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Solvability of Coupled Systems of Generalized Hammerstein-Type Integral Equations in the Real Line

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Abstract: In this work, we consider a generalized coupled system of integral equations of Hammerstein-type with, eventually, discontinuous nonlinearities. The main existence tool is Schauder's fixed point theorem in the space of bounded and continuous functions with bounded and continuous derivatives on $\mathbb{R}$, combined with the equiconvergence at $\pm\infty$ to recover the compactness of the correspondent operators. To the best of our knowledge, it is the first time where coupled Hammerstein-type integral equations in real line are considered with nonlinearities depending on several derivatives of both variables and, moreover, the derivatives can be of different order on each variable and each equation. On the other hand, we emphasize that the kernel functions can change sign and their derivatives in order to the first variable may be discontinuous. The last section contains an application to a model to study the deflection of a coupled system of infinite beams.

Keywords: coupled systems; Hammerstein integral equations; real line; L^∞-Carathéodory functions; Schauder's fixed point Theorem; infinite beams

MSC: 45G15; 34B15; 47H30; 34B27; 34L30

1. Introduction

Integral equations are of many types and Hammerstein-type is a particular case of them. These equations appear naturally in inverse problems, fluid dynamics, potential theory, spread of interdependent epidemics, elasticity, . . . (see References [1–3]). Hammerstein-type integral equations usually arise from the reformulation of boundary value problem associated of partial or ordinary differential equations.

Hammerstein-type integral equations in real line play an important role in physical problems and are often used to reformulate or rewrite mathematical problems. For example, the propagation of mono-frequency acoustic or electromagnetic waves over flat nonhomogeneous terrain modeled by the Helmholtz equation

$$\triangle\varphi + k^2\varphi = 0,$$

in the upper half plane $D = \{(x_1, x_2) \subset \mathbb{R}^2 : x_2 > 0\}$ with a Robin or impedance condition

$$\frac{\partial \triangle}{\partial x_2} + ik\beta\varphi = \varphi_0$$

on the boundary line ∂D, where k is a wave number, $\beta \in L^\infty(\partial D)$ is the surface admittance describing the local properties of the ground surface ∂D and $\varphi_0 \in L^\infty(\partial D)$ the inhomogeneous term, can be reformulated as Hammerstein-type integral equations (see Reference [4]). In fact, the authors have shown that the above problem is equivalent to Hammerstein-type integral equations in the real line

$$u(x) - \int_{-\infty}^{+\infty} \tilde{k}(x-y)z(y)u(y)dy = \psi(x), \ x \in \mathbb{R},$$

where ψ is given, $\tilde{k} \in L^1 \cap C(\mathbb{R} \setminus \{0\})$, $z \in L^\infty$ is closely connected with the surface admittance β ($z = i(1-\beta)$) and $u \in BC$ is to be determined. In Reference [5] new variants of some nonlinear alternatives of Leray–Schauder and Krasnosel'skij type were introduced, involving the weak topology of Banach spaces. Along with the proof of theorems on the existence of solutions, profound constructive solvability theorems were proposed with analysis of branching solutions of nonlinear Hammerstein integral equation presented in Reference [6]. Interested readers can find explicit and implicit parameterizations in the construction of branching solutions by iterative methods in Reference [7].

However, due to the lack of compactness, there are only a few studies in the literature on Hammerstein integral equations in the real line compared to works in bounded domains.

By reviewing existence results for various types of functional, differential, and integral equations, in Reference [8], Banaś and Sadarangani use arguments associated with the measure of non-compactness and illustrate applications in proving existence for some integro-functional equations in the set of continuous functions.

Information on the utility, and some mathematical tools used to address Hammerstein-type integral equations in real line or half-line can be found in References [9–14].

We also highlight recent works, not necessarily in real line or half-line, on Hammerstein-type integral equations, with several approaches and applications in References [13,15–22] and the references therein.

On the other hand, Cabada et al. [23] deal with Hammerstein-type integral equations in unbounded domains via spectral theory. More concretely they study the existence of fixed points of the integral operator

$$Tu(t) := \int_{-\infty}^{+\infty} k(t,s)\eta(s)f(s,u(s))ds,$$

where $f : \mathbb{R}^2 \to [0, +\infty)$ satisfies a sort of $L^\infty-$Carathéodory conditions, $k : \mathbb{R}^2 \to \mathbb{R}$ is the kernel function and $\eta(t) \geq 0$ for a.e. $t \in \mathbb{R}$.

IIhan and Ozdemir [24], study the nonlinear perturbed integral equation

$$x(t) = (T_1x)(t) + (T_2x)(t) \int_0^{+\infty} u(t,s,x(s))ds, \quad t \in \mathbb{R}^+,$$

where the functions $u(t,s,x)$ and the operators T_ix, $(i = 1,2)$ are given, while $x = x(t)$ is an unknown function. Using the technique of a suitable measure of noncompactness, they prove an existence theorem for the mentioned system.

Based on several fundamental assumptions and some necessary and sufficient conditions under which the solution blows up in finite time, in Reference [25], Brunner and Yang investigate the blow-up behaviors of solutions of Hammerstein-Volterra-type equations

$$u(t) = \phi(t) + \int_0^t k(t-s)G(s,u(s))ds,$$

where $f : [0, \infty) \to [0, \infty)$ and $G : [0, \infty) \times \mathbb{R} \to [0, \infty)$ are continuous functions, the kernel $k : (0, \infty) \to [0, \infty)$ is a locally integrable function and u is an unknown continuous solution.

Motivated by the works above, we consider the following generalized coupled systems of integral equations of Hammerstein-type

$$\begin{cases} u_1(t) = \int_{-\infty}^{+\infty} k_1(t,s)\, g_1(s)\, f_1(s, u_1(s), \dots, u_1^{(m_1)}(s), u_2(s), \dots, u_2^{(n_1)}(s))\, ds, \\ \\ u_2(t) = \int_{-\infty}^{+\infty} k_2(t,s)\, g_2(s)\, f_2(s, u_1(s), \dots, u_1^{(m_2)}(s), u_2(s), \dots, u_2^{(n_2)}(s)) ds, \end{cases} \tag{1}$$

where $k_\iota : \mathbb{R}^2 \to \mathbb{R}$, $\iota = 1, 2$, are the kernel functions such that $k_\iota \in W^{r_\iota, 1}(\mathbb{R}^2)$, $r_\iota = \max\{m_\iota, n_\iota\}$, with $m_\iota, n_\iota \geq 0$, $g_\iota \in L^1(\mathbb{R})$ with $g_\iota(t) \geq 0$ for a.e. $t \in \mathbb{R}$ integrable, and $f_\iota : \mathbb{R}^{m_\iota + n_\iota + 3} \to \mathbb{R}$ are $L^\infty-$Carathéodory functions.

The main existence tool is Schauder's fixed point theorem in the space of bounded and continuous functions with bounded and continuous derivatives on $\mathbb{R}$, combined with the equiconvergence at $\pm\infty$ to recover the compactness of the correspondent operators. To the best of our knowledge, it is the first time where coupled Hammerstein-type integral equations in real line are considered with nonlinearities depending on several derivatives of both variables and, moreover, the derivatives can be of different order on each variable and each equation. On the other hand, we emphasize that the kernel functions can change sign and their derivatives in order to the first variable may be discontinuous.

The outline of the present paper is as follows—Section 2 contains auxiliary results and assumptions of this paper. In Section 3, we present an existence result. Lastly, an application to a real phenomenon is shown—a model to study the deflection of a coupled system of infinite beams.

2. Auxiliary Results and Assumptions

For $\iota = 1, 2$, let $r_\iota = \max\{m_\iota, n_\iota\}$ and consider the Banach spaces defined by $E_\iota := BC^{r_\iota}(\mathbb{R})$ (space of bounded and continuous functions with bounded and continuous derivatives on $\mathbb{R}$, till order r_ι).

The spaces E_ι defined above are equipped with the norms $\|\cdot\|_{E_\iota}$, where

$$\|w\|_{E_\iota} := \max\left\{\|w^{(i)}\|_\infty,\, i = 0, 1, \dots, r_\iota\right\}$$

and

$$\|w\|_\infty := \sup_{t \in \mathbb{R}} |w(t)|.$$

Besides, $E := E_1 \times E_2$ with the norm

$$\|(u_1, u_2)\|_E := \max\left\{\|u_1\|_{E_1}, \|u_2\|_{E_2}\right\},$$

is also a Banach space.

Definition 1. *A function $h : \mathbb{R} \times \mathbb{R}^q \to [0, \infty)$, for q a positive integer, is $L^\infty-$Carathéodory if*

(i) $h(\cdot, y)$ is measurable for each fixed $y \in \mathbb{R}^q$;
(ii) $h(t, \cdot)$ is continuous for a.e. $t \in \mathbb{R}$;
(iii) for each $\rho > 0$, there exists a function $\varphi_\rho \in L^\infty(\mathbb{R})$ such that,

$$|h(t, y)| \leq \varphi_\rho(t) \text{ for } y \in [-\rho, \rho] \text{ and a.e. } t \in \mathbb{R}.$$

Next lemma and theorem give, respectively, a criterion to guarantee the compacity on $\mathbb{R}$ and the existence of solution via Schauder's fixed point (see References [26,27]).

Lemma 1. *A set $M \subset X$ is relatively compact if the following conditions hold:*

(i) *all functions from M are uniformly bounded;*
(ii) *all functions from M are equicontinuous on any compact interval of $\mathbb{R}$;*
(iii) *all functions from M are equiconvergent at $\pm\infty$, that is, for any given $\epsilon > 0$, there exists $t_\epsilon > 0$ such that*

$$\left| x^{(i)}(t) - \lim_{t\to\pm\infty} x^{(i)}(t) \right| < \epsilon,\ \forall |t| > t_\epsilon,\ i = 0,1,\ldots,r_\iota \text{ and } x \in M.$$

Theorem 1. *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P as at least one fixed point in Y.*

In this paper we consider the following assumptions:

(A1) For $\iota = 1,2$, the function $k_\iota : \mathbb{R}^2 \to \mathbb{R}$, $k_\iota \in W^{r_\iota,1}\left(\mathbb{R}^2\right)$, verify for all $\tau \in \mathbb{R}$,

$$\lim_{t\to\pm\infty} k_\iota(t,s) \in \mathbb{R},\ \lim_{t\to\pm\infty} \left| \frac{\partial^i k_\iota}{\partial t^i}(t,s) \right| \in \mathbb{R},$$

for $i = 1,\ldots,r_\iota,\ s \in \mathbb{R}$, and

$$\lim_{t\to\tau} \left| \frac{\partial^i k_\iota}{\partial t^i}(t,s) - \frac{\partial^i k_\iota}{\partial t^i}(\tau,s) \right| = 0, \text{ for a.e. } s \in \mathbb{R} \text{ and } i = 0,1,\ldots,r_\iota.$$

(A2) For $\iota = 1,2$, there are positive continuous functions $\phi_{\iota j}$, $j = 0,1,\ldots,r_\iota$, such that

$$\left| \frac{\partial^j k_\iota}{\partial t^j}(t,s) \right| \le \phi_{\iota j}(s) \text{ for } t \in \mathbb{R}, \text{ a.e. } s \in \mathbb{R}$$

and

$$\int_{-\infty}^{+\infty} \phi_{\iota j}(s) g_\iota(s) \varphi_{\iota\rho}(s) ds < +\infty \text{ for } j = 0,1,\ldots,r_\iota,$$

with $\varphi_{\iota\rho}$ given by Definition 1.

3. Main Theorem

This section is dedicated to the main result of this article, that is, its statement and its proof and provides the existence of solution of problem (1).

Theorem 2. *Let for $\iota = 1,2$, $f_\iota : \mathbb{R}^{m_\iota+n_\iota+3} \to \mathbb{R}$ be $L^\infty-$Carathéodory functions, such that, for some $t \in \mathbb{R}$, $f_\iota(t,0,\ldots,0) \neq 0$, and $g_\iota \in L^1(\mathbb{R})$ with $g_\iota(t) \ge 0$ for a.e. $t \in \mathbb{R}$.*

Consider that assumptions $(A1)$, $(A2)$ hold, and, moreover, assume that there is $R > 0$, such that, for $j = 0,1,\ldots,r_\iota$,

$$R > \max \left\{ \int_{-\infty}^{+\infty} \phi_{\iota j}(s) g_\iota(s) \varphi_R(s) ds \right\}, \tag{2}$$

where $\varphi_R \in L^\infty(\mathbb{R})$, $y \in \mathbb{R}^{m_\iota+n_\iota+2}$ with $|f_\iota(t,y)| \le \varphi_R(t)$, a.e. $t \in \mathbb{R}$. Then problem (1) has a nontrivial solution $(u, v) \in E_1 \times E_2$.

Proof. Consider the operators $T_1 : E \to E_1$ and $T_2 : E \to E_2$ such that

$$\begin{cases} T_1(u_1,u_2)(t) = \int_{-\infty}^{+\infty} k_1(t,s)\, g_1(s)\, f_1(s,u_1(s),\ldots,u_1^{(m_1)}(s),u_2(s),\ldots,u_2^{(n_1)}(s))\, ds, \\ T_2(u_1,u_2)(t) = \int_{-\infty}^{+\infty} k_2(t,s)\, g_2(s)\, f_2(s,u_1(s),\ldots,u_1^{(m_2)}(s),u_2(s),\ldots,u_2^{(n_2)}(s))\, ds. \end{cases} \tag{3}$$

Next, we will show that the operator $T : E \to E$ defined by $T = (T_1, T_2)$ has a fixed point on E.

The proof will follow several steps, presented in detail for operator $T_1(u,v)$. The technique for operator $T_2(u,v)$ is similar.

Step 1: T *is well defined and uniformly bounded in* E.

Consider a bounded set $D \subseteq E$ and $(u,v) \in D$. Therefore, there is $\rho_1 > 0$ such that

$$\|(u,v)\|_E \leq \rho_1. \tag{4}$$

By the Lebesgue Dominated Theorem, $(A1)$, $(A2)$ and because f_1 is $L^\infty-$Carathéodory function, follow that, for $i = 0,1,\ldots,r_1$,

$$\begin{aligned}
\|T_1\,(u_1,u_2)^{(i)}\|_\infty &= \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{+\infty} \frac{\partial^i k_1}{\partial t^i}(t,s)\, g_1(s)\, f_1 \left(\begin{array}{c} s, u_1(s),\ldots,u_1^{(m_1)}(s), \\ u_2(s),\ldots,u_2^{(n_1)}(s) \end{array} \right) ds \right| \\
&\leq \int_{-\infty}^{+\infty} \phi_{1i}(s) g_1(s) \left| f_1 \left(\begin{array}{c} s, u_1(s),\ldots,u_1^{(m_1)}(s), \\ u_2(s),\ldots,u_2^{(n_1)}(s) \end{array} \right) \right| ds \\
&\leq \int_{-\infty}^{+\infty} \phi_{1i}(s) \varphi_{1\rho_1}(s) g_1(s)\, ds < +\infty.
\end{aligned}$$

Taking into account these arguments, T_2 verifies similar bounds and $\|T(u,v)\|_E < +\infty$, that is $TD \subseteq E$ is uniformly bounded.

Step 2: T *is equicontinuous in* E.

Consider $t_1,\ t_2 \in \mathbb{R}$ and suppose without loss of generality, that $t_1 \leq t_2$. So, by $(A1)$, for $i = 0,1,\ldots,r_1$,

$$\begin{aligned}
& |(T_1\,(u_1,u_2))^{(i)}(t_1) - (T_1\,(u_1,u_2))^{(i)}(t_2)| \\
\leq\ & \int_{-\infty}^{+\infty} \left| \frac{\partial^i k_1}{\partial t^i}(t_1,s) - \frac{\partial^i k_1}{\partial t^i}(t_2,s) \right| |g_1(s)| \left| f_1 \left(\begin{array}{c} s, u_1(s),\ldots,u_1^{(m_1)}(s), \\ u_2(s),\ldots,u_2^{(n_1)}(s) \end{array} \right) \right| ds \\
\leq\ & \int_{-\infty}^{+\infty} \left| \frac{\partial^i k_1}{\partial t^i}(t_1,s) - \frac{\partial^i k_1}{\partial t^i}(t_2,s) \right| g_1(s)\ \varphi_{1\rho_1}(s) ds \to 0, \text{ as } t_1 \to t_2.
\end{aligned}$$

Therefore, T_1D is equicontinuous in E_1. In the same way it can be shown that T_2D is equicontinuous on E_2. Thus, TD is equicontinuous on E.

Step 3: TD *is equiconvergent at* $\pm\infty$.

Consider $(u,v) \in D \subseteq E$ and $i = 0,1,\ldots,r_1$. For the operator T_1, we have by (A1),

$$\begin{aligned}
& |(T_1\,(u_1,u_2))^{(i)}(t) - \lim_{t\to\pm\infty} (T_1\,(u_1,u_2))^{(i)}(t)| \\
\leq\ & \int_{-\infty}^{+\infty} \left| \frac{\partial^i k_1}{\partial t^i}(t,s) - \frac{\partial^i k_1}{\partial t^i}(\pm\infty,s) \right| |g_1(s)| \left| f_1 \left(\begin{array}{c} s, u_1(s),\ldots,u_1^{(m_1)}(s), \\ u_2(s),\ldots,u_2^{(n_1)}(s) \end{array} \right) \right| ds \\
\leq\ & \int_{-\infty}^{+\infty} \left| \frac{\partial^i k_1}{\partial t^i}(t,s) - \frac{\partial^i k_1}{\partial t^i}(\pm\infty,s) \right| g_1(s)\ \varphi_{1\rho_1}(s) ds \to 0, \text{ as } t \to \pm\infty.
\end{aligned}$$

T_1D is equiconvergent at $\pm\infty$. By similar arguments, it can be proved that T_2D is equiconvergent at $\pm\infty$. Moreover, TD is equiconvergent at $\pm\infty$. By Lemma 1, TD is relatively compact .

Step 4: $T\Omega \subset \Omega$ for some $\Omega \subset E$ *a closed, bounded and convex set.*

Consider

$$\Omega := \{(u,v) \in E : \|(u,v)\|_E \leq \rho_2\},$$

with $\rho_2 > 0$ such that, for $\iota = 1,2$ and $i = 0,\dots,r_\iota$,

$$\rho_2 := \max\left\{\rho_1, \int_{-\infty}^{+\infty} \phi_{\iota j}(s) g_\iota(s) \varphi_{\rho_1}(s) ds\right\}$$

with $\rho_1 > 0$ given by (4).

Following the arguments used in Step 1 we have, for $(u,v) \in \Omega$,

$$\begin{aligned} \|T(u,v)\|_E &= \|(T_1(u,v), T_2(u,v))\|_E \\ &= \max\left\{\|T_1(u,v)\|_{E_1}, \|T_2(u,v)\|_{E_2}\right\} \leq \rho_2. \end{aligned}$$

So, $T\Omega \subset \Omega$, and by Theorem 1, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$, has a fixed point $(u, v) \in E_1 \times E_2$, that is, the problem (1) has at least one solution. □

Remark 1. *If, for $\iota = 1,2$,*

$$\lim_{t\to-\infty} k_\iota(t,s) = \lim_{t\to+\infty} k_\iota(t,s),$$

then the solution of (1) is a homoclinic solution, otherwise it is a heteroclinic solution.

4. Application to Fourth Order Coupled Systems of Infinite Beams Deflection Model

Recently, in Reference [28], the authors studied two-beam coupled structure as two infinite beams and considering the coupling between the bending wave and the torsion, the conversion of wave types at the coupled interface, as well as others details on the coupling of beams.

Motivated by the concept of very large floating structures and ice plates in waves, in Reference [29], Jang et al. consider the inverse loading distribution from measured deflection of an infinite beam on elastic foundation. They express the relationship between the loading distribution and vertical deflection of the beam in the form of an integral equation of the first kind.

An efficient method for the static deflection analysis of an infinite beam on a nonlinear elastic foundation is developed in Reference [30], where the authors combine the quasilinear method and Green's functions to obtain the approximate solution.

Motivated by the works above and, specifically, in Reference [31], where the authors analyze of moderately large deflections of infinite nonlinear beams resting on elastic foundations under localized external loads, we consider an arbitrary family of nonlinear coupled system of Bernoulli–Euler–v. Karman problem composed by two fourth order differential equation

$$\begin{cases} E_1 I_1 u^{(4)}(x) + \eta_1 u(x) = \frac{1}{(1+x^2)^2}\left[\frac{3}{2}E_1 A_1 (u'(x))^2 v''(x) + \omega_1(x)\right], \quad x \in \mathbb{R} \\ \\ E_2 I_2 v^{(4)}(x) + \eta_2 v(x) = \frac{1}{(1+x^2)^2}\left[\frac{3}{2}E_2 A_2 (v'(x) u'(x))^2 u''(x) + \omega_2(x)\right] \end{cases} \tag{5}$$

and the boundary conditions

$$\begin{cases} u(\pm\infty) = 0, \; v(\pm\infty) = 0, \\ u'(\pm\infty) = 0, \; v'(\pm\infty) = 0, \end{cases} \tag{6}$$

where

$$w(\pm\infty) := \lim_{x\to\pm\infty} w(x).$$

We also emphasize that:

- E_i, I_i, $i = 1,2$ are the Young's modulus (the elastic modulus of the material) and the mass moment of inertia of the cross section of beam 1 and beam 2, respectively;
- $\eta_1 u(x)$, $\eta_2 v(x)$ are the spring force upward of first and second beams, respectively;
- A_1, A_2 are the cross-sectional area of first and second beams, respectively;
- $\omega_1(x)$, $\omega_2(x)$ are the positive localized applied loading downward on the corresponding beams.

In fact, the differential system (5) and (6) can be rewritten as the following system of integral equations

$$\begin{cases} u(x) = \int_{-\infty}^{+\infty} k_1(x,s)\ \frac{1}{E_1 I_1} \frac{1}{(1+s^2)^2} \left[\frac{3}{2} E_1 A_1 \left(u'(s)\right)^2 v''(s) + \omega_1(s) + \eta_1 u(s)\right] ds, \\ v(x) = \int_{-\infty}^{+\infty} k_2(x,s)\ \frac{1}{E_2 I_2} \frac{1}{(1+s^2)^2} \left[\frac{3}{2} E_2 A_2 \left(v'(s) u'(s)\right)^2 u''(s) + \omega_2(s) + \eta_2 v(s)\right] ds, \end{cases} \tag{7}$$

where the kernel functions $k_1(x,s)$ and $k_2(x,s)$ are given, respectively, by the corresponding Green's functions

$$k_\iota(x,s) = \frac{e^{-\alpha_\iota |s-x|}}{\sqrt{2}^5 \alpha_\iota^3} \sin\left(\alpha_\iota |s-x| + \frac{\pi}{4}\right), \tag{8}$$

with $\alpha_\iota = \frac{\sqrt{2}}{2} \sqrt[4]{\frac{\eta_\iota}{E_\iota I_\iota}}$ *for* $\iota = 1,2$.

For $j = 0,1,2$, $\iota = 1,2$, and defining

$$k_{\iota j}^-(x,s) := \frac{e^{\alpha_\iota (s-x)}}{\sqrt{2}^{5-j} \alpha_\iota^{3-j}} \sin\left(\alpha_\iota (x-s) + \frac{\pi(3j+1)}{4}\right),$$

$$k_{\iota j}^+(x,s) := \frac{e^{\alpha_\iota (x-s)}}{\sqrt{2}^{5-j} \alpha_\iota^{3-j}} \sin\left(\alpha_\iota (s-x) + \frac{\pi(3j+1)}{4}\right),$$

we have

$$u^{(j)}(x) = \int_{-\infty}^{x} k_{1j}^-(x,s) F_1(s) ds + (-1)^j \int_{x}^{+\infty} k_{1j}^+(x,s) F_1(s) ds, \tag{9}$$

and

$$v^{(j)}(x) = \int_{-\infty}^{x} k_{2j}^-(x,s) F_2(s) ds + (-1)^j \int_{x}^{+\infty} k_{2j}^+(x,s) F_2(s) ds,$$

with

$$\begin{aligned} F_1(s) &= \frac{1}{1+s^2} \left[\frac{3}{2} E_1 A_1 \left(u'(s)\right)^2 v''(s) + \omega_1(s) + \eta_1 u(s)\right], \\ F_2(s) &= \frac{1}{1+s^2} \left[\frac{3}{2} E_2 A_2 \left(v'(s) u'(s)\right)^2 u''(s) + \omega_2(s) + \eta_2 v(s)\right]. \end{aligned}$$

The system (7) is a particular case of (1) with $r_1 = r_2 = m_1 = m_2 = n_1 = n_2 = 2$, $g_1(x) = \frac{1}{1+x^2} \frac{1}{E_1 I_1}$, $g_2(x) = \frac{1}{1+x^2} \frac{1}{E_2 I_2}$, $E_1 I_1 > 0$, $E_2 I_2 > 0$ and

$$\begin{aligned} f_1(x, y_1, \ldots, y_6) &= \frac{1}{1+x^2} \left[\frac{3}{2} E_1 A_1 y_2^2 y_6 + \omega_1(x) + \eta_1 y_1\right], \\ f_2(x, y_1, \ldots, y_6) &= \frac{1}{1+x^2} \left[\frac{3}{2} E_2 A_2 \left(y_5 y_2\right)^2 y_3 + \omega_2(x) + \eta_2 y_4\right]. \end{aligned}$$

The functions $f_1,\ f_2 : \mathbb{R}^7 \to \mathbb{R}$, respectively, are $L^\infty-$Carathéodory functions, as, for $\rho > 0$ such that

$$\max\left\{|y_1|, |y_2|, |y_3|, |y_4|, |y_5|, |y_6|\right\} < \rho,$$

there exist functions $\varphi_{1\rho}, \varphi_{2\rho} \in L^\infty(\mathbb{R})$ verifying

$$f_1(x, y_1, \ldots, y_6) \le \frac{1}{1+x^2}\left[\frac{3}{2}E_1 A_1 \rho^3 + \sup_{x\in\mathbb{R}} \omega_1(x) + \eta_1\rho\right] := \varphi_{1\rho}(x),$$

$$f_2(x, y_1, \ldots, y_6) \le \frac{1}{1+x^2}\left[\frac{3}{2}E_2 A_2 \rho^3 + \sup_{x\in\mathbb{R}} \omega_2(x) + \eta_2\rho\right] := \varphi_{2\rho}(x).$$

Note also that (A1) and (A2) are satisfied, since, for $\iota = 1, 2$ and $j = 0, 1, 2$,

$$\lim_{x\to\pm\infty} \frac{\partial^j k_\iota}{\partial x^j}(x,s) = 0,$$

$$\left|\frac{\partial^j k_\iota}{\partial x^j}(x,s)\right| \le \frac{1}{\sqrt{2}^{5-j}\alpha_\iota^{3-j}} := \phi_{\iota j},\ \forall s \in \mathbb{R},$$

and

$$\int_{-\infty}^{+\infty} \phi_{\iota j}(s) g_\iota(s) \varphi_{\iota\rho}(s) ds < +\infty.$$

So, by Theorem 2, there is at least a nontrivial solution $(u, v) \in E_1 \times E_2$ of problems (5) and (6). In addition, by Remark 3 the solution is a nontrivial homoclinic solution.

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