A Directional Curvature Formula for Convex Bodies in $\mathbb{R}^n$ *

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Abstract

Given a compact convex set $F \subset \mathbb{R}^n$, with the origin in its interior, and a point on its boundary, near which it is given by an implicit equation, we present a formula to compute the curvature in the direction of any tangent vector. For this we consider the intersection curve between the boundary of F and a suitable plane, but without using the plane equations or the curve expression. Furthermore we see that, when we use the equations of the plane and the equation that define the boundary of F near the fixed point, the formula that we obtain is equivalent to the existing ones, but it is easier to use.

Key words: *convex set; curvature; implicit function theorem; tangent vector.*
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1 Introduction

In [9] the authors proposed some concepts concerning the geometric structure of a closed convex bounded set F, with zero in its interior, in a Hilbert space H. Inspired essentially from the geometry of Banach spaces (see [13]), they introduced three moduli of local rotundity for the set F, one symmetrical (using the norm of H) and two asymmetric (using the "asymmetric norm" given by the Minkowski functional of F). Using the symmetrical modulus the authors defined the concept of strict convexity graduated by some parameter $\alpha > 0$. The main numerical characteristic resulting from these considerations is the curvature (and the respective curvature radius) of F, which shows how rotund the set F is near a fixed boundary point ξ watching along a given direction ξ^*. Considering the polar set of F, F^o, they defined also the modulus of local smoothness and the local smoothness of F^o. As well-known (see, for example, [12, 13, 15, 16]) the strict convexity of a convex closed bounded set F with zero in its interior is strongly related to the smoothness of F^o, but in [9] that relation was quantified. In particular, a local asymmetric version of the Lindenstrauss duality theorem [9, Proposition 4.2] was proved there, which

*This paper is dedicated to the memory of dear colleague Vladimir V. Goncharov (University of Évora, Portugal), who passed away when this work was just beginning, we only had an idea of how to prove the main result in $\mathbb{R}^3$.

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quantitatively establishes the duality between local smoothness and local rotundity. Thus, the curvature of F can be considered also as a numerical characteristic of F^o, showing how sleek F^o is in a neighbourhood of a boundary point ξ^* if you look along a direction ξ. Applying this theorem, it was obtained a characterization of the curvature of F in terms of the second derivative of its dual Minkowski functional [9, Proposition 4.4]. From what we have just said, and not only (for more results see [9, 10]), the formula for curvature is, from a theoretical point of view, very useful, but in practice it is very difficult to use even in $\mathbb{R}^2$ as we can see in [9, Example 8.4]. Then, in this paper, we propose, in some sense and for some kind of convex bodies F (compact convex sets with interior points) in $\mathbb{R}^n$, $n \geq 2$, an equivalent formula to compute its curvature but easier to use. Namely, in Theorem 1 below, given ξ on the boundary of F, ∂F, near which it is given by an implicit equation, we present a formula for the curvature of F at ξ in the direction of any tangent vector. For this, and for a fixed tangent vector, we will consider the intersection curve between ∂F and a suitable plane, but without using the plane equations or the curve expression. In a few words, we can say that [9] gives us an approximate idea of the shape of F in a global neighbourhood of ξ, while in this paper the exact shape of F near ξ in each tangent direction is obtained.

Before moving on to the work itself, let us review more precisely what is already done in this area.

A definition for curvature similar to the formula that will be obtained here, and called directional curvature, appears in [1] for a (not necessarily convex) C^2-manifold embedded in a Hilbert space. In [5, p.14] (see also [2, 14]), for a convex body F in $\mathbb{R}^n$ ($n \geq 2$), a smooth point ξ in ∂F (smooth means that at ξ there exists only one supporting hyperplane to F), an interior unit normal vector ξ^* of F at ξ, and an unit vector ξ^{**} orthogonal to ξ^*, H. Busemann considered the 2-dimensional halfplane

$$H\left(\xi, \xi^*, \xi^{**}\right) = \left\{\eta \in \mathbb{R}^n : \eta = \xi + \lambda \xi^* + \mu \xi^{**} \text{ with } \lambda, \mu \in \mathbb{R} \text{ and } \mu \geq 0\right\}, \tag{1}$$

which intersects ∂F in a plane convex curve. Denoting by r_η, for $\eta \in H\left(\xi, \xi^*, \xi^{**}\right) \cap \partial F$ near ξ, the radius of the circle with centre on the normal line $\xi + \mathbb{R}^+ \xi^*$ containing both ξ and η, the author defined

$$\rho_l^{\xi^{**}}(\xi) := \liminf_{\eta \to \xi} r_\eta, \quad \rho_u^{\xi^{**}}(\xi) := \limsup_{\eta \to \xi} r_\eta$$

as the lower and upper curvature radius, respectivelly. If the numbers $\gamma_l^{\xi^{**}}(\xi) := \left(\rho_l^{\xi^{**}}(\xi)\right)^{-1}$ and $\gamma_u^{\xi^{**}}(\xi) := \left(\rho_u^{\xi^{**}}(\xi)\right)^{-1}$ (called lower and upper curvature, respectivelly) are equal and finite, he says that the curvature of F at ξ in direction ξ^{**} exists and is equal to the common value.

Differential geometry of intersection curves of two (or more) surfaces in $\mathbb{R}^3$ (or higher dimension) were studied by many authors (see, for example, [3, 8, 17] and the bibliography therein). There are studies in which all the surfaces are defined implicitly, others in which all are parametrically defined, and others in which there are surfaces of both types. For this work, we are only interested in those defined implicitly. At [3, 8, 17] the authors present formulas (or algorithms) for computing differential geometric properties (such as tangent vector, normal vector, curvatures and torsion) of the intersection curve. In [8, (5.4)] the author derives a formula for the curvature of the curve defined by the intersection of $n-1$ implicit surfaces in $\mathbb{R}^n$. This formula is laborious to apply when the space has dimension $n \geq 4$, because we need to do several operations with the gradients of all functions that implicity define the surfaces. But, when the curve is obtained by the intersection between the surface of a convex body (given locally by an implicit equation) and a 2-dimensional plane (as defined in (2)) that formula can be rewritten in a very simple way, as we will see in Section 6.

This paper is organized as follows. In Section 2 we introduce some notations, definitions and one example where we can see that the definition of curvature presented in [9] can give us only an approximate value. In Section 3, we present the conditions on F, the normal cone to F at a convenient $\xi \in \partial F$, the tangent hyperplane of F at ξ, the definition of directional curvature, some its properties and its relation to the definition of [9]. Section 4 is dedicated to the main result of this paper and its proof. In Section 5 we relate the directional curvature of F with the radius of a suitable sphere. The relationship between our formula and Goldman's, when applied to our case, is proved in Section 6. Finally, the Section 7 is dedicated to the examples.

2 Basic notations and definitions

We will consider in the space $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, with the usual inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, a compact convex set F with the vector null of $\mathbb{R}^n$ (represented by $\mathbf{0}$) in its interior $\text{int}F$. We denote by F^o the *polar set* of F, i.e.,

$$F^o := \{\xi^* \in \mathbb{R}^n : \langle \xi, \xi^* \rangle \leq 1 \;\; \forall \xi \in F\}.$$

Together with the *Minkowski functional* $\rho_F(\cdot)$ defined by

$$\rho_F(\xi) := \inf\{\lambda > 0 : \xi \in \lambda F\}$$

we introduce the *support function* $\sigma_F : \mathbb{R}^n \to \mathbb{R}^+$,

$$\sigma_F(\xi^*) := \sup\{\langle \xi, \xi^* \rangle : \xi \in F\}.$$

Observe that

$$\rho_F(\xi) = \sigma_{F^o}(\xi),$$

and, consequently,

$$\frac{1}{\|F\|}\|\xi\| \leq \rho_F(\xi) \leq \|F^o\|\,\|\xi\|, \quad \xi \in \mathbb{R}^n, \tag{2}$$

where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$. The inequalities (2) mean that $\rho_F(\cdot)$ is a sublinear functional "equivalent" to the norm $\|\cdot\|$. It is not a norm since $-F \neq F$ in general. As usual, we represent by ∂F the *boundary* of F. In what follows we will use the so-called *duality mapping* $\mathfrak{J}_F : \partial F^o \to \partial F$ that associates the set

$$\mathfrak{J}_F(\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}$$

with each $\xi^* \in \partial F^o$. We say that (ξ, ξ^*) is a *dual pair* when $\xi^* \in \partial F^o$ and $\xi \in \mathfrak{J}_F(\xi^*)$. The *normal cone* to F at ξ, in the sense of Convex Analysis, is given by

$$\mathbf{N}_F(\xi) := \{\zeta^* \in \mathbb{R}^n : \langle \eta - \xi, \zeta^* \rangle \leq 0 \text{ for every } \eta \in F\},$$

and the *proximal normal cone* to F at ξ is

$$\mathbf{N}_F^P(\xi) := \left\{\zeta^* \in \mathbb{R}^n : \text{there exists } \sigma \geq 0 \text{ such that } \langle \eta - \xi, \zeta^* \rangle \leq \sigma \|\eta - \xi\|^2 \text{ for every } \eta \in F\right\}.$$

Since F is closed and convex we have (see [6, Proposition 1.1.10])

$$\mathbf{N}_F^P(\xi) = \mathbf{N}_F(\xi). \tag{3}$$

It is easy to show that $\mathbf{N}_F(\xi) \cap \partial F^o$ is the pre-image of the mapping $\mathfrak{J}_F(\xi)$, $\mathfrak{J}_F^{-1}(\cdot)$, calculated at ξ. The *tangent cone* to F at ξ is the polar of $\mathbf{N}_F(\xi)$, since $\mathbf{N}_F(\xi)$ is, in fact, a cone, it is given by

$$\{u \in \mathbb{R}^n : \langle u, \zeta^* \rangle \leq 0 \text{ for every } \zeta^* \in \mathbf{N}_F(\xi)\}.$$

We will only work with the *hyperplane tangent* to the set F at the point ξ:

$$\mathbf{T}_F(\xi) := \{u \in \mathbb{R}^n : \langle u, \zeta^* \rangle = 0 \text{ for every } \zeta^* \in \mathbf{N}_F(\xi)\}. \tag{4}$$

Following [9, Definition 3.2], for each dual pair (ξ, ξ^*) the *modulus of strict convexity of* F at ξ with respect to (w.r.t.) ξ^* is

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) := \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \|\xi - \eta\| \geq r\}, \quad r > 0, \tag{5}$$

and F is said to be *strictly convex* (or *rotund*) at ξ w.r.t. ξ^* if

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0 \quad \text{for all} \quad r > 0. \tag{6}$$

If (6) is fulfilled then ξ is an *exposed point* of F and the vector ξ^* *exposes* ξ in the sense that the hyperplane $\{\eta \in \mathbb{R}^n : \langle \eta, \xi^* \rangle = \sigma_F(\xi^*)\}$ touches F only at the point ξ, or, in other words, $\mathfrak{J}_F(\xi^*) = \{\xi\}$. So, in this case, ξ is well defined whenever ξ^* is fixed.

Definition 1 ([9]) *Fix $\xi^* \in \partial F^o$, and let ξ be the unique element of $\mathfrak{J}_F(\xi^*)$. The set F is said to be* strictly convex of order 2 *(*at the point ξ*)* w.r.t. ξ^* *if*

$$\hat{\gamma}_F(\xi, \xi^*) = \liminf_{\substack{(r,\eta,\eta^*)\to(0+,\xi,\xi^*) \\ \eta\in\mathfrak{J}_F(\eta^*),\, \eta^*\in\partial F^o}} \frac{2\ \widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^2} > 0. \tag{7}$$

The number

$$\hat{\varkappa}_F(\xi, \xi^*) = \frac{1}{\|\xi^*\|}\ \hat{\gamma}_F(\xi, \xi^*)$$

*is said to be the (*square*)* curvature *of F at $\xi \in \partial F$* w.r.t. ξ^*.

An example Consider the compact convex set

$$F := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le 1 - \xi_1^4, \ -1 \le \xi_1 \le 1 \right\}.$$

For any arbitrary dual pair (ξ, ξ^*), with $\xi := (\xi_1, \xi_2)$, by the symmetry of F, we just consider the case $\xi_1, \xi_2 \ge 0$. Using [9, Example 8.3] we obtain:

(i) If $\xi_2 > 0$ then the (unique) normal vector ξ^* to F at ξ, such that $\langle \xi, \xi^* \rangle = 1$ is given by

$$\xi^* = \frac{1}{1 + 3\xi_1^4} \left(4\xi_1^3, 1 \right).$$

After a hard work, we obtained

$$\hat{\varkappa}_F(\xi, \xi^*) = \frac{\hat{\gamma}_F(\xi, \xi^*)}{\|\xi^*\|} \le \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}} \tag{8}$$

and

$$\hat{\varkappa}_F(\xi, \xi^*) \ge \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6 \, \Sigma^2(\xi_1)}}, \tag{9}$$

where $\Sigma(\xi_1) := \sqrt{1 + \left(\sum_{k=0}^{3} |\xi_1|^k \right)^2}$. Combining the estimates (8) and (9) we see that the curvature $\hat{\varkappa}(\xi, \xi^*)$ is of order $O\left(\xi_1^2\right)$ (as $|\xi_1| \to 0$). In particular, $\hat{\varkappa}_F$ is equal to zero at the points $(0, \pm 1)$.

(ii) If $\xi := (1, 0)$ we have

$$\mathbf{N}_F(\xi) = \left\{ (v_1, v_2) \in \mathbb{R}^2 : v_1 \le 4|v_2| \right\}.$$

For $\xi^* \in \partial \mathbf{N}_F(\xi)$, by the lower semicontinuity of the function $(\xi, \xi^*) \mapsto \hat{\gamma}_F(\xi, \xi^*)$, we can apply the same reasoning as above, but not for $\xi^* \in \mathrm{int}\mathbf{N}_F(\xi)$. In this last case we have $\hat{\varkappa}_F(\xi, \xi^*) = +\infty$ (see [9, Proposition 3.8]).

In the previous example we obtained only the estimates (8) and (9), but using the theory developed in this paper we will obtain an equality (see Example 1).

3 Directional curvature

In everything that follows we consider a compact convex set $F \subset \mathbb{R}^n$, $n \ge 2$, with $\mathbf{0} \in \mathrm{int}F$. Fixed $\xi \in \partial F$ we assume that there are $\delta > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^2$ at $\xi + \delta \mathbf{B}$ ($\mathbf{B} \subset \mathbb{R}^n$ represents the open unit ball), such that

$$F \subset \{ x \in \mathbb{R}^n : f(x) \le 0 \},$$

$$\langle \xi, \nabla f(\xi) \rangle > 0, \tag{10}$$

where $\nabla f(\xi)$ means the gradient vector of f at ξ, and such that, for $x \in \xi + \delta \mathbf{B}$, we have $x \in \partial F$ if and only if $f(x) = 0$.

Remark 1 *Thanks to (10) and the continuity of $\nabla f(\cdot)$ at ξ there is $0<\delta'\leq\delta$ such that*

$$\inf_{\eta\in\xi+\delta'\mathbf{B}} \langle\eta,\nabla f(\eta)\rangle>0. \tag{11}$$

In particular, we have $\nabla f(\eta)\neq\mathbf{0}$ for any $\eta\in\xi+\delta'\mathbf{B}$.

Proposition 1 *We have*

$$\mathbf{N}_F(\eta)=\bigcup_{\lambda\geq 0}\lambda\nabla f(\eta), \qquad \eta\in\partial F\cap\left(\xi+\delta'\mathbf{B}\right).$$

Proof. By (3), for an arbitrary $\eta\in\partial F\cap(\xi+\delta'\mathbf{B})$, it's enough to prove that

$$\mathbf{N}_F^p(\eta)=\bigcup_{\lambda\geq 0}\lambda\nabla f(\eta). \tag{12}$$

Since f is of classe $\mathcal{C}^2$ at $\xi+\delta\mathbf{B}$, by [6, Theorem 1.2.5 and Corolary 1.2.6], there are $\sigma,\rho>0$ such that $(\eta+\rho\mathbf{B})\subset(\xi+\delta'\mathbf{B})$ and

$$f(y)\geq f(\eta)+\langle\nabla f(\eta),y-\eta\rangle-\sigma\|y-\eta\|^2, \qquad \forall y\in\eta+\rho\mathbf{B},$$

and consequently

$$\langle\nabla f(\eta),y-\eta\rangle\leq\sigma\|y-\eta\|^2, \qquad \forall y\in(\eta+\rho\mathbf{B})\cap F.$$

Thanks to [6, Proposition 1.1.5] $\nabla f(\eta)\in\mathbf{N}_F^p(\eta)$. Since $\mathbf{N}_F^p(\eta)$ is a cone we have, in fact, $\lambda\nabla f(\eta)\in\mathbf{N}_F^p(\eta)$, $\lambda\geq 0$.
To prove the other inclusion at (12) fix $\zeta\in\mathbf{N}_F^p(\eta)$. By [6, Proposition 1.1.5] there is a constant $\sigma>0$ such that

$$\langle\zeta,y-\eta\rangle\leq\sigma\|y-\eta\|^2,$$

whenever y belongs to $\partial F\cap(\xi+\delta\mathbf{B})$. Put another way, this is equivalent to say that the point η minimizes the function $y\mapsto\langle-\zeta,y\rangle+\sigma\|y-\eta\|^2$ over all points y satisfying $f(y)=0$ and $\|y-\xi\|<\delta$. The Lagrange Multiplier Rule of classical calculus provides a scalar $\lambda\geq 0$ such that $\zeta=\lambda\nabla f(\eta)$, which completes the proof. ■

Consequently, for any $\eta\in\partial F\cap(\xi+\delta'\mathbf{B})$ fixed, $\mathfrak{J}_F^{-1}(\eta)=\mathbf{N}_F(\eta)\cap\partial F^o$ is a singleton, and the unique $\eta^*\in\mathfrak{J}_F^{-1}(\eta)$ is given by

$$\eta^*=\frac{1}{\langle\eta,\nabla f(\eta)\rangle}\nabla f(\eta). \tag{13}$$

This means that η^* is well defined whenever η is fixed. On the other hand, $\nabla f(\eta)\neq\mathbf{0}$ implies that there is a first $i\in I:=\{1,...,n\}$ such that

$$f_{x_i}(\eta):=\frac{\partial f}{\partial x_i}(\eta)\neq 0. \tag{14}$$

Fixed such i the hyperplane tangent to F at η is given by (see (4))

$$\begin{aligned}\mathbf{T}_F(\eta) &= \{v \in \mathbb{R}^n : \langle v, \nabla f(\eta)\rangle = 0\} \\ &= \left\{(v_1, ..., v_n) \in \mathbb{R}^n : v_i = -\sum_{j=1, j\neq i}^{n} \frac{f_{x_j}(\eta)}{f_{x_i}(\eta)} v_j\right\}.\end{aligned}$$

Denote by $u^j(\eta)$, $j \in I\backslash\{i\}$, the vector of $\mathbb{R}^n$ with 1 in the jth coordinate, $-\frac{f_{x_j}(\eta)}{f_{x_i}(\eta)}$ in the ith coordinate and 0 in the others. Since f is of class $\mathcal{C}^2$ at $\xi + \delta\mathbf{B}$, $u^j(\eta)$ will be close to $u^j(\xi)$, whenever η is close to ξ.

For our results we need to introduce the following. Given $\eta \in \xi + \delta'\mathbf{B}$ and $u(\eta) \in \mathbf{T}_F(\eta)$, $u(\eta) \neq \mathbf{0}$, consider the subset of $\mathbb{R}^n$

$$P(\eta, u(\eta)) := \operatorname{span}\{\nabla f(\eta), u(\eta)\} + \eta,$$

where $\operatorname{span}\{\nabla f(\eta), u(\eta)\}$ means the generated space by the vectors $\nabla f(\eta)$ and $u(\eta)$. Note that the vectors $\nabla f(\eta)$ and $u(\eta)$ are linearly independent, so the set $P(\eta, u(\eta))$ is, in fact, a 2-dimensional plane in $\mathbb{R}^n$ (it will simply be called a plane).

Bellow we introduce some directional notions, based on the respective notions presented in [9], and already seen here in Section 2. To simplify the notation, in general, we will not refer to the unique ξ^*, given by (13).

Definition 2 *The* 2-dimensional modulus of strict convexity of F at $\xi \in \partial F$ *(with respect to* ξ^**)* in the direction of the vector $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$ *is given by*

$$\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) = \inf\{\langle \xi - \eta, \xi^*\rangle : \eta \in F \cap P(\xi, u(\xi)), \|\xi - \eta\| \geq r\}, \qquad r > 0.$$

The set F is strictly convex at ξ in the direction of $u(\xi)$ *if* $\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) > 0$ *for all* $r > 0$.

Remark 2 *Since the set $F \subset \mathbb{R}^n$ is compact and convex we have the equalities*

$$\begin{aligned}\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) &= \inf\{\langle \xi - \eta, \xi^*\rangle : \eta \in F \cap P(\xi, u(\xi)), \|\xi - \eta\| = r\} \\ &= \inf\{\langle \xi - \eta, \xi^*\rangle : \eta \in \partial F \cap P(\xi, u(\xi)), \|\xi - \eta\| = r\},\end{aligned}$$

for any $r > 0$ and $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$.

Proposition 2 *Let $\xi \in \partial F$, $\xi^* \in \partial F^o$ given by (13) and $u(\xi) \in \mathbf{T}_F(\xi)$, $u(\xi) \neq \mathbf{0}$. If $\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) > 0$ for all $r > 0$ then $\mathfrak{J}_F(\xi^*) \cap P(\xi, u(\xi)) = \{\xi\}$.*

Proof. By construction $\xi \in \mathfrak{J}_F(\xi^*) \cap P(\xi, u(\xi))$. If there was $\overline{\xi} \in \mathfrak{J}_F(\xi^*) \cap P(\xi, u(\xi))$ with $\overline{\xi} \neq \xi$, we would have

$$\langle \xi - \overline{\xi}, \xi^*\rangle = 0,$$

and consequently

$$\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) = 0,$$

for $r := \|\xi - \overline{\xi}\| > 0$, which is absurd. ■

Definition 3 *The* 2-dimensional curvature of F at $\xi \in \partial F$ *(w.r.t. ξ^*)* in the direction of $u^j(\xi)$, $j \in I \backslash \{i\}$, *is given by*

$$\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) = \frac{1}{\|\xi^*\|}\ \hat{\gamma}_F\left(\xi, u^j(\xi)\right),$$

where

$$\hat{\gamma}_F\left(\xi, u^j(\xi)\right) = \liminf_{\substack{(r,\eta)\to\left(0^+,\xi\right)\\ \eta\in\partial F}} \frac{2\ \widehat{\mathfrak{C}}_F\left(r,\eta,u^j(\eta)\right)}{r^2}.$$

The set F is said to be strictly convex of the second order at ξ in the direction of $u^j(\xi)$ *when* $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) > 0$.

Such at [9, Proposition 3.7] we may extend the concept of directional strict convexity for the case of an arbitrary compact convex solid (do not assuming that $\mathbf{0} \in \mathrm{int}F$). For this, we need to remember that the interior of any convex set C in $\mathbb{R}^n$ relative to its affine hull (the smallest affine set that includes C) is the *relative interior* of C, denoted by $\mathrm{rint}C$.

Proposition 3 *Let $\xi \in \partial F$, $i \in I$ as above, $j \in I\backslash\{i\}$, $y_1, y_2 \in \mathrm{rint}(F \cap P(\xi, u(\xi)))$ and $\xi_1^* \in \mathfrak{J}_{F-y_1}^{-1}(\xi - y_1)$. Then there is an unique $\xi_2^* \in \mathfrak{J}_{F-y_2}^{-1}(\xi - y_2)$ colinear with ξ_1^* and such that*

$$\frac{1}{\|\xi_1^*\|}\hat{\gamma}_{F-y_1}\left(\xi - y_1, u^j(\xi)\right) = \frac{1}{\|\xi_2^*\|}\hat{\gamma}_{F-y_2}\left(\xi - y_2, u^j(\xi)\right). \tag{15}$$

Proof. First, notice that ξ_1^* is unique and, by (13), is given by $\frac{1}{\langle \xi - y_1, \nabla f(\xi)\rangle}\nabla f(\xi)$. As the same reason the unique $\xi_2^* \in \mathfrak{J}_{F-y_2}^{-1}(\xi - y_2)$ is given by $\frac{1}{\langle \xi - y_2, \nabla f(\xi)\rangle}\nabla f(\xi)$, and it is colinear with ξ_1^*. Now, let us fix $\eta \in \partial F$ close to ξ, and the corresponding vectors η_1^* and η_2^* (which are close to ξ_1^* and ξ_2^*, respectively). Notice that $\eta_1^* \in \mathfrak{J}_{F-y_1}^{-1}(\eta - y_1)$ implies $\langle y - y_1, \eta_1^*\rangle < 1$ for any $y \in \mathrm{int}F$, and we can write

$$\eta_2^* = \frac{1}{1 + \langle y_1 - y_2, \eta_1^*\rangle}\eta_1^*.$$

So, from Definition 2, we obtain

$$\begin{aligned}
& \frac{1}{\|\eta_2^*\|}\widehat{\mathfrak{C}}_{F-y_2}\left(r, \eta - y_2, u^j(\eta)\right) \\
= \ & \frac{1}{\|\eta_2^*\|}\inf\left\{\langle \eta - y_2 - \zeta, \eta_2^*\rangle : \zeta \in \left(F \cap P(\eta, u(\eta)) - y_2\right), \|\eta - y_2 - \zeta\| \geq r\right\} \\
= \ & \frac{1}{\|\eta_2^*\|}\inf\left\{\langle \eta - y, \eta_2^*\rangle : y \in F \cap P(\eta, u(\eta)), \|\eta - y\| \geq r\right\} \\
= \ & \frac{1}{\|\eta_1^*\|}\inf\left\{\langle \eta - y, \eta_1^*\rangle : y \in F \cap P(\eta, u(\eta)), \|\eta - y\| \geq r\right\} \\
= \ & \frac{1}{\|\eta_1^*\|}\widehat{\mathfrak{C}}_{F-y_1}\left(r, \eta - y_1, u^j(\eta)\right),
\end{aligned}$$

i.e.,

$$\frac{1}{\|\eta_2^*\|}\widehat{\mathfrak{C}}_{F-y_2}\left(r,\eta-y_2,u^j(\eta)\right)=\frac{1}{\|\eta_1^*\|}\widehat{\mathfrak{C}}_{F-y_1}\left(r,\eta-y_1,u^j(\eta)\right)$$

for all $r>0$. Dividing both parts of the last equality by r^2 and passing to $\liminf$ as $r\to 0^+$, $\eta\to\xi$ we easily come to (15). ■

In the last proof we used the known fact that $\mathbf{N}_{F-y_i}(\xi-y_i)=\mathbf{N}_F(\xi)$.

Remember that $u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}$ if there are $n-1$ real numbers α_j, $j\in I\setminus\{i\}$, not simultaneously null, such that

$$u(\xi)=\sum_{j=1,\ j\neq i}^{n}\alpha_j u^j(\xi).$$

This means that $u(\xi)$ is a vector of $\mathbb{R}^n$ with

$$-\sum_{j=1,\ j\neq i}^{n}\alpha_j\frac{f_{x_j}(\xi)}{f_{x_i}(\xi)}$$

at the ith coordinate and α_j, $j\in I\setminus\{i\}$, at the jth coordinate. If for any $\eta\in\partial F$ near ξ we define $u(\eta)$ as the non-zero vector of $\mathbb{R}^n$ corresponding to $u(\xi)$, i.e., the jth coordinates, $j\in I\setminus\{i\}$, are the same in both vectors, and the ith coordinate of $u(\eta)$ is given by

$$-\sum_{j=1,\ j\neq i}^{n}\alpha_j\frac{f_{x_j}(\eta)}{f_{x_i}(\eta)},$$

then it will be possible to put

$$\hat{\varkappa}_F(\xi,u(\xi))=\frac{1}{\|\xi^*\|}\hat{\gamma}_F(\xi,u(\xi))=\frac{1}{\|\xi^*\|}\liminf_{\substack{(r,\eta)\to(0^+,\xi)\\ \eta\in\partial F}}\frac{2\,\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))}{r^2}.$$

Note that such vector $u(\eta)$ is, in fact, in $\mathbf{T}_F(\eta)$.

Proposition 4 *Let $\xi\in\partial F$. If there is $u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}$ such that $\hat{\gamma}_F(\xi,u(\xi))>0$, then we will have $\mathfrak{J}_F(\eta^*)\cap P(\eta,u(\eta))=\{\eta\}$ for every η close enough to ξ (and respective η^* given by (13)).*

Proof. The condition $\hat{\gamma}_F(\xi,u(\xi))>0$ means that for some $\theta>0$ and $\rho>0$ the inequality

$$\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))\geq\theta r^2 \tag{16}$$

takes place whenever $\|\xi-\eta\|\leq\rho$, $\eta\in\partial F$ and $0<r\leq\rho$. Thanks to the monotony of the function $r\mapsto\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))$, decreasing if necessary the constant $\theta>0$, we can assume that (16) is valid for all positive r. In fact, $\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))=+\infty$ whenever $r>2\|F\|$ and for $\rho\leq r\leq 2\|F\|$ we have

$$\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))\geq\widehat{\mathfrak{C}}_F(\rho,\eta,u(\eta))\geq\theta\left(\frac{\rho}{r}\right)^2 r^2\geq\theta\left(\frac{\rho}{2\|F\|}\right)^2 r^2.$$

Hence, $\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))>0$, for all $r>0$, and the conclusion follows from Proposition 2. ■

In the next proposition $\hat{\gamma}_F(\xi)$ represents $\hat{\gamma}_F(\xi,\xi^*)$, given by (7), for the unique $\xi^*=\frac{1}{\langle\xi,\nabla f(\xi)\rangle}\nabla f(\xi)$.

Proposition 5 *We have*

$$\hat{\gamma}_F(\xi,u(\xi))\geq\hat{\gamma}_F(\xi),\qquad\forall u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}.\tag{17}$$

Furthermore, if $\hat{\gamma}_F(\xi,u(\xi))=0$ *for some* $u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}$*, we have* $\hat{\gamma}_F(\xi)=0$ *too.*

Proof. In fact, by (5),

$$\begin{aligned}\liminf_{\substack{(r,\eta)\to(0^+,\xi)\\ \eta\in\partial F}}\frac{\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))}{r^2} &= \liminf_{\substack{(r,\eta)\to(0^+,\xi)\\ \eta\in\partial F,\ \eta^*=\frac{1}{\langle\eta,\nabla f(\eta)\rangle}\nabla f(\eta)}}\frac{\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))}{r^2}\\ &\geq \liminf_{\substack{(r,\eta)\to(0^+,\xi)\\ \eta\in\partial F,\ \eta^*=\frac{1}{\langle\eta,\nabla f(\eta)\rangle}\nabla f(\eta)}}\frac{\widehat{\mathfrak{C}}_F(r,\eta,\eta^*)}{r^2} &(18)\\ &\geq \liminf_{\substack{(r,\eta,\eta^*)\to(0^+,\xi,\xi^*)\\ \eta\in\mathfrak{J}_F(\eta^*),\ \eta^*\in\partial F^o}}\frac{\widehat{\mathfrak{C}}_F(r,\eta,\eta^*)}{r^2}, &(19)\end{aligned}$$

which implies (17).
Now, recalling that we always have $\hat{\gamma}_F(\xi)\geq0$, if there is $u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}$ such that $\hat{\gamma}_F(\xi,u(\xi))=0$, we will have $\hat{\gamma}_F(\xi)=0$. ■

Since, when $n=2$, we have $\mathbf{T}_F(\xi)=\operatorname{span}\{t_F(\xi)\}=\{\lambda t_F(\xi):\lambda\in\mathbb{R}\}$ for $t_F(\xi):=(-f_{x_2}(\xi),f_{x_1}(\xi))$ and $P(\xi,\lambda t_F(\xi))=\mathbb{R}^2$ for every $\lambda\in\mathbb{R}\setminus\{0\}$, then we have the following result.

Proposition 6 *For* $n=2$ *the equality holds at (17) if* $\hat{\gamma}_F(\xi,\lambda t_F(\xi))>0$*, for some* $\lambda\in\mathbb{R}\setminus\{0\}$*.*

Proof. If $\lambda\in\mathbb{R}\setminus\{0\}$ is such that $\hat{\gamma}_F(\xi,\lambda t_F(\xi))>0$ then, by Proposition 4, $\mathfrak{J}_F(\eta^*)\cap P(\eta,\lambda t_F(\eta))=\{\eta\}$ for every η close enough to ξ (and respective η^*). Since, for each such η we have $P(\eta,u(\eta))=\mathbb{R}^2$, we obtain the equality at (19). On the other hand, also for each such η, we have

$$\widehat{\mathfrak{C}}_F(r,\eta,u(\eta))=\widehat{\mathfrak{C}}_F(r,\eta,\eta^*),\quad\forall r>0$$

(see Definition 2 and (5)). Therefore there is also the equality in (18), and consequently

$$\hat{\gamma}_F(\xi,\lambda t_F(\xi))=\hat{\gamma}_F(\xi).\tag{20}$$

■

The last proposition together with Proposition 5 imply that $\hat{\gamma}_F(\xi)=\hat{\gamma}_F(\xi,u(\xi))$, for every $u(\xi)\in\mathbf{T}_F(\xi)\setminus\{\mathbf{0}\}$. This conclusion was already expected, since in $\mathbb{R}^2$ there is only one tangent direction.

4 The main result

In this section we prove that, under our conditions, the directional curvature $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right)$, $j \in I\backslash\{i\}$, can be calculated very easily. For this we need to use the Hessian matrix of f calculated at ξ, $\nabla^2 f(\xi)$, that is, the $n \times n$ matrix with

$$f_{x_r x_s}(\xi) := \frac{\partial^2 f}{\partial x_r \partial x_s}(\xi) \tag{21}$$

at the row r and column s, for every $r, s \in \{1, ..., n\}$.

Theorem 1 *Let a compact convex set $F \subset \mathbb{R}^n$, $n \geq 2$, with $\mathbf{0} \in \mathbb{R}^n$ in its interior, and a point $\xi \in \partial F$. Assume that there are $\delta > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^2$ at $\xi + \delta\mathbf{B}$, such that*

$$F \subset \{x \in \mathbb{R}^n : f(x) \leq 0\}, \qquad \langle \xi, \nabla f(\xi)\rangle > 0,$$

and such that, for $x \in \xi + \delta\mathbf{B}$, we have $x \in \partial F$ if and only if $f(x) = 0$. Then we have

$$\hat{\gamma}_F\left(\xi, u^j(\xi)\right) = \frac{1}{\langle \xi, \nabla f(\xi)\rangle \left\|u^j(\xi)\right\|^2}\left\langle \nabla^2 f(\xi)\, u^j(\xi), u^j(\xi)\right\rangle, \qquad j \in I\backslash\{i\}. \tag{22}$$

Proof. By hypothesis $\langle \xi, \nabla f(\xi)\rangle > 0$, so let us fix the first $i \in I$ such that $f_{x_i}(\xi) \neq 0$ (see Remark 1). For any $\eta \in \mathbb{R}^n$, let $\eta^i \in \mathbb{R}^{n-1}$ the vector η without the ith coordinate. Thanks to the Implicit Function Theorem there are a neighbourhood $U := \xi^i + \delta_1\mathbf{B} \subset \mathbb{R}^{n-1}$, $0 < \delta_1 \leq \delta$, and a $\mathcal{C}^2$ function $g : U \to \mathbb{R}$ such that:

(i) $f\left(\eta^i, g\left(\eta^i\right)\right) = 0$, for any $\eta^i \in U$,

(ii) for $\eta^i \in U$ such that $f(\eta) = 0$ we have $\eta_i = g\left(\eta^i\right)$, and

(iii) for any $\eta^i \in U$ we have

$$g_{x_j}\left(\eta^i\right) = -\frac{f_{x_j}\left(\eta^i, g\left(\eta^i\right)\right)}{f_{x_i}\left(\eta^i, g\left(\eta^i\right)\right)} \quad (j \in I\backslash\{i\}),$$

where $\left(\eta^i, g\left(\eta^i\right)\right) \in \mathbb{R}^n$ represents the vector η with $g\left(\eta^i\right)$ instead of η_i.

Let $j \in I\backslash\{i\}$. By (11) for each $\varepsilon > 0$ there is $0 < \overline{\delta} = \overline{\delta}(\varepsilon) \leq \min\{\delta', \delta_1\}$such that

$$\left\| \frac{1}{\langle \eta, \nabla f(\eta)\rangle}\nabla f(\eta) - \frac{1}{\langle \xi, \nabla f(\xi)\rangle}\nabla f(\xi)\right\| < \varepsilon \tag{23}$$

holds for any $\eta \in \xi + \overline{\delta}\mathbf{B}$ (by the continuity of $\nabla f(\cdot)$ at ξ),

$$\left| \frac{\left\langle \nabla^2 f(\zeta)\, v, v\right\rangle}{\langle \eta, \nabla f(\eta)\rangle} - \frac{\left\langle \nabla^2 f(\xi)\, v, v\right\rangle}{\langle \xi, \nabla f(\xi)\rangle}\right| < \frac{\varepsilon}{2}, \tag{24}$$

for any $\eta, \zeta \in \xi + \overline{\delta}\mathbf{B}$ and any $v \in \mathbb{R}^n$, $\|v\| = 1$ (by the continuity of $\nabla^2 f(\cdot)$ at ξ), and such that

$$\left| \left\langle \nabla^2 f(\xi) \frac{y-\eta}{\|y-\eta\|}, \frac{y-\eta}{\|y-\eta\|} \right\rangle - \left\langle \nabla^2 f(\xi) \frac{u^j(\xi)}{\|u^j(\xi)\|}, \frac{u^j(\xi)}{\|u^j(\xi)\|} \right\rangle \right| < \langle \xi, \nabla f(\xi) \rangle \varepsilon, \tag{25}$$

holds for any $\eta, y \in \xi + \overline{\delta}\mathbf{B}$, $y \in P\left(\eta, u^j(\eta)\right)$, $y \neq \eta$, with $f(y) = f(\eta) = 0$ (using the continuity of $\nabla g(\cdot)$ and $\nabla f(\cdot)$ at ξ^i and ξ, respectively, and using the Lagrange Mean Value Theorem).

Let us prove the inequality " $\geq$ " in (22), assuming that $\hat{\gamma}_F\left(\xi, u^j(\xi)\right) < +\infty$, because in the other case there is nothing to prove. Let us fix $\varepsilon > 0$, the corresponding $\overline{\delta} > 0$, $0 < r < \frac{\overline{\delta}}{2}$ and $\eta \in \left(\xi + \frac{\overline{\delta}}{2}\mathbf{B}\right) \cap \partial F$. We want to prove that

$$\frac{\widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right)}{r^2} > \frac{1}{\langle \xi, \nabla f(\xi) \rangle \|u^j(\xi)\|^2} \left\langle \nabla^2 f(\xi) u^j(\xi), u^j(\xi) \right\rangle - \varepsilon.$$

Remember that $\eta^* = \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \nabla f(\eta)$. By Remark 2 there is $y \in \partial F \cap P\left(\eta, u^j(\eta)\right)$ with $\|\eta - y\| = r$, such that

$$\widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right) > \langle \eta - y, \eta^* \rangle - \frac{\varepsilon}{4} r^2. \tag{26}$$

Notice that $\|\eta - y\| = r > 0$ implies $y \neq \eta$. Putting

$$v := \frac{y-\eta}{\|y-\eta\|}, \tag{27}$$

then $\eta + rv = y$. Thanks to the Taylor's formula (see, e.g., [4, p.75])

$$f(\eta + rv) = f(\eta) + \langle rv, \nabla f(\eta) \rangle + \int_0^r \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle (r - \tau) \, d\tau,$$

and by the definition of v

$$f(y) = f(\eta) + \langle y - \eta, \nabla f(\eta) \rangle + \int_0^r \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle (r - \tau) \, d\tau.$$

Hence, by using the Mean Value Theorem for integrals and remembering that $f(\eta) = f(y) = 0$, we obtain

$$\begin{aligned} \langle \eta - y, \nabla f(\eta) \rangle &= \int_0^r \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle (r - \tau) \, d\tau \\ &= \frac{r^2}{2} \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle, \end{aligned} \tag{28}$$

for some $\tau = \tau(r, v) \in]0, r[$. Let us fix such τ. By (26), (13) and (28), respectively, we have

$$\begin{aligned} \widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right) &> \langle \eta - y, \eta^* \rangle - \frac{\varepsilon}{4} r^2 \\ &= \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \langle \eta - y, \nabla f(\eta) \rangle - \frac{\varepsilon}{4} r^2 \\ &\geq \frac{r^2}{2 \langle \eta, \nabla f(\eta) \rangle} \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle - \frac{\varepsilon}{4} r^2. \end{aligned} \tag{29}$$

Since

$$\|\eta + \tau v - \xi\| = \left\| \eta + \tau \frac{y-\eta}{\|y-\eta\|} - \xi \right\| \leq \|\eta - \xi\| + \tau < \overline{\delta}, \tag{30}$$

by (24) we obtain

$$\left| \frac{\left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle}{\langle \eta, \nabla f(\eta) \rangle} - \frac{\left\langle \nabla^2 f(\xi) v, v \right\rangle}{\langle \xi, \nabla f(\xi) \rangle} \right| < \frac{\varepsilon}{2},$$

and using (25) and (29) we conclude

$$\begin{aligned} \frac{\widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right)}{r^2} &> \frac{1}{2\langle \eta, \nabla f(\eta) \rangle} \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle - \frac{\varepsilon}{4} \\ &> \frac{1}{2\langle \xi, \nabla f(\xi) \rangle} \left\langle \nabla^2 f(\xi) v, v \right\rangle - \frac{\varepsilon}{2} \\ &> \frac{1}{2\langle \xi, \nabla f(\xi) \rangle} \left\langle \nabla^2 f(\xi) \frac{u^j(\xi)}{\|u^j(\xi)\|}, \frac{u^j(\xi)}{\|u^j(\xi)\|} \right\rangle - \varepsilon. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0^+$ we obtain the desired inequality:

$$\hat{\gamma}_F\left(\xi, u^j(\xi)\right) \geq \frac{1}{\langle \xi, \nabla f(\xi) \rangle \|u^j(\xi)\|^2} \left\langle \nabla^2 f(\xi) u^j(\xi), u^j(\xi) \right\rangle.$$

In order to show the opposite inequality let us assume that $\hat{\gamma}_F\left(\xi, u^j(\xi)\right) > 0$ (the case $\hat{\gamma}_F\left(\xi, u^j(\xi)\right) = 0$ is trivial). Let us fix $\varepsilon > 0$ and $0 < \overline{\delta} = \overline{\delta}(\varepsilon) \leq \min\{\delta', \delta_1\}$ such that (23), (24), (25) and

$$\frac{\widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right)}{r^2} > \hat{\gamma}_F\left(\xi, u^j(\xi)\right) - \frac{\varepsilon}{4} \tag{31}$$

holds for every $0 < r < \overline{\delta}$ and $\eta \in \partial F$ with $\|\eta - \xi\| < \overline{\delta}$. Let us fix $0 < r < \frac{\overline{\delta}}{2}$, $\eta \in \left(\xi + \frac{\overline{\delta}}{2}\mathbf{B}\right) \cap \partial F$ and respective η^*. We have

$$\begin{aligned} \widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\eta)\right) &= \inf\left\{\langle \eta - y, \eta^* \rangle : y \in \partial F \cap P\left(\eta, u^j(\eta)\right), \|\eta - y\| = r\right\} \\ &= \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \inf\left\{\langle \eta - y, \nabla f(\eta) \rangle : y \in \partial F \cap P\left(\eta, u^j(\eta)\right), \|\eta - y\| = r\right\}. \end{aligned} \tag{32}$$

Now let us fix $y \in \partial F \cap P\left(\eta, u^j(\eta)\right)$ with $\|\eta - y\| = r$ and define v as in (27). Proceeding as above we obtain (28) for some $\tau = \tau(r, \eta) \in]0, r[$. Let us fix this τ. Using (28), (30), (24) and (25), respectively, we obtain

$$\begin{aligned} \frac{1}{r^2} \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \langle \eta - y, \nabla f(\eta) \rangle &= \frac{1}{2\langle \eta, \nabla f(\eta) \rangle} \left\langle \nabla^2 f(\eta + \tau v) v, v \right\rangle \\ &< \frac{1}{2\langle \xi, \nabla f(\xi) \rangle} \left\langle \nabla^2 f(\xi) v, v \right\rangle + \frac{\varepsilon}{4} \\ &< \frac{1}{2\langle \xi, \nabla f(\xi) \rangle} \left\langle \nabla^2 f(\xi) \frac{u^j(\xi)}{\|u^j(\xi)\|}, \frac{u^j(\xi)}{\|u^j(\xi)\|} \right\rangle + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\ &= \frac{1}{2\langle \xi, \nabla f(\xi) \rangle} \frac{1}{\|u^j(\xi)\|^2} \left\langle \nabla^2 f(\xi) u^j(\xi), u^j(\xi) \right\rangle + \frac{3\varepsilon}{4}. \end{aligned}$$

Consequently (see (32))

$$\frac{\widehat{\mathfrak{C}}_F\left(r,\eta,u^j\left(\eta\right)\right)}{r^2}<\frac{1}{2\left\langle \xi,\nabla f\left(\xi\right)\right\rangle}\frac{1}{\left\|u^j\left(\xi\right)\right\|^2}\left\langle \nabla^2 f\left(\xi\right)u^j\left(\xi\right),u^j\left(\xi\right)\right\rangle+\frac{3\varepsilon}{4},$$

and by (31)

$$\hat{\gamma}_F\left(\xi,u^j\left(\xi\right)\right)<\frac{1}{\left\langle \xi,\nabla f\left(\xi\right)\right\rangle}\frac{1}{\left\|u^j\left(\xi\right)\right\|^2}\left\langle \nabla^2 f\left(\xi\right)u^j\left(\xi\right),u^j\left(\xi\right)\right\rangle+2\varepsilon.$$

Passing to the limit as $\varepsilon\to 0^+$ we obtain the inequality " $\leq$ " in (22). ■

Remembering the Definition 3 we conclude that the 2-dimensional curvature of F at $\xi\in\partial F$ in the direction of $u^j\left(\xi\right)$, $j\in I\setminus\{i\}$, is given by

$$\hat{\varkappa}_F\left(\xi,u^j\left(\xi\right)\right)=\frac{1}{\left\|\nabla f\left(\xi\right)\right\|\left\|u^j\left(\xi\right)\right\|^2}\left\langle \nabla^2 f\left(\xi\right)u^j\left(\xi\right),u^j\left(\xi\right)\right\rangle. \tag{33}$$

Note that, for a fixed $j\in I\setminus\{i\}$ and $\lambda\in\mathbb{R}\setminus\{0\}$ we have

$$\hat{\varkappa}_F\left(\xi,\lambda u^j\left(\xi\right)\right)=\frac{1}{\left\|\nabla f\left(\xi\right)\right\|\left\|u^j\left(\xi\right)\right\|^2}\left\langle \nabla^2 f\left(\xi\right)u^j\left(\xi\right),u^j\left(\xi\right)\right\rangle, \tag{34}$$

as it would be expected. Moreover, following the proof of Theorem 1 it is possible to prove that

Corollary 2 *We have*

$$\hat{\varkappa}_F\left(\xi,u\left(\xi\right)\right)=\frac{1}{\left\|\nabla f\left(\xi\right)\right\|\left\|u\left(\xi\right)\right\|^2}\left\langle \nabla^2 f\left(\xi\right)u\left(\xi\right),u\left(\xi\right)\right\rangle,$$

for any $u\left(\xi\right)\in\mathbf{T}_F\left(\xi\right)\setminus\{\mathbf{0}\}$.

Notice that, by (34), for $n=2$, to say that $\hat{\varkappa}_F\left(\xi,u\left(\xi\right)\right)>0$ for some $u\left(\xi\right)\in\mathbf{T}_F\left(\xi\right)\setminus\{\mathbf{0}\}$, is the same as saying that $\hat{\varkappa}_F\left(\xi,\left(-f_{x_2}\left(\xi\right),f_{x_1}\left(\xi\right)\right)\right)>0$. Therefore, by Proposition 6, if there is $\lambda\in\mathbb{R}\setminus\{0\}$ such that $\hat{\varkappa}_F\left(\xi,\lambda\left(-f_{x_2}\left(\xi\right),f_{x_1}\left(\xi\right)\right)\right)>0$, we will have

$$\hat{\varkappa}_F\left(\xi,u\left(\xi\right)\right)=\hat{\varkappa}_F\left(\xi\right),\qquad\forall u\left(\xi\right)\in\mathbf{T}_F\left(\xi\right)\setminus\{\mathbf{0}\}.$$

Since f is of class $\mathcal{C}^2$ in $\xi+\delta\mathbf{B}$, all the conclusions will remain valid if we replace ξ with any $\eta\in\partial F\cap\left(\xi+\delta\mathbf{B}\right)$.

Following the idea of G. Crasta and A. Malusa presented in [7, pg.5749], we have the following result.

Theorem 3 *Let $\xi \in \partial F$ and $\hat{\varkappa}_F\left(\xi, u^{j_1}(\xi)\right) \leq \cdots \leq \hat{\varkappa}_F\left(\xi, u^{j_{n-1}}(\xi)\right)$ be the curvatures of F at ξ in the direction of the $n-1$ vectors that generate $\mathbf{T}_F(\xi)$. If*

$$\left\|\nabla^2 f(\xi)\right\| := \sup_{\substack{u,v\in\mathbb{R}^n \\ \|u\|=\|v\|=1}} \left|\left\langle \nabla^2 f(\xi)\, u, v\right\rangle\right| < \infty, \tag{35}$$

then

$$\hat{\varkappa}_F\left(\xi, u^{j_1}(\xi)\right) = \min_{u\in U_\xi} \hat{\varkappa}(u) \quad \text{and} \quad \hat{\varkappa}_F\left(\xi, u^{j_{n-1}}(\xi)\right) = \max_{u\in U_\xi} \hat{\varkappa}(u),$$

where $\hat{\varkappa}(u) := \frac{1}{\|\nabla f(\xi)\|}\left\langle \nabla^2 f(\xi)\, u, u\right\rangle$ and $U_\xi := \{v \in \mathbf{T}_F(\xi) : \|v\| = 1\}$.

Proof. Assuming (35) it is easy to show that the application $u \mapsto \hat{\varkappa}(u)$ is continuous in $\mathbf{S} := \{x \in \mathbb{R}^n : \|x\| = 1\}$, and in particular in U_ξ. Hence it admits a maximum and a minimum on U_ξ. Let $\overline{u} \in U_\xi$ be a maximum point. Then, by (33),

$$\hat{\varkappa}(\overline{u}) = \max_{u\in U_\xi} \hat{\varkappa}(u) \geq \hat{\varkappa}\left(\frac{u^{j_{n-1}}(\xi)}{\|u^{j_{n-1}}(\xi)\|}\right) = \hat{\varkappa}_F\left(\xi, u^{j_{n-1}}(\xi)\right).$$

On the other hand, since $\overline{u} \in \mathbf{T}_F(\xi)$, then $\hat{\varkappa}_F(\xi, \overline{u}) \leq \hat{\varkappa}_F\left(\xi, u^{j_{n-1}}(\xi)\right)$, and consequently $\hat{\varkappa}(\overline{u}) = \hat{\varkappa}_F\left(\xi, u^{j_{n-1}}(\xi)\right)$. Reasoning as above, if $\overline{v}$ is a minimum on U_ξ, we deduce that $\hat{\varkappa}(\overline{v}) = \hat{\varkappa}_F\left(\xi, u^{j_1}(\xi)\right)$. ■

5 Directional curvature radius

As in [5, p.14] (see also [14, 2]) we also relate the directional curvature to the radius of some ball.

Definition 4 *The* 2-dimensional curvature radius of F at $\xi \in \partial F$ *(w.r.t. ξ^*)* in the direction of $u^j(\xi)$, $j \in I\backslash\{i\}$, *is given by*

$$\widehat{\mathfrak{R}}_F\left(\xi, u^j(\xi)\right) = \frac{1}{\hat{\varkappa}_F\left(\xi, u^j(\xi)\right)}. \tag{36}$$

Roughly speaking, the directional curvature $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right)$ shows how rotund the boundary ∂F is in a neighbourhood of ξ (watching from the end of the vector $u^j(\xi)$) when we "cut" F with the plane $P\left(\xi, u^j(\xi)\right)$. As follows from Proposition 3 it does not depend on the position of the origin in $\mathrm{int}F$ and can be defined also when $0 \notin \mathrm{int}F$. By using (36) we give the following geometric characterization of the directional curvature radius.

Proposition 7 *Fixed $j \in I\backslash\{i\}$, we have*

$$\frac{\widehat{\mathfrak{R}}_F\left(\xi, u^j(\xi)\right)}{\|\xi^*\|} = \limsup_{\substack{(\varepsilon,\eta)\to(0^+,\xi) \\ \eta\in\partial F}} \inf\left\{r > 0 : F \cap P\left(\eta, u^j(\eta)\right) \cap \left(\eta + \varepsilon\overline{\mathbf{B}}\right) \subset \eta - r\eta^* + r\|\eta^*\|\,\overline{\mathbf{B}}\right\}. \tag{37}$$

Proof. Let us prove first the inequality " $\leq$ " in (37) assuming without loss of generality that the right-hand side (further denoted by R) is finite. Taking an arbitrary $\rho > R$, by the definition of lim sup, we can afirm that for each $\varepsilon > 0$ small enough and for each $\eta \in \partial F$ from a neighbourhood of ξ, the relation

$$\inf \left\{ r > 0 : F \cap P\left(\eta, u^j(\eta)\right) \cap \left(\eta + \varepsilon \overline{\mathbf{B}}\right) \subset \eta - r\eta^* + r \left\|\eta^*\right\| \overline{\mathbf{B}} \right\} < \rho$$

holds. In particular,

$$F \cap P\left(\eta, u^j(\eta)\right) \cap \left(\eta + \varepsilon \overline{\mathbf{B}}\right) \subset \eta - \rho\eta^* + \rho \left\|\eta^*\right\| \overline{\mathbf{B}},$$

which implies

$$\left\|\zeta - \eta + \rho\eta^*\right\|^2 \leq \rho^2 \left\|\eta^*\right\|^2,$$

whenever $\zeta \in F \cap P\left(\eta, u^j(\eta)\right)$ with $\|\zeta - \eta\| = \varepsilon$, or, in another form,

$$\langle \zeta - \eta, \eta^* \rangle \leq -\frac{\varepsilon^2}{2\rho}. \tag{38}$$

If $w \in F \cap P\left(\eta, u^j(\eta)\right)$ is an arbitrary point with $\|w - \eta\| \geq \varepsilon$ then setting $\zeta := \lambda w + (1 - \lambda)\eta$ in $F \cap P\left(\eta, u^j(\eta)\right)$, where $\lambda := \frac{\varepsilon}{\|w - \eta\|} \leq 1$, we have

$$\|\zeta - \eta\| = \|\lambda w + (1 - \lambda)\eta - \eta\| = \lambda \|w - \eta\| = \varepsilon$$

and

$$\langle \eta - \zeta, \eta^* \rangle = \lambda \langle \eta - w, \eta^* \rangle.$$

By (38) we obtain

$$\frac{\varepsilon^2}{2\rho} \leq \langle \eta - \zeta, \eta^* \rangle = \lambda \langle \eta - w, \eta^* \rangle \leq \langle \eta - w, \eta^* \rangle,$$

so

$$\frac{\varepsilon^2}{2\rho} \leq \inf \left\{ \langle \eta - w, \eta^* \rangle : w \in F \cap P\left(\eta, u^j(\eta)\right), \|w - \eta\| \geq \varepsilon \right\}.$$

Hence, passing to lim inf as $\varepsilon \to 0^+, \eta \to \xi$ and $\rho \to R^+$ we conclude the first part of the proof.

In order to show the opposite inequality let us assume that $R > 0$ (the case $R = 0$ is trivial). If $0 < \rho < R$ then, by the definition of lim sup there are $\varepsilon > 0$ arbitrarily small and $\eta \in \partial F$ arbitrarily closed to ξ, such that

$$\inf \left\{ r > 0 : F \cap P\left(\eta, u^j(\eta)\right) \cap \left(\eta + \varepsilon \overline{\mathbf{B}}\right) \subset \eta - r\eta^* + r \left\|\eta^*\right\| \overline{\mathbf{B}} \right\} > \rho.$$

Then the set $F \cap P\left(\eta, u^j(\eta)\right) \cap \left(\eta + \varepsilon \overline{\mathbf{B}}\right)$ is not contained in $\eta - \rho\eta^* + \rho \|\eta^*\| \overline{\mathbf{B}}$, or, in other words, there is $\zeta \in F \cap P\left(\eta, u^j(\eta)\right)$ with $\|\zeta - \eta\| \leq \varepsilon$ such that

$$\left\|\zeta - \eta + \rho\eta^*\right\| > \rho \left\|\eta^*\right\|.$$

Consequently, setting, $r := \|\zeta - \eta\| \leq \varepsilon$ we have

$$2\rho \langle \eta - \zeta, \eta^* \rangle < \|\zeta - \eta\|^2 = r^2.$$

So

$$\begin{aligned}\widehat{\mathfrak{C}}_F\left(r,\eta,u^j(\eta)\right) &= \inf\left\{\langle \eta - w, \eta^*\rangle : w \in F \cap P\left(\eta, u^j(\eta)\right), \|w-\eta\| \geq r\right\} \\ &\leq \langle \eta - \zeta, \eta^*\rangle < \frac{r^2}{2\rho}.\end{aligned}$$

Passing to $\liminf$ as $r \to 0^+$, $\eta \to \xi$ and then to $\lim$ as $\rho \to R^-$ we conclude the proof. ■

6 Relation with the usual curvature formula for implicit space curves

In this section we see that it's possible to compute directional curvatures using the usual curvature formula for implicit space curves (that is, curves in $\mathbb{R}^n$ given by the intersection of $n-1$ implicit equations). In fact, fixed $\xi \in \partial F$, $i \in I$ (given by (14)) and $j \in I \setminus \{i\}$ and considering the cases $n = 2$ and $n \geq 3$ separately, we prove that the formula in (33) coincides with the formula obtained by R. Goldman in [8] for the curve $\partial F \cap P\left(\xi, u^j(\xi)\right)$.

For $n = 2$, near a fixed $\xi \in \partial F$ the curve $\partial F \cap P(\xi, u(\xi))$, for any $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$ (where $\mathbf{T}_F(\xi) = \{\lambda(-f_{x_2}(\xi), f_{x_1}(\xi)) : \lambda \in \mathbb{R}\}$ - see before Proposition 6), is given by $f(\eta) = 0$. Therefore, by (34), we have

$$\begin{aligned}\hat{\varkappa}_F(\xi, u(\xi)) &= \frac{1}{\|\nabla f(\xi)\| \|u(\xi)\|^2}\left\langle \nabla^2 f(\xi) u(\xi), u(\xi)\right\rangle \\ &= \frac{f_{x_1}^2(\xi) f_{x_2x_2}(\xi) - 2 f_{x_2}(\xi) f_{x_1}(\xi) f_{x_1x_2}(\xi) + f_{x_2}^2(\xi) f_{x_1x_1}(\xi)}{\left(f_{x_1}^2(\xi) + f_{x_2}^2(\xi)\right)^{\frac{3}{2}}} \\ &= \frac{\begin{bmatrix} -f_{x_2}(\xi) & f_{x_1}(\xi)\end{bmatrix}\begin{bmatrix} f_{x_1x_1}(\xi) & f_{x_1x_2}(\xi) \\ f_{x_1x_2}(\xi) & f_{x_2x_2}(\xi)\end{bmatrix}\begin{bmatrix} -f_{x_2}(\xi) \\ f_{x_1}(\xi)\end{bmatrix}}{\left(f_{x_1}^2(\xi) + f_{x_2}^2(\xi)\right)^{\frac{3}{2}}} = k_G(\xi),\end{aligned}$$

where $k_G(\xi)$ is the curvature given by R. Goldman in [8, (3.4)]. So, when $n = 2$, we have $\hat{\varkappa}_F(\xi, u(\xi)) = k_G(\xi)$, and this means that we can obtain the directional curvature $\hat{\varkappa}_F(\xi, u(\xi))$ calculating $k_G(\xi)$ for the respective curve (given implicitly).

When $n \geq 3$ note that, fixed $\xi \in \partial F$, $i \in I$ and $j \in I \setminus \{i\}$, the curve $\partial F \cap P\left(\xi, u^j(\xi)\right)$, near ξ, is given by the intersection of the $n-1$ implicit equations:

$$f(\eta) = 0, \quad p_{k_1\xi}(\eta) = 0, ..., p_{k_{n-2}\xi}(\eta) = 0, \tag{39}$$

$k_1, ..., k_{n-2} \in I \setminus \{i, j\}$ and $k_1 < ... < k_{n-2}$. In fact, if we put

$$a_{kl}(\xi) := -\frac{f_{x_l}(\xi) f_{x_k}(\xi)}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)}, \quad l = i, j,$$

$$p_{k\xi}(\eta_1, ..., \eta_n) := \eta_k - \xi_k + a_{ki}(\xi)(\eta_i - \xi_i) + a_{kj}(\xi)(\eta_j - \xi_j), \qquad (\eta_1, ..., \eta_n) \in \mathbb{R}^n,$$

and use the definition of generated space, it is easy to show that

$$P\left(\xi, u^j(\xi)\right) = \bigcap_{k\in I\backslash\{i,j\}} \{\eta \in \mathbb{R}^n : p_{k\xi}(\eta) = 0\}.$$

In order to present the Goldman's curvature formula for $n \geq 3$ we first need to introduce the generalization to the cross product from 3-dimensions to n-dimensions:

Definition 5 *([11, p.165]) The* external product *of two vectors in an n-dimensional space, $n \geq 3$, spanned by $e_1, ..., e_n$ is a vector in a space of dimension $\frac{n(n-1)}{2}$ spanned by a new collection of vectors denoted by $\{e_i \wedge e_j\}$, where $i < j$. Let $u = u_1e_1 + ... + u_ne_n$ and $v = v_1e_1 + ... + v_ne_n$ then*

$$u \wedge v = \sum_{i<j} \det \begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix} (e_i \wedge e_j).$$

For the next definition, as well as for the rest of the work, we just need to compute the magnitude of the external product, which is given by the formula

$$\|u \wedge v\|^2 = \sum_{i<j} \left(\det \begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix}\right)^2. \tag{40}$$

Assuming that e_i, $i = 1, ..., n$, is the vector of $\mathbb{R}^n$ with one in the ith position and zero everywhere else, and that $e := (e_1, ..., e_n)$ is the canonical basis of $\mathbb{R}^n$, we are in conditions to see the usual curvature formula for implicit space curves (see [8, (5.4)]):

Definition 6 *The curvature formula for a point ξ on a curve defined by the intersection of $n-1$ implicit hypersurfaces $F_1(x_1, ..., x_n) = 0, ..., F_{n-1}(x_1, ..., x_n) = 0$ is*

$$k_G(\xi) = \frac{\left\|\left(Tan\left(F_1, ..., F_{n-1}\right)(\xi) * \nabla\left(Tan\left(F_1, ..., F_{n-1}\right)\right)(\xi)\right) \wedge Tan\left(F_1, ..., F_{n-1}\right)(\xi)\right\|}{\left\|Tan\left(F_1, ..., F_{n-1}\right)(\xi)\right\|^3}, \tag{41}$$

where $Tan\left(F_1, ..., F_{n-1}\right)(\xi)$ is the tangent to the intersection curve at ξ given by

$$\begin{aligned} Tan\left(F_1, ..., F_{n-1}\right)(\xi) &= \det \begin{bmatrix} e \\ \nabla F_1(\xi) \\ \vdots \\ \nabla F_{n-1}(\xi) \end{bmatrix} \\ &= \det \begin{bmatrix} e_1 & \cdots & e_n \\ F_{1x_1}(\xi) & \cdots & F_{1x_n}(\xi) \\ \vdots & \vdots & \vdots \\ F_{n-1x_1}(\xi) & \cdots & F_{n-1x_n}(\xi) \end{bmatrix}, \end{aligned}$$

$\nabla\left(Tan\left(F_1, ..., F_{n-1}\right)\right)(\xi)$ is the $n \times n$ matrix in where each column is the gradiente of the respective component of the row matrix $Tan\left(F_1, ..., F_{n-1}\right)$, with the derivatives calculated at ξ, and $*$ represents the product between matrices.

Next we calculate the Goldman's curvature for the curve $\partial F \cap P\left(\xi, u^j(\xi)\right)$, at ξ, using its implicit equations (see (39)). We will denote it by $k_G^j(\xi)$ to distinguish it from the curvature of a general (implicit) curve. For that we need to remember the notation in (21).

Theorem 4 *We have*

$$k_G^j(\xi) = \frac{\left| f_{x_i x_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_j}(\xi) f_{x_i}^2(\xi) \right|}{\|\nabla f(\xi)\| \left(f_{x_i}^2(\xi) + f_{x_j}^2(\xi) \right)}.$$

Proof. For $k \in I \setminus \{i, j\}$ and $x := (x_1, ..., x_n) \in \mathbb{R}^n$ fixed, we have

$$\frac{\partial p_{k\xi}}{\partial x_m}(x) = \begin{cases} 1, & \text{if } m = k \\ a_{km}(\xi), & \text{if } m = i \text{ or } m = j \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

So the tangent to the intersection curve $\partial F \cap P\left(\xi, u^j(\xi)\right)$ at $\eta \in (\xi + \delta' \mathbf{B}) \cap \partial F \cap P\left(\xi, u^j(\xi)\right)$ is given by

$$\begin{aligned} \operatorname{Tan}\left(f, p_{k_1\xi}, ..., p_{k_{n-2}\xi}\right)(\eta) &= \det \begin{bmatrix} e \\ \nabla f(\eta) \\ \nabla p_{k_1\xi}(\eta) \\ \vdots \\ \nabla p_{k_{n-2}\xi}(\eta) \end{bmatrix} \\ &= \sum_{m=1}^{n} (-1)^{1+m} \det A_{m\xi}(\eta) e_m, \end{aligned}$$

where, for each $m \in I$, $A_{m\xi}(\eta)$ is the matrix obtained from $\begin{bmatrix} e \\ \nabla f(\eta) \\ \nabla p_{k_1\xi}(\eta) \\ \vdots \\ \nabla p_{k_{n-2}\xi}(\eta) \end{bmatrix}$ eliminating the first line and the mth column. Remembering that we have (42) for each $r \in \{1, n-2\}$, then

$$(-1)^{1+m} \det A_{m\xi}(\eta) = (-1)^{i+j} \begin{cases} \sum_{r=1}^{n-2} f_{x_{k_r}}(\eta) a_{k_r j}(\xi) - f_{x_j}(\eta), & \text{if } i < j \text{ and } m = i \\ f_{x_i}(\eta) - \sum_{r=1}^{n-2} f_{x_{k_r}}(\eta) a_{k_r i}(\xi), & \text{if } i < j \text{ and } m = j \\ f_{x_j}(\eta) - \sum_{r=1}^{n-2} f_{x_{k_r}}(\eta) a_{k_r j}(\xi), & \text{if } i > j \text{ and } m = i \\ \sum_{r=1}^{n-2} f_{x_{k_r}}(\eta) a_{k_r i}(\xi) - f_{x_i}(\eta), & \text{if } i > j \text{ and } m = j \\ f_{x_j}(\eta) a_{k_m i}(\xi) - f_{x_i}(\eta) a_{k_m j}(\xi), & \text{otherwise.} \end{cases}$$

Let $\omega^j(\xi) \in \mathbb{R}^n$ the vector with $(-1)^{i+j+1} f_{x_j}(\xi)$ in the ith coordinate, $(-1)^{i+j} f_{x_i}(\xi)$ at the jth coordinate, if $i < j$, or with symmetrical values if $i > j$, and 0 elsewhere. It is easy to show that

$$\sum_{m=1}^{n} (-1)^{1+m} \det A_m(\xi)\, e_m = \frac{\|\nabla f(\xi)\|^2}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)}\, \omega^j(\xi)$$
$$= (-1)^{i+j} \frac{\|\nabla f(\xi)\|^2 f_{x_i}(\xi)}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)}\, u^j(\xi),$$

that is

$$\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi) = (-1)^{i+j} \frac{\|\nabla f(\xi)\|^2 f_{x_i}(\xi)}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)}\, u^j(\xi). \tag{43}$$

After some calculations we conclude that

$$\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi) * \nabla\left(\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)\right)(\xi) = \frac{\|\nabla f(\xi)\|^2}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)}\, M,$$

where M is a line matrix. Using (40) we obtain

$$\left\|\left(\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi) * \nabla\left(\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)\right)(\xi)\right) \wedge \operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi)\right\|^2$$
$$= \left(\frac{\|\nabla f(\xi)\|^2}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)}\right)^5 \left(f_{x_ix_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_ix_j}(\xi) + f_{x_jx_j}(\xi) f_{x_i}^2(\xi)\right)^2.$$

Which implies

$$\left\|\left(\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi) * \nabla\left(\operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)\right)(\xi)\right) \wedge \operatorname{Tan}\left(f, p_{k_1}, ..., p_{k_{n-2}}\right)(\xi)\right\|$$
$$= \frac{\|\nabla f(\xi)\|^5}{\left(f_{x_i}^2(\xi)+f_{x_j}^2(\xi)\right)^{\frac{5}{2}}} \left|f_{x_ix_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_ix_j}(\xi) + f_{x_jx_j}(\xi) f_{x_i}^2(\xi)\right|,$$

and consequently (see (43))

$$k_G^j(\xi) = \frac{\frac{\|\nabla f(\xi)\|^5}{\left(f_{x_i}^2(\xi)+f_{x_j}^2(\xi)\right)^{\frac{5}{2}}} \left|f_{x_ix_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_ix_j}(\xi) + f_{x_jx_j}(\xi) f_{x_i}^2(\xi)\right|}{\left(\frac{\|\nabla f(\xi)\|^2 \left|f_{x_i}(\xi)\right|}{f_{x_i}^2(\xi)+f_{x_j}^2(\xi)} \left\|u^j(\xi)\right\|\right)^3}$$
$$= \frac{\left|f_{x_ix_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_ix_j}(\xi) + f_{x_jx_j}(\xi) f_{x_i}^2(\xi)\right|}{\|\nabla f(\xi)\| \left(f_{x_i}^2(\xi)+f_{x_j}^2(\xi)\right)}.$$

■

Remembering (33), and that $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) \geq 0$ because $\widehat{\mathfrak{C}}_F\left(r, \eta, u^j(\xi)\right) \geq 0$ for every $r > 0$ and every $\eta \in \partial F$ close enough to ξ, it is easy to show the next result.

Corollary 5 *We have*

$$\begin{aligned}\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) &= \frac{f_{x_i x_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_j}(\xi) f_{x_i}^2(\xi)}{\|\nabla f(\xi)\| \left(f_{x_i}^2(\xi) + f_{x_j}^2(\xi)\right)} \\ &= k_G^j(\xi).\end{aligned}$$

So, we have the equality $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) = k_G^j(\xi)$, for every $n \in \mathbb{N}$. Therefore, when we want to calculate directional curvatures of a convex body in $\mathbb{R}^n$ at a point ξ on its boundary, both checking our conditions, we can calculate $k_G(\xi)$ for the respective curve (given implicitly). But, as we can see following the proof of Theorem 4, it is faster to use (33) than to use (41), mainly for $n \geq 3$.

7 Examples

1. Consider the compact convex set $F \subset \mathbb{R}^2$, with $(0,0)$ in its interior,

$$F = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1 - x_1^4,\ -1 \leq x_1 \leq 1\right\}.$$

Close to $\xi = (\xi_1, \xi_2) \in \partial F$ with $\xi_2 > 0$ (the case $\xi_2 < 0$ is analogous) we have $f(x_1, x_2) := x_2 - 1 + x_1^4$,

$$\nabla f(\xi) = \left(4\xi_1^3, 1\right), \quad \nabla^2 f(\xi) = \left[\begin{array}{cc} 12\xi_1^2 & 0 \\ 0 & 0 \end{array}\right],$$

and

$$\mathbf{T}_F(\xi) = \operatorname{span}\left\{\left(1, -4\xi_1^3\right)\right\}.$$

Consequently, for any $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{(0,0)\}$,

$$\hat{\varkappa}_F(\xi, u(\xi)) = \frac{12\xi_1^2}{\sqrt{\left(4\xi_1^3\right)^2 + 1}\ \left(\left(4\xi_1^3\right)^2 + 1\right)} = \frac{12\xi_1^2}{\left(16\xi_1^6 + 1\right)^{\frac{3}{2}}}.$$

Recalling Proposition 6, we can see that Theorem 1 allows us to obtain the following equality

$$\hat{\varkappa}_F(\xi) = \frac{12\xi_1^2}{\left(16\xi_1^6 + 1\right)^{\frac{3}{2}}},$$

whereas in [9, Example 8.3] we had obtained only the inequalities

$$\frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}\ \Sigma^2(\xi_1)} \leq \hat{\varkappa}_F(\xi) \leq \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}},$$

where $\Sigma(\xi_1) := \sqrt{1 + \left(\sum_{k=0}^{3} |\xi_1|^k\right)^2}$.

Note that at $\xi = (0, \pm 1)$ we have $\hat{\varkappa}_F(\xi, u(\xi)) = 0$, as would be expected. Here we can't calculate the curvature at $\xi = (\pm 1, 0)$ because there isn't a $\mathcal{C}^2$ function f checking our conditions, but in [9, Example 8.3] there is an estimate for the curvature at such points.

2. Let F a sphere in $\mathbb{R}^n$
$$\left\{x=(x_1,...,x_n):\sum_{t=1}^{n}x_t^2\leq R^2\right\}.$$
Consider $f(x)=\sum_{t=1}^{n}x_t^2-R^2$ for x near a fixed $\xi\in\partial F$. We have
$$\nabla f(\xi)=2\xi,\quad \nabla^2 f(\xi)=2\mathbf{I}_n,$$
where $\mathbf{I}_n$ is the identity matrix of the order n. Fix the first $i\in I:=\{1,...,n\}$ such that $f_{x_i}(\xi)=2\xi_i\neq 0$, then $\mathbf{T}_F(\xi)$ is spanned by $n-1$ vectors $u^j(\xi)\in\mathbb{R}^n$, $j\in I\backslash\{i\}$, with 1 in the jth coordinate, $-\frac{\xi_j}{\xi_i}$ in the ith coordinate and 0 in the others. Therefore
$$\hat{\varkappa}_F\left(\xi,u^j(\xi)\right)=\frac{\left(\left(\frac{\xi_j}{\xi_i}\right)^2+1\right)^2}{\|\xi\|\left\|u^j(\xi)\right\|^2}=\frac{1}{R},$$
which means that the curvature at any point on the boundary of the sphere, in the direction of any vector of its hyperplane tangent, is equal to $\frac{1}{R}$.

3. Consider a cylinder
$$F_{a,b}=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3: x_1^2+x_3^2\leq a^2,\ |x_2|\leq b\right\},\quad a,b\in\mathbb{R}^+.$$
Near $\xi=(\xi_1,\xi_2,\xi_3)\in\partial F_{a,b}$, with $\xi_1^2+\xi_3^2=a^2,\ |\xi_2|<b$, put $f(x_1,x_2,x_3):=x_1^2+x_3^2-a^2$. We have
$$\nabla f(\xi)=2(\xi_1,0,\xi_3),\quad \nabla^2 f(\xi)=\begin{bmatrix}2&0&0\\0&0&0\\0&0&2\end{bmatrix},$$
and, fixed the first $i\in\{1,3\}$ such that $f_{x_i}(\xi)\neq 0$, we have
$$\mathbf{T}_{F_{a,b}}(\xi)=\left\{(v_1,v_2,v_3): v_i=-\frac{\xi_j}{\xi_i}v_j,\ j\in\{1,3\}\setminus\{i\},\ v_2\in\mathbb{R}\right\}.$$
Then
$$\left\langle\nabla^2 f(\xi)\,u^j(\xi),u^j(\xi)\right\rangle=\begin{cases}0, & \text{if } j=2\\ 2\left(\frac{\xi_j}{\xi_i}\right)^2+2, & \text{if } j\in\{1,3\}\setminus\{i\}\end{cases}$$
and consequently
$$\hat{\varkappa}_{F_{a,b}}\left(\xi,u^j(\xi)\right)=\begin{cases}0, & \text{if } j=2\\ \frac{1}{a}, & \text{if } j\in\{1,3\}\setminus\{i\}\end{cases}.$$
If we consider $u(\xi)=\alpha u^2(\xi)+\beta u^j(\xi)$, for any $\alpha,\beta\in\mathbb{R}$ we will obtain
$$\hat{\varkappa}_{F_{a,b}}(\xi,u(\xi))=\frac{\beta^2 a}{\left(a^2\beta^2+\xi_1^2\alpha^2\right)}\in\left]0,\frac{1}{a}\right[.$$

Now fix $\xi = (\xi_1, \xi_2, \xi_3) \in \partial F_{a,b}$ with $\xi_1^2 + \xi_3^2 < a^2$ and $\xi_2 = b$ (the case $\xi_2 = -b$ is analogous). Near ξ we have $f(x_1, x_2, x_3) := x_2 - b$,

$$\nabla f(\xi) = (0, 1, 0), \quad \mathbf{T}_{F_{a,b}}(\xi) = \operatorname{span}\{(1,0,0), (0,0,1)\}$$

and $\nabla^2 f(\xi)$ is the zero matrix of the order n. So

$$\hat{\varkappa}_{F_{a,b}}(\xi, u(\xi)) = 0, \quad \forall u(\xi) \in \mathbf{T}_{F_{a,b}}(\xi) \setminus \{(0,0,0)\}.$$

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