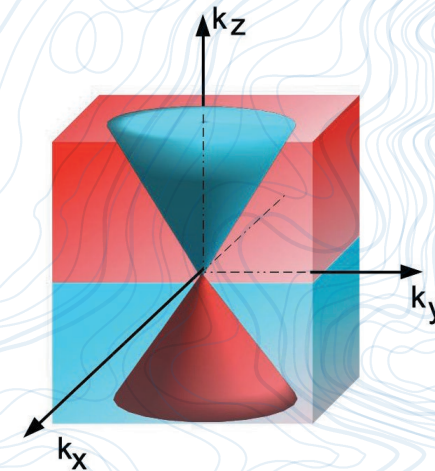


# Topology in Condensed Matter

## An Introduction



Miguel Araújo · Pedro Sacramento

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This text serves as a pedagogical introduction to the theoretical concepts on application of topology in condensed matter systems. It covers an introduction to basic concepts of topology, emphasizes the relation of geometric concepts such as the Berry phase to topology, having in mind applications in condensed matter. In addition to describing two basic systems such as topological insulators and topological superconductors, it also reviews topological spin systems and photonic systems. It also describes the use of quantum information concepts in the context of topological phases and phase transitions, and the effect of non-equilibrium perturbations on topological systems.

This book provides a comprehensive introduction to topological insulators, topological superconductors and topological semimetals. It includes all the mathematical background required for the subject. There are very few books with such a coverage in the market.

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## Preface to the Portuguese Edition

The introduction of the first theoretical model of a topological superconductor by Kitaev in 2001, and the invention of the topological insulator by Kane and Mele in 2005 have led to a renewed interest in condensed matter topological systems. This followed earlier studies of magnetic moments in low dimensional systems, such as in the context of the Kosterlitz-Thouless transition and spin chains, and also in the context of the quantum Hall effect. Indeed, we have witnessed a great proliferation of scientific papers in the last 10 years on the subject, which has been one of the main research areas in the theoretical group of the Center for Physics and Engineering of Advanced Materials at Instituto Superior Técnico (IST) in Lisbon.

Over the years, we increasingly felt the need to provide graduate students at IST with a text allowing them to quickly learn recently developed concepts that are still very dispersed in the literature.

Chapters 1 and 2 present the basic concepts relevant to the classification of topological properties. While the first chapter may be omitted on a first reading, we consider chapter 2 indispensable for the understanding of the remainder of the book. The remaining chapters do not need to be read sequentially, so the reader may directly choose to read those which most interest him or her. The first five chapters contain exercises that will help consolidate the understanding of the concepts and techniques. The remaining chapters contain applications of the basic concepts.

Chapters 3, 4, and 5 are devoted to the three classes of fermionic topological systems this book is mostly concerned with, namely, the topological insulators, superconductors and semimetals.

Spin and photonic systems are discussed in chapters 6 and 7, respectively. Under certain conditions, these systems also have topological properties. In chapter 8 we discuss the application of quantum information methods as an alternative way to understand the properties of topological systems. Finally, the robustness of out-of-equilibrium systems' topological properties is studied in chapter 9. The book includes two appendices discussing some complementary aspects to the presented subjects.

Interacting systems have supposedly been left out of this introductory text, owing both to their complexity and to the fact that they are a still developing subject.

This text serves as a pedagogical introduction to the theoretical concepts on the subject, allowing the advanced student or researcher to acquire the basic knowledge necessary to access the specialized literature. It does

not attempt to provide a general review of this extensive field. Instead, we refer the interested reader to the review articles which include recent experimental and theoretical developments.

The bibliography at the end of each chapter does not aim to be, and could not be, comprehensive. It includes those papers which most influenced the authors in the writing process, or which were found to best serve as complementary reading.

There are several books and review articles which cover topological systems and can be used as complementary reading to this introductory text. We suggest, for instance,

- D.J. Thouless, *Topological quantum numbers in nonrelativistic physics*, World Scientific, (1998).
- B.A. Bernevig, T.L. Hughes, *Topological insulators and topological superconductors*, Princeton University Press (2013).
- S.-Q. Shen, *Topological Insulators* (Springer, Berlin, 2012).
- J. K. Asbóth, L. Oroszlány, and A. P. Pályi, *A Short Course on Topological Insulators*, (Springer, 2016).
- M. Z. Hasan, and C. L. Kane, *Rev. Mod. Phys.* 82, 3045 (2010)
- X.-L. Qi, and S.-C. Zhang, *Rev. Mod. Phys.* 83, 1057 (2011).

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