de Caparica, Lisbon, Portugal
e-mail: fac@fct.unl.pt

M. I. Gomes
Faculty of Science of Lisbon (FCUL/DEIO) and CEAUL, University of Lisbon, Cidade Universitária, Campo Grande, Lisbon, Portugal
e-mail: ivette.gomes@fc.ul.pt

L. Henriques-Rodrigues
School of Science and Technology (ECT-UE) and CIMA, University of Évora, Évora, Portugal
e-mail: ligiahr@uevora.pt

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1 A Brief Introduction

Extreme value theory (EVT) and *statistics of extremes* help us to control potentially disastrous events, of high relevance to society and with a high social impact. Domains of application of EVT are quite diverse. We mention *biostatistics, finance, insurance, structural engineering* and also *environment, hydrology, meteorology* and *seismology*. Earthquakes, fires, floods and other extreme events have provided impetus for several recent re-developments of *extreme value analysis* (EVA), of *statistics of univariate extremes* (SUE) and also multivariate and spatial extremes.

By the late seventies, it was common to work in the field of parametric statistics of extremes, essentially through the use of the limiting models for extremes. The developments of the asymptotic EVT led researchers to work under semi-parametric and non-parametric frameworks. Nowadays, with the increase in computational resources, the parametric modelling gained a new dynamism with the use of Bayesian and spatial techniques.

Apart from the estimation of the *extreme value index* (EVI), one of the primary parameters in EVA, the reliable estimation of other important parameters of rare events, like the *Weibull tail coefficient* (WTC), the shape parameter in a Weibull-type right-tail, will be among the topics to be addressed. Among a large variety of Weibull-type right-tails, we mention the Exponential, the Gamma, the Logistic and the Normal tails. They thus form an important and large subgroup of light and exponential right-tailed distributions of a Gumbel type, being of high interest in hydrology, meteorology, environmental and actuarial science, among other areas of application. As mentioned above, we shall emphasize the use of generalized means (GMs) in the WTC-estimation.

2 A Brief Motivation for the Need of EVT

To motivate the interest for this area, and despite the great variety of disasters that have happened recently, we merely mention the historical floods in the North Sea, on February 1, 1953. According to Encyclopaedia Britannica [1], this was the worst storm recorded in the North Sea with extensive floodings in several North sea countries that caused 2551 deaths and vast destruction.

As a way of preventing future floods, the Dutch government created the Delta project, to determine the height of the dikes and dams so that the probability of flooding in a future year would be extremely small [1]. And EVT was used as a tool to reliably answer this question.

When dealing with extreme or rare events, we are interested in working with maximum or minimum values and we want to characterize the tails' behaviour. For this, we need to use asymptotic methods, being necessary to make a compromise since there are generally not many observations in the tails of the distributions and extrapolation upwards or downwards of the observed sample is required.

EVT is a Statistics' branch that provides the probabilistic tools to fully characterize and understand extreme and rare events. Even when dealing with 'big data', the tails are scarce, and just as mentioned above it is often required an estimation beyond the sample extremes. The answer to the question, '*Is there a hidden pattern underlying this type of extreme events?*', is positive, being next partially and briefly provided.

3 A Brief Touch on Asymptotical EVT

Some of the key tools that have led to the way statistical EVT has been exploding in these last decades are the following ones: 1 – The key result obtained by Fréchet [2], on the functional equation of stability for maxima, which led him to the now rightly called Fréchet law; 2 – Such a functional equation was later solved, still with some restrictions, by Fisher and Tippett [3], who derived the possible non-degenerate limiting laws of the linearly normalized sample maxima,

$$\frac{X_{n:n} - b_n}{a_n}, \quad a_n > 0,\ b_n \in \mathbb{R}, \quad X_{n:n} := \max(X_1, \ldots; X_n), \tag{1}$$

associated with an *independent and identically distributed* (IID) random sample, $\underline{X}_n := (X_1, \ldots, X_n)$ from a *cumulative distribution function* (CDF) F. They then arrived at the *extreme value* (EV) CDFs,

$$\text{Type I}: \quad \Lambda(x) = e^{-e^{-x}},\ x \in \mathbb{R} \quad [Gumbel], \tag{2}$$

$$\text{Type II}: \quad \Phi_\alpha(x) = e^{-x^{-\alpha}},\ x > 0,\ \alpha > 0 \quad [Fréchet], \tag{3}$$

$$\text{Type III}: \quad \Psi_\alpha(x) = e^{-(-x)^\alpha},\ x < 0,\ \alpha > 0 \quad [Max - Weibull]; \tag{4}$$

3 – Such a result was initially formalized by Gnedenko [4], used by Gumbel [5], for applications of EVT in engineering and hydrology, and finally formalized by de Haan [6].

SUE is thus mainly based on the aforementioned EV models, also called *max-stable* laws, related to the non-degenerate limiting behaviour of the sequence of linearly normalized maximum values, as in (1). SUE deals thus essentially with the above-mentioned EV CDFs, in (2), (3) and (4), which can be encompassed in the *general extreme value* (GEV) CDF,

$$G_\xi(x) \equiv \text{GEV}_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-1/\xi}},\ 1+\xi x \geq 0, & \text{if } \xi \neq 0, \\ e^{-e^{-x}},\ x \in \mathbb{R}, & \text{if } \xi = 0, \end{cases} \tag{5}$$

with ξ the so-called EVI, the primary parameter in SUE. But SUE is also based on asymptotic results related to the non-degenerate limiting behaviour of a set of upper *order statistics* (OSs), either individually or jointly (Weissman [7, 8]; Pickands [9]; Gomes [10–12]; Smith [13]), or of excesses over high thresholds (Davison [14];

Smith [15]; Davison and Smith [16]), linked to *generalized Pareto* CDFs ($\mathrm{GP}_\xi(\cdot) = 1 + \ln\ \mathrm{GEV}_\xi(\cdot)$). And the fact that $\min(X_1, \ldots, X_n) = -\max(-X_1, \ldots, -X_n)$ enables the derivation of analogous results for minima and lower OSs.

The aforementioned main theoretical result on the non-degenerate limiting behaviour of the linearly normalized maximum in (1) is commonly known as the Fisher–Tippett–Gnedenko's theorem, also called *extremal types theorem* (ETT), and has a role similar to the *central limit theorem* (CLT) for averages (or sums). The CDF F is then said to belong to the *max-domain of attraction* (MDA) of GEV_ξ, and we write $F \in \mathcal{D_M}(\mathrm{GEV}_\xi)$. The EVI measures the heaviness of the *right-tail function* (RTF), $\overline{F}(x) := 1 - F(x)$. The heavier the right-tail, the larger ξ is.

Statistical applications of EVT have given emphasis to the relaxation of the independence condition and homoscedasticity, to the consideration of multidimensional and spatial frameworks and from a theoretical point of view, to a deeper and deeper use of regular variation and point processes.

4 Semi-parametric Estimation in SUE

The crucial parameter in SUE is the already defined EVI, denoted by ξ ($\in \mathbb{R}$). For dependent samples, we also have the extremal index, related to the mean size of clusters of extreme events. Under a semi-parametric framework, there is no fitting of an adequate parametric model. It is only assumed that $F \in \mathcal{D_M}(\mathrm{GEV}_\xi)$, with $\mathrm{GEV}_\xi(\cdot)$ given in (5), ξ being the unique primary parameter of extreme events to be initially estimated, on the basis of a few upper observations, and according to an adequate methodology.

It is then common to consider the k upper observations above the random threshold $X_{n-k:n}$, i.e. $X_{n:n} \geq \cdots \geq X_{n-k+1:n}$. Such a threshold needs to be an upper intermediate OS, i.e.

$$k = k_n \to \infty, \quad k \in [1, n), \qquad k = o(n) \qquad \text{as } n \to \infty. \tag{6}$$

Let $F^{\leftarrow}$ denote the generalized inverse function associated with the underlying CDF, F. Let U be the associated *reciprocal tail quantile function*:

$$U(t) := F^{\leftarrow}(1 - 1/t), \quad t \in [1, \infty]. \tag{7}$$

The model F is commonly said to have a heavy right-tail if and only if there exists a positive real ξ such that

$$\overline{F} = 1 - F \in RV_{-1/\xi} \quad \text{if and only if} \quad U \in RV_\xi, \tag{8}$$

with $U(\cdot)$ defined in (7) and where the notation RV_β stands for the class of regularly varying functions at infinity with an index of regular variation equal to β, i.e. positive measurable functions $g(\cdot)$ such that $\lim_{t\to\infty} g(tx)/g(t) = x^\beta$, for all $x > 0$.

Since risks are more dangerous when we deal with a heavy RTF, we often consider heavy-tailed models, i.e. Pareto-type underlying CDFs, with a positive EVI, working thus in

$$\mathcal{D_M}^+ := \mathcal{D_M}\left(\mathrm{GEV}_{\xi>0}\right), \tag{9}$$

or equivalently, models F such that (8) holds.

4.1 A Class of GM EVI-Estimators

Among the large variety of EVI-estimators, we mention the Hill (H) estimators [17]. The H EVI-estimators are the average of the log-excesses, $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \leq i \leq k < n$, i.e.

$$\mathrm{H}_{k,n} \equiv \mathrm{H}(k) \equiv \mathrm{H}(k; \underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \qquad 1 \leq k < n. \tag{10}$$

We further mention one of the competitive generalizations of $\mathrm{H}(k)$, recently introduced in the literature, and based on a simple GM.

First, note that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k} \frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}.$$

The H EVI-estimator in (10) is thus the logarithm of the *geometric mean* (or *power mean-of-order*-0) of

$$U_{ik} := \frac{X_{n-i+1:n}}{X_{n-k:n}}, \quad 1 \leq i \leq k < n. \tag{11}$$

Almost simultaneously, Brilhante et al. [18], Paulauskas and Vaičiulis [19] and Beran et al. [20] (see also [21]) considered as basic statistics, the power mean-of-order-p (MO_p) of U_{ik}, $1 \leq i \leq k$, in (11), for $p \geq 0$. More generally, Caeiro et al. [22] considered the same statistics for any $p \in \mathbb{R}$, i.e.

$$\mathrm{M}_p(k) = \begin{cases} \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p\right)^{1/p}, & \text{if } p \neq 0, \\ \left(\prod_{i=1}^{k} U_{ik}\right)^{1/k}, & \text{if } p = 0, \end{cases}$$

and the associated class of MO_p EVI-estimators:

$$\mathrm{H}_{k,n,p} \equiv \mathrm{H}_p(k) = \mathrm{H}_p(k; \underline{X}_n) := \begin{cases} \left(1 - \mathrm{M}_p^{-p}(k)\right)/p, & \text{if } p < 1/\xi,\ p \neq 0, \\ \ln \mathrm{M}_0(k) = \mathrm{H}(k), & \text{if } p = 0. \end{cases} \tag{12}$$

The use of the extra tuning parameter $p \in \mathbb{R}$ and the MO_p methodology can thus provide a much more adequate EVI-estimation. Asymptotic normality is achieved for $p \leq 1/(2\xi)$. But, on the basis of Gomes et al. [23] (see also [24]), we can now go up to $p = 1/\xi$, getting then a sum-stable behaviour, with an index of sum-stability $\alpha = 1/(p\xi)$. And for $p = 1/\xi$, we get, for $\mathrm{H}_p(k) - \xi$, a deterministic dominant component, of the order of $1/\ln k$.

4.2 Semi-parametric Estimation of the WTC

The WTC is the parameter θ in an RTF of the type

$$\overline{F}(x) = 1 - F(x) =: e^{-H(x)}, \quad H \in RV_{1/\theta},\ \theta \in \mathbb{R}^+. \tag{13}$$

Equivalently to (13), we can say that

$$U(e^t) = H^{\leftarrow}(t) \in RV_\theta \quad \Longleftrightarrow \quad U(t) =: (\ln t)^\theta L(\ln t), \tag{14}$$

with $L \in RV_0$, a slowly varying function.

In a context of EVT for maxima, it is possible to prove that we have an EVI $\xi = 0$, but usually a very slow rate of convergence. We are working with those tails, like the Normal RTF, in the MDA of Gumbel's law $\Lambda(\cdot)$, in (2), which can exhibit a penultimate (or pre-asymptotic) behaviour, a concept introduced in the aforementioned seminal paper by Fisher and Tippett, [3]. Such RTFs, despite double-exponential, look more similar either to

- Max-Weibull, $\Psi_\alpha(x) = \exp(-(-x)^\alpha), x < 0\ (\xi = -1/\alpha < 0)$
- or to Fréchet, $\Phi_\alpha(x) = \exp(-x^{-\alpha}), x > 0\ (\xi = 1/\alpha > 0)$

right-tails, according to $\theta < 1$ or $\theta > 1$, respectively. Details on penultimate behaviour can be found in Gomes [10, 25] and Gomes and de Haan [26], among others.

Here, we merely mention the most relevant WTC-estimators in Gardes and Girard [27], which are given by

$$\widehat{\theta}^{\mathrm{H}}_{k,n} := \frac{\ln(n/k)}{k} \sum_{i=1}^{k} V_{ik} = \ln(n/k)\mathrm{H}_{k,n}, \tag{15}$$

with $\mathrm{H}_{k,n}$ the already defined H EVI-estimators, in (10). More generally than $\widehat{\theta}^{\mathrm{H}}_{k,n}$, we now suggest the consideration of MO_p WTC-estimators, based on the aforementioned GM EVI-estimators, in (12), i.e.

$$\widehat{\theta}^{\mathrm{MO}_p}_{k,n} := \ln(n/k)\mathrm{H}_{k,n,p}. \tag{16}$$

And recently, Lehmer's mean-of-order-p EVI-estimators (Penalva et al. [28–30]) have revealed even a higher efficiency, but have not yet been considered for the WTC-estimation.

4.3 Consistency of the WTC-Estimators

To achieve the consistency of the new class of WTC-estimators, we just need to consider $p \neq 0$, in (16), since the case $p = 0$ that corresponds to the class $\widehat{\theta}^{\mathrm{H}}_{k,n}$, in (15), was already studied in Gardes and Girard [27]. We start by observing that, for any $p \neq 0$, and with $U(\cdot)$ defined in (7),

$$\left(\frac{U(tx)}{U(t)}\right)^p = \left(1 + \frac{\ln x}{\ln t}\right)^{p\theta} \left(\frac{L(\ln t + \ln x)}{L(\ln t)}\right)^p.$$

Since $L(\cdot)$, defined in (14), is in RV_0, and applying a first-order Taylor expansion to the first term, we can write

$$\left(\frac{U(tx)}{U(t)}\right)^p \sim 1 + p\,\theta \frac{\ln x}{\ln t}.$$

Let $Y_{1:n}, Y_{2:n}, \ldots, Y_{n:n}$ denote the OSs associated with a random sample of n independent standard Pareto random variables with CDF $F_Y(y) = 1 - 1/y, y \geq 1$. Then $X_{i:n} \stackrel{d}{=} U(Y_{i:n}), 1 \leq i \leq n$ and $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$. In this case, the following distributional representation holds, with U_{ik} defined in (11),

$$\begin{aligned} U_{ik}^p &\stackrel{d}{=} \left(\frac{U(Y_{n-i+1:n})}{U(Y_{n-k:n})}\right)^p \\ &\stackrel{d}{=} \left(\frac{U(Y_{n-k:n}Y_{k-i+1:k})}{U(Y_{n-k:n})}\right)^p \sim 1 + \frac{p\,\theta \ln Y_{k-i+1:k}}{\ln(n/k)}. \end{aligned}$$

Since $E_i = \ln Y_i$ are IID exponentially random variables with mean value 1 and $E_{n-k:n} \sim \ln(n/k) \to \infty$, for intermediate sequences of OSs satisfying (6), we then get

$$\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p \stackrel{d}{=} 1 + \frac{p\,\theta}{\ln(n/k)}(1 + o_{\mathbb{P}}(1)), \quad p \neq 0,$$

with $o_p(1)$ uniform in i, $1 \le i \le k$ (see [22]). From (12) and (16), the consistency of the MO_p WTC-estimators in (16) follows, in the whole $\mathcal{D}_{\mathcal{M}}{}^{+}$, in (9), provided that (6) holds.

5 Finite Sample Behaviour with Simulated Data

In this section, we evaluate the finite sample performance of the class of estimators $\widehat{\theta}_{k,n}^{MO_p}$, in (16), through a Monte Carlo simulation study. The values for the tuning parameter p were selected from a preliminary simulation study. The value $p = 0$ was always used, since it provides the estimator in (15). The value $p = 1$ was also used as an example of a positive tuning parameter. We have considered the following typical distributions within the class of Weibull-type models: the Gamma distribution with a shape parameter equal to 0.75 ($\theta = 1$) and the Half-Normal model ($\theta = 0.5$). In Figs. 1 and 2, we present, at the left, the simulated mean value and, at the right, the corresponding simulated *root mean squared error* (RMSE), as a function of k, provided by the aforementioned class of WTC-estimators and 20,000 samples of size $n = 1000$. The horizontal solid line, at the left plot, indicates the true WTC value. Similar patterns were obtained for other simulated models and sample sizes.

In Table 1, we present the simulated values of the RMSE at the simulated optimal level, for samples of sizes 100, 200, 500, 1000, 2000 and 5000. For each model and sample size, the smallest RMSE is written in **bold**. The smallest RMSE is always achieved by $\widehat{\theta}_{k,n}^{MO_p} := \ln(n/k)\mathrm{H}_{k,n,p}$, in (16), with $p < 0$. Moreover, the optimal p decreases, as the sample size n increases. For large sample sizes, the choices -3 and -1.5 seem to provide an overall good performance for the Gamma and Half-Normal models, respectively.

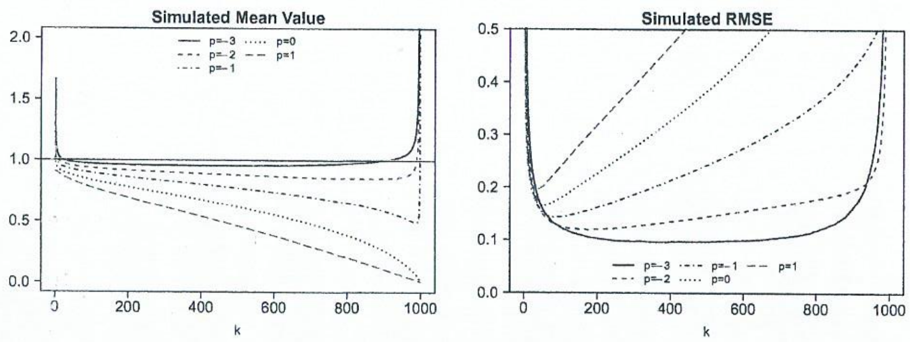

Fig. 1 Simulated Mean values (left) and RMSEs (right) of the WTC-estimators under study from samples of size $n = 1000$ from a Gamma(0.75, 1) parent ($\theta = 1$)

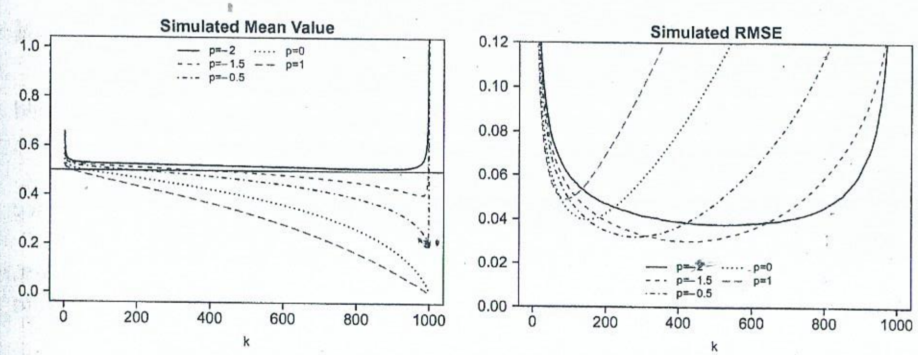

Fig. 2 Simulated Mean values (left) and RMSEs (right) of the WTC-estimators under study from samples of size $n = 1000$ from a Half-Normal parent ($\theta = 0.5$)

Table 1 Simulated RMSE at the simulated optimal level

Sample size:	100	200	500	1000	2000	5000
	Gamma(0.75, 1)					
$p = -3$	0.2808	0.1868	**0.1206**	**0.0942**	**0.0781**	**0.0653**
$p = -2$	0.2311	**0.1768**	0.1369	0.1173	0.1023	0.0867
$p = -1$	**0.2302**	0.1948	0.1619	0.1413	0.1248	0.1068
$p = 0$	0.2547	0.2242	0.1880	0.1651	0.1478	0.1273
$p = 1$	0.2910	0.2573	0.2180	0.1936	0.1738	0.1494
	Half-Normal					
$p = -2$	0.1191	0.0814	0.0512	0.0377	0.0280	0.0195
$p = -1.5$	0.0985	0.0678	**0.0419**	**0.0300**	**0.0215**	**0.0137**
$p = -1$	0.0878	**0.0637**	0.0430	0.0320	0.0237	0.0157
$p = 0$	**0.0873**	0.0694	0.0507	0.0398	0.0311	0.0220
$p = 1$	0.0961	0.0792	0.0605	0.0489	0.0396	0.0295

A few general comments:

- For all simulated parents, we could always find a value of p (negative, contrary to what happens with the MO_p EVI-estimation), such that, for adequate k-values, there is a reduction in RMSE, as well as in bias, and for such a value of p, the MO_p often strongly beats the $\mathrm{H} \equiv MO_0$ WTC-estimators.
- Algorithmic details on the choice of tuning parameters under play are still under progress, but can be easily devised, similar to what has been done for an EVI-estimation in Caeiro and Gomes [31] and Gomes et al. [32], where R-scripts are provided.

6 Overall Conclusions

- Risk analyses related to extreme events are challenging and require the combined expertise of statisticians and domain experts in climatology, hydrology, finance, insurance, medicine, sports and other fields.
- In our opinion, even SUE is still a quite lively topic of research, of high relevance in risk modelling.
- Important developments have appeared recently in the area of *spatial extremes*, where *parametric models*, both asymptotic and pre-asymptotic, became again quite relevant.
- And in a semi-parametric framework, topics like *threshold selection* and the PORT methodology, with PORT standing for *peaks over random thresholds*, a terminology coined in Araújo Santos et al. [33], are still quite challenging.
- The lack of efficiency of the MO_p WTC-estimators for $p > 0$, and of the MO_p EVI-estimators for $p < 0$, together with the results in Stehlík et al. [34], related to the robustness of the MO_{-1} EVI-estimators, deserves a further discussion of the topic 'robustness versus efficiency'.
- Related statistical research with critical risk assessment applications can be found in several books, like Embrechts et al. [35], Beirlant et al. [36], Gomes et al. [37] and Dey and Yan [38], among others. For recent overviews on SUE and its possible application in risk modelling, see Davison and Huser [39] and Gomes and Guillou [40].
- We have here considered the univariate case only, but EVT is of high relevance both in the multivariate and in the spatial setup, whenever dealing with the modelling of extreme events or equivalently the modelling of risk.
- A comparative study with other WTC-estimators, like the ones in Diebolt et al. [41], Gardes and Girard [42, 43], Goegebeur et al. [44], Gong and Ling [45] and Kpanzou et al. [46], among others, is expected to be developed in the near future.
- In a way similar to what has been done in Worms and Worms [47], the new estimator can be developed for censored data.
- Also corresponding estimators of extreme quantiles can be developed either for complete or censored (mild/heavy) settings.

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