Neighbourhood retractions of nonconvex sets in a Hilbert space via sublinear functionals*

Vladimir V. Goncharov†and Fátima F. Pereira
CIMA-UE, Rua Romão Ramalho 59, 7000-671, Évora, Portugal‡

Abstract

For a closed subset C of a Hilbert space $(H, \|\cdot\|)$ and for a sublinear functional $\rho : H \to \mathbb{R}^+$, which is equivalent to the norm $\|\cdot\|$, we give conditions guaranteeing existence and uniqueness of the nearest points to C in the sense of the semidistance generated by ρ. This permits us to construct a continuous retraction onto C well defined in a neighbourhood $\mathcal{U} \supset C$. In particular, according to one of the conditions, $\mathcal{U}$ can be represented in terms of balance between the local strict convexity modulus of ρ and the measure of nonconvexity of the set C at each point.

Key words: *Time-minimum problem, Minkowski functional, generalized projection, strict convexity, curvature, proximal normals*

Mathematical Subject Classification(2000**)**: 49J52, 49N15

1 Introduction

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. As well known, for any convex closed set $C \subset H$ each $x \in H$ admits the unique nearest point (so called *metric projection*) $\pi_C(x) \in C$, i.e., such that $\|x - \pi_C(x)\| = \mathrm{d}_C(x) := \inf\{\|x - y\| : y \in C\}$ (*Chebyshev property* of convex sets), and, moreover, the mapping $x \mapsto \pi_C(x)$ is continuous (even Lipschitzean with the Lipschitz constant 1). On the other hand, the class of sets admitting such type continuous retraction $\pi_C : H \to C$ consists just of convex closed sets (see [1, 5]).

The further natural question is to describe the class of sets for which the continuous projection is well defined not on the whole space H but on some neighbourhood $\mathcal{U}$ of C. Such sets were studied by many authors starting from the pioneer work by H. Federer [19] (see, e.g., [27, 7, 26, 10, 25, 12, 13, 3]

*Work supported by: projecto POCI/MAT/56727/2004 and Financiamento Programático Especial do CIMA-UE (Centro de Investigação em Matemática e Aplicações da Universidade de Évora, Portugal) both of FCT (Fundação para Ciência e Tecnologia, Portugal)

†Corresponding author. Tel.: +351 936514934; Fax: +351 266745393

‡E-mail: goncha@uevora.pt (V.Goncharov); fmfp@uevora.pt (F.Pereira)

and the bibliography therein). They appear in the literature under various names such as the *sets with positive reach* [19], *p-convex* [7] or *φ-convex* [12, 13] *sets*, *proximally smooth sets* [10], *$O(2)$-convex sets* [26] and so on. Roughly speaking, these sets could be characterized by the following geometric property: given $\bar{x} \in \partial C$ for any $x, y \in C$ near $\bar{x}$ a convex combination $\lambda x + (1-\lambda) y$ (not necessarily belonging to C) is distant from C not more than of the order $O\left(\|x-y\|^2\right)$. An exact (analytic) definition will be given in sequel. Here and further on ∂C stands for the *boundary* of the set C. Notice that φ-convexity is equivalent to a series of other properties such as smoothness of the distance function $\mathrm{d}_C(\cdot)$ in the open domain $\mathcal{U} \setminus C$, and the choice of name depends on which of them one wishes to emphasize.

Observe that the distance $\mathrm{d}_C(x)$ can be seen as the minimum time necessary to reach the boundary ∂C starting from the point $x \notin C$ by trajectories of the control system

$$\dot{x}(t) = v(t), \quad \|v(t)\| \leq 1, \tag{1.1}$$

and the projection $\pi_C(x)$ is nothing else than the point on the target set attainable for this time. As already said, the well-posedness of $x \mapsto \pi_C(x)$ is equivalent to the regularity of the minimum time function $x \mapsto \mathrm{d}_C(x)$ whose gradient is equal to $(x - \pi_C(x)) / \|x - \pi_C(x)\|$. Moreover, $\mathrm{d}_C(\cdot)$ is the (unique) *viscosity solution* of the *Hamilton-Jacobi equation*

$$\|\nabla u(x)\| = 1, \quad u\,|_{\partial C} = 0, \tag{1.2}$$

in the sense of M. Crandall and P.-L. Lions [16] (see also [4]).

Slightly extending this problem (see [14, 15]) we can consider instead of the closed unit ball in (1.1) (denoted further by $\overline{B}$) an arbitrary closed convex bounded subset $F \subset H$, containing the origin in its interior (we need the last condition in order to guarantee controllability). So that, given a point $x \in H$ we are led to study the following time optimal control problem:

$$\min \left\{ T > 0 : \exists x(\cdot),\ x(T) \in C,\ x(0) = x, \right. \\ \left. \textit{and}\ \dot{x}(t) \in F \textit{ a.e. in } [0,T] \right\}. \tag{1.3}$$

The set of terminal points $x(T)$ for all functions $x(\cdot)$, which are minimizers in (1.3) (if any), is called further the *time-minimum projection* of x onto C (with respect to F) and is denoted by $\pi_C^F(x)$. We keep the same name and notation for the unique element of $\pi_C^F(x)$ in the case when it is a singleton. Taking into account the fact that each terminal point can be achieved by an affine trajectory (due to convexity of F), we represent the *minimum time function* (value function in (1.3)) as

$$\mathfrak{T}_C^F(x) = \inf_{y \in C} \rho_F(y - x),$$

where $\rho_F(\cdot)$ is the *Minkowski functional* of the set F,

$$\rho_F(\xi) := \inf \{\lambda > 0 : \xi \in \lambda F\}. \tag{1.4}$$

Therefore,

$$\pi_C^F(x) = \left\{ y \in C : \rho_F(y - x) = \mathfrak{T}_C^F(x) \right\}.$$

Earlier some generic properties of this best approximation problem were studied (see [17, 8]), while in [9] a relationship between the local well-posedness of the time-minimum projection and the directional derivatives of the function $\mathfrak{T}_C^F(\cdot)$ (slightly different from the respective relationship in the case of usual metric projection) was proved. The later papers [14, 15, 28] instead were devoted to characterization of various kinds of subdifferentials of $x \mapsto \mathfrak{T}_C^F(x)$ in terms of the normal cones to the set C (in [28] this problem was considered in an arbitrary Banach space). Furthermore, in [15] some conditions guaranteeing the well-posedness of the time-minimum projection were obtained (see Theorem 5.6). They are suitable also for the regularity of the value function $\mathfrak{T}_C^F(\cdot)$, which, similarly to the case $F = \overline{B}$, can be interepreted as the (unique) viscosity solution of the boundary value problem

$$\rho_{F^0}(-\nabla u(x)) = 1, \quad u|_{\partial C} = 0$$

(compare with (1.2)). Under these conditions, requiring φ-convexity of the target set C (with $\varphi = \text{const}$) and some type of uniform strict convexity of F controllable with a parameter $\gamma > 0$, the mapping $x \mapsto \pi_C^F(x)$ is defined and single-valued on a neighbourhood of C, given by some relation between φ and γ. However, these hypotheses are not so sharp as for the usual metric projections and can be essentially refined.

In our paper we propose some way to generalize the well-posedness result of [15]. Namely, under certain assumptions we wish to construct an open neighbourhood of the closed set C basing on a balance between the "scaled" curvatures of C and F, where the existence, the uniqueness (and the continuity as well) of the time-minimum projection $\pi_C^F(\cdot)$ take place. To this end we introduce first (in Sections 3 and 4) some concepts concerning the local structure of a convex body F (and of its polar set F^0) such as moduli of strict convexity (local uniform rotundity) and of uniform smoothness taken essentially from the geometry of Banach spaces (see, e.g., [22, Ch. 5]) and adapted to the case of "asymmetric" norms. Here some concepts of curvature naturally appear. We study their properties and prove a local asymmetric version of the Lindenstrauss duality theorem, which permits, in particular, to obtain a characterization of the curvatures in terms of the second derivative of the dual Minkowski functional.

The main results follow from the fact that under suitable assumptions each minimizing sequence of the functional $x \mapsto \rho_F(x - z)$ on the set C (z belongs to a neighbourhood of C) is a Cauchy sequence. The proof is based on an important property obtained in Section 5 by using the Ekeland's variational principle. Namely, we show, roughly speaking, that given an arbitrary minimizing sequence $\{x_n\}$ one may find sequences $\{x_n'\}$ and $\{x_n''\}$ which are close to $\{x_n\}$ and such that the difference between some outward normal vector to the set C at the point $x_n' \in C$ and an inward normal to a suitable homothetic transformation of F at x_n'' tends to zero.

In Section 6 we prove the general retraction theorem (Theorem 6.1), presenting two types of sufficient conditions. One of them does not use neither φ-convexity of the set C (its boundary can even have "inward corner" points) nor some kind of uniform rotundity of F, and the other essentially generalizes the known hypotheses. Next (Theorem 6.2) we give an explicit formula for the neighbourhood of C where the retraction is defined. These results are then concretized for the case of a target set with smooth boundary (Theorems 7.1-7.3) as well as under the second order differentiability hypothesis for the polar set F^0 (Theorem 7.4). Finally, in the last section we join some examples illustrating the obtained results.

2 Basic notations and definitions

We consider a convex closed bounded set $F \subset H$ such that $0 \in \operatorname{int} F$ ("int" stands for the *interior* of F), and denote by F^0 its *polar set*, i.e.,

$$F^0 := \{\xi^* \in H : \langle \xi, \xi^* \rangle \leq 1 \ \ \forall \xi \in F\}.$$

Together with the *Minkowski functional* $\rho_F(\xi)$ defined by (1.4) we introduce the *support function* $\sigma_F : H \to \mathbb{R}^+$, $\sigma_F(\xi^*) := \sup\{\langle \xi, \xi^* \rangle : \xi \in F\}$, and observe that

$$\rho_F(\xi) = \sigma_{F^0}(\xi), \tag{2.1}$$

and, consequently,

$$\frac{1}{\|F\|}\|\xi\| \leq \rho_F(\xi) \leq \left\|F^0\right\| \|\xi\|, \quad \xi \in H, \tag{2.2}$$

where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$. The inequalities (2.2) mean that $\rho_F(\cdot)$ is a sublinear functional "equivalent" to the norm $\|\cdot\|$. It is not a norm since $-F \neq F$ in general. As a consequence of (2.1) and (2.2) we have the Lipschitz property

$$|\rho_F(\xi_1) - \rho_F(\xi_2)| \leq \left\|F^0\right\| \|\xi_1 - \xi_2\|. \tag{2.3}$$

In what follows we use the so-called *duality mapping* $\mathfrak{J}_F : \partial F^0 \to \partial F$ that associates with each $\xi^* \in \partial F^0$ the set

$$\mathfrak{J}_F(\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}.$$

If there is no ambiguity (the set F is fixed) then we denote the duality mapping simply by $\mathfrak{J}(\cdot)$. We say also that (ξ, ξ^*) is the *dual pair* when $\xi^* \in \partial F^0$ and $\xi \in \mathfrak{J}_F(\xi^*)$.

Let us denote by $\mathbf{N}_F(\xi)$ the *normal cone* to F at the point $\xi \in F$ and by $\partial\rho_F(\xi)$ the *subdifferential* of the function $\rho_F(\cdot)$ in the sense of Convex Analysis. Notice that for each $\xi^* \in \partial F^0$ the set $\mathfrak{J}_F(\xi^*)$ is nothing else than $\partial\rho_{F^0}(\xi^*)$, and $\mathfrak{J}_F^{-1}(\xi) = \mathbf{N}_F(\xi) \cap \partial F^0$, $\xi \in \partial F$. As well known, the mapping $v \mapsto \sigma_{\partial\rho_F(\xi)}(v)$ coincides with the *directional derivative* of $\rho_F(\cdot)$ at $\xi \in H$ defined by

$$\mathbf{D}\rho_F(\xi)(v) := \lim_{\lambda \to 0+} \frac{\rho_F(\xi + \lambda v) - \rho_F(\xi)}{\lambda}, \quad v \in H. \tag{2.4}$$

If $\mathbf{D}\rho_F(\xi)(-v) = -\mathbf{D}\rho_F(\xi)(v)$, and the convergence in (2.4) is uniform with respect to v from each bounded subset of H, then the function $\rho_F(\cdot)$ is *Fréchet differentiable* at the point ξ. In this case $\partial\rho_F(\xi) = \{\nabla\rho_F(\xi)\}$ where the *Fréchet derivative* (or *gradient*) $\nabla\rho_F(\xi)$ is the unique vector such that $\mathbf{D}\rho_F(\xi)(v) = \langle\nabla\rho_F(\xi), v\rangle$, $v \in H$.

On the other hand, a target set $C \subset H$ is assumed only to be nonempty and closed. Various concepts of normal (and tangent) cones to C at a point $x \in C$ can be found, e.g., in [11, 23]. However, in sequel we use mainly the *proximal normal cone*

$$\mathbf{N}_C^p(x) := \left\{ v : \exists\sigma > 0 \textit{ such that } \langle v, y - x\rangle \leq \sigma\|y - x\|^2 \textit{ for all } y \in C \right\}. \quad (2.5)$$

Denote also by $\mathbf{N}_C^l(x)$ the so-called *Mordukhovich* (or *limiting*) *normal cone*, which in the case of Hilbert space consists of all weak limits of the sequences $v_n \in \mathbf{N}_C^p(x_n)$ such that $x_n \to x$, $x_n \in C$ (see [23, p.240]).

For each $v \in \mathbf{N}_C^p(x)$, $v \neq 0$, let us define

$$\psi_C(x, v) := \frac{1}{\|v\|} \sup_{y \in C\setminus\{x\}} \frac{\langle v, y - x\rangle}{\|y - x\|^2} < +\infty$$

that measures degree of "prominence" (or "cavity") of the set C at the point x with respect to the direction v. In particular, if $\psi_C(x, v) > 0$ then we have another representation:

$$\frac{1}{2\|v\|\psi_C(x, v)} = \sup\{\lambda > 0 : \mathrm{d}_C(x + \lambda v) = \lambda\|v\|\},$$

i.e., each sphere centred on the half-line $\{x + \lambda v : \lambda > 0\}$ and touching the boundary ∂C at x only has a radius $r \leq \frac{1}{2\psi_C(x,v)}$. Otherwise ($\psi_C(x, v) \leq 0$) such sphere can have a radius arbitrarily large. Setting

$$\hat{\psi}_C(x, v) := \frac{1}{\|v\|} \limsup_{C \ni y \to x} \frac{\langle v, y - x\rangle}{\|y - x\|^2}$$

we get a local characteristic of the set C. Observe that C is "concave" at x with respect to the direction v whenever $\hat{\psi}_C(x, v) > 0$, and $\frac{1}{2\hat{\psi}_C(x,v)}$ is the "concavity radius". For some purposes (compare, for instance, with the definitions of Section 3) the number $-\hat{\psi}_C(x, v)$ can be interpreted as *exterior (negative) curvature* of the (nonconvex) set C. It is convenient to set also $\psi_C(x, 0) = \hat{\psi}_C(x, 0) = 0$. Since $\hat{\psi}_C(x, v) < +\infty$ iff $\psi_C(x, v) < +\infty$ (see [11, p.25]), we have

$$\mathbf{N}_C^p(x) = \left\{ v \in H : \hat{\psi}_C(x, v) < +\infty \right\}.$$

Let us define the *"reduced" boundary*

$$\partial^* C := \{x \in \partial C : \mathbf{N}_C^p(x) \neq \{0\}\},$$

which is dense in ∂C (see [11, p. 49]).

If $\psi_C(x,v)$ is majorized by some continuous nonnegative function (say $\varphi(\cdot)$) uniformly in $v \in \mathbf{N}_C^p(x)$ (i.e., $\psi_C(x,v) \leq \varphi(x)$ for all $x \in \partial C$ and $v \in \mathbf{N}_C^p(x)$) then the set C is said to be *φ-convex* (or *proximally smooth*). Another definition in terms of "almost monotonicity" of the normal cone can be given. Namely, a closed set $C \subset H$ is φ-convex iff for some continuous function $\varphi : C \to \mathbb{R}^+$ the inequality

$$\langle v-w, x-y\rangle \geq -\left(\varphi(x)\|v\| + \varphi(y)\|w\|\right)\|x-y\|^2$$

holds whenever $x, y \in C$, $v \in \mathbf{N}_C^p(x)$ and $w \in \mathbf{N}_C^p(y)$. If C is convex then we clearly set $\varphi(x) \equiv 0$. Since all basic normal cones to a φ-convex set coincide (see, e.g., [13, Proposition 6.2]), there is no ambiguity to write $\mathbf{N}_C(x)$ in the place of $\mathbf{N}_C^p(x)$.

Finally, we say that a closed set $C \subset H$ *has smooth* (or $\mathcal{C}^1$) *boundary* at the point $x_0 \in \partial C$ if there exist $\varepsilon > 0$ and a continuous mapping $\mathfrak{n}$: $\partial C \cap \left(x_0 + \varepsilon\overline{B}\right) \to \partial\overline{B}$ such that $\mathfrak{n}(x)$ is a unique vector from $\mathbf{N}_C^l(x)$ with $\|\mathfrak{n}(x)\| = 1$. If this property is satisfied globally (i.e., $\mathbf{N}_C^l(x) \cap \partial\overline{B}$ is a singleton continuously depending on $x \in \partial C$) then the boundary of C is said to be *smooth*.

3 Moduli of local strict convexity and curvatures

Let us recall first the *modulus of local uniform rotundity* (or *local uniform convexity*) well known in the geometry of Banach spaces [22, p.460] but applied here to convex sets and to their Minkowski functionals (see also [17, 8]).

Definition 3.1 *Given a convex closed bounded set $F \subset H$ with $0 \in \operatorname{int} F$, $\xi \in \partial F$ and $r > 0$ we put*

$$\delta_F(r,\xi) := \inf\left\{\rho_F(\xi) + \rho_F(\eta) - \rho_F(\xi+\eta) : \ \eta \in F, \ \rho_F(\eta-\xi) \geq r\right\}. \quad (3.1)$$

Taking into account the agreement $\inf\varnothing := +\infty$, we can assume that r in this definition admits any positive value.

The modulus $\delta_F(\cdot,\xi)$ shows, in fact, how far the sublinear functional $\rho_F(\cdot)$ is from a linear one in a neighbourhood of the point $\xi \in \partial F$. It is clear that always $\delta_F(r,\xi) \geq 0$, and, following the tradition (see [21]), the set F is said to be *uniformly rotund* (or *uniformly strictly convex*) at the point ξ if $\delta_F(r,\xi) > 0$ for all $r > 0$.

However, as we'll see later, the moduli suggested below are more suitable for the asymmetric case than (3.1).

Definition 3.2 *Let $\xi \in \partial F$ and $\xi^* \in \partial F^0$ be points such that $\langle \xi, \xi^* \rangle = 1$ (or, in other words, $\xi \in \mathfrak{J}(\xi^*)$). We define three* moduli of strict convexity *of the set F at the point ξ with respect to (w.r.t.) the direction ξ^* by the formulas:*

$$\begin{aligned} \mathfrak{C}_F^+(r, \xi, \xi^*) &:= \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \rho_F(\eta - \xi) \geq r\}\,; \\ \mathfrak{C}_F^-(r, \xi, \xi^*) &:= \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \rho_F(\xi - \eta) \geq r\}\,; \\ \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) &:= \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F,\ \|\xi - \eta\| \geq r\}\,, \end{aligned} \tag{3.2}$$

$r > 0$.

Observe that for all $r > 0$ the inequality $\mathfrak{C}_F^+(r, \xi, \xi^*) \geq \delta_F(r, \xi)$ holds. Indeed, for each $\eta \in F$ with $\rho_F(\eta - \xi) \geq r$ by (2.1) we have

$$\delta_F(r, \xi) \leq 2 - \rho_F(\xi + \eta) \leq 2\langle \xi, \xi^* \rangle - \langle \xi + \eta, \xi^* \rangle = \langle \xi - \eta, \xi^* \rangle\,.$$

But the opposite inequality is violated even in the simplest cases. For example, if $F = \overline{B}$, $\|\xi\| = 1$ and $\xi^* = \xi$ ($\mathfrak{J}^{-1}(\xi) = \{\xi\}$ is singleton) then the direct calculations give $\delta_F(r, \xi) = \frac{r^2}{2+\sqrt{4-r^2}}$ while $\mathfrak{C}_F^{\pm}(r, \xi, \xi^*) = \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = r^2/2$, $0 < r \leq 2$.

Due to (2.2) we also have the following inequalities:

$$\mathfrak{C}_F^{\pm}\left(\frac{r}{\|F\|}, \xi, \xi^*\right) \leq \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) \leq \mathfrak{C}_F^{\pm}\left(\|F^0\|\, r, \xi, \xi^*\right), \quad r > 0. \tag{3.3}$$

Definition 3.2 suggests another concept of strict convexity. Namely, the set F is said to be (*locally*) *strictly convex at the point $\xi \in \partial F$ w.r.t. $\xi^* \in \mathfrak{J}^{-1}(\xi)$* if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0$ for all $r > 0$. The modulus $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*)$ here can be, certainly, substituted by $\mathfrak{C}_F^{\pm}(r, \xi, \xi^*)$ (see (3.3)). This, obviously, implies that ξ is an *exposed point* of F, and the vector ξ^* *exposes* ξ in the sense that the hyperplane $\{\eta \in H : \langle \eta, \xi^* \rangle = \sigma_F(\xi^*)\}$ touches F at the point ξ only, or, in other words, that $\mathfrak{J}(\xi^*) = \{\xi\}$. Therefore, we could speak just about the (*local*) *strict convexity w.r.t. the vector ξ^** (do not refering to the unique $\xi \in \mathfrak{J}(\xi^*)$).

From Definition 3.2 we get also a "strict monotonicity" inequality:

$$\langle \eta - \xi, \eta^* - \xi^* \rangle \geq \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) + \widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \tag{3.4}$$

whenever $\xi \in \mathfrak{J}(\xi^*)$ and $\eta \in \mathfrak{J}(\eta^*)$ with $\|\xi - \eta\| \geq r$, which permits to characterize the local strict convexity in terms of the duality mapping (and in terms of the dual Minkowski functional as well).

Proposition 3.1 *The set F is strictly convex w.r.t. $\xi^* \in \partial F^0$ if and only if one of the following assertions holds:*

(i) *ξ is a strongly exposed point of F w.r.t. ξ^*, i.e., $\mathfrak{J}(\xi^*) = \{\xi\}$ is a singleton, and each sequence $\{\xi_n\} \subset F$ such that $\langle \xi_n, \xi^* \rangle \to \langle \xi, \xi^* \rangle = 1$, $n \to \infty$, converges to ξ strongly ($\|\xi_n - \xi\| \to 0$ as $n \to \infty$);*

(ii) *the duality mapping $\mathfrak{J} : \partial F^0 \to \partial F$ is Hausdorff continuous at ξ^* with $\mathfrak{J}(\xi^*) = \{\xi\}$, which in this case means*

$$\sup_{\eta \in \mathfrak{J}(\eta^*)} \|\eta - \xi\| \to 0 \text{ as } \eta^* \to \xi^*, \ \eta^* \in \partial F^0;$$

(iii) *the function $\rho_{F^0}(\cdot)$ is Fréchet differentiable at ξ^*, and $\nabla \rho_{F^0}(\xi^*) = \xi$.*

Proof. Let us show that the strict convexity of F w.r.t. ξ^* is equivalent to the property (**i**). Assuming that a unique point $\xi \in \mathfrak{J}(\xi^*)$ (here and further on we write $\xi = \mathfrak{J}(\xi^*)$) is not strongly exposed for F (w.r.t. ξ^*) we can choose $\varepsilon > 0$ and a sequence $\{\xi_n\} \subset F$ with $\|\xi_n - \xi\| \geq \varepsilon$ such that $\langle \xi - \xi_n, \xi^* \rangle \to 0$ as $n \to \infty$. Hence, $\widehat{\mathfrak{C}}_F(\varepsilon, \xi, \xi^*) \leq \langle \xi - \xi_n, \xi^* \rangle \to 0$, and the strict convexity is violated. On the other hand, if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = 0$ for some $r > 0$ then by Definition 3.2 there exists a sequence $\{\xi_n\} \subset F$ such that $\|\xi_n - \xi\| \geq r$ and $\langle \xi - \xi_n, \xi^* \rangle \to 0$ as $n \to \infty$. But this is impossible if ξ^* strongly exposes $\xi \in \partial F$.

Equivalence of the conditions (**ii**) and (**iii**) follows from Corollary 2 [2, p. 460], while (**iii**) $\Longleftrightarrow$ (**i**) was proved in [24, Proposition 5.11]. ■

We will need sometimes uniformity in the assumption of the local strict convexity. Namely, given $U \subset \partial F^0$ let us call the set F *uniformly strictly convex w.r.t. the set U* if

$$\beta_U(r) := \inf \left\{ \widehat{\mathfrak{C}}_F(r, \xi, \xi^*) : \xi^* \in U \right\} > 0$$

for all $r > 0$. Here as usual ξ denotes the point $\mathfrak{J}(\xi^*)$ for respective $\xi^* \in U$. If in the definition above U is a neighbourhood of a point $\xi_0^* \in \partial F^0$ then we say that F is *uniformly strictly convex w.r.t. ξ_0^**. This property makes sense mainly in infinite dimensional spaces, where it is stronger than the strict convexity w.r.t. all the vectors near ξ_0^*. By arguing as in Proposition 3.1 we obtain

Proposition 3.2 *If the set F is uniformly strictly convex w.r.t. $U \subset \partial F^0$ then the duality mapping $\mathfrak{J}_F(\cdot)$ is single-valued and uniformly continuous on U.*

Proof. Assuming that the uniform continuity on U does not hold, we find $\varepsilon > 0$ and two sequences $\{\xi_n^*\}$, $\{\eta_n^*\} \subset U$ such that $\|\xi_n^* - \eta_n^*\| \to 0$ as $n \to \infty$ but $\|\mathfrak{J}(\xi_n^*) - \mathfrak{J}(\eta_n^*)\| \geq \varepsilon$, $n = 1, 2, \ldots$. Denoting by $\xi_n := \mathfrak{J}(\xi_n^*)$ and $\eta_n := \mathfrak{J}(\eta_n^*)$, it follows from (3.4) that

$$\langle \eta_n - \xi_n, \eta_n^* - \xi_n^* \rangle \geq \widehat{\mathfrak{C}}_F(\varepsilon, \xi_n, \xi_n^*) + \widehat{\mathfrak{C}}_F(\varepsilon, \eta_n, \eta_n^*) \geq 2\beta_U(\varepsilon) > 0,$$

which is a contradiction. ■

Let us give now a stronger concept of (local) strict convexity.

Definition 3.3 *Fix $\xi^* \in \partial F^0$, and let ξ be the unique element of $\mathfrak{J}(\xi^*)$. The set F is said to be* strictly convex of the order $\alpha > 0$ *(*at the point ξ*)* with respect to ξ^* *if*

$$\hat{\gamma}_{F,\alpha}(\xi, \xi^*) := \liminf_{\substack{(r,\eta,\eta^*) \to (0+,\xi,\xi^*) \\ \eta \in \mathfrak{J}(\eta^*),\, \eta^* \in \partial F^0}} \frac{\widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{r^\alpha} > 0. \tag{3.5}$$

Remark 3.1 *The condition (3.5) means that for some $\theta > 0$ and $\delta > 0$ the inequality*

$$\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \geq \theta r^\alpha \tag{3.6}$$

takes place whenever $\|\eta^* - \xi^*\| \leq \delta$, $\|\eta - \xi\| \leq \delta$, $\eta \in \mathfrak{J}(\eta^*)$, $\eta^* \in \partial F^0$ and $0 < r \leq \delta$. By the monotonicity of the function $r \mapsto \widehat{\mathfrak{C}}_F(r, \eta, \eta^*)$, diminishing if necessary the constant $\theta > 0$, we may suppose that (3.6) is valid for all positive r. Hence, F is uniformly strictly convex w.r.t. ξ^*, and by Proposition 3.2 the duality mapping is single-valued and uniformly continuous in a neighbourhood of ξ^*. In particular, the condition $\eta \to \xi$ in (3.5) is superfluous.

The numbers (3.5) possess the following invariantness property (we do not assume here that $0 \in \operatorname{int} F$).

Proposition 3.3 *Let $y_1, y_2 \in \operatorname{int} F$, $\xi \in \partial F$ and $\xi_1^* \in \mathfrak{J}^{-1}_{F-y_1}(\xi - y_1)$. Then there exists a unique $\xi_2^* \in \mathfrak{J}^{-1}_{F-y_2}(\xi - y_2)$ colinear with ξ_1^* and such that*

$$\frac{1}{\|\xi_1^*\|} \hat{\gamma}_{F-y_1,\alpha}(\xi - y_1, \xi_1^*) = \frac{1}{\|\xi_2^*\|} \hat{\gamma}_{F-y_2,\alpha}(\xi - y_2, \xi_2^*) \tag{3.7}$$

for each $\alpha > 0$.

Proof. Setting $\xi_2^* := \frac{\xi_1^*}{1 + \langle y_1 - y_2, \xi_1^* \rangle}$ we see that ξ_2^* has the same direction as ξ_1^*, $\xi_2^* \in \partial(F - y_2)^0$ and $\langle \xi - y_2, \xi_2^* \rangle = 1$, i.e., $\xi_2^* \in \mathfrak{J}^{-1}_{F-y_2}(\xi - y_2)$. Given $\eta \in \partial F$ close to ξ and $\eta_1^* \in \mathfrak{J}^{-1}_{F-y_1}(\eta - y_1)$ close to ξ_1^* directly from Definition 3.2 we obtain

$$\frac{1}{\|\eta_1^*\|} \widehat{\mathfrak{C}}_{F-y_1}(r, \eta - y_1, \eta_1^*) = \frac{1}{\|\eta_2^*\|} \widehat{\mathfrak{C}}_{F-y_2}(r, \eta - y_2, \eta_2^*), \tag{3.8}$$

$r > 0$, where $\eta_2^* := \frac{\eta_1^*}{1 + \langle y_1 - y_2, \eta_1^* \rangle}$ belongs, obviously, to some neighbourhood of ξ_2^*. Dividing both parts of (3.8) by r^α and passing to $\liminf$ as $r \to 0+$, $\eta \to \xi$, $\eta_1^* \to \xi_1^*$ (and, consequently, $\eta_2^* \to \xi_2^*$) we easily come to (3.7) (see (3.5)). ■

Observing that the common direction of the vectors ξ_1^* and ξ_2^* from Proposition 3.3 is normal to the set F at the point ξ, we may extend the concept of strict convexity for the case of a closed convex bounded body (do not assuming that $0 \in \operatorname{int} F$). Indeed, given $\xi \in \partial F$ and $\nu \in \mathbf{N}_F(\xi)$, $\|\nu\| = 1$, we say that F is *strictly convex of the order $\alpha > 0$ (at the point ξ) w.r.t. the vector ν* if the translated set $F - y$ is strictly convex of the order α (at the point $\xi - y$)

w.r.t. the same direction ν (or w.r.t. $\nu/\rho_{(F-y)^0}(\nu) \in \partial (F-y)^0$, see Definition 3.3), where y is an arbitrary element from $\operatorname{int} F$. We use such generalization in Section 7 (see Proposition 7.1(i)). Furthermore, since this is a local property, it can be extended also for the case of an unbounded set.

In what follows we use the strict convexity of order $\alpha = 2$ only denoting $\hat{\gamma}_{F,2}(\xi,\xi^*)$ simply by $\hat{\gamma}_F(\xi,\xi^*)$. Let us define (*square*) *curvature*

$$\hat{\varkappa}_F(\xi,\xi^*) := \frac{1}{\|\xi^*\|}\hat{\gamma}_F(\xi,\xi^*)$$

of the set F at the point $\xi \in \partial F$ w.r.t. ξ^* (or with respect to the normal direction $\nu = \xi^*/\|\xi^*\| \in \mathbf{N}_F(\xi) \cap \partial\overline{B}$), which shows how rotund the boundary ∂F is in a neighbourhood of ξ. As follows from Proposition 3.3 the curvature does not depend on position of the origin in $\operatorname{int} F$ and can be defined also when $0 \notin \operatorname{int} F$. Another characterization of the curvature can be given via radius of the smallest ball centred on the half-line opposite to the vector ξ^*, which touches the boundary ∂F at ξ and contains a part of the set F near this point. Exactly, denoting by

$$\widehat{\mathfrak{R}}_F(\xi,\xi^*) := \frac{1}{2\hat{\varkappa}_F(\xi,\xi^*)}$$

(the so-called *curvature radius* of F) we have

Proposition 3.4 *Given $\xi^* \in \partial F^0$ and $\xi \in \mathfrak{J}(\xi^*)$,*

$$\frac{\widehat{\mathfrak{R}}_F(\xi,\xi^*)}{\|\xi^*\|} = \limsup_{\substack{(\varepsilon,\eta,\eta^*)\to(0+,\xi,\xi^*)\\ \eta\in\mathfrak{J}(\eta^*),\,\eta^*\in\partial F^0}} \inf\left\{r>0 : F\cap\left(\eta+\varepsilon\overline{B}\right) \subset \right.$$
$$\left. \subset \eta - r\eta^* + r\|\eta^*\|\,\overline{B}\right\}. \tag{3.9}$$

Proof. Let us prove the inequality "$\le$" in (3.9) assuming without loss of generality that the right-hand side (further denoted by R) is finite. Taking an arbitrary $\rho > R$ we can afirm that for each $\varepsilon > 0$ small enough and for each dual pair (η,η^*) from a neighbourhood of (ξ,ξ^*) the relation

$$F\cap\left(\eta+\varepsilon\overline{B}\right) \subset \eta - \rho\eta^* + \rho\|\eta^*\|\,\overline{B}$$

holds. In particular,

$$\|\zeta-\eta+\rho\eta^*\|^2 \le \rho^2\|\eta^*\|^2$$

whenever $\zeta \in F$ with $\|\zeta-\eta\| = \varepsilon$, or, in another form,

$$\langle \eta-\zeta,\eta^*\rangle \ge \frac{\varepsilon^2}{2\rho}. \tag{3.10}$$

If $w \in F$ is an arbitrary point with $\|w-\eta\| \ge \varepsilon$ then setting $\zeta := \lambda w + (1-\lambda)\eta \in F$, where $\lambda := \varepsilon/\|w-\eta\| \le 1$, we have $\|\zeta-\eta\| = \varepsilon$ and $\langle\eta-\zeta,\eta^*\rangle = \lambda\langle\eta-w,\eta^*\rangle$. Comparing with (3.10) we obtain that (see Definition 3.2)

$$\frac{1}{2\rho} \le \frac{\widehat{\mathfrak{C}}_F(\varepsilon,\eta,\eta^*)}{\varepsilon^2}.$$

Hence, passing to $\liminf$ as $\varepsilon \to 0+$, $(\eta, \eta^*) \to (\xi, \xi^*)$ and $\rho \to R+$ we conclude the fist part of the proof.

In order to show the opposite inequality let us assume that $R > 0$ (in the case $R = 0$ it is trivial). If now $0 < \rho < R$ then by the definition of $\limsup$ there exist an arbitrarily small $\varepsilon > 0$ and a dual pair (η, η^*) arbitrarily near (ξ, ξ^*) such that the relation $F \cap \left(\eta + \varepsilon\overline{B}\right) \subset \eta - r\eta^* + r\,\|\eta^*\|\,\overline{B}$ implies $\rho < r$. In particular, $\|\zeta - \eta + \rho\eta^*\|^2 > \rho^2\,\|\eta^*\|^2$ for some $\zeta \in F$ with $\|\zeta - \eta\| \le \varepsilon$, and, consequently, setting $r := \|\zeta - \eta\| \le \varepsilon$ we have

$$\frac{\widehat{\mathfrak{C}}_F\,(r, \eta, \eta^*)}{r^2} < \frac{1}{2\rho}. \tag{3.11}$$

Passing in (3.11) to $\liminf$ as $r \to 0+$, $(\eta, \eta^*) \to (\xi, \xi^*)$ and to limit as $\rho \to R-$ we prove the inequality "$\geq$" in (3.9). $\blacksquare$

Besides of $\hat{\gamma}_F\,(\xi, \xi^*)$ in what follows we use also one-sided characteristics $\gamma_F^+\,(\xi, \xi^*)$ and $\gamma_F^-\,(\xi, \xi^*)$ defined by the same way as (3.5), $\alpha = 2$, but with the modulus $\widehat{\mathfrak{C}}_F\,(r, \eta, \eta^*)$ substituted by $\mathfrak{C}_F^{\pm}\,(r, \eta, \eta^*)$, respectively. However, they do not satisfy the invariantness property given by Proposition 3.3 (see Example 4.1), being connected with the "true" curvature through the inequalities

$$\frac{1}{\|F^0\|^2}\,\hat{\varkappa}_F\,(\xi, \xi^*) \le \frac{\gamma_F^{\pm}\,(\xi, \xi^*)}{\|\xi^*\|} \le \|F\|^2\,\hat{\varkappa}_F\,(\xi, \xi^*)\,, \tag{3.12}$$

(see (3.3)).

According to Remark 3.1 it makes sense to define

$$\begin{aligned}\gamma_F\,(\xi, \xi^*) := \sup \Big\{&\theta > 0 : \exists \varepsilon > 0 \ \text{ such that } \ \widehat{\mathfrak{C}}_F\,(r, \eta, \eta^*) \ge \theta r^2 \\ &\text{whenever } \ \|\eta - \xi\| \le \varepsilon,\ \|\eta^* - \xi^*\| \le \varepsilon, \\ &\eta \in \mathfrak{J}\,(\eta^*)\,,\, \eta^* \in \partial F^0 \quad \text{and} \quad r > 0\Big\}\,,\end{aligned} \tag{3.13}$$

or, in a compact form,

$$\gamma_F\,(\xi, \xi^*) = \liminf_{\substack{(\eta,\eta^*)\to(\xi,\xi^*) \\ \eta\in\mathfrak{J}(\eta^*),\,\eta^*\in\partial F^0}} \ \inf_{r>0} \frac{\widehat{\mathfrak{C}}_F\,(r, \eta, \eta^*)}{r^2}. \tag{3.14}$$

We see directly from the definition that the function $(\xi, \xi^*) \mapsto \gamma_F\,(\xi, \xi^*)$ is lower semicontinuous (and $(\xi, \xi^*) \mapsto \hat{\gamma}_F\,(\xi, \xi^*)$ as well). Furthermore, arguing as in Proposition 3.4 we have

$$\frac{1}{2\gamma_F\,(\xi, \xi^*)} = \limsup_{\substack{(\eta,\eta^*)\to(\xi,\xi^*) \\ \eta\in\mathfrak{J}(\eta^*),\,\eta^*\in\partial F^0}} \ \inf\left\{r > 0 : F \subset \eta - r\eta^* + r\,\|\eta^*\|\,\overline{B}\right\}. \tag{3.15}$$

It follows readily from (3.8) that $\varkappa_F\,(\xi, \xi^*) := \gamma_F\,(\xi, \xi^*)\,/\,\|\xi^*\|$ is invariant with respect to translations similarly to the curvature $\hat{\varkappa}_F\,(\xi, \xi^*)$. On the other hand,

$\varkappa_F(\xi,\xi^*)$ and $\mathfrak{R}_F(\xi,\xi^*) := \frac{1}{2\varkappa_F(\xi,\xi^*)}$ are not only local characteristics of the boundary at the point ξ but depend also on the size of the set F. In particular, $\mathfrak{R}_F(\xi,\xi^*)$ can not be too small, namely (see (3.15)),

$$\mathfrak{R}_F(\xi,\xi^*) \geq \mathfrak{r}_F, \tag{3.16}$$

where $\mathfrak{r}_F > 0$ is the *Chebyshev radius* of the convex set F. This distinguishes it from the "true" curvature radius $\widehat{\mathfrak{R}}_F(\xi,\xi^*)$. In what follows we call $\varkappa_F(\xi,\xi^*)$ and $\mathfrak{R}_F(\xi,\xi^*)$ *scaled curvature* and *scaled curvature radius*, respectively.

4 Modulus of local smoothness. Dual statements

As well-known (see [2], [21]−[24] and others) the strict convexity of a convex closed bounded set F with $0 \in \operatorname{int} F$ is strongly related to smoothness of its polar set F^0. We are interested now in quantitative aspect of such connection. In particular, we would like to find some relationships between the functions $\gamma_F^{\pm}(\xi,\xi^*)$ introduced in the previous section and the local characteristics of F^0.

Definition 4.1 *Let us fix $\xi^* \in \partial F^0$ and $\xi \in \mathfrak{J}_F(\xi^*) \subset \partial F$. For $t \in \mathbb{R}$ we define a* modulus of (uniform) smoothness of the set F^0 at the point ξ^* w.r.t. ξ *by*

$$\mathfrak{S}_{F^0}(t,\xi^*,\xi) := \sup\left\{\rho_{F^0}(\xi^* + t\eta^*) - \rho_{F^0}(\xi^*) - t\langle\xi,\eta^*\rangle : \ \eta^* \in F^0\right\}. \tag{4.1}$$

Since $\xi \in \partial\rho_{F^0}(\xi^*)$, we always have $\mathfrak{S}_{F^0}(t,\xi^*,\xi) \geq 0$. By Proposition 3.1(iii), if F is strictly convex w.r.t. ξ^* then

$$\lim_{t\to 0}\frac{\mathfrak{S}_{F^0}(t,\xi^*,\xi)}{t} = 0, \tag{4.2}$$

where ξ is the unique element of $\mathfrak{J}_F(\xi^*)$. Moreover, there is a relationship between the modulus of uniform smoothness and the modulus of strict convexity given by the following statement, which is nothing else than a one-sided local version of *Lindenstrauss duality theorem* (see [20, Theorem 1]).

Proposition 4.1 *Let $\xi \in \partial F$ and $\xi^* \in \partial F^0$ be such that $\langle\xi,\xi^*\rangle = 1$. Then for each $t > 0$ the equalities*

$$\mathfrak{S}_{F^0}(\pm t,\xi^*,\xi) = \sup\left\{tr - \mathfrak{C}_F^{\pm}(r,\xi,\xi^*) : \ r > 0\right\} \tag{4.3}$$

hold.

Proof. Let us prove the equality (4.3) for $\mathfrak{C}_F^{+}(r,\xi,\xi^*)$ only. The other one can be proved similarly.

Given $\varepsilon > 0$, from (4.1) we choose $\eta^* \in F^0$ and $\eta \in F$ such that

$$\begin{aligned} \mathfrak{S}_{F^0}(t,\xi^*,\xi) &\le \langle \eta, \xi^* + t\eta^*\rangle - \langle \xi,\xi^*\rangle - t\langle \xi,\eta^*\rangle + \varepsilon \le \\ &\le \langle \eta - \xi, \xi^*\rangle + t\rho_F(\eta-\xi) + \varepsilon \le \\ &\le \sup_{\eta\in F}\{t\rho_F(\eta-\xi) - \langle \xi-\eta,\xi^*\rangle\} + \varepsilon \le \\ &\le \sup_{r>0}\left\{tr - \mathfrak{C}_F^+(r,\xi,\xi^*)\right\} + \varepsilon, \end{aligned}$$

and the inequality "$\le$" in (4.3) follows.

In order to prove the opposite inequality let us fix $\varepsilon > 0$ and choose first $r > 0$, $\eta \in F$ with $\rho_F(\eta-\xi) \ge r$ and then $\eta^* \in F^0$ such that

$$\begin{aligned} \sup_{r>0}\left\{tr - \mathfrak{C}_F^+(r,\xi,\xi^*)\right\} &\le t\rho_F(\eta-\xi) - \langle \xi-\eta,\xi^*\rangle + \varepsilon \le \\ &\le \langle \eta, t\eta^*\rangle - \langle \xi, t\eta^*\rangle + \langle \eta,\xi^*\rangle - \rho_{F^0}(\xi^*) + \varepsilon \le \\ &\le \sigma_F(\xi^* + t\eta^*) - t\langle \xi,\eta^*\rangle - \rho_{F^0}(\xi^*) + \varepsilon \le \\ &\le \mathfrak{S}_{F^0}(t,\xi^*,\xi) + \varepsilon, \end{aligned}$$

and the proof is concluded. ■

If we put

$$\mathfrak{C}_F(r,\xi,\xi^*) := \begin{cases} \mathfrak{C}_F^+(r,\xi,\xi^*) & \text{if} \quad r>0; \\ 0 & \text{if} \quad r=0; \\ \mathfrak{C}_F^-(-r,\xi,\xi^*) & \text{if} \quad r<0 \end{cases}$$

then (4.3) can be written in a more symmetric form

$$\mathfrak{S}_{F^0}(\cdot,\xi^*,\xi) = \mathfrak{C}_F^{\bigstar}(\cdot,\xi,\xi^*), \tag{4.4}$$

where "$\bigstar$" means the *Legendre-Fenchel transform.*

Now, by using Proposition 4.1, we obtain a dual characterization of the second order strict convexity, which makes more precise the equality (4.2).

Proposition 4.2 *Let (ξ,ξ^*) be a dual pair of elements: $\xi \in \partial F$, $\xi^* \in \partial F^0$, $\xi \in \mathfrak{J}(\xi^*)$. Then*

$$\frac{1}{4\gamma_F^{\pm}(\xi,\xi^*)} = \limsup_{\substack{(t,\eta,\eta^*)\to(0\pm,\xi,\xi^*) \\ \eta\in\mathfrak{J}(\eta^*),\,\eta^*\in\partial F^0}} \frac{\mathfrak{S}_{F^0}(t,\eta^*,\eta)}{t^2}. \tag{4.5}$$

Proof. We prove the formula (4.5) for $\gamma_F^+(\xi,\xi^*)$. The respective proof for $\gamma_F^-(\xi,\xi^*)$ is similar.

While proving the inequality "$\ge$" in (4.5) we can assume without loss of generality that $\gamma_F^+(\xi,\xi^*) > 0$ (i.e., F is strictly convex of the second order w.r.t. ξ^*). Then the mapping $\mathfrak{J}(\cdot)$ is single-valued and continuous in a neighbourhood of ξ^* (see Remark 3.1), and taking an arbitrary $0 < \beta < \gamma_F^+(\xi,\xi^*)$ one can choose $\varepsilon > 0$ such that

$$\mathfrak{C}_F^+(r,\mathfrak{J}(\eta^*),\eta^*) > \beta r^2$$

for all $0 < r \leq \varepsilon$ and $\eta^* \in \partial F^0$ with $\|\eta^* - \xi^*\| \leq \varepsilon$. As it is easy to see,

$$\sup\left\{tr - \beta r^2 : 0 < r \leq \varepsilon\right\} = \frac{t^2}{4\beta} \tag{4.6}$$

for all $0 < t \leq 2\varepsilon\beta$. On the other hand, observing that $\mathfrak{C}_F^+\left(r, \mathfrak{J}\left(\eta^*\right), \eta^*\right) = +\infty$ whenever $r > D := 2\left\|F^0\right\| \|F\|$, by the monotonicity of $\mathfrak{C}_F^+\left(\cdot, \mathfrak{J}\left(\eta^*\right), \eta^*\right)$, we have

$$\sup\left\{tr - \mathfrak{C}_F^+\left(r, \mathfrak{J}\left(\eta^*\right), \eta^*\right) : r > \varepsilon\right\} \leq tD - \beta\varepsilon^2 \leq \frac{t^2}{4\beta} \tag{4.7}$$

for all $0 < t \leq 2\beta\left(D - \sqrt{D^2 - \varepsilon^2}\right)$. Thus, applying the duality formula (4.3), we obtain from (4.6) and (4.7)

$$\frac{\mathfrak{S}_{F^0}\left(t, \eta^*, \mathfrak{J}\left(\eta^*\right)\right)}{t^2} \leq \frac{1}{4\beta}.$$

Hence, passing to $\limsup$ as $t \to 0+$, $\eta^* \to \xi^*$ and to limit as $\beta \to \gamma_F^+\left(\xi, \xi^*\right) -$ we conclude the first part of the proof.

In order to prove the converse inequality let us suppose that the right-hand side of (4.5) (further denoted by L) is finite. Then, taking any $\beta > L$ we can find $\varepsilon > 0$ such that

$$\mathfrak{S}_{F^0}\left(t, \eta^*, \eta\right) < \beta t^2 \tag{4.8}$$

for all $0 < t \leq \varepsilon$ and for each dual pair (η, η^*) such that $\|\eta - \xi\| \leq \varepsilon$, $\|\eta^* - \xi^*\| \leq \varepsilon$. Applying the Legendre-Fenchel transform to (4.8) we have

$$\mathfrak{S}_{F^0}^{\bigstar}\left(r, \eta^*, \eta\right) \geq \sup\left\{tr - \beta t^2 : 0 < t \leq \varepsilon\right\} = \frac{r^2}{4\beta}, \tag{4.9}$$

$0 < r \leq 2\varepsilon\beta$. Since the double conjugate function is always below the original one, it follows from (4.9) and (4.4) that

$$\mathfrak{C}_F^+\left(r, \eta, \eta^*\right) \geq \frac{r^2}{4\beta}.$$

Dividing by r^2 and passing to $\liminf$ we obtain now the desired inequality. ■

Let us concretize now the formula (4.5) in the case when F^0 has the second order smooth boundary.

As we know (see Remark 3.1 and Proposition 3.1 (iii)) if $\gamma_F^+\left(\xi, \xi^*\right) > 0$ then $\rho_{F^0}(\cdot)$ is Fréchet differentiable on $\partial F^0 \cap \left(\xi^* + \varepsilon\overline{B}\right)$ for some $\varepsilon > 0$, and, furthermore, the Fréchet derivative $\nabla\rho_{F^0}(\cdot)$ is (uniformly) continuous on a neighbourhood of ξ^*. Remind that the functional $\rho_{F^0}(\cdot)$ is said to be *(Fréchet) twice differentiable at* $\xi^* \in \partial F^0$ if there exists a (self-adjoint) linear bounded operator $\nabla^2\rho_{F^0}\left(\xi^*\right) : H \to H$ (called *Fréchet second derivative*) such that

$$\frac{\nabla\rho_{F^0}\left(\xi^* + tv^*\right) - \nabla\rho_{F^0}\left(\xi^*\right)}{t} \to \nabla^2\rho_{F^0}\left(\xi^*\right) v^* \quad \text{as} \quad t \to 0+$$

uniformly in $v^* \in F^0$. Let us define the F^0-*norm* of the operator $\nabla^2 \rho_{F^0}(\xi^*)$ by

$$\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} := \sup_{v^* \in F^0} \left\langle \nabla^2 \rho_{F^0}(\xi^*)\, v^*, v^* \right\rangle. \tag{4.10}$$

Finally, the boundary ∂F^0 is said to be *of class* $\mathcal{C}^2$ *(second order smooth) at the point* $\xi^* \in \partial F^0$ if $\rho_{F^0}(\cdot)$ is twice differentiable at each point of a neighbourhood of ξ^*, and the mapping $\eta^* \mapsto \nabla^2 \rho_{F^0}(\eta^*)$ is continuous near ξ^* with respect to the operator topology. This is the same to require the continuous differentiability of the (unique) unit normal vector to F^0 near the point ξ^*. Hence, in particular, the continuity of the functional $\eta^* \mapsto \left\|\nabla^2 \rho_{F^0}(\eta^*)\right\|_{F^0}$ in a neighbourhood of ξ^* follows.

Proposition 4.3 *Assume that the boundary of the set* F^0 *is of class* $\mathcal{C}^2$ *at a point* $\xi^* \in \partial F^0$*, and* $\xi \in \partial F$ *is the unique element of* $\mathfrak{J}(\xi^*)$ *(in other words* $\xi = \nabla \rho_{F^0}(\xi^*)$*). Then*

$$\gamma_F^+(\xi, \xi^*) = \gamma_F^-(\xi, \xi^*) = \frac{1}{2\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0}}. \tag{4.11}$$

Proof. Given $\eta^* \in \partial F^0$ in a neghbourhood of the point ξ^*, by the Taylor formula (see, e.g., [6, p.75]) for each $v^* \in F^0$ and $t > 0$ small enough we have

$$\begin{aligned}\rho_{F^0}(\eta^* + tv^*) &= \rho_{F^0}(\eta^*) + t\langle \eta, v^*\rangle + \\ &\quad + \int_0^t \left\langle \nabla^2 \rho_{F^0}(\eta^* + \tau v^*)\, v^*, v^* \right\rangle (t - \tau)\; d\tau,\end{aligned} \tag{4.12}$$

where $\eta := \nabla \rho_{F^0}(\eta^*) = \mathfrak{J}(\eta^*)$. Hence, by using the mean value theorem, given $t > 0$ and $v^* \in F^0$ we find $\tau^* = \tau(t, v^*)$, $0 < \tau^* < t$, such that (see (4.1))

$$\frac{\mathfrak{S}_{F^0}(t, \eta^*, \eta)}{t^2} = \frac{1}{2} \sup_{v^* \in F^0} \left\langle \nabla^2 \rho_{F^0}(\eta^* + \tau^* v^*)\, v^*, v^* \right\rangle. \tag{4.13}$$

By continuity of the second derivative we have the convergence

$$\nabla^2 \rho_{F^0}(\eta^* + \tau^* v^*) \to \nabla^2 \rho_{F^0}(\xi^*)$$

as $\eta^* \to \xi^*$, $\eta^* \in \partial F^0$, and as $t \to 0+$ in the operator topology, which is uniform in $v^* \in F^0$. Therefore, the right-hand side of (4.13) converges to $\frac{1}{2}\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0}$. Remind now the formula (4.5) and obtain

$$\frac{1}{4\gamma_F^+(\xi, \xi^*)} = \frac{1}{2}\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0}.$$

In order to find the same representation for $\gamma_F^-(\xi, \xi^*)$ it is enough to apply the Taylor formula (4.12) for the vector $-v^*$ instead of v^*, $v^* \in F^0$. ■

Let us give a simple example illustrating the last proposition.

Example 4.1 *Fix $a \in H$ with $\|a\| < 1$ and consider the set*

$$F := \{\xi \in H : \|\xi - a\| \le 1\}.$$

It is easy to see that

$$\rho_{F^0}(\xi^*) = \sigma_F(\xi^*) = \langle \xi^*, a\rangle + \|\xi^*\|, \quad \xi^* \in H.$$

This function is twice continuously differentiable at each $\xi^* \neq 0$, and taking $\xi^* \in \partial F^0$ we have

$$\nabla^2 \rho_{F^0}(\xi^*) v^* = \frac{\|\xi^*\|^2 v^* - \langle \xi^*, v^*\rangle \xi^*}{\|\xi^*\|^3}, \quad v^* \in H.$$

Applying Lagrange multipliers we find the $\|F^0\|$-norm of this operator (see (4.10)):

$$\begin{aligned}
&\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} = \\
= \ & \frac{1}{\|\xi^*\|^3} \sup\left\{\|\xi^*\|^2 \|v^*\|^2 - \langle \xi^*, v^*\rangle^2 : \langle v^*, a\rangle + \|v^*\| \le 1\right\} = \\
= \ & \frac{1}{\left(1 - \|a\|^2\right)^2 \|\xi^*\|^3} \left(\|\xi^*\|^2 \left(1 + \|a\|^2\right) - 2\langle \xi^*, a\rangle^2 + \right. \\
& \left. +2\sqrt{\left(\|\xi^*\|^2 - \langle \xi^*, a\rangle^2\right)\left(\|a\|^2 \|\xi^*\|^2 - \langle \xi^*, a\rangle^2\right)}\right), \qquad (4.14)
\end{aligned}$$

and the rotundity characteristics $\gamma_F^{\pm}(\xi, \xi^*)$ should be found from (4.11). Here as usual $\xi \in \partial F$ is the unique point with $\langle \xi, \xi^*\rangle = 1$. In particular cases when ξ^* is colinear to a the square root in (4.14) vanishes, and we obtain

$$\gamma_F^{\pm}(\xi, \xi^*) = \begin{cases} \frac{1-\|a\|}{2} & \text{if} \quad \xi^* = \frac{a}{\|a\|(1+\|a\|)}; \\ \frac{1+\|a\|}{2} & \text{if} \quad \xi^* = -\frac{a}{\|a\|(1-\|a\|)}. \end{cases}$$

Thus, $\gamma_F^{\pm}(\xi, \xi^*)$ depend essentially on a (on position of the origin inside the ball). Namely, they tend either to 0 or to 1 as $\|a\| \to 1$ whenever the origin is either more distant from the point ξ or more close to ξ, respectively. This distinguishes $\gamma_F^{\pm}(\xi, \xi^*)$ from $\hat{\gamma}_F(\xi, \xi^*)$ (see Proposition 3.3). Observe that in the case $a = 0$ the formula (4.14) gives $\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} = 1$, and $\gamma_F^{\pm}(\xi, \xi^*) = 1/2$ for each $\xi^* \in H$ with $\|\xi^*\| = 1$ and $\xi = \xi^*$ (see the remark after Definition 3.2).

5 A property of minimizing sequences

Let us return now to the minimum time problem (1.3), where the value function $\mathfrak{T}_C^F(z)$ is always finite and strictly positive for all $z \notin C$. We consider minimizing sequences $\{x_n\} \subset C$ for the mapping $x \mapsto \rho_F(x - z)$ on C, i.e., such that $\rho_F(x_n - z) \to \mathfrak{T}_C^F(z)+$ as $n \to \infty$. The following statement is crucial for proving of the main theorems contained in the next sections.

Lemma 5.1 *Let $C \subset H$ be a nonempty closed set, $z \in H \setminus C$, and $\{x_n\} \subset C$ be a minimizing sequence for $x \mapsto \rho_F(x-z)$ on C. Then there exist another minimizing sequence $\{x'_n\} \subset C$ and sequences $\{x''_n\}$, $\{v_n\}$, $\{\xi^*_n\}$ such that $v_n \in \mathbf{N}^p_C(x'_n)$, $\xi^*_n \in \partial\rho_F(x''_n - z)$ and*

$$\begin{aligned} \|x'_n - x_n\| + \|x''_n - x_n\| &\to 0, &(5.1)\\ \|v_n + \rho_F(x''_n - z)\,\xi^*_n\| &\to 0, &(5.2)\end{aligned}$$

as $n \to \infty$.

Proof. Given an arbitrary sequence $\varepsilon_n \to 0+$ with $\rho^2_F(x_n - z) \leq \left(\mathfrak{T}^F_C(z)\right)^2 + \varepsilon^2_n$, by the Ekeland's variational principle (see [18, Corollary 11]) there exists $\{y_n\} \subset C$ satisfying the conditions

$$\begin{aligned} \rho^2_F(y_n - z) &\leq \left(\mathfrak{T}^F_C(z)\right)^2 + \varepsilon^2_n; &(5.3)\\ \|x_n - y_n\| &\leq \varepsilon_n; &(5.4)\\ \rho^2_F(y_n - z) &\leq \rho^2_F(y - z) + \varepsilon_n\|y - y_n\| \qquad \forall y \in C, &(5.5)\end{aligned}$$

$n = 1, 2, \ldots$. The inequality (5.5), in particular, means that y_n minimizes the functional

$$F(y) := \rho^2_F(y - z) + \varepsilon_n\|y - y_n\| + \mathbf{I}_C(y) \tag{5.6}$$

on H, where $\mathbf{I}_C(\cdot)$ is the *indicator function* of the set C (it is equal to zero on C and to $+\infty$ elsewhere). Denoting by $\partial^p F(y)$ the *proximal subdifferential* of (5.6) (see [11, p. 29]) we obviously have $0 \in \partial^p F(y_n)$. According to the fuzzy sum rule (see Theorem 8.3 [11, p. 56]),

$$0 \in \mathbf{N}^p_C(x'_n) + \partial\left(\rho^2_F(x''_n - z) + \varepsilon_n\|x''_n - y_n\|\right) + \varepsilon_n\overline{B} \tag{5.7}$$

for some sequences $\{x'_n\} \subset C$ and $\{x''_n\} \subset H$, $\|x'_n - y_n\| \leq \varepsilon_n$, $\|x''_n - y_n\| \leq \varepsilon_n$, $n = 1, 2, \ldots$. Since the subdifferential in the right-hand side of (5.7) is contained in $2\rho_F(x''_n - z)\,\partial\rho_F(x''_n - z) + \varepsilon_n\overline{B}$, one can find vectors $v_n \in \mathbf{N}^p_C(x'_n)$ and $\xi^*_n \in \partial\rho_F(x''_n - z)$ with the property (5.2). It follows from (5.3) that $\{x'_n\}$ is a minimizing sequence of $x \mapsto \rho_F(x - z)$ on C, and (5.1) also holds. ■

Remark 5.1 *The relation (5.2), in particular, shows that x'_n belong to $\partial^* C$ for all n large enough, since otherwise $\xi^*_n \to 0$. But this is impossible because $\xi^*_n \in \partial F^0$ (see [15, Corollary 2.3]).*

Remark 5.2 *The vectors v_n in Lemma 5.1 can be chosen such that*

$$\rho_{F^0}(-v_n) = \rho_F(x''_n - z), \tag{5.8}$$

$n = 1, 2, \ldots$. Indeed, setting $v'_n := v_n \frac{\rho_F(x''_n - z)}{\rho_{F^0}(-v_n)} \in \mathbf{N}^p_C(x'_n)$ we have, by the Lipschitz continuity of $\rho_{F^0}(\cdot)$ (see (2.3)) and by (5.2),

$$\begin{aligned} \|v_n - v'_n\| &= \frac{\|v_n\|}{\rho_{F^0}(-v_n)}\left|\rho_F(x''_n - z)\,\rho_{F^0}(\xi^*_n) - \rho_{F^0}(-v_n)\right| \leq \\ &\leq \|F\|\left\|F^0\right\|\|\rho_F(x''_n - z)\,\xi^*_n + v_n\| \to 0,\end{aligned}$$

and, therefore, v_n can be substituted by v'_n.

Remark 5.3 *In the case when all the basic normal cones to the set C coincide (e.g., if C is φ-convex), in the proof of Lemma 5.1 we may use the limiting subdifferential ([23, p.82]) in the place of $\partial^p F(y)$, and apply the precise sum rule instead of the fuzzy one (see [11, p. 62]). In this way we obtain a stronger statement of Lemma 5.1, which gives $x_n' = x_n''$.*

6 Well-posedness of the time-minimum projection

Further on we always assume the dynamic set $F \subset H$ to be nonempty closed convex bounded with $0 \in \operatorname{int} F$ and the target $C \subset H$ to be an arbitrary nonempty closed set. Let us introduce two local hypotheses.

We say that the pair of sets (F, C) satisfies *the condition* $(\mathbf{A})$ at a point $x_0 \in \partial C$ if there exists $\delta > 0$ such that

$(\mathbf{A}_1)$ the mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right)$ is single-valued and Lipschitz continuous on

$$C_\delta(x_0) := \{x \in \partial^* C : \|x - x_0\| \leq \delta\};$$

$(\mathbf{A}_2)$ F is uniformly strictly convex with respect to

$$U_{\delta,\delta'}(x_0) := \partial F^0 \cap \bigcup_{x \in C_\delta(x_0)} \left[-\mathbf{N}_C^p(x) \cap \partial F^0 + \delta' \overline{B}\right] \tag{6.1}$$

for some $\delta' > 0$.

Alternatively, we say that (F, C) satisfies *the condition* $(\mathbf{B})$ at $x_0 \in \partial C$ if for some $\delta > 0$

$(\mathbf{B}_1)$ the function $\psi_C(x, v)$ is upper bounded on the set

$$\{(x, v) : x \in C_\delta(x_0),\, v \in \mathbf{N}_C^p(x)\}$$

(or, in other words, C is proximally smooth in a neighbourhood of the point x_0);

$(\mathbf{B}_2)$ there exist $\delta' > 0$ and $K > 0$ such that

$$\varkappa_F\left(\mathfrak{J}_F(\xi^*), \xi^*\right) \geq K \quad \text{for all } \xi^* \in \hat{U}_{\delta,\delta'}(x_0),$$

where

$$\hat{U}_{\delta,\delta'}(x_0) := \partial F^0 \cap \bigcup_{x \in C_\delta(x_0) \setminus \{x_0\}} \left[-\mathbf{N}_C^p(x) \cap \partial F^0 + \delta' \overline{B}\right]. \tag{6.2}$$

We are ready now to formulate the main result.

Theorem 6.1 *Assume that at each point $x_0 \in \partial C$ the pair of sets (F, C) satisfies either the condition* $(\mathbf{A})$ *or* $(\mathbf{B})$*. Then there exists an open set $\mathcal{U} \supset C$ such that for each $z \in \mathcal{U}$ the time-minimum projection $\pi_C^F(z)$ is a singleton, and the mapping $z \mapsto \pi_C^F(z)$ is continuous on $\mathcal{U}$.*

Proof. We prove first that given $x_0 \in \partial C$ one can find an (open) neighbourhood $\mathcal{U}(x_0)$ such that for an arbitrary $z \in \mathcal{U}(x_0)$ each minimizing sequence $\{x_n\}$ of $x \mapsto \rho_F(x - z)$ on the set C is a Cauchy sequence.

Case 1. The condition $(\mathbf{A})$ holds at the point x_0. Then we set

$$\mathcal{U}(x_0) := \left\{ z \in H : \|z - x_0\| < \frac{\delta}{\|F\| \|F^0\| + 1}, \quad \mathfrak{T}_C^F(z) < \frac{1}{L} \right\}, \tag{6.3}$$

where $L > 0$ is the Lipschitz constant of $x \mapsto \mathfrak{J}_F\left(-N_C^p(x) \cap \partial F^0\right)$ on $C_\delta(x_0)$ (see $(\mathbf{A}_1)$). Fix $z \in \mathcal{U}(x_0) \setminus C$ and a minimizing sequence $\{x_n\} \subset C$. Let us choose $\{x_n'\} \subset \partial^* C$, $\{x_n''\}$, $v_n \in \mathbf{N}_C^p(x_n')$ and $\xi_n^* \in \partial \rho_F(x_n'' - z)$ as in Lemma 5.1 and such that $\rho_F(x_n'' - z) = \rho_{F^0}(-v_n)$, $n = 1, 2, \ldots$ (see Remarks 5.1 and 5.2). Since by (2.2)

$$\begin{aligned} &\|x_n - x_0\| \le \|F\| \rho_F(x_n - z) + \|z - x_0\| \le \\ &\le \left(\|F\| \left\|F^0\right\| + 1\right) \|z - x_0\| + \|F\| \left(\rho_F(x_n - z) - \mathfrak{T}_C^F(z)\right), \end{aligned} \tag{6.4}$$

and $\rho_F(x_n - z) - \mathfrak{T}_C^F(z) \to 0+$, $\|x_n - x_n'\| \to 0$ as $n \to \infty$, we can suppose without loss of generality that $x_n' \in C_\delta(x_0)$ for all $n = 1, 2, \ldots$. Consider a decreasing sequence $\nu_n \to 0+$ such that

$$\|x_n' - x_n\| + \|x_n'' - x_n\| \le \nu_n; \tag{6.5}$$

$$\rho_F(x_n' - z) \le \mathfrak{T}_C^F(z) + \nu_n; \tag{6.6}$$

$$\|v_n + \rho_F(x_n'' - z)\, \xi_n^*\| \le \frac{1}{2} \mathfrak{T}_C^F(z)\, \nu_n, \tag{6.7}$$

$n = 1, 2, \ldots$. It follows, in particular, from (6.7) and (6.5) that

$$\left\| \frac{v_n}{\rho_{F^0}(-v_n)} + \xi_n^* \right\| \le \frac{\mathfrak{T}_C^F(z)}{2\left(\mathfrak{T}_C^F(z) - \|F^0\| \|x_n'' - x_n\|\right)} \nu_n \le \nu_n. \tag{6.8}$$

Furthermore, (see Proposition 3.2) the hypothesis $(\mathbf{A}_2)$ implies that the (single-valued) mapping $\mathfrak{J}_F : U_{\delta, \delta'}(x_0) \to \partial F$ is uniformly continuous, and, therefore, the sequence

$$\beta_n := \sup_{\substack{\|\xi^* - \eta^*\| \le \nu_n \\ \xi^*, \eta^* \in U_{\delta, \delta'}(x_0)}} \|\mathfrak{J}_F(\xi^*) - \mathfrak{J}_F(\eta^*)\|$$

tends to zero as $n \to \infty$.

Observe that $\xi_n^* \in \mathbf{N}_F\left(\frac{x_n'' - z}{\rho_F(x_n'' - z)}\right) \cap \partial F^0$ (see [15, Corollary 2.3]), and hence, as it is easy to see,

$$\frac{x_n'' - z}{\rho_F(x_n'' - z)} = \mathfrak{J}_F(\xi_n^*). \tag{6.9}$$

By (6.8) we have $\xi_n^*, -\frac{v_n}{\rho_{F^0}(-v_n)} \in U_{\delta,\delta'}(x_0)$, and, consequently,

$$\left\| \mathfrak{J}_F(\xi_n^*) - \mathfrak{J}_F\left(-\frac{v_n}{\rho_{F^0}(-v_n)}\right) \right\| \le \beta_n,\ n = 1, 2, \dots . \tag{6.10}$$

Given $m \ge n$ we obtain from (6.9) and (6.10) (see also (6.5) and (6.6)):

$$\begin{aligned} \|x_m'' - x_n''\| \le{}& \rho_F(x_m'' - z) \|\mathfrak{J}_F(\xi_m^*) - \mathfrak{J}_F(\xi_n^*)\| + \\ &+ |\rho_F(x_m'' - z) - \rho_F(x_n'' - z)| \|F\| \le \\ &\le \mathfrak{T}_C^F(z) \|\mathfrak{J}_F(\xi_m^*) - \mathfrak{J}_F(\xi_n^*)\| + 4\nu_n \|F\| \left(\|F^0\| + 1\right) \le \\ &\le \mathfrak{T}_C^F(z) \left\| \mathfrak{J}_F\left(-\frac{v_m}{\rho_{F^0}(-v_m)}\right) - \mathfrak{J}_F\left(-\frac{v_n}{\rho_{F^0}(-v_n)}\right) \right\| + \\ &+ 2\mathfrak{T}_C^F(z)\beta_n + 4\nu_n \|F\| \left(\|F^0\| + 1\right). \end{aligned} \tag{6.11}$$

Since $-\frac{v_n}{\rho_{F^0}(-v_n)} \in -\mathbf{N}_C^p(x_n') \cap \partial F^0$, applying the condition $(\mathbf{A}_1)$ we find from (6.11) that

$$\left(1 - L\mathfrak{T}_C^F(z)\right) \|x_m' - x_n'\| \le \nu_n'$$

for some sequence $\nu_n' \to 0+,\ n \to \infty$. Hence, by the choice of z (see (6.3)) we conclude that $\{x_n'\}$ (and $\{x_n\}$ as well) is a Cauchy sequence.

Case 2. If at the point x_0 the condition $(\mathbf{B})$ holds then we set

$$\mathcal{U}(x_0) := \left\{ z \in H : \|z - x_0\| < \frac{\delta}{\|F\| \|F^0\| + 1},\ \ \mathfrak{T}_C^F(z) < \frac{K}{M} \right\}, \tag{6.12}$$

where the constant $M > 0$ is such that $\psi_C(x, v) \le M$ for all $x \in C_\delta(x_0)$ and $v \in \mathbf{N}_C^p(x)$. Let $z \in \mathcal{U}(x_0) \setminus C$, and $\{x_n\} \subset C$ be a minimizing sequence of $x \mapsto \rho_F(x - z)$ on C. Everything is already proved if $x_n \to x_0$, $n \to \infty$. Otherwise, as we'll see in sequel, there is no loss of generality to suppose that x_0 is not a cluster point of $\{x_n\}$, and that the number sequence $\{\rho_F(x_n - z)\}$ is nonincreasing. By using Lemma 5.1 similarly to the Case 1 we choose sequences $\{x_n'\} \subset \partial^* C$, $\{x_n''\}$, $v_n \in \mathbf{N}_C^p(x_n')$ and $\xi_n^* \in \partial\rho_F(x_n'' - z)$ satisfying (5.1), (5.2) and (5.8). Observe that in virtue of the hypothesis $(\mathbf{B}_1)$ a simpler version of Lemma 5.1 holds that gives $x_n' = x_n''$ (see Remark 5.3). But, for the sake of uniformity, we prefer to keep all the notations. We can assume, certainly, that $0 < \|x_n' - x_0\| < \delta$, $n = 1, 2, \dots$ (see (5.1) and (6.4)). Let us choose a decreasing sequence $\nu_n \to 0+$ satisfying the inequalities (6.5)-(6.8), and assume that $\nu_n \le \delta'$, $n = 1, 2, \dots$. Since $x_n' \in C_\delta(x_0) \backslash \{x_0\}$ and $-v_n/\rho_{F^0}(-v_n) \in -\mathbf{N}_C^p(x_n') \cap \partial F^0$, we obtain from (6.8) that $\xi_n^* \in \hat{U}_{\delta,\delta'}(x_0)$ (see (6.2)).

For convenience sake let us introduce the following notations:

$$\begin{aligned} &\rho_n := \rho_F(x_n'' - z); \\ &G_n := z + \rho_n F; \\ &R_n := \frac{1}{\|\xi_n^*\|}\mathfrak{R}_F(\xi_n, \xi_n^*) = \frac{1}{2\gamma_F(\xi_n, \xi_n^*)} \text{ (see (3.14), (3.15))}; \\ &\psi_n := \psi_C(x_n', v_n). \end{aligned}$$

Here $\xi_n := \mathfrak{J}_F(\xi_n^*)$ can be found as in the Case 1 (see (6.9)). Combining the hypotheses ($\mathbf{B}_1$) and ($\mathbf{B}_2$) we have from the above arguments:

$$\frac{1}{2R_n} - \mathfrak{T}_C^F(z)\|\xi_n^*\|\psi_n \geq 2\nu := \frac{1}{\|F\|}\left(K - \mathfrak{T}_C^F(z)\,M\right) > 0.$$

Since $\|v_n\| - \mathfrak{T}_C^F(z)\|\xi_n^*\| \to 0+$ as $n \to \infty$ (see (6.7) and (6.6)), using again boundedness of the sequence $\{\psi_n\}$, we can choose $\nu_n' > 0$ such that

$$\frac{1}{2(R_n + \nu_n')} - \|v_n\|\psi_n \geq \nu \tag{6.13}$$

for $n = 1, 2, ...$ large enough (assume that for all n).

Let us consider the approximate *curvature centre* of the set G_n (at the point x_n'')

$$z_n := x_n'' - \rho_n(R_n + \nu_n')\,\xi_n^*. \tag{6.14}$$

We claim that for each $m \geq n$

$$\|z_n - x_m''\| \leq \|z_n - x_n''\| + 2\left\|F^0\right\|\|F\|\,\nu_n. \tag{6.15}$$

Indeed, monotonicity of the sequence $\{\rho_F(x_n - z)\}$ implies $\rho_m \leq \rho_n + 2\left\|F^0\right\|\nu_n$. On the other hand, from the definition of G_n, from (3.15) and (6.9) we obtain:

$$\begin{aligned} G_n &\subset z + \rho_n\left(\mathfrak{J}(\xi_n^*) - (R_n + \nu_n')\,\xi_n^* + (R_n + \nu_n')\|\xi_n^*\|\,\overline{B}\right) = \\ &= x_n'' - \rho_n(R_n + \nu_n')\,\xi_n^* + \rho_n(R_n + \nu_n')\|\xi_n^*\|\,\overline{B} = \\ &= z_n + \|x_n'' - z_n\|\,\overline{B}. \end{aligned}$$

Thus,

$$\begin{aligned} x_m'' &\in G_m \subset G_n + 2\left\|F^0\right\|\|F\|\,\nu_n\overline{B} \subset \\ &\subset z_n + \left(\|x_n'' - z_n\| + 2\left\|F^0\right\|\|F\|\,\nu_n\right)\overline{B}, \end{aligned}$$

and the inequality (6.15) follows.

Given arbitrary $m \geq n$, by (6.14), (6.7) and by the definition of proximal normals we find:

$$\begin{aligned} &\langle z_n - x_n'', x_m'' - x_n''\rangle = \\ =\;& \rho_n(R_n + \nu_n')\langle -\xi_n^*, x_m'' - x_n''\rangle \leq \\ \leq\;& (R_n + \nu_n')\langle v_n, x_m'' - x_n''\rangle + \frac{1}{2}\mathfrak{T}_C^F(z)(R_n + \nu_n')\,\nu_n\|x_m'' - x_n''\| \leq \\ \leq\;& (R_n + \nu_n')\,\psi_n\|v_n\|\|x_m' - x_n'\|^2 + (R_n + \nu_n')\,\mu_n, \end{aligned}$$

where $\mu_n \to 0+$, $n \to \infty$. Hence,

$$\begin{aligned} &\left\|\frac{z_n - x_m''}{2} + \frac{z_n - x_n''}{2}\right\|^2 = \left\|z_n - x_n'' + \frac{x_n'' - x_m''}{2}\right\|^2 \geq \\ \geq\;& \|z_n - x_n''\|^2 + \frac{1}{4}\|x_m'' - x_n''\|^2 - (R_n + \nu_n')\,\psi_n\|v_n\|\|x_m' - x_n'\|^2 - \\ &-(R_n + \nu_n')\,\mu_n. \end{aligned} \tag{6.16}$$

Applying the parallelogram identity and combining with (6.16) we obtain:

$$
\begin{aligned}
& \frac{1}{4}\left\|x_m''-x_n''\right\|^2=\left\|\frac{z_n-x_m''}{2}-\frac{z_n-x_n''}{2}\right\|^2= \\
= \quad & \frac{1}{2}\left\|z_n-x_m''\right\|^2+\frac{1}{2}\left\|z_n-x_n''\right\|^2-\left\|\frac{z_n-x_m''}{2}+\frac{z_n-x_n''}{2}\right\|^2 \leq \\
\leq \quad & \frac{1}{2}\left(\left\|z_n-x_m''\right\|^2-\left\|z_n-x_n''\right\|^2\right)-\frac{1}{4}\left\|x_m''-x_n''\right\|^2+ \\
& +\left(R_n+\nu_n'\right)\psi_n\left\|v_n\right\|\left\|x_m'-x_n'\right\|^2+\mu_n\left(R_n+\nu_n'\right).
\end{aligned}
$$

Therefore, by using the claim above (see (6.15)), (6.5), the hypothesis ($\mathbf{B}_2$) and the *a priori* estimate (3.16)) we conclude that

$$
\left[\frac{1}{2\left(R_n+\nu_n'\right)}-\left\|v_n\right\|\psi_n\right]\left\|x_m'-x_n'\right\|^2\leq\mu_n'
$$

for some $\mu_n'\to 0$ as $n\to\infty$. The Cauchy property of the sequence $\{x_n'\}$ (and of $\{x_n\}$ as well) follows from this inequality together with (6.13).

Let us pass now to the second part of the proof. Denote by

$$
\mathcal{U}=\bigcup_{x_0\in C}\mathcal{U}\left(x_0\right)\supset C,
$$

where we put $\mathcal{U}\left(x_0\right):=\operatorname{int}C$ for $x_0\in\operatorname{int}C$. Given $x_0\in\partial C$, $z\in\mathcal{U}\left(x_0\right)\setminus C$ and a minimizing sequence $\{x_n\}\subset C$ of $x\mapsto\rho_F\left(x-z\right)$ on C, in the Case 1 (the condition ($\mathbf{A}$) valid at x_0) we immediately find the (unique) projection $\pi_C^F\left(z\right)$ as limit of $\{x_n\}$, existing since it is a Cauchy sequence. Otherwise (the condition ($\mathbf{B}$) holds) we choose first a subsequence $\{x_{k_n}\}$ such that $\{\rho_F\left(x_{k_n}-z\right)\}$ is nonincreasing, and x_0 is not a cluster point of $\{x_{k_n}\}$. Being a Cauchy sequence it converges to an element $x\in\pi_C^F\left(z\right)$. Assuming that $x,y\in\pi_C^F\left(z\right)$ with $x\neq y$ we consider the sequence $\{x_n\}$ whose odd terms are equal to x and all even terms are equal to y. Since $\{\rho_F\left(x_n-z\right)\}$ is now stationary, we can again apply the first part of the proof and conclude the convergence of $\{x_n\}$ to $x=y$. Notice that the above arguments are applicable also if one of the points x or y coincides with x_0 (because for a pair of natural numbers n and m with $m\geq n$ we utilize the hypothesis ($\mathbf{B}_2$) at the point x_n'' only). In order to show continuity at the point $z\in\mathcal{U}$ let us observe that for each $\{z_n\}\subset\mathcal{U}$ converging to z the sequence $\left\{\pi_C^F\left(z_n\right)\right\}$ minimizes $x\mapsto\rho_F\left(x-z\right)$ on C. Indeed,

$$
\begin{aligned}
\rho_F\left(\pi_C^F\left(z_n\right)-z\right) \quad & \leq \quad \rho_F\left(\pi_C^F\left(z_n\right)-z_n\right)+\rho_F\left(z_n-z\right)\leq \\
& \leq \quad \mathfrak{T}_C^F\left(z\right)+2\left\|F^0\right\|\left\|z_n-z\right\|\to\mathfrak{T}_C^F\left(z\right)+.
\end{aligned}
$$

Thus, by the same reasons as above, each subsequence of $\left\{\pi_C^F\left(z_n\right)\right\}$ admits a subsequence converging to $\pi_C^F\left(z\right)$. So $\pi_C^F\left(z_n\right)\to\pi_C^F\left(z\right)$, and theorem is completely proved. ■

Thus we have two types of local assumptions guaranteeing the well-posedness of the time-minimum projection in a neighbourhood of a fixed point $x_0 \in \partial C$. The first one (the condition (**A**)) provides regularity of the superposition operator involving both the proximal normal cone to C and the gradient $\nabla \rho_{F^0}(\cdot)$, while the other involves the curvatures of F and C being square characteristics of these sets. Therefore, we can refer to (**A**) and (**B**) as to the first and to the second order condition, respectively. Although there is a large class of problems, which satisfy both hypotheses (for instance, if $F = \overline{B}$ and $C = \{x \in H : f(x) \leq 0\}$, where $f(\cdot)$ is a locally $\mathcal{C}^{1,1}$ function with $\nabla f(x) \neq 0$), simple examples show (see Section 8) that none of the two ((**A**) and (**B**)) implies the other. In the end of Section 7 we amplify a little bit this list of local conditions including some extreme cases.

If the set C is proximally smooth then we can give an explicit formula for a neighbourhood where the continuous retraction $\pi_C^F(\cdot)$ is defined, which has, however, mainly theoretic interest due to the fact that it involves approximations to the projection itself. To this end let us consider a slightly stronger hypothesis than $(\mathbf{B}_2)$. Namely, we say that (F, C) satisfies the *condition* $(\mathbf{B}_2')$ at a point $x_0 \in \partial C$ if there exist $\delta, \delta' > 0$ and $K > 0$ such that

$$\varkappa_F(\mathfrak{J}(\xi^*), \xi^*) \geq K \quad \text{for all } \xi^* \in U_{\delta,\delta'}(x_0),$$

where the set $U_{\delta,\delta'}(x_0)$ is defined by (6.1).

Theorem 6.2 *Assume that $C \subset H$ is φ-convex with a continuous function $\varphi : C \to \mathbb{R}^+$, and at each point $x_0 \in \partial C$ the pair (F, C) satisfies the condition $(\mathbf{B}_2')$. Then the mapping $z \mapsto \pi_C^F(z)$ is single-valued and continuous on the open set $\mathfrak{A}(C)$ of all points $z \in H$, which either belong to C or satisfy the inequality*

$$\liminf_{\mathfrak{F}(z)} \left\{ \varkappa_F(\mathfrak{J}(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\, \varphi(x) \right\} > 0. \tag{6.17}$$

Here $\mathfrak{F}(z)$, $z \notin C$, is the filter in H^3 generated by the sets

$$\left\{ (x, v, \xi^*) : \rho_F(x - z) < \mathfrak{T}_C^F(z) + \varepsilon,\ x \in \partial C; \right.$$
$$\left. v \in \mathbf{N}_C(x),\ \|\xi^* + v\| < \varepsilon,\ \xi^*, -v \in \partial F^0 \right\}, \quad \varepsilon > 0.$$

Proof. In order to prove openess of $\mathfrak{A}(C)$ let us take first $z \in \mathfrak{A}(C) \setminus C$ and choose $\nu > 0$, $\varepsilon > 0$ such that

$$\varkappa_F(\mathfrak{J}(\xi^*), \xi^*) - \mathfrak{T}_C^F(z)\, \varphi(x) \geq \nu \tag{6.18}$$

whenever $x \in \partial C$ with $\rho_F(x - z) \leq \mathfrak{T}_C^F(z) + \varepsilon$ and $v \in \mathbf{N}_C(x)$, $\xi^* \in \partial F^0$ with $\|\xi^* + v\| \leq \varepsilon$, $\rho_{F^0}(-v) = 1$. By the *a priori* estimate (3.16) the function $\varphi(\cdot)$ is bounded on the set of x satisfying (6.18), say $\varphi(x) \leq M$ with some $M > 0$. Set

$$\varepsilon' := \min\left(\frac{\varepsilon}{4\|F^0\|}, \frac{\varepsilon}{2}, \frac{\nu}{2M\|F^0\|} \right).$$

Assuming, moreover, that $\left(z+\varepsilon'\overline{B}\right)\cap C=\varnothing$, for each $z'\in z+\varepsilon'\overline{B}$ let us define the set

$$P\left(z'\right):=\left\{x\in\partial C:\rho_F\left(x-z'\right)\le\mathfrak{T}_C^F\left(z'\right)+\varepsilon'\right\}\neq\varnothing. \tag{6.19}$$

Then, by the choice of ε', for arbitrary vectors $\xi^*,-v\in\partial F^0$ with $v\in\mathbf{N}_C\left(x\right)$, $x\in P\left(z'\right)$ and $\|\xi^*+v\|\le\varepsilon'$ the inequality

$$\varkappa_F\left(\mathfrak{J}\left(\xi^*\right),\xi^*\right)-\mathfrak{T}_C^F\left(z'\right)\varphi\left(x\right)\ge\frac{\nu}{2} \tag{6.20}$$

holds, implying that $z'\in\mathfrak{A}\left(C\right)$.

Let now $z:=x_0\in\partial C$. By the hypothesis ($\mathbf{B}_2'$) and continuity of the function $\varphi\left(\cdot\right)$ there exist $\delta,\delta'>0$ and positive constants K, M such that $\varkappa_F\left(\mathfrak{J}\left(\xi^*\right),\xi^*\right)\ge K$ for all $\xi^*\in U_{\delta,\delta'}\left(x_0\right)$ and $\varphi\left(x\right)\le M$ for all $x\in C_\delta\left(x_0\right)$. Set

$$\varepsilon':=\frac{1}{2}\min\left(\frac{K}{\|F^0\|\,M},\frac{\delta}{\|F\|\left(\|F^0\|+1\right)}\right).$$

Taking $z'\notin C$ with $\|z'-x_0\|\le\varepsilon'$ and $x\in P\left(z'\right)$ we have that

$$\begin{aligned}\|x-x_0\| &\le \|x-z'\|+\|z'-x_0\|\le\\ &\le \|F\|\left(\mathfrak{T}_C^F\left(z'\right)+\varepsilon'\right)+\varepsilon'\le\delta,\end{aligned}$$

and $\mathfrak{T}_C^F\left(z'\right)\varphi\left(x\right)\le K/2$. If, furthermore, $v\in\mathbf{N}_C\left(x\right)$, $\rho_{F^0}\left(-v\right)=1$, and $\xi^*\in\partial F^0$ with $\|\xi^*+v\|\le\varepsilon'<\delta'$ then clearly $\xi^*\in U_{\delta,\delta'}\left(x_0\right)$, and we obtain the inequality (6.20) with $\nu=K$. Consequently, $z'\in\mathfrak{A}\left(C\right)$.

Proving the well-posedness of the projection $\pi_C^F\left(\cdot\right)$ we can proceed as in the proof of Theorem 6.1 with some minor changements. Let us fix $z\in\mathfrak{A}\left(C\right)$, $z\notin C$, and take a minimizing sequence $\{x_n\}$ for $x\mapsto\rho_F\left(x-z\right)$ on the set C, assuming that $\{\rho_F\left(x_n-z\right)\}$ decreases (may be not strictly). Choosing then the sequences $\{x_n'\}\subset\partial C$, $v_n\in\mathbf{N}_C\left(x_n'\right)$, $\xi_n^*\in\partial\rho_F\left(x_n'-z\right)$ from Lemma 5.1 (see Remarks 5.2 and 5.3) and a decreasing number sequence $\nu_n\to 0+$, which satisfies the inequalities $\|x_n'-x_n\|\le\nu_n$, $\rho_F\left(x_n'-z\right)\le\mathfrak{T}_C^F\left(z\right)+\nu_n$ and

$$\left|\mathfrak{T}_C^F\left(z\right)\|\xi_n^*\|-\|v_n\|\right|\le\nu_n \tag{6.21}$$

(see (5.2)), we find then (see (6.17)) a number $\nu>0$ such that

$$\varkappa_F\left(\mathfrak{J}\left(\xi_n^*\right),\xi_n^*\right)-\mathfrak{T}_C^F\left(z\right)\varphi\left(x_n'\right)\ge 2\nu\,\|F\| \tag{6.22}$$

for $n\ge 1$ large enough (assume that for all n). Denoting as earlier

$$R_n:=\frac{1}{2\,\|\xi_n^*\|\,\varkappa_F\left(\mathfrak{J}\left(\xi_n^*\right),\xi_n^*\right)}$$

and $\psi_n:=\varphi\left(x_n'\right)$ we rewrite (6.22) in the form

$$\frac{1}{2R_n}-\mathfrak{T}_C^F\left(z\right)\|\xi_n^*\|\,\psi_n\ge 2\nu. \tag{6.23}$$

Due to the estimate (3.16) the sequence $\{1/R_n\}$ is bounded (and $\{\psi_n\}$ is bounded too as follows from (6.23)). Taking into account the inequality (6.21), we come to (6.13), and the remainder of the proof is exactly the same as respective reasoning in Theorem 6.1. ■

In a finite dimensional space due to the compactness of the set $U_{\delta,\delta'}(x_0)$, the condition $(\mathbf{B}_2')$ can be substituted by the second order strict convexity of F w.r.t. each vector $\xi^* \in -\mathbf{N}_C(x_0)$. However, in general, we have to require the local uniformity of this property through lack of the strong convergence of normals. A global version of the uniform strict convexity was introduced in [15] (see Definition 5.2). Notice that the γ-*strict convexity* considered there (with $\gamma > 0$) is nothing else than the inequality $\varkappa_F(\mathfrak{J}(\xi^*), \xi^*) \geq \gamma/2$ valid simultaneously for all $\xi^* \in \partial F^0$. In this case as an immediate consequence of Theorem 6.2 we obtain the following well-posedness result.

Corollary 6.1 *Let $F \subset H$ be a closed bounded γ-strictly convex set with $0 \in \operatorname{int} F$, and let $C \subset H$ be nonempty closed and φ-convex set with a continuous function $\varphi : C \to \mathbb{R}^+$. Then the projection $\pi_C^F(z)$ is a singleton continuously depending on $z \in \mathfrak{B}(C)$, where*

$$\mathfrak{B}(C) := \left\{ z \in H : \limsup_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ x \in \partial C}} \rho_F(x-z)\,\varphi(x) < \frac{\gamma}{2} \right\} \tag{6.24}$$

is an open set containing C.

The set (6.24), which is clearly smaller than the neighbourhood given by (6.17), can be written in terms of the projection as

$$\mathfrak{B}(C) = \left\{ z \in H : \mathfrak{T}_C^F(z)\,\varphi\left(\pi_C^F(z)\right) < \frac{\gamma}{2} \right\}.$$

Notice that the unit ball $\overline{B}$ is γ-strictly convex with $\gamma = 1$, and the set (6.24) in the case $F = \overline{B}$ is exactly the same as constructed in [7] (see Definition 2.5 and Proposition 2.6). On the other hand, if the set C is φ-convex with $\varphi = \text{const}$ then the well-posedness condition given by Corollary 6.1 admits the form $2\varphi\mathfrak{T}_C^F(z) < \gamma$, which is slightly weaker than the hypotheses of Theorem 5.6 [15].

7 Some particular and special cases

Let us concretize the results obtained in the previous section. First, we consider the case of a φ-convex target set with smooth boundary, denoting by $\mathfrak{n}(x)$ the unit normal vector to C at the point $x \in \partial C$ and setting

$$\mathfrak{v}(x) := -\frac{\mathfrak{n}(x)}{\rho_{F^0}(-\mathfrak{n}(x))}. \tag{7.1}$$

Theorem 7.1 *Let C be a closed set with smooth boundary, which is φ-convex with a continuous function $\varphi : C \to \mathbb{R}^+$, and let F be a closed bounded set with $0 \in \operatorname{int} F$, which is strictly convex of the second order w.r.t. each vector $\mathfrak{v}(x)$, $x \in \partial C$. Then the time-minimum projection $\pi_C^F(\cdot)$ is well defined on the (open) set $\mathfrak{A}(C)$ (see (6.17)), which in this case admits the form*

$$\left\{ z \in H : \liminf_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ \xi^* - \mathfrak{v}(x) \to 0 \\ x \in \partial C,\, \xi^* \in \partial F^0}} \left\{ \varkappa_F\left(\mathfrak{J}(\xi^*), \xi^*\right) - \mathfrak{T}_C^F(z)\,\varphi(x) \right\} > 0 \right\}. \tag{7.2}$$

We put naturally $\liminf$ *in (7.2) to be equal to $+\infty$ whenever $z \in \operatorname{int} C$.*

Proof. It is immediate consequence of Theorem 6.2, since the second order strict convexity of F together with the lower semicontinuity of $\xi^* \mapsto \varkappa_F(\mathfrak{J}(\xi^*), \xi^*)$ at $\mathfrak{v}(x)$, $x \in \partial C$, and with the continuity of $\mathfrak{v}(\cdot)$ imply the condition $(\mathbf{B}_2')$. ■

In terms of the time-minimum projection itself (already defined and single-valued on $\mathfrak{A}(C)$) we can represent this neighbourhood as

$$\mathfrak{A}(C) = C \cup \left\{ z \in H \setminus C : \mathfrak{T}_C^F(z)\,\varphi(\bar{x}) < \varkappa_F\left(\mathfrak{J}(\mathfrak{v}(\bar{x})), \mathfrak{v}(\bar{x})\right) \right\}$$

where $\bar{x} := \pi_C^F(z)$.

Remark 7.1 *If $\dim H < \infty$ then each minimizing sequence has a cluster point, and, consequently, the neighbourhood (7.2) can be written in a simpler form*

$$\mathfrak{A}(C) = \left\{ z : \liminf_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ x \in \partial C}} \left\{ \varkappa_F\left(\mathfrak{J}(\mathfrak{v}(x)), \mathfrak{v}(x)\right) - \mathfrak{T}_C^F(z)\,\varphi(x) \right\} > 0 \right\}.$$

Concretizing now the local result given by Theorem 6.1 we have

Theorem 7.2 *Let C be a closed φ-convex set with smooth boundary and such that for each point $x_0 \in \partial C$ one of the assumptions holds:*

(**i**) *the set F is uniformly strictly convex w.r.t. the vector $\mathfrak{v}(x_0)$, and the (single-valued) mapping $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ is Lipschitz continuous near x_0;*

(**ii**) *the set F is strictly convex of the second order w.r.t. $\mathfrak{v}(x_0)$.*

Then $z \mapsto \pi_C^F(z)$ is a neighbourhood retraction of the set C.

Proof. The hypothesis (**i**) is nothing else than the condition (**A**) at the point x_0 specified for the case of smooth boundary, while (**ii**) implies the condition $(\mathbf{B}_2')$ at x_0, which is equivalent to $(\mathbf{B}_2)$ in this case. Thus, we are able to apply directly Theorem 6.1. ■

Notice that if at each point $x_0 \in \partial C$ the dynamics satisfies the hypothesis (**i**) from the above theorem then we can entirely avoid the φ-convexity assumption for the target set.

Theorem 7.3 *Let C be a closed set with smooth boundary, and let F be uniformly strictly convex w.r.t. each vector $\mathfrak{v}(x)$, $x \in \partial C$. If, moreover, the (single-valued) mapping $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ is locally lipschitzean on ∂C then the statement of Theorem 7.2 holds.*

On the other hand, we can obtain the well-posedness of $\pi_C^F(\cdot)$ in a neighbourhood of a φ-convex set C even with lack of the strict convexity of F w.r.t. $\xi^* \in -\mathbf{N}_C^p(x) \cap \partial F^0$ for some isolated points $x \in \partial C$ where smoothness of the boundary is also violated (see Example 8.4).

Observe that the formulas (6.17), (7.2) as well as neighbourhoods $\mathcal{U}(x_0)$ given by Theorem 6.1 (see (6.12)) involve the function $\varkappa_F(\xi, \xi^*)$, which can not be substituted, in general, by the "true" curvature $\hat{\varkappa}_F(\xi, \xi^*)$. Let us propose a method to estimate $\varkappa_F(\xi, \xi^*)$ from below basing on the differentiability properties of the duality mapping $\mathfrak{J}_F(\cdot)$ similarly as it was done for $\gamma_F^{\pm}(\xi, \xi^*)$ (see (4.11)). To this end we assume that the set F^0 has second order smooth boundary (at $\xi^* \in \partial F^0$) and associate to each $\delta > 0$ some positive number $\beta(\delta, \xi^*)$ such that

$$\left\|\nabla^2 \rho_{F^0}(\eta^*) - \nabla^2 \rho_{F^0}(\xi^*)\right\| \le \delta$$

whenever $\eta^* \in \partial F^0$ with $\|\eta^* - \xi^*\| \le \beta(\delta, \xi^*)$. Then, given $\delta > 0$ and $0 < \lambda < 1$ the inequality

$$\left\|\nabla^2 \rho_{F^0}(\eta^* + tv^*) - \nabla^2 \rho_{F^0}(\xi^*)\right\| \le \delta \tag{7.3}$$

holds for all $0 < t \le (1-\lambda)\beta(\delta, \xi^*) / \left\|F^0\right\|$, $v^* \in \partial F^0$ and $\eta^* \in \partial F^0$ with $\|\eta^* - \xi^*\| < \lambda\beta(\delta, \xi^*)$. Recalling the proof of Proposition 4.3 we obtain from (4.13) and (7.3) that

$$\mathfrak{S}_{F^0}(t, \eta^*, \eta) \le \frac{1}{2}\left(\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} + \delta\left\|F^0\right\|^2\right) t^2 \tag{7.4}$$

where as usual $\eta := \mathfrak{J}(\eta^*)$. Applying the Legendre-Fenchel transform to both parts of (7.4) we come to the inequality

$$\begin{aligned} &\mathfrak{S}_{F^0}^{\bigstar}(r, \eta^*, \eta) \ge \\ \ge\ &\sup\left\{tr - \frac{1}{2}\left(\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} + \delta\left\|F^0\right\|^2\right) t^2 : \right. \\ &\left. 0 < t \le (1-\lambda)\frac{\beta(\delta, \xi^*)}{\|F^0\|}\right\} = \frac{r^2}{2\left(\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} + \delta\|F^0\|^2\right)}, \end{aligned} \tag{7.5}$$

which holds true for all $0 < r \le (1-\lambda)\, q(\delta, \xi^*)$, where

$$q(\delta, \xi^*) := \beta(\delta, \xi^*)\frac{\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} + \delta\left\|F^0\right\|^2}{\|F^0\|}.$$

By using the duality between the moduli of local smoothness and of local strict convexity (see (4.4)) we obtain from (7.5) and (3.3) that

$$\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \ge \frac{r^2}{2\|F\|^2\left(\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} + \delta\|F^0\|^2\right)} \tag{7.6}$$

whenever $0 < r \leq (1-\lambda)\, q(\delta, \xi^*) \|F\|$. Obviously, $\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) = +\infty$ for $r > 2\|F\|$, while in the case $(1-\lambda)\, q(\delta, \xi^*) \|F\| < r \leq 2\|F\|$, by the monotonicity of the function $\widehat{\mathfrak{C}}_F(\cdot, \eta, \eta^*)$, we have

$$\widehat{\mathfrak{C}}_F(r, \eta, \eta^*) \geq (1-\lambda)^2 \frac{q^2(\delta, \xi^*)}{8\|F\|^2 \left(\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \|F^0\|^2 \right)} r^2. \tag{7.7}$$

Finally, comparing the inequalities (7.6) and (7.7), which hold for all η^* near ξ^*, by arbitrarity of λ, $0 < \lambda < 1$, we obtain (see (3.14)):

$$\gamma_F(\xi, \xi^*) \geq \frac{1}{2\|F\|^2 \left(\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \|F^0\|^2 \right)} \min \left(\frac{q^2(\delta, \xi^*)}{4}, 1 \right). \tag{7.8}$$

This estimate together with Theorem 7.1 permit us to formulate the following result.

Theorem 7.4 *In addition to the hypotheses of Theorem 7.1 let us suppose that the polar set F^0 has boundary of class $\mathcal{C}^2$ near $\mathfrak{v}(x)$ for each $x \in \partial C$. Then for a given $\delta > 0$ the time-minimum projection $\pi_C^F(\cdot)$ is well-defined on the (open) set $\mathfrak{A}_\delta(C)$ of all $z \in H$, which either belong to C or satisfy the inequality*

$$\liminf_{\substack{\rho_F(x-z) \to \mathfrak{T}_C^F(z)+ \\ \xi^* - \mathfrak{v}(x) \to 0 \\ x \in \partial C,\ \xi^* \in \partial F^0}} \left\{ \mathfrak{Q}(\delta, \xi^*) - \mathfrak{T}_C^F(z)\, \varphi(x) \right\} > 0,$$

where

$$\mathfrak{Q}(\delta, \xi^*) := \frac{1}{2\|F\|^2 \|\xi^*\|} \min \left[\frac{\beta^2(\delta, \xi^*) \left(\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \|F^0\|^2 \right)}{4\|F^0\|^2}, \right.$$
$$\left. \frac{1}{\left\| \nabla^2 \rho_{F^0}(\xi^*) \right\|_{F^0} + \delta \|F^0\|^2} \right]. \tag{7.9}$$

Remark 7.2 *It is seen from (7.9) and from the definition of $\beta(\delta, \xi^*)$ that the neighbourhood $\mathfrak{A}_\delta(C)$ is larger whenever the second derivative $\nabla^2 \rho_{F^0}(\cdot)$ grows slower. Theorem 7.4 perfectly works, in particular, when $\nabla^2 \rho_{F^0}(\cdot)$ is Lipschitz continuous locally at each point $\xi^* \in \partial F^0$ (in a ε_{ξ^*}-neighbourhood of ξ^*) with Lipschitz constant L_{ξ^*}, in which case we can choose $\beta(\delta, \xi^*)$ equal to* $\min(\delta / L_{\xi^*}, \varepsilon_{\xi^*})$ *(see Example 8.3).*

Concluding this section let us give two special hypotheses involving local convexity of the target set, which also guarantee the well-posedness of the projection.

Proposition 7.1 *Suppose that for a given $x_0 \in \partial C$ one of the following conditions holds:*

(**i**) *C has smooth boundary at x_0, and for some $\varepsilon > 0$ the set $C \cap \left(x_0 + \varepsilon \overline{B}\right)$ has nonempty interior, and it is strictly convex of the second order at x_0 (w.r.t. the corresponding normal vector);*

(**ii**) *for some $\varepsilon > 0$ the set $C \cap \left(x_0 + \varepsilon \overline{B}\right)$ is convex, and F is strictly convex of the second order w.r.t. each $v \in -\mathbf{N}_C(x) \cap \partial F^0$, where $x \in \partial C$ with $\|x - x_0\| \le \varepsilon$.*

Then the function $z \mapsto \pi_C^F(z)$ is single-valued and continuous in a neighbourhood of the point x_0.

Proof. Let us consider each case separately.

(**i**) Without loss of generality (translating if necesary the set C) we can suppose that $0 \in \operatorname{int} G$, where $G := C \cap \left(x_0 + \varepsilon \overline{B}\right)$. Let us denote by $v_0 := \mathfrak{n}(x_0) / \rho_{G^0}(\mathfrak{n}(x_0))$, where $\mathfrak{n}(x_0)$ is the unit normal vector to C (as well as to G, certainly) at the point x_0. Since $\nu := \gamma_G(x_0, v_0) > 0$, by (3.13) and by the continuity of the mapping $x \mapsto \mathfrak{n}(x)$ in a neighbourhood of x_0 there exist $0 < \delta \le \varepsilon$ and $\theta \ge \nu/2$ such that $\widehat{\mathfrak{C}}_G(r, x, v) \ge \theta r^2$ whenever $x \in \partial C$, $\|x - x_0\| \le \delta$, $v = \mathfrak{n}(x) / \rho_{G^0}(\mathfrak{n}(x))$ and $r > 0$.

Setting now $\mathcal{U}(x_0) := x_0 + \frac{\delta}{D} \overline{B}$, where $D := 2 \left\|F^0\right\| \|F\|$, take $z \in \mathcal{U}(x_0)$ and a minimizing sequence $\{x_n\} \subset \partial C$ of the function $x \mapsto \rho_F(x - z)$ on C. Similarly as in the proof of Theorem 6.1 we see that $\|x_n - x_0\| \le \delta$, and hence, by Definition 3.2,

$$\langle x_n - x_m, \mathfrak{n}(x_n) \rangle \ge \frac{\nu}{2} \rho_{G^0}(\mathfrak{n}(x_n)) \|x_m - x_n\|^2 \tag{7.10}$$

for all $m \ge n \ge 1$ sufficiently large. In accordance with Lemma 5.1 and Remarks 5.2, 5.3 we do not lose generality if suppose that for some vectors $\xi_n^* \in \mathbf{N}_F\left(\frac{x_n - z}{\rho_F(x_n - z)}\right) \cap \partial F^0$ and for some sequence $\nu_n \to 0+$ the inequality

$$\|\mathfrak{v}(x_n) - \xi_n^*\| \le \frac{\nu_n}{\mathfrak{T}_C^F(z)}, \tag{7.11}$$

$n = 1, 2, ...,$ takes place, where $\mathfrak{v}(x_n)$ is given by (7.1). Let us set $\lambda_n := \frac{\rho_{F^0}(-\mathfrak{n}(x_n))}{\rho_{G^0}(\mathfrak{n}(x_n))}$ and $z_n := x_n + \lambda_n \xi_n^*$. By using (7.11) and (7.10) we obtain that

$$\langle z_n - x_n, x_m - x_n \rangle \ge -\frac{\nu_n}{\mathfrak{T}_C^F(z)} \lambda_n \|x_m - x_n\| + \frac{\nu}{2} \|x_m - x_n\|^2 \tag{7.12}$$

for all $m \ge n \ge 1$. On the other hand, ξ_n^* is a normal vector to the set $z + \rho_F(x_n - z) F$ at the point x_n, and x_m belongs to this set by the eventual monotonicity of $\{\rho_F(x_n - z)\}$. Therefore, $\langle z_n - x_n, x_m - x_n \rangle \le 0$, and combining this with (7.12) we find

$$\frac{\nu}{2} \|x_m - x_n\| \le \frac{\nu_n}{\mathfrak{T}_C^F(z)} \lambda_n.$$

Hence $\{x_n\}$ is a Cauchy sequence because $\{\lambda_n\}$ is bounded, and the remainder follows by the same line as in the proof of Theorem 6.1.

(**ii**) In this case we set $\mathcal{U}(x_0) := x_0 + \frac{\varepsilon}{2(\|F\|\|F^0\|+1)}\overline{B}$ and show directly that $\pi_C^F(z) \neq \varnothing$ for each $z \in \mathcal{U}(x_0)$. Indeed, if $\{x_n\} \subset C$ is a sequence with $\rho_F(x_n - z) \leq \mathfrak{T}_C^F(z) + 1/n$ then by the boundedness there exists its subsequence converging weakly to some $x \in H$. Since

$$x_n \in C \cap \left(z + \left(\mathfrak{T}_C^F(z) + \frac{1}{n}\right) F\right) \subset C \cap \left(x_0 + \varepsilon\overline{B}\right)$$

for $n \geq 1$ large enough, and the last set is weakly closed, we have $x \in C$. On the other hand, choosing a sequence $y_n \in z + \mathfrak{T}_C^F(z) F$ such that $\rho_F(x_n - y_n) \leq 1/n$ we observe that the weak limit of some its subsequence is equal to x too. Hence $x \in \left(z + \mathfrak{T}_C^F(z) F\right) \cap C = \pi_C^F(z)$. This simple argument was used earlier, e.g., in [15, Theorem 4.2(b)].

Let us assume now that the projection $\pi_C^F(z)$ consists at least of two different points, say x and y, both clearly belonging to $G := C \cap \left(x_0 + \varepsilon\overline{B}\right)$. Then $\pi_C^F(z)$ contains the whole segment $\{\lambda x + (1-\lambda) y : \lambda \in [0,1]\}$. Fix some $\hat{x} := \hat{\lambda} x + \left(1 - \hat{\lambda}\right) y$ with $0 < \hat{\lambda} < 1$. Then we have

$$0 \in \partial\left(\rho_F(\hat{x} - z) + \mathbf{I}_G(\hat{x})\right) = \partial\rho_F(\hat{x} - z) + \mathbf{N}_C(\hat{x}), \tag{7.13}$$

and there exists a unit normal vector $\hat{\mathfrak{n}} \in \mathbf{N}_C(\hat{x})$ such that $\hat{\mathfrak{v}} := -\hat{\mathfrak{n}}/\rho_{F^0}(-\hat{\mathfrak{n}}) \in \partial\rho_F(\hat{x} - z) \subset \partial F^0$, or, equivalently, $(\hat{x} - z)/\mathfrak{T}_C^F(z) \in \partial\rho_{F^0}(\hat{\mathfrak{v}}) = \mathfrak{J}_F(\hat{\mathfrak{v}})$. In fact, $\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}$ is the unique element of $\mathfrak{J}_F(\hat{\mathfrak{v}})$, and $\hat{\mathfrak{v}} \in \mathbf{N}_F\left(\frac{\hat{x}-z}{\mathfrak{T}_C^F(z)}\right)$. It easily follows now that the vector $\hat{\mathfrak{v}}$ is orthogonal to the line

$$L := \left\{\lambda \frac{x - z}{\mathfrak{T}_C^F(z)} + (1 - \lambda) \frac{y - z}{\mathfrak{T}_C^F(z)} : \lambda \in \mathbb{R}\right\}.$$

Hence $\mathfrak{R}_F(\mathfrak{J}_F(\hat{\mathfrak{v}}), \hat{\mathfrak{v}}) = +\infty$ contradicting the condition of theorem.

Finally, let us consider a sequence $\{z_n\} \subset \mathcal{U}(x_0)$ converging to some $z \in \mathcal{U}(x_0)$. By the arguments above, without loss of generality we may suppose that $\left\{\pi_C^F(z_n)\right\}$, being a minimizing sequence for $x \mapsto \rho_F(x - z)$ on C, converges weakly to the unique projection $\pi_C^F(z)$. Setting $\hat{x} := \pi_C^F(z)$, from the relation (7.13) we find again a normal vector $\hat{\mathfrak{n}} \in \mathbf{N}_C(\hat{x})$ such that $(\hat{x} - z)/\mathfrak{T}_C^F(z)$ is the unique element of $\mathfrak{J}_F(\hat{\mathfrak{v}})$ where $\hat{\mathfrak{v}} := -\hat{\mathfrak{n}}/\rho_{F^0}(-\hat{\mathfrak{n}})$. Therefore, it is a strongly exposed point of F w.r.t. $\hat{\mathfrak{v}}$ (see Proposition 3.1(i)). In particular, the weak convergence of $\left\{\left(\pi_C^F(z_n) - z_n\right)/\mathfrak{T}_C^F(z_n)\right\}$ to $\left(\pi_C^F(z) - z\right)/\mathfrak{T}_C^F(z)$ implies the strong convergence, and the continuity of the mapping $z \mapsto \pi_C^F(z)$ follows. ■

8 Examples

Example 8.1 *In a Hilbert space H for a fixed $v \in H$, $\|v\| = 1$, and $0 < \theta < 1$ let us consider the convex closed cone*

$$K_{v,\theta} := \{x \in H : \langle v, x\rangle \geq \theta \|x\|\},$$

whose polar cone is

$$K^0_{v,\theta} = \left\{ x \in H : \langle -v, x \rangle \geq \sqrt{1-\theta^2}\, \|x\| \right\}.$$

Taking now $0 < \theta_1 < \theta_2 < 1$*, we define* $C := \overline{H \setminus K_{v,\theta_1}}$ *and* $F := (K_{v,\theta_2} - v) \cap \overline{B}$*.*

The set C neither has smooth boundary, nor is φ-convex, and, moreover, the origin is its "inward corner" point, $\mathbf{N}^p_C(0) = \{0\}$. On the other hand, F is not strictly convex, because the boundary ∂F contains a lot of linear segments.

However, the hypotheses of Theorem 6.1 are fulfilled, and $\pi^F_C(\cdot)$ is a (global) continuous retraction of C. Indeed, let us represent the target set in the form $C = \{x \in H : f(x) \leq 0\}$ where $f(x) := \langle v, x \rangle - \theta_1 \|x\|$. Then $\mathbf{N}^p_C(x) = \nabla f(x)\, \mathbb{R}^+ = \left(v - \theta_1 \frac{x}{\|x\|} \right) \mathbb{R}^+$ for each $x \in \partial^* C = \partial C \setminus \{0\}$. In particular, taking $\xi^* \in -\mathbf{N}^p_C(x) \cap \partial F^0$ we have

$$\langle -v, \xi^* \rangle = \sqrt{1-\theta_1^2}\, \|\xi^*\| > \sqrt{1-\theta_2^2}\, \|\xi^*\|,$$

i.e., $\xi^* \in \operatorname{int} K^0_{v,\theta_2}$, implying obviously $\mathfrak{J}_F(\xi^*) = \{-v\}$. Therefore, the condition $(\mathbf{A}_1)$ is satisfied trivially at the point $x_0 = 0$ (with arbitrary $\delta > 0$). In order to justify $(\mathbf{A}_2)$ let us choose $\delta' > 0$ and σ, $\sqrt{1-\theta_2^2} < \sigma < 1$, such that

$$\langle -v, \eta^* \rangle \geq \sigma\, \|\eta^*\|$$

whenever $\eta^* \in \partial F^0$ with $\|\eta^* - \xi^*\| \leq 2\delta'$. Hence, for each $\eta^* \in U_{\delta,\delta'}(0)$ (see (6.1)) and each $\eta \in F$ by duality of the cones we have $\left\langle \eta^* + \delta' v, v + \eta \right\rangle \leq 0$, and recalling that $\mathfrak{J}_F(\eta^*) = \{-v\}$ we obtain

$$\begin{aligned} \widehat{\mathfrak{C}}_F(r, -v, \eta^*) &= \inf\{\langle -v - \eta, \eta^* \rangle : \eta \in F,\ \|v + \eta\| \geq r\} \geq \\ &\geq \delta' \inf\{\langle v + \eta, v \rangle : \eta \in F,\ \|v + \eta\| \geq r\} \geq \delta' \theta_2 r > 0, \end{aligned}$$

which means the uniform strict convexity w.r.t. the set of directions $U_{\delta,\delta'}(0)$. In this example, certainly, it is easier to observe directly the uniform continuity of the mapping $\mathfrak{J}_F(\eta^*) \equiv -v$ on $U_{\delta,\delta'}(0)$ (this is what we really need proving Theorem 6.1) than to construct an estimate of the modulus $\widehat{\mathfrak{C}}_F$.

Example 8.2 *Let us modify slightly the previous example, taking arbitrary* $v \in H$ *with* $\|v\| = 1$*;* $0 < \theta_1, \theta_2 < 1$*;* $1 < \alpha < 2$ *and setting*

$$\begin{aligned} &C := \{x \in H : \langle v, x \rangle \leq \theta_1 \|x\|^\alpha\}; \\ &F := \{\xi \in H : \langle v, \xi + v \rangle \geq \theta_2 \|\xi + v\|^\alpha\}. \end{aligned}$$

Clearly, F is convex closed bounded with $0 \in \operatorname{int} F$, and C is closed admitting at each point $x \in \partial C$, $x \neq 0$, the unique unit normal vector directed as $\nabla f(x) = v - \alpha\theta_1 \frac{x}{\|x\|^{2-\alpha}}$ (here $f(x) := \langle v, x \rangle - \theta_1 \|x\|^\alpha$), which is also continuously extendable up to the origin (we have $\nabla f(0) = v$). So that $\mathbf{N}_C^l(x) = \nabla f(x)\mathbb{R}^+$, $x \in \partial C$, and the boundary of C is smooth. However, $\mathbf{N}_C^p(0) = \{0\}$ (as one easily verifies there is no point except the origin itself whose metric projection onto C is 0), while $\mathbf{N}_C^p(x) = \mathbf{N}_C^l(x)$ at other points $x \in \partial C$. Therefore, C is not φ-convex, and the condition (**B**) can not be applied (at least in a neighbourhood of the point 0).

Let us verify the hypothesis (**A**). First of all, F is uniformly strictly convex (w.r.t. the whole ∂F^0). It is even strictly convex of the second order with the curvature uniformly bounded from below (γ-strictly convex). Indeed, for the point $\bar{\xi} := -v \in \partial F$ setting $\bar{\xi}^* := -v/\rho_{F^0}(-v) = -v$ we directly have

$$\widehat{\mathfrak{C}}_F\left(r, \bar{\xi}, \bar{\xi}^*\right) = \theta_2 r^\alpha \geq \frac{\theta_2}{\left(\|F\| + 1\right)^{2-\alpha}} r^2, \quad r > 0, \tag{8.1}$$

while for each fixed $\xi \in \partial F$, $\xi \neq -v$, and $\eta \in F$ close to ξ by the second order Taylor formula (see, e.g, [6, p.75]) we obtain:

$$\begin{aligned}
&\|\eta + v\|^\alpha - \|\xi + v\|^\alpha - \frac{\alpha}{\|\xi + v\|^{2-\alpha}} \langle \eta - \xi, \xi + v \rangle = \\
= \;& \alpha \int_0^1 \left[\frac{\|\eta - \xi\|^2}{\|\eta_\tau + v\|^{2-\alpha}} - (2 - \alpha) \frac{\langle \eta_\tau + v, \eta - \xi \rangle^2}{\|\eta_\tau + v\|^{4-\alpha}} \right] (1 - \tau)\; d\tau \geq \\
\geq \;& \alpha(\alpha - 1) \|\eta - \xi\|^2 \int_0^1 \frac{1 - \tau}{\|\eta_\tau + v\|^{2-\alpha}}\; d\tau \geq \frac{\alpha(\alpha - 1)\|\eta - \xi\|^2}{2\left(\|F\| + 1\right)^{2-\alpha}},
\end{aligned} \tag{8.2}$$

where $\eta_\tau := \tau\eta + (1 - \tau)\xi$, $\tau \in [0, 1]$. Observe that $\mathbf{N}_F(\xi) = \nabla g(\xi)\mathbb{R}^+$ where $g(\xi) := \theta_2 \|\xi + v\|^\alpha - \langle v, \xi + v \rangle$, and as follows from (8.2)

$$\begin{aligned}
\langle \xi - \eta, \nabla g(\xi) \rangle &= \left\langle \xi - \eta, \alpha\theta_2 \frac{\xi + v}{\|\xi + v\|^{2-\alpha}} - v \right\rangle \geq \\
&\geq \theta_2 \left[\|\eta + v\|^\alpha - \|\xi + v\|^\alpha - \frac{\alpha}{\|\xi + v\|^{2-\alpha}} \langle \eta - \xi, \xi + v \rangle \right] \geq \\
&\geq \frac{\theta_2 \alpha(\alpha - 1)}{2\left(\|F\| + 1\right)^{2-\alpha}} \|\eta - \xi\|^2 .
\end{aligned}$$

Since $\|\nabla g(\xi)\| \leq 1$, from this inequality and from (3.2) we obtain that

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) \geq \frac{\theta_2 \alpha(\alpha - 1)}{2\|F\|\left(\|F\| + 1\right)^{2-\alpha}} r^2, \quad r > 0,$$

where $\xi^* := \nabla g(\xi)/\rho_{F^0}(\nabla g(\xi))$. Recalling (8.1) we conclude that F is γ-strictly convex with some $\gamma > 0$.

In order to verify the hypothesis ($\mathbf{A}_1$) let us fix an arbitrary point $x \in \partial C$, $x \neq 0$, with the proximal normal vector $\nabla f(x)$ and determine a (unique) $\xi \in \partial F$

such that $-\nabla f(x)$ is normal to F at ξ. Since $\mathbf{N}_F(\xi) = \nabla g(\xi)\,\mathbb{R}^+$, solving the equation $-\nabla f(x) = \lambda \nabla g(\xi)$, $\lambda > 0$, we find immediately that $\lambda = 1$ and

$$\xi = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\alpha-1}} x - v.$$

Thus, the (single-valued) mapping $x \mapsto \mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right)$ is Lipschitz continuous on $C_\delta(0)$ with $\delta > 0$ arbitrarily large, and the Lipschitz constant is $L = \left(\frac{\theta_1}{\theta_2}\right)^{\frac{1}{\alpha-1}}$. Applying now Theorem 6.1 we can affirm that $\pi_C^F(\cdot)$ is a neighbourhood retraction defined on the open set (see (6.3))

$$\mathcal{U} = \left\{ z \in H : \mathfrak{T}_C^F(z) < \left(\frac{\theta_2}{\theta_1}\right)^{\frac{1}{\alpha-1}} \right\}.$$

The following example (in the space $H = \mathbb{R}^2$ for the sake of clarity) illustrates the second order condition (balance between the curvatures).

Example 8.3 *Let* $F := \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \le 1 - \xi_1^4,\ -1 \le \xi_1 \le 1\right\}$ *and* $C := \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le x_2^2\right\}$.

Observe that F is closed convex bounded with $0 \in \operatorname{int} F$, and C is closed, φ-convex with

$$\varphi(x) = \frac{1}{\sqrt{1 + 4x_2^2}}, \quad x = (x_1, x_2) \in \partial C, \tag{8.3}$$

and has smooth boundary with the unit normal vector

$$\mathfrak{n}(x) = \frac{1}{\sqrt{1 + 4x_2^2}}(1, -2x_2), \quad x \in \partial C.$$

Let us estimate the curvature $\varkappa_F(\xi, \xi^*)$ for an arbitrary dual pair (ξ, ξ^*), $\xi \in \mathfrak{J}_F(\xi^*)$, $\xi^* \in \partial F^0$. Setting $\xi = (\xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R}$, by the symmetry reason we can consider, clearly, only the case when $\xi_2 \ge 0$ (and $\xi_1 < 0$). If $\xi_2 > 0$ then the (unique) normal vector ξ^* to F at ξ such that $\rho_{F^0}(\xi^*) = 1$ is given by

$$\xi^* = \frac{1}{1 + 3\xi_1^4}\left(4\xi_1^3, 1\right).$$

From Definition 3.2 after the simple transformations we have

$$\begin{aligned}\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = \frac{1}{1 + 3\xi_1^4} \inf \Big\{ (\eta_1 - \xi_1)^2 \Big[(\eta_1 - \xi_1)^2 + \\ + 4\xi_1(\eta_1 - \xi_1) + 6\xi_1^2 \Big] : \|\xi - \eta\| \ge r, \quad -1 \le \eta_1 \le 1 \Big\}, \quad r > 0.\end{aligned} \tag{8.4}$$

In virtue of the inequality

$$\|\xi - \eta\| \le |\eta_1 - \xi_1| \sqrt{1 + \left(1 + |\xi_1| + |\xi_1|^2 + |\xi_1|^3\right)^2}, \tag{8.5}$$

it follows (see (8.4)) that

$$\frac{\widehat{\mathfrak{C}}_F\left(r,\xi,\xi^*\right)}{r^2} \geq \frac{1}{\left(1+3\xi_1^4\right)\Sigma^2\left(\xi_1\right)}\left[\frac{r^2}{\Sigma^2\left(\xi_1\right)}+4r\frac{\xi_1}{\Sigma\left(\xi_1\right)}+6\xi_1^2\right], \tag{8.6}$$

where $\Sigma\left(\xi_1\right) := \sqrt{1+\left(\sum_{k=0}^{3}\left|\xi_1\right|^k\right)^2}$. Notice that the right-hand side in (8.6) is continuous in ξ. Therefore, in order to obtain an estimate of the scaled curvature from below it is enough only to pass to infimum in (8.6) for $r>0$ (see (3.14)), while for the "true" (local) curvature we let $r \to 0+$ (see (3.5)). Thus

$$\varkappa_F\left(\xi,\xi^*\right) = \frac{\gamma_F\left(\xi,\xi^*\right)}{\left\|\xi^*\right\|} \geq K\left(\xi_1\right) := \frac{2\xi_1^2}{\sqrt{1+16\xi_1^6\Sigma^2\left(\xi_1\right)}}, \tag{8.7}$$

and $\hat{\varkappa}_F\left(\xi,\xi^*\right) \geq 3K\left(\xi_1\right)$. In the same way (employing the inequality $\|\xi-\eta\| \geq \left|\xi_1-\eta_1\right|$ instead of (8.5)) we find upper bounds of the curvatures, which are of the order $O\left(\xi_1^2\right)$ as well. In particular, both $\varkappa_F$ and $\hat{\varkappa}_F$ are equal to zero at the points $(0,\pm1)$. Therefore, the set F is not γ-strictly convex, and the results of [15] can not be applied here.

However, there is a *local* uniform rotundity along the boundary of C that permits us to apply Theorem 7.2 (ii). To be more precise let us estimate the respective curvatures. Considering $x=(x_1,x_2)\in\partial C$ with $|x_2|\geq 1/8$ we see that for the vector

$$\mathfrak{v}\left(x\right) := -\frac{\mathfrak{n}\left(x\right)}{\rho_{F^0}\left(-\mathfrak{n}\left(x\right)\right)} = \frac{8\left|x_2\right|^{1/3}}{3+16x_2^{4/3}}\left(-1,2x_2\right) \tag{8.8}$$

belonging to ∂F^0 there is a unique $\xi=\left(\xi_1,\xi_2\right)\in\mathfrak{J}_F\left(\mathfrak{v}\left(x\right)\right)$. Namely,

$$\xi_1 = -\frac{1}{2\left|x_2\right|^{1/3}} \in [-1,0[\quad \text{and} \quad \xi_2 = \left(1-\xi_1^4\right)\operatorname{sgn}\left(x_2\right). \tag{8.9}$$

Setting for simplicity $\Sigma^2\left(\xi_1\right)\leq 17$, from (8.7) we have at this point:

$$\varkappa_F\left(\xi,\mathfrak{v}\left(x\right)\right) \geq \frac{1}{17}\frac{\left|x_2\right|^{1/3}}{\sqrt{1+4x_2^2}}. \tag{8.10}$$

Otherwise (if $|x_2|<1/8$) the vector $\mathfrak{v}\left(x\right)$ belongs to the interior of the normal cone

$$\mathbf{N}_F\left(-1,0\right) = \left\{\left(v_1,v_2\right)\in\mathbb{R}^2 : v_1 \leq -4\left|v_2\right|\right\}, \tag{8.11}$$

and the second order strict convexity also follows. In this case the curvature $\hat{\varkappa}_F$ at $\bar{\xi}=(-1,0)$ w.r.t. the vector $\mathfrak{v}\left(x\right)$ is equal to $+\infty$, while $\varkappa_F\left(\bar{\xi},\mathfrak{v}\left(x\right)\right)$ is a finite positive number depending on the size of both sets F and F^0, and on the proximity of $\mathfrak{v}\left(x\right)$ to the boundary $\partial\mathbf{N}_F\left(\bar{\xi}\right)$. To obtain a precise estimate we can proceed, e.g., as in the proof of Theorem 7.4. Namely, let us denote by $d(x)$ the minimal distance of $\mathfrak{v}(x)$ (see (8.8)) from $e^{\pm} := (-1, \pm 1/4)$ that are extreme vectors among those $\xi^* \in \partial F^0$ with $\mathfrak{J}_F(\xi^*) = \bar{\xi}$. Therefore, the function $\xi^* \mapsto \nabla^2 \rho_{F^0}(\xi^*)$ is Lipschitz continuous (it is identical zero) on $\partial F^0 \cap \left(\mathfrak{v}(x) + d(x)\overline{B}\right)$. Substituting $\left\|\nabla^2 \rho_{F^0}(\xi^*)\right\|_{F^0} = 0$; $\beta(\delta, \mathfrak{v}(x)) = d(x)$ (see (4.10) and Remark 7.2) and choosing a suitable $\delta > 0$, from the inequality (7.8) we obtain

$$\varkappa_F\left(\bar{\xi}, \mathfrak{v}(x)\right) \geq \frac{d(x)}{4\|F\|^2\|F^0\|\|\mathfrak{v}(x)\|}, \tag{8.12}$$

where $\|F\|$ and $\left\|F^0\right\|$ can be found through the radii of two balls: one containing the set F and another contained in it. In our case, for instance, $\|F\| \leq 7/6$ and $\left\|F^0\right\| \leq 9/8$.

Summarizing everything said above, we affirm that the time-minimum projection $\pi_C^F(\cdot)$ is well-posed locally (near C), and, furthermore, the estimates (8.10) and (8.12) together with (8.3) allow us to evaluate the radius $r(x)$ of a ball centred at a given $x \in \partial C$ where such well-posedness takes place. In particular (see (6.12)), $r(x) = O\left(|x_2|^{1/3}\right)$ as $|x_2| \to \infty$.

Notice that in this example the mapping $x \mapsto \mathfrak{J}_F(\mathfrak{v}(x))$ is locally Lipschitzean, and so we are able to apply the condition (**A**) as well (see Theorem 7.3), which gives even a larger radius $r(x) = O\left(|x_2|^{4/3}\right)$ as $|x_2| \to \infty$ (see (6.3) and (8.9)).

In the conclusion let us consider the mixed case (when there are points of both types: either satisfying the condition (**B**) only, or the condition (**A**)) emphasizing the situation when the boundedness of the curvature from below should be verified only in a neighbourhood of a given point $x_0 \in \partial C$ but not at x_0 itself.

Example 8.4 *Let us define two continuous real functions $f : [-1, 1] \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ as follows:*

$$f(t) := \begin{cases} 1 - t^4 & \text{if } t \notin \left[-\frac{1}{\sqrt[3]{3}}, -\frac{1}{\sqrt[3]{5}}\right]; \\ \text{affine} & \text{otherwise,} \end{cases}$$

$$g(t) := \begin{cases} \frac{1}{5}t - \frac{1}{100} & \text{if } 0 \leq t \leq \frac{1}{10}; \\ t^2 & \text{if } \frac{1}{10} < t < \frac{3}{4}; \\ \left(t - \frac{1}{2}\right)^2 + \frac{1}{2} & \text{if } t \geq \frac{3}{4}. \end{cases}$$

Set

$$\begin{aligned} F &:= \left\{(\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq f(\xi_1),\ -1 \leq \xi_1 \leq 1\right\}; \\ C &:= \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq g(|x_2|)\right\}. \end{aligned}$$

In this modification of the previous example the boundary ∂F has two affine pieces, and the target set is neither φ-convex (because it has an "inward corner" point $a = \left(-\frac{1}{100}, 0\right)\big)$, nor smooth (besides of the point a it has multiple normals at $b^{\pm} = \left(\frac{9}{16}, \pm\frac{3}{4}\right)\big)$.

For each $x_0 \in \partial C$, $x_0 \neq a, b^{\pm}$, we may proceed as in Example 8.3 since at these points both conditions (**A**) and (**B**) hold. If $x_0 = a$ then we can not apply (**B**) because the boundedness of $\psi_C(\cdot)$ near a fails. However, for each $x \in \partial C$ close to a the (nontrivial) cone $-\mathbf{N}_C^p(x)$ is contained in the interior of $\mathbf{N}_F(-1,0)$. In particular, $\mathfrak{J}_F\left(-\mathbf{N}_C^p(x) \cap \partial F^0\right) \equiv (-1,0)$, and the condition (**A**) follows (compare with Example 8.1).

Let now $x_0 = b^+$ (the symmetric point is considered similarly). Although at this point ∂C is not smooth (the normal cone is generated by two noncolinear vectors $e_1 = (1, -1/2)$ and $e_2 = (1, -3/2)$), the function $\psi_C(\cdot)$ is upper bounded in a neighbourhood of x_0, namely,

$$\psi_C(x, v) \le \max\left\{\frac{1}{\sqrt{1+4x_2^2}}, \frac{1}{\sqrt{1+(2x_2-1)^2}}\right\}, \tag{8.13}$$

$x = (x_1, x_2) \in C_\delta(x_0)$, $v \in \mathbf{N}_C^p(x)$, for some $\delta > 0$. Notice that $\mathfrak{J}_F\left(-\frac{e_1}{\rho_{F^0}(-e_1)}\right)$ and $\mathfrak{J}_F\left(-\frac{e_2}{\rho_{F^0}(-e_2)}\right)$ are different, hence the condition ($\mathbf{A}_1$) is violated. Also we have no strict convexity of the set F with respect to the vector $-e/\rho_{F^0}(-e)$, where

$$e := \left(\left(\frac{1}{3}\right)^{4/3} - \left(\frac{1}{5}\right)^{4/3}, \left(\frac{1}{5}\right)^{1/3} - \left(\frac{1}{3}\right)^{1/3}\right)$$

belongs to the interior of $\mathbf{N}_C^p(b^+)$, impeding to apply the condition ($\mathbf{B}_2'$). Nevertheless, for each $x \in C_\delta(b^+) \setminus \{b^+\}$ the (unique) unit normal vector $\mathfrak{n}(x)$ to C (also belonging to $\mathbf{N}_C^p(b^+)$) is far enough from $e/\|e\|$, and F is strictly convex of the second order w.r.t. $\mathfrak{v}(x) := -\mathfrak{n}(x)/\rho_{F^0}(-\mathfrak{n}(x))$. Moreover, the curvature is uniformly bounded from below, and the hypothesis ($\mathbf{B}_2$) holds. In such a way constructing a neighbourhood of x_0, where $\pi_C^F(\cdot)$ is well defined, we may take into account balance between (8.13) at the points $x \in \partial C$ near x_0 and the curvature of F only at $(\xi_1, \xi_2) \in \partial F$ with $\xi_1 \in \left[-\left(\frac{1}{2}\right)^{1/3}, -\left(\frac{1}{6}\right)^{1/3}\right]$, which are close to the end-points of the respective arc.

Acknowledgements. The authors are very indebted to the referees for precious comments and suggestions, in particular, concerning with the proof of Lemma 5.1 that permitted to improve essentially the main results and some proofs.

References

[1] E. Asplund, Chebyshev sets in Hilbert space, *Trans. Amer. Math.* Soc. **144** (1969), 235-240.

[2] E. Asplund and R.T. Rockafellar, Gradients of convex functions, *Trans. Amer. Math.* Soc. **139** (1969), 443-467.

[3] M. V. Balashov and G. E. Ivanov, Properties of the metric projection on weakly Vial-convex sets and parametrization of set-valued mappings with weakly convex images, *Math. Notes* **80** (2006), 461-467.

[4] M. Bardi, A boundary value problem for the minimal time function, *SIAM J. Control Optimization* **27** (1989), 776-785.

[5] H. Berens and U. Westphal, Kodissipative metrische projektionen in normierten linearen räumen, in: *Lineare Räume and Approximation* (Oberwolfach, 1977), P. L. Butzer, B. Sz.-Nagy (eds.), *Int. Ser. Numer. Math.* **40**, Birkhäuser, Basel (1978), 119-130.

[6] M. S. Berger, Nonlineraity and Functional Analysis, *Lectures on Nonlinear Problems in Mathematical Analysis*, Academic Press, New York (1977).

[7] A. Canino, On p-convex sets and geodesics, *J. Differ. Equations* **75** (1988), 118-157.

[8] Chong Li, On well posed generalized best approximation problems, *J. Approximation Theory* **107** (2000), 96-108.

[9] Chong Li and Renxing Ni, Derivatives of generalized distance functions and existence of generalized nearest points, *J. Approximation Theory* **115** (2002), 44-55.

[10] F. H. Clarke, R. J. Stern and P. R. Wolenski, Proximal smoothness and the lower-$\mathcal{C}^2$ property, *J. Convex Analysis* **2** (1995), 117-144.

[11] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern and P. R. Wolenski, Nonsmooth Analysis and Control Theory, Springer, New York (1998).

[12] G. Colombo and V. V. Goncharov, A class of non-convex sets with locally well-posed projection, *preprint S.I.S.S.A.* **78/99/M** (1999).

[13] G. Colombo and V. V. Goncharov, Variational inequalities and regularity properties of closed sets in Hilbert spaces, *J. Convex Analysis* **8** (2001), 197-221.

[14] G. Colombo and P. R. Wolenski, The subgradient formula for the minimal time function in the case of constant dynamics in Hilbert space, *J. Glob. Optim.* **28** (2004), 269-282.

[15] G. Colombo and P. R. Wolenski, Variational Analysis for a class of minimal time functions in Hilbert spaces, *J. Convex Analysis* **11** (2004), 335-361.

[16] M. Crandall and P.-L. Lions, Hamilton-Jacobi equations in infinite dimensions. I: Uniqueness of viscosity solutions, *J. Funct. Anal.* **62** (1985), 379-396.

[17] F. S. De Blasi and J. Myjak, On a generalized best approximation problem, *J. Approximation Theory* **94** (1998), 54-72.

[18] I. Ekeland, Nonconvex minimization problems, *Bull. Amer. Math. Soc.* **1** (1979), 443-474.

[19] H. Federer, Curvature measures, *Trans. Amer. Math. Soc.* **93** (1959), 418-491.

[20] J. Lindenstrauss, On the modulus of smoothness and divergent series in Banach spaces, *Mich. Math. J.* **10** (1963), 241-252.

[21] A. R. Lovaglia, Locally uniformly convex Banach spaces, *Trans. Amer. Math. Soc.* **78** (1955), 225-238.

[22] R. E. Megginson, An Introduction to Banach Space Theory, *Graduate Texts in Mathematics* **183**, Springer, New York (1991).

[23] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation **I**: *Basic Theory*, Springer, Berlin (2006).

[24] R. R. Phelps, Convex Functions, Monotone Operators and Differentiability, *Lecture Notes in Mathematics* **1364**, Springer, Berlin (1989).

[25] R. A. Poliquin, R. T. Rockafellar and L. Thibault, Local differentiability of distance functions, *Trans. Amer. Math. Soc.* **352** (2000), 5231-5249.

[26] A. Shapiro, Existence and differentiability of metric projections in Hilbert spaces, *SIAM J. Optim.* **4** (1994), 130-141.

[27] J.-P. Vial, Strong and weak convexity of sets and functions, *Math. Oper. Res.* **8** (1983), 231-259.

[28] Yiran He and Kung Fu Ng, Subdifferentials of a minimum time function in Banach spaces, *J. Math. Anal. Appl.* **321** (2006), 896-910.