NONSTANDARD LINEAR ALGEBRA WITH ERROR ANALYSIS

Júlia Maria da Rocha Vilaverde Justino

Tese apresentada à Universidade de Évora
para obtenção do Grau de Doutor em Matemática
Especialidade: Álgebra e Lógica

ORIENTADOR: *Imme van den Berg*

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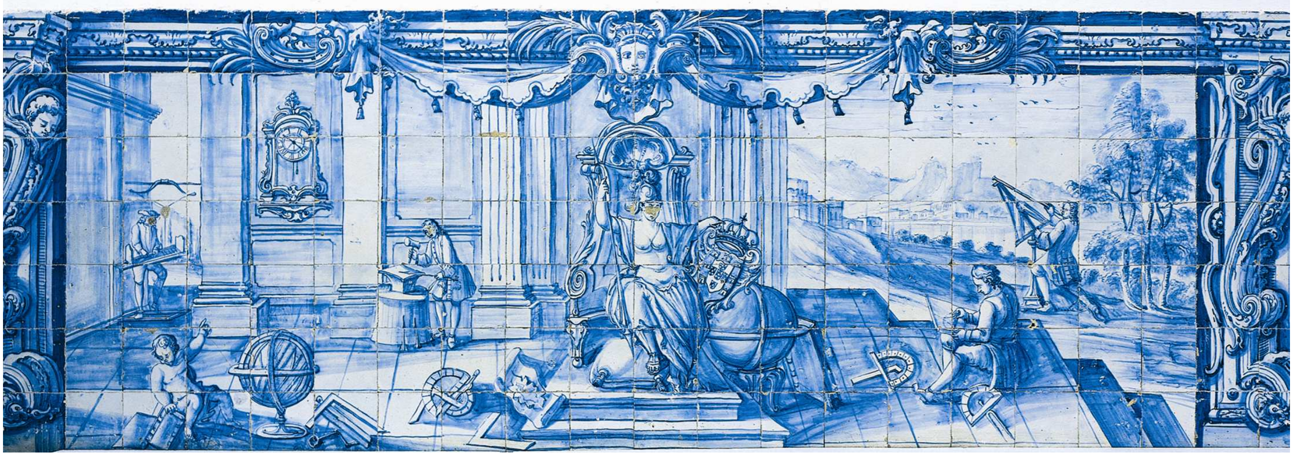

INSTITUTO DE INVESTIGAÇÃO E FORMAÇÃO AVANÇADA

Universidade de Évora

Instituto de Investigação e Formação Avançada

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ABSTRACT

Systems of linear equations, called flexible systems, with coefficients having uncertainties of type $o(.)$ or $O(.)$ are studied from the point of view of nonstandard analysis. Then uncertainties of the afore-mentioned kind will be given in the form of so-called neutrices, for instance the set of all infinitesimals. In some cases an exact solution of a flexible system may not exist. In this work conditions are presented that guarantee the existence of an admissible solution, in terms of inclusion, and also conditions that guarantee the existence of a maximal solution. These conditions concern restrictions on the size of the uncertainties appearing in the matrix of coefficients and in the constant term vector of the system. Applying Cramer's rule under these conditions, one obtains, at least, an admissible solution of the system. In the case a maximal solution is produced by Cramer's rule, one proves that it is the same solution produced by Gauss-Jordan elimination.

KEYWORDS: Cramer´s rule, Gauss-Jordan elimination, neutrices, external numbers, nonstandard analysis.

Álgebra linear não standard e gestão de incertezas

RESUMO

Neste trabalho consideramos sistemas de equações lineares flexíveis, sistemas de equações lineares cujos coeficientes têm incertezas de tipo $o\,(.)$ ou $O\,(.)$. Este tipo de incertezas irá ser analisado, à luz da análise não standard, como conjuntos de infinitesimais conhecidos como neutrizes. Em sistemas de equações lineares flexíveis nem sempre existe uma solução exata. No entanto, neste trabalho apresentam-se condições que garantem a existência de pelo menos uma solução admissível, no sentido de inclusão, e as condições que garantem a existência de solução maximal nesse tipo de sistemas. Tais condições são restrições àcerca da ordem de grandeza do tipo de incertezas existentes, tanto na matriz dos coeficientes do sistema como na respetiva matriz dos termos independentes. Utilizando a regra de Cramer sob essas condições é possível produzir, pelo menos, uma solução admissível do sistema. No caso em que se garante a obtenção da solução maximal do sistema pela regra de Cramer, prova-se que essa solução corresponde à solução obtida pelo método de eliminação de Gauss.

PALAVRAS-CHAVE: Regra de Cramer, método de eliminação de Gauss, neutrizes, números externos, análise não standard.

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Dedico esta tese ao meu pai, com saudade, um ser extraordinário que me ensinou que nunca é tarde demais para se lutar por aquilo em que se acredita.

Contents

Chapter 1

Introduction

All measurements of physical quantities are subject to uncertainties for it is never possible to measure anything exactly. One may try to make the error as small as possible but the error is always there and to draw valid conclusions the error must be dealt with properly. Bad things can happen if error analysis is ignored.

The derailment at Gare Montparnasse, Paris, 1895.

In classical mathematics does not exist a very highly developed algebra of propagation of errors. One drawback of the functional $o(.)$ and $O(.)$ calculus [4] is the absence of total ordering which leads to some complexity in calculus. In interval calculus [8], error operations are well defined but at a certain point the error bounds are so large that they no longer have practical value. In statistics [11], confidence intervals are used to find probabilistic upper bounds of the errors, but so many times they are too far from the real size of the actual errors. In numerical analysis [20][17][9], the solution of a practical problem is produced by numerical methods which many times depend on functional analysis, that by their nature are not so obviously implemented in actual computing with numbers. In computation one of the important problems is the existence of some mismatch between the theoretical calculation with real numbers and the practical calculation with computerized numbers.

Let us illustrate the latter with the search for valid solutions of a practical problem of computation with matrices, which is one of the main objectives of numerical analysis [21][3] and is related to the principal topic of this study. In the formulation of a computed problem and its solution it is essential to estimate the effect of the various errors induced by the following considerations:

1. The coefficients of a given matrix may have been determined directly from physical measurements and therefore the represented matrix is an approximation of the matrix which corresponds to the exact measurements.

2. The coefficients of a given matrix may be defined exactly by mathematical formulae but if any of those coefficients is irrational or too large to fit in the floating-point system of the computer, once more we have to work with an approximation of the exact matrix.

3. Even if the matrix implemented in the computer is exact, the same may not be true for the computed solutions because some operations increase the number of digits in such an amount that the floating-point system stars to round off, producing errors.

By the previous considerations there is a need for some error analysis concerning a substantial amount of algebraic properties. An approach within nonstandard analysis may reach this goal for we can model errors by infinitesimals, which are numbers, so there is no need to work with functions. Within the infinitesimals we may distinguish various convex groups, called *neutrices*, that correspond to different sizes of errors. The term neutrices is borrowed from Van der Corput [5] who had also in mind an efficient theory of neglecting, partly realized, where the neutrices are certain groups of functions. The fact that all neutrices are sets of numbers instead of sets of functions leads to more powerful algebraic properties. Also the neutrices of numbers are totally ordered. *External numbers* are the sum of a neutrix and a real (nonstandard) number. The algebraic laws of external numbers are completely characterized [15][6]. In a sense, within this approach we work directly with the order of magnitude of errors leading to substantial efficiency and simplifications in calculations.

The aim of this work is to find conditions that guarantee the existence of a maximal solution, in terms of inclusion, for systems of linear equations with external numbers. The kind of systems under consideration will be called *flexible systems of linear equations*.

We will show that the maximal solution of a non-singular non-homogeneous flexible system of linear equations is, like in usual linear algebra, given by Cramer's rule, with some restrictions induced by the size of the uncertainties of the system. If not all of those restrictions are satisfied, it is still possible, in some cases, to produce an admissible solution by adapting Cramer's rule. When fitting Cramer's rule to a flexible system, the condition that the determinant of the matrix of coefficients is non-zero is substituted by a condition stating that the determinant of the matrix should not be too small. As we will see this can be concretized in terms of the so-called *absorbers* of neutrices.

We relate this theoretical result on the maximal solution produced by Cramer's rule to the procedure of Gauss-Jordan elimination, which is the basis of numerical methods on solving systems of linear equations. In fact, we formulate conditions such that both methods lead to the same solution.

This thesis has the following structure. In Chapter 2 we recall the notions of neutrix and external number, their operations and some useful properties. In Chapter 3 we define flexible systems of linear equations and introduce the notions of admissible, maximal and exact solutions. In Chapter 4 we present the conditions upon the size of the uncertainties appearing in a flexible system of linear equations that guarantee that a maximal solution is produced by Cramer's rule. We also investigate appropriate adaptations under weaker conditions so that an admissible solution is given. We illustrate Cramer's rule and its weakenings by some examples. In Chapter 5 we define appropriate Gauss-operations and the notion of Gauss-solution and show that, under suitable conditions, the maximal solution given by Cramer´s rule and the set of all Gauss-solutions are identical. The 2 by 2 case was already published in [12] but, as we will see, the general case is much more involved due to the presence of minors. The results and proofs will also be illustrated by concrete cases.

For a review of Cramer's rule we refer to [19], [10] and [2]. For a review of Gauss-Jordan elimination we refer to [17] and [18].

To indicate strict set identity we will use the symbol "$=$". The symbol "$\subseteq$" represents inclusion. Strict inclusion is denoted by "$\subset$".

Chapter 2

Neutrices and External numbers

The setting of this thesis is the axiomatic nonstandard analysis IST as presented by Nelson in [16]. A recent introduction to IST is contained in [7]. We use freely external sets where we follow the approach HST as indicated in [13]; this is an extension of an essential part of IST. For a thorough introduction to external numbers with proofs we refer to [14] and [15].

We recall that within IST the nonstandard numbers are already present in the standard set $\mathbb{R}$. *Infinitesimal numbers* (or *infinitesimals*) are real numbers that are smaller, in absolute value, than any positive standard real number. *Infinitely large numbers* are reciprocals of infinitesimals, i.e. real numbers larger than any standard real number. *Limited numbers* are real numbers which are not infinitely large and *appreciable numbers* are limited numbers which are not infinitesimals. The external set of all infinitesimal numbers is denoted by $\oslash$, the external set of all limited numbers is denoted by $\pounds$, the external set of all positive appreciable numbers is denoted by @ and the external set of all positive infinitely large numbers by $\not\infty$.

A *neutrix* is an additive convex subgroup of $\mathbb{R}$. Except for $\{0\}$ and $\mathbb{R}$, all neutrices are external sets. The most common neutrices are $\oslash$ and $\pounds$. All other neutrices contain $\pounds$ or are contained in $\oslash$. Let ε be a positive infinitesimal. Examples of neutrices contained in $\oslash$ are $\varepsilon\pounds$, $\varepsilon\oslash$, $\pounds\varepsilon^{\not\infty}$, numbers smaller than any standard power of ε, and $\pounds e^{-\frac{@}{\varepsilon}}$, the numbers which are exponential small with respect to ε. Examples of neutrices that contain $\pounds$ are $\omega\pounds$, $\omega\oslash$ and $\omega^2\pounds$, where ω is an infinitely large number. It is clear that $\pounds$, $\omega\pounds$ and $\varepsilon\pounds$ are isomorphic groups and also that $\oslash$, $\omega\oslash$ and $\varepsilon\oslash$ are isomorphic. However it can be shown [1] that the neutrices $\pounds$, $\oslash$, $\pounds\varepsilon^{\not\infty}$ and $\pounds e^{-\frac{@}{\varepsilon}}$ are not isomorphic by internal isomorphism. The external class of all neutrices is denoted by $\mathcal{N}$. Neutrices are totally ordered by inclusion. Addition and multiplication on $\mathcal{N}$ are defined by the Minkowski operations as it follows:

$$A + B = \{a + b \mid (a, b) \in A \times B\}$$

and

$$AB = \{ab \mid (a, b) \in A \times B\},$$

for $A, B \in \mathcal{N}$.

The sum of two neutrices is the largest one for inclusion.

Proposition 2.1 *If $A, B \in \mathcal{N}$, then $A + B = \max(A, B)$.*

Neutrices are invariant under multiplication by appreciable numbers.

Proposition 2.2 *If $A \in \mathcal{N}$, then $\pm @A = A$.*

An *external number* is the algebraic sum of a real number and a neutrix. The external class of all external numbers is denoted by $\mathbb{E}$. If $a \in \mathbb{R}$ and $A \in \mathcal{N}$, then $\alpha \equiv a + A \in \mathbb{E}$

and A is called the *neutrix part of* α, being denoted as $N(\alpha)$; $N(\alpha)$ is unique but a is not because for all $c \in \alpha$, $\alpha = c + N(\alpha)$. We then say that c is a *representative of* α. Clearly, neutrices are external numbers such that the representative may be chosen equal to 0. All classical real numbers are external numbers with the neutrix part equal to $\{0\}$. An external number α is called *zeroless*, if $0 \notin \alpha$. Let $\alpha = a + A$ be zeroless. Then its *relative uncertainty* $R(\alpha)$ is defined by the neutrix A/a. Notice that $A/a = A/\alpha$, hence $R(\alpha)$ is independent of the choice of a; also $R(\alpha) \subseteq \oslash$ (see Lemmas 2.5 and 2.6).

Let $\alpha = a + A$ and $\beta = b + B$ be two external numbers. Then α and β are either disjoint or one contains the other, indeed

$$\alpha \cap \beta = \emptyset \vee \beta \subseteq \alpha \vee \alpha \subseteq \beta. \tag{2.1}$$

Addition, subtraction, multiplication and division of α with β are given by Minkowski operations. One shows that

$$\begin{aligned}
\alpha + \beta &= a + b + \max(A, B); \\
\alpha - \beta &= a - b + \max(A, B); \\
\alpha\beta &= ab + \max(aB, bA, AB) \\
&= ab + \max(aB, bA) \text{ if } \alpha \text{ or } \beta \text{ is zeroless;} \\
\frac{\alpha}{\beta} &= \frac{a}{b} + \frac{1}{b^2}\max(aB, bA) = \frac{\alpha\beta}{b^2}, \text{ with } \beta \text{ zeroless.}
\end{aligned}$$

The relation $\alpha \leqslant \beta$ if and only if $\forall x \in \alpha \exists y \in \beta (x \leqslant y)$ is a relation of total order compatible with addition and multiplication. Observe that with this rule, one has $0 \leqslant \oslash \leqslant \pounds$. The *absolute value* of α is then defined by $|\alpha| = \begin{cases} \alpha & \text{if } A \leqslant \alpha \\ -\alpha & \text{if } \alpha < A \end{cases}$.

The next tables present the principal rules of external calculus used in this thesis:

$\pm$	$\oslash$	@	£
$\oslash$	$\oslash$	@	£
@	@	@	£
£	£	£	£

$\times$	$\oslash$	@	£
$\oslash$	$\oslash$	$\oslash$	$\oslash$
@	$\oslash$	@	£
£	$\oslash$	£	£

.

In practice, calculations with external numbers tend to be rather straightforward as it will be illustrated by the following examples.

Let ε be a positive infinitesimal. Then

$$(6+\oslash)+(-2+\varepsilon £)=(6-2)+(\oslash+\varepsilon £)=4+\oslash;$$

$$\begin{aligned}(6+\oslash)(-2+\varepsilon £) &= 6\,(-2)+(-2)\oslash+6\varepsilon £+\oslash\varepsilon £\\ &= -12+\oslash+\varepsilon £+\varepsilon\oslash=-12+\oslash;\end{aligned}$$

$$\begin{aligned}\frac{6+\oslash}{-2+\varepsilon £} &= \frac{6}{-2}\cdot\frac{1+\oslash/6}{1+\varepsilon £/2}=(-3)\,\frac{1+\oslash}{1+\varepsilon £}\\ &= (-3)\,(1+\oslash)(1+\varepsilon £)=-3+\oslash.\end{aligned}$$

However, multiplication of external numbers is not fully distributive, for instance

$$\oslash\varepsilon=\oslash(1+\varepsilon-1)\subset\oslash(1+\varepsilon)-\oslash\cdot 1=\oslash+\oslash=\oslash.$$

Yet distributivity can be entirely characterized [6]. Let $\alpha=a+A$, β and γ be external numbers, where $a\in\mathbb{R}$ and A is a neutrix. Important cases where distributivity is verified are

$$a(\beta+\gamma)=a\beta+a\gamma \tag{2.2}$$

and

$$(a+A)\beta=a\beta+A\beta. \tag{2.3}$$

Also subdistributivity always holds, this means that $\alpha(\beta+\gamma) \subseteq \alpha\beta + \alpha\gamma$; the property follows from the well-known property of subdistributivity of interval calculus.

Definition 2.3 Let A be a neutrix and α be an external number. We say that α is an *absorber* of A if $\alpha A \subset A$.

Example 2.4 According to Proposition 2.2, appreciable numbers are not absorbers. So an absorber must be an infinitesimal. Let ε be a positive infinitesimal. Then ε is an absorber of $\oslash$ because $\varepsilon\oslash \subset \oslash$. However, not necessarily all infinitesimals are absorbers of a given neutrix, for instance $\varepsilon \pounds \varepsilon^{-\not\infty} = \pounds \varepsilon^{-\not\infty}$.

We now show some simple results about calculation properties of external numbers that will be used in the next chapters.

Lemma 2.5 *Let $\alpha = a + A$ be a zeroless external number. Then its relative uncertainty $R(\alpha) = A/a$ satisfies*

$$\frac{A}{a} \subseteq \oslash.$$

Proof. Since $\alpha = a + A$ is zeroless, one has $0 \notin \alpha$ and so $|a| > A$. Hence $\frac{A}{a} < 1$ and so $\frac{A}{a} \subseteq \oslash$ because there is no neutrix strictly included in $\pounds$ and which strictly contains $\oslash$. ■

Lemma 2.6 *Let A be a neutrix and $\beta = b + B$ be a zeroless external number. Then $\frac{A}{\beta} = \frac{A}{b}$ and $A\beta = Ab$.*

Proof. Since $B \subseteq b\oslash$ by Lemma 2.5, $AB \subseteq \oslash bA \subseteq bA$. Hence $\frac{A}{\beta} = \frac{0+A}{b+B} = \frac{bA}{b^2} = \frac{A}{b}$ and $A\beta = (0+A)(b+B) = \max(bA, AB) = Ab$. ■

Lemma 2.7 *Let $a \in \mathbb{R}$, $A \in \mathcal{N}$ and $n \in \mathbb{N}$ be standard. If $|a| > A$, then*

$$N\left((a+A)^n\right) = a^{n-1}A.$$

Proof. Since $|a| > A$, by Lemma 2.6, we have $(a+A)^2 = (a+A)(a+A) = a^2 + aA$. So $(a+A)^3 = (a+A)(a+A)^2 = (a+A)\left(a^2+aA\right) = a^3 + a^2A$. Using external induction, we conclude that

$$(a+A)^n = a^n + a^{n-1}A.$$

Hence $N\left((a+A)^n\right) = a^{n-1}A$. ■

Lemma 2.8 *Let $\alpha = a + A$ be a zeroless external number. Then*

$$\alpha \cap \oslash\alpha = \emptyset.$$

Proof. Because α is zeroless, $0 \notin \alpha$ and $\oslash\alpha = \oslash a$, with $|a| > A$. Yet $0 \in \oslash\alpha \subseteq \oslash$ and so $\oslash\alpha \nsubseteq \alpha$. On the other hand, $a \in \alpha$ but $a \notin \oslash a = \oslash\alpha$. So $\alpha \nsubseteq \oslash\alpha$. Hence α and $\oslash\alpha$ are disjoint by (2.1). ■

Chapter 3

Flexible systems of linear equations

In this chapter we introduce some notations and define the flexible systems and some related notions.

Notation 3.1 Let $m, n \in \mathbb{N}$ be standard. For $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$, let $\alpha_{ij} = a_{ij} + A_{ij}$, with $a_{ij} \in \mathbb{R}$ and $A_{ij} \in \mathcal{N}$. We denote

1. $\mathcal{A} = [\alpha_{ij}]$, an $m \times n$ matrix

2. $\overline{\alpha} = \max\limits_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} |\alpha_{ij}|$

3. $\overline{a} = \max\limits_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} |a_{ij}|$

4. $\overline{A} = \max\limits_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} A_{ij}$

5. $\underline{A} = \min\limits_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} A_{ij}$.

In particular, for a column vector $\mathcal{B} = [\beta_i]$, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i \leqslant n$, we denote $\overline{\beta} = \max\limits_{1 \leqslant i \leqslant n} |\beta_i|$, $\overline{b} = \max\limits_{1 \leqslant i \leqslant n} |b_i|$, $\overline{B} = \max\limits_{1 \leqslant i \leqslant n} B_i$ and $\underline{B} = \min\limits_{1 \leqslant i \leqslant n} B_i$.

Definition 3.2 Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} \in \mathbb{E}$ for all $i, j \in \{1, \ldots, n\}$. We call *determinant of* $\mathcal{A}$ to the external number given by

$$\det \mathcal{A} \equiv \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix} \equiv \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdots \alpha_{np_n},$$

where S_n denote the set of all permutations of the set $\{1, \ldots, n\}$ and $\sigma = (p_1, \ldots, p_n) \in S_n$.

We observe that not all equations with external numbers can be solved in terms of equalities. For instance, no external number, or even set of external numbers, satisfies the equation $\oslash\xi = \pounds$ since one should have $\xi \subseteq \pounds$ and $\oslash\pounds = \oslash \subset \pounds$. So we will study inclusions instead of equalities.

Definition 3.3 Let $m, n \in \mathbb{N}$ be standard and $\alpha_{ij} = a_{ij} + A_{ij}, \beta_i = b_i + B_i$, $\xi_j = x_j + X_j \in \mathbb{E}$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$. We call

$$\begin{cases} \alpha_{11}\xi_1 + & \cdots & +\alpha_{1j}\xi_j + & \cdots & +\alpha_{1n}\xi_n & \subseteq \beta_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{m1}\xi_1 + & \cdots & +\alpha_{mj}\xi_j + & \cdots & +\alpha_{mn}\xi_n & \subseteq \beta_m \end{cases}$$

a *flexible system of linear equations.*

Definition 3.4 Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and let $\mathcal{B} = [\beta_i]$ be a column vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for all $i, j \in \{1, \ldots, n\}$.

1. $\mathcal{A}$ is called a *non-singular matrix* if $\Delta = \det \mathcal{A}$ is zeroless.

2. $\mathcal{B}$ is called an *upper zeroless vector* if $\overline{\beta}$ is zeroless.

Definition 3.5 Let $n \in \mathbb{N}$ be standard and $\alpha_{ij} = a_{ij} + A_{ij}, \beta_i = b_i + B_i$, $\xi_j = x_j + X_j \in \mathbb{E}$ for all $i, j \in \{1, \ldots, n\}$. Consider the square flexible system of linear equations

$$\begin{cases} \alpha_{11}\xi_1 + & \cdots & +\alpha_{1j}\xi_j + & \cdots & +\alpha_{1n}\xi_n & \subseteq \beta_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{n1}\xi_1 + & \cdots & +\alpha_{nj}\xi_j + & \cdots & +\alpha_{nn}\xi_n & \subseteq \beta_n \end{cases}, \tag{3.1}$$

with matrix representation given by $\mathcal{AX} \subseteq \mathcal{B}$. If $\mathcal{A}$ is a non-singular matrix, the system is called *non-singular*. If $\mathcal{B}$ is an upper zeroless vector, the system is called *non-homogeneous*. Moreover, if 1 is a representative of $\overline{\alpha}$, $\mathcal{A}$ is called a *reduced matrix* and we speak about a *reduced system*. If external numbers $\xi_1, \ldots, \xi_n$ can actually be found to satisfy (3.1), the column vector $(\xi_1, \ldots, \xi_n)^T$ is called an *admissible solution* of $\mathcal{AX} \subseteq \mathcal{B}$. A solution $\xi = (\xi_1, \ldots, \xi_n)^T$ of the system (3.1) is *maximal* if no (external) set $\eta \supset \xi$ satisfies this flexible system. If $\xi_1, \ldots, \xi_n$ satisfy the system (3.1) with equalities, the column vector $(\xi_1, \ldots, \xi_n)^T$ is called the *exact solution* of $\mathcal{AX} \subseteq \mathcal{B}$.

Chapter 4

Cramer's rule

Not all non-singular non-homogeneous flexible systems of linear equations can be resolved by Cramer's rule. We need to control the uncertainties of the system in order to guarantee that Cramer's rule produces a valid solution and, if necessary, to make some adaptations. The matrix $\mathcal{A}$ of coefficients has to be more precise, in a sense, than the constant term vector $\mathcal{B}$. The general theorem presented in this chapter shows that, under certain conditions upon the size of the uncertainties appearing in a non-singular non-homogeneous flexible system of linear equations, it is possible to guarantee the existence of a maximal solution by Cramer's rule. Even when not all of those conditions are satisfied it is still possible, in some cases, to obtain an admissible solution given by adapting Cramer's rule, where we neglect some uncertainties of the system.

From now on we will simply call a non-singular non-homogeneous flexible system of linear equations *flexible system* and a reduced non-singular non-homogeneous flexible system of linear equations *reduced flexible system.*

We start by defining the kind of precision needed in order to control the uncertainties appearing in a flexible system.

Definition 4.1 Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ be a non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and $\mathcal{B} = [\beta_i]_{n \times 1}$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$.

We define the *relative uncertainty of* $\mathcal{A}$ by

$$R(\mathcal{A}) = \overline{A}\overline{\alpha}^{n-1} \diagup \Delta.$$

We define the *relative precision of* $\mathcal{B}$ by

$$P(\mathcal{B}) = \underline{B} \diagup \overline{\beta}.$$

Remark 4.2 If $\mathcal{A} = [\alpha]$, with $\alpha = a + A$ zeroless, the relative uncertainty of $\mathcal{A}$ reduces to A/a, the relative uncertainty of the external number $\det \mathcal{A} = \alpha$. In general $R(\mathcal{A})$ gives an upper bound of the relative uncertainty of $\det \mathcal{A}$. Note that if $\overline{\alpha} \subseteq @$ we simply have $R(\mathcal{A}) = \overline{A} \diagup \Delta$.

Notation 4.3 Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, and $\mathcal{B} = [\beta_i]$ be a column vector, with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. We denote

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & \beta_1 & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & \beta_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}$$

$$\mathcal{M}_j(b) = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & b_1 & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & b_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}$$

$$\mathcal{M}_j(a,b) = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}.$$

4.1 Existence of admissible and maximal solution

We now present the Main Theorem that states all the needed conditions to guarantee that Cramer's rule produces a maximal solution. Even when not all of those conditions are satisfied it is still possible, in some cases, to obtain an admissible solution by adapting Cramer's rule.

Theorem 4.4 (Main Theorem) *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\Delta = \det \mathcal{A} = d + D$, and let $\mathcal{B} = [\beta_i]$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$. Consider the flexible system $\mathcal{AX} \subseteq \mathcal{B}$ where $\mathcal{X} = [\xi_i]$, with $\xi_i = x_i + X_i \in \mathbb{E}$ for all $i \in \{1, \ldots, n\}$.*

1. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$, then*
$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1(b)}{d} \\ \vdots \\ \frac{\det \mathcal{M}_n(b)}{d} \end{bmatrix}$$
is an admissible solution of $\mathcal{AX} \subseteq \mathcal{B}$.

2. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$ and Δ is not an absorber of $\underline{B}$, then*
$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1(b)}{\Delta} \\ \vdots \\ \frac{\det \mathcal{M}_n(b)}{\Delta} \end{bmatrix}$$
is an admissible solution of $\mathcal{AX} \subseteq \mathcal{B}$.

3. *If $R(\mathcal{A}) \subseteq P(\mathcal{B})$, Δ is not an absorber of $\underline{B}$ and $\underline{B} = \overline{B}$, then*
$$\mathcal{X} = \begin{bmatrix} \frac{\det \mathcal{M}_1}{\Delta} \\ \vdots \\ \frac{\det \mathcal{M}_n}{\Delta} \end{bmatrix}$$
is an admissible and maximal solution of $\mathcal{AX} \subseteq \mathcal{B}$.

We will call $\left(\frac{\det \mathcal{M}_1}{\Delta}, ..., \frac{\det \mathcal{M}_n}{\Delta}\right)^T$ the *Cramer-solution* of the flexible system (3.1).

So Part 3 of Theorem 4.4 states conditions guaranteeing that the Cramer-solution maximally satisfies (3.1).

Under the weaker conditions of Part 2, one is forced to substitute the constant term vector $\mathcal{B}$ by a representative, the uncertainties occurring in $\mathcal{B}$ possibly being too large. If only the condition on the relative precision $R(\mathcal{A}) \subseteq P(\mathcal{B})$ is known to hold, also the determinant Δ must be substituted by a representative.

The condition that Δ should not be so small as to be an absorber of $\underline{B}$ may be seen, in a sense, as a generalization of the usual condition on non-singularity of determinant of the matrix of coefficients, i.e. that this determinant should be non-zero.

The condition that all the uncertainties of $\mathcal{B}$ should be equal is not usually satisfied, but if the flexible system does not verify the condition $N(\beta_i) = B$, for all $i \in \{1, \ldots, n\}$, one may solve the flexible system, now with $\underline{B} = \min_{1 \leqslant i \leqslant n} B_i$ instead of the $N(\beta_i)$. If for this new system we have $R(\mathcal{A}) \subseteq P(\mathcal{B})$ and also that Δ is not an absorber of $\underline{B}$, by Cramer's rule one obtains the maximal solution of the modified flexible system. Clearly this is an admissible solution of the original system.

We show now some examples which illustrate the role of the conditions presented in Theorem 4.4.

The first two examples show that not all flexible systems can be resolved by Cramer's rule and also illustrate the importance of the condition on precision in a flexible system.

Example 4.5 Let ε be a positive infinitesimal. Consider the following non-homogeneous flexible system of linear equations

$$\begin{cases} (3+\varepsilon\oslash)\,\xi_1 + (-1+\oslash)\,\xi_2 \subseteq 1+\varepsilon\pounds \\ (2+\varepsilon\pounds)\,\xi_1 + (1+\varepsilon\oslash)\,\xi_2 \subseteq \varepsilon\pounds. \end{cases}$$

A real part of this system is given by $\begin{cases} 3x - y = 1 \\ 2x + y = 0 \end{cases}$ which has the exact solution

$$\begin{cases} x = \frac{1}{5} \\ y = -\frac{2}{5} \end{cases}.$$

The matrix representation of the system is given by $\mathcal{AX} = \mathcal{B}$, with

$$\mathcal{X} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon\pounds & 1+\varepsilon\oslash \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\varepsilon\pounds \\ \varepsilon\pounds \end{bmatrix}.$$

We have $\Delta = \det\mathcal{A} = \begin{vmatrix} 3+\varepsilon\oslash & -1+\oslash \\ 2+\varepsilon\pounds & 1+\varepsilon\oslash \end{vmatrix} = 5+\oslash$, which is zeroless. So the initial system is non-singular. When we apply Cramer's rule, we get

$$\begin{aligned} \xi_1 &= \frac{\begin{vmatrix} 1+\varepsilon\pounds & -1+\oslash \\ \varepsilon\pounds & 1+\varepsilon\oslash \end{vmatrix}}{\Delta} = \frac{1+\varepsilon\pounds}{5+\oslash} = \frac{1}{5}+\oslash \\ \xi_2 &= \frac{\begin{vmatrix} 3+\varepsilon\oslash & 1+\varepsilon\pounds \\ 2+\varepsilon\pounds & \varepsilon\pounds \end{vmatrix}}{\Delta} = \frac{-2+\varepsilon\pounds}{5+\oslash} = -\frac{2}{5}+\oslash. \end{aligned}$$

However, this is not a valid solution because

$$\begin{aligned} (3+\varepsilon\oslash)\,\xi_1 + (-1+\oslash)\,\xi_2 &= (3+\varepsilon\oslash)\left(\frac{1}{5}+\oslash\right) + (-1+\oslash)\left(-\frac{2}{5}+\oslash\right) \\ &= 1+\oslash \supset 1+\varepsilon\pounds \end{aligned}$$

and

$$(2+\varepsilon\pounds)\,\xi_1 + (1+\varepsilon\oslash)\,\xi_2 = (2+\varepsilon\pounds)\left(\frac{1}{5}+\oslash\right) + (1+\varepsilon\oslash)\left(-\frac{2}{5}+\oslash\right) = \oslash \supset \varepsilon\pounds.$$

In fact, using representatives, it is easy to show that this system does not have solutions at all.

Notice that $R(\mathcal{A}) = \overline{A}\overline{\alpha}/\Delta = \frac{3\oslash}{5+\oslash} = \oslash$ and $P(\mathcal{B}) = \underline{B}/\overline{\beta} = \frac{\varepsilon\pounds}{1+\varepsilon\pounds} = \varepsilon\pounds$. So $R(\mathcal{A}) \nsubseteq P(\mathcal{B})$ and Theorem 4.4 cannot be applied, although Δ is not an absorber of $\underline{B}$, since $\Delta\underline{B} = \varepsilon\pounds = \underline{B}$, and $\underline{B} = \overline{B} = \varepsilon\pounds$.

Example 4.6 Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3\xi_1 + (-1+\varepsilon\oslash)\,\xi_2 \subseteq 1+\varepsilon\pounds \\ 2\xi_1 + \xi_2 \subseteq \varepsilon\pounds. \end{cases}$$

Its matrix representation is given by $\mathcal{AX} = \mathcal{B}$, where

$$\mathcal{X} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\varepsilon\pounds \\ \varepsilon\pounds \end{bmatrix}.$$

We have $\overline{A} = \varepsilon\oslash$, $\underline{B} = \varepsilon\pounds$ and $\Delta = \det\mathcal{A} = \begin{vmatrix} 3 & -1+\varepsilon\oslash \\ 2 & 1 \end{vmatrix} = 5+\varepsilon\oslash$ zeroless. Also (i) $R(\mathcal{A}) = \varepsilon\oslash \subseteq \varepsilon\pounds = P(\mathcal{B})$, (ii) Δ is not an absorber of $\underline{B}$ since $\Delta\underline{B} = \varepsilon\pounds = \underline{B}$ and (iii) $\underline{B} = \varepsilon\pounds = \overline{B}$. Hence all the conditions of Part 3 of Theorem 4.4 are satisfied. Applying Cramer's rule we get

$$\begin{aligned} \xi_1 &= \frac{\begin{vmatrix} 1+\varepsilon\pounds & -1+\varepsilon\oslash \\ \varepsilon\pounds & 1 \end{vmatrix}}{\Delta} = \frac{1+\varepsilon\pounds}{5+\varepsilon\oslash} = \frac{1}{5} + \varepsilon\pounds \\ \xi_2 &= \frac{\begin{vmatrix} 3 & 1+\varepsilon\pounds \\ 2 & \varepsilon\pounds \end{vmatrix}}{\Delta} = \frac{-2+\varepsilon\pounds}{5+\varepsilon\oslash} = -\frac{2}{5} + \varepsilon\pounds. \end{aligned}$$

When testing the validity of this solution, we have indeed that

$$3\xi_1 + (-1+\varepsilon\oslash)\,\xi_2 = 3\left(\frac{1}{5} + \varepsilon\pounds\right) + (-1+\varepsilon\oslash)\left(-\frac{2}{5} + \varepsilon\pounds\right) = 1+\varepsilon\pounds$$

and

$$2\xi_1 + \xi_2 = 2\left(\frac{1}{5} + \varepsilon\pounds\right) + \left(-\frac{2}{5} + \varepsilon\pounds\right) = \varepsilon\pounds.$$

Notice that this system has the same real part as the previous system, to which Cramer's rule could not be applied.

The following example also satisfies the conditions of Part 3 of Theorem 4.4, which guarantee the validity of the solution produced by Cramer's rule.

Example 4.7 Let ε be a positive infinitesimal. Consider the following flexible system

$$\begin{cases} \left(1+\varepsilon^2\oslash\right)\xi_1+\xi_2+\left(1+\varepsilon^3\pounds\right)\xi_3 \subseteq \frac{1}{\varepsilon}+\varepsilon\oslash \\ \left(2+\varepsilon^3\pounds\right)\xi_1+\left(-1+\varepsilon^2\oslash\right)\xi_2-\xi_3 \subseteq \varepsilon\oslash \\ \left(\varepsilon+\varepsilon^3\oslash\right)\xi_1+\xi_2+\left(2+\varepsilon^2\oslash\right)\xi_3 \subseteq 1+\varepsilon\oslash\,. \end{cases}$$

Here the matrix representation is given by $\mathcal{A}\mathcal{X}=\mathcal{B}$, with

$$\mathcal{X}=\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix},\ \mathcal{A}=\begin{bmatrix} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & -1 \\ \varepsilon+\varepsilon^3\oslash & 1 & 2+\varepsilon^2\oslash \end{bmatrix},\ \mathcal{B}=\begin{bmatrix} \frac{1}{\varepsilon}+\varepsilon\oslash \\ \varepsilon\oslash \\ 1+\varepsilon\oslash \end{bmatrix}.$$

One has

$$\Delta=\det\mathcal{A}=\begin{vmatrix} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & -1 \\ \varepsilon+\varepsilon^3\oslash & 1 & 2+\varepsilon^2\oslash \end{vmatrix}=-3+\varepsilon^2\oslash\in @.$$

Also $R(\mathcal{A})=\overline{A}\overline{\alpha}^2\diagup\Delta=\frac{4\varepsilon^2\oslash}{-3+\varepsilon^2\oslash}=\varepsilon^2\oslash$, $P(\mathcal{B})=\underline{B}\diagup\overline{\beta}=\frac{\varepsilon\oslash}{\frac{1}{\varepsilon}+\varepsilon\oslash}=\varepsilon^2\oslash$ and $\Delta\underline{B}=\varepsilon\oslash=\underline{B}$. So (i) $R(\mathcal{A})\subseteq P(\mathcal{B})$, (ii) Δ is not an absorber of $\underline{B}$ and (iii) $\underline{B}=\overline{B}=\varepsilon\oslash$. When we apply Cramer's rule, we get

$$\xi_1 = \frac{\begin{vmatrix} \frac{1}{\varepsilon}+\varepsilon\oslash & 1 & 1+\varepsilon^3\pounds \\ \varepsilon\oslash & -1+\varepsilon^2\oslash & -1 \\ 1+\varepsilon\oslash & 1 & 2+\varepsilon^2\oslash \end{vmatrix}}{\Delta}=\frac{-\frac{1}{\varepsilon}+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=\frac{1}{3\varepsilon}+\varepsilon\oslash$$

$$\xi_2 = \frac{\begin{vmatrix} 1+\varepsilon^2\oslash & \frac{1}{\varepsilon}+\varepsilon\oslash & 1+\varepsilon^3\pounds \\ 2+\varepsilon^3\pounds & \varepsilon\oslash & -1 \\ \varepsilon+\varepsilon^3\oslash & 1+\varepsilon\oslash & 2+\varepsilon^2\oslash \end{vmatrix}}{\Delta}=\frac{2-\frac{4}{\varepsilon}+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash$$

$$\xi_3 = \frac{\begin{vmatrix} 1+\varepsilon^2\oslash & 1 & \frac{1}{\varepsilon}+\varepsilon\oslash \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & \varepsilon\oslash \\ \varepsilon+\varepsilon^3\oslash & 1 & 1+\varepsilon\oslash \end{vmatrix}}{\Delta}=\frac{\frac{2}{\varepsilon}-2+\varepsilon\oslash}{-3+\varepsilon^2\oslash}=-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\,.$$

When testing the validity, we find that $(\xi_1, \xi_2, \xi_3)^T$ satisfies the equations. Indeed

$$\begin{aligned} & \left(1+\varepsilon^2\oslash\right)\xi_1+\xi_2+\left(1+\varepsilon^3\pounds\right)\xi_3 \\ = & \left(1+\varepsilon^2\oslash\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)+\left(1+\varepsilon^3\pounds\right)\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=\frac{1}{\varepsilon}+\varepsilon\oslash \end{aligned}$$

$$\begin{aligned} & \left(2+\varepsilon^3\pounds\right)\xi_1+\left(-1+\varepsilon^2\oslash\right)\xi_2-\xi_3 \\ = & \left(2+\varepsilon^3\pounds\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(-1+\varepsilon^2\oslash\right)\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)-\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=\varepsilon\oslash \end{aligned}$$

$$\begin{aligned} & \left(\varepsilon+\varepsilon^3\oslash\right)\xi_1+\xi_2+\left(2+\varepsilon^2\oslash\right)\xi_3 \\ = & \left(\varepsilon+\varepsilon^3\oslash\right)\left(\frac{1}{3\varepsilon}+\varepsilon\oslash\right)+\left(\frac{4}{3\varepsilon}-\frac{2}{3}+\varepsilon\oslash\right)+\left(2+\varepsilon^2\oslash\right)\left(-\frac{2}{3\varepsilon}+\frac{2}{3}+\varepsilon\oslash\right)=1+\varepsilon\oslash. \end{aligned}$$

The next example refers to Part 2 of Theorem 4.4.

Example 4.8 Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3\xi_1+(-1+\varepsilon\oslash)\xi_2\subseteq 1+\oslash \\ 2\xi_1+\xi_2\subseteq\varepsilon\pounds. \end{cases}$$

Its matrix representation is given by $\mathcal{AX}=\mathcal{B}$, where

$$\mathcal{X}=\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix},\ \mathcal{A}=\begin{bmatrix}3 & -1+\varepsilon\oslash\\ 2 & 1\end{bmatrix},\ \mathcal{B}=\begin{bmatrix}1+\oslash\\ \varepsilon\pounds\end{bmatrix}.$$

We have $\overline{A}=\varepsilon\oslash$ and $\underline{B}=\varepsilon\pounds$. The determinant $\Delta=\det\mathcal{A}=\begin{vmatrix}3 & -1+\varepsilon\oslash\\ 2 & 1\end{vmatrix}=5+\varepsilon\oslash$ is zeroless. One has $R(\mathcal{A})=\varepsilon\oslash\subseteq\varepsilon\pounds=P(\mathcal{B})$ and Δ is not an absorber of $\underline{B}$. However $\underline{B}=\varepsilon\pounds\neq\oslash=\overline{B}$. So this system satisfies only the conditions of Part 2 of Theorem 4.4.

Cramer's rule yields

$$\begin{aligned} \xi_1 &= \frac{\begin{vmatrix}1+\oslash & -1+\varepsilon\oslash\\ \varepsilon\pounds & 1\end{vmatrix}}{\Delta}=\frac{1+\oslash}{5+\varepsilon\oslash}=\frac{1}{5}+\oslash \\ \xi_2 &= \frac{\begin{vmatrix}3 & 1+\oslash\\ 2 & \varepsilon\pounds\end{vmatrix}}{\Delta}=\frac{-2+\oslash}{5+\varepsilon\oslash}=-\frac{2}{5}+\oslash. \end{aligned}$$

This is not a valid solution. Indeed

$$2\xi_1 + \xi_2 = \frac{2}{5} + \oslash + \left(-\frac{2}{5} + \oslash\right) = \oslash \supset \varepsilon \pounds.$$

If we ignore the uncertainties of the constant term vector in $\det \mathcal{M}_1$ and $\det \mathcal{M}_2$, by Part 2 of Theorem 4.4, Cramer's rule produces an admissible solution:

$$\begin{aligned} x &= \frac{\begin{vmatrix} 1 & -1+\varepsilon\oslash \\ 0 & 1 \end{vmatrix}}{\Delta} = \frac{1}{5+\varepsilon\oslash} = \frac{1}{5} + \varepsilon \oslash \\ y &= \frac{\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}}{\Delta} = \frac{-2}{5+\varepsilon\oslash} = -\frac{2}{5} + \varepsilon \oslash . \end{aligned}$$

When testing the validity of this solution, we have indeed that

$$3x + (-1+\varepsilon\oslash)\, y = \frac{3}{5} + \varepsilon \oslash + \frac{2}{5} + \varepsilon\oslash = 1 + \varepsilon\oslash \subseteq 1 + \oslash$$

and

$$2x + y = \frac{2}{5} + \varepsilon \oslash - \frac{2}{5} + \varepsilon\oslash = \varepsilon\oslash \subseteq \varepsilon \pounds.$$

In the last example we may apply only Part 1 of Theorem 4.4.

Example 4.9 Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 3\xi_1 + \left(-1+\varepsilon^2\oslash\right)\xi_2 \subseteq 1 + \oslash \\ 2\varepsilon\xi_1 + \varepsilon\xi_2 \subseteq \varepsilon\pounds. \end{cases}$$

Here the matrix representation is given by $\mathcal{AX} = \mathcal{B}$, with

$$\mathcal{X} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 3 & -1+\varepsilon^2\oslash \\ 2\varepsilon & \varepsilon \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1+\oslash \\ \varepsilon\pounds \end{bmatrix}.$$

We have $\overline{A} = \varepsilon^2\oslash$ and $\underline{B} = \varepsilon\pounds$. The determinant $\Delta = \det \mathcal{A} = \begin{vmatrix} 3 & -1+\varepsilon^2\oslash \\ 2\varepsilon & \varepsilon \end{vmatrix} = 5\varepsilon + \varepsilon^3\oslash$ is infinitesimal, yet zeroless. It holds that $R(\mathcal{A}) = \varepsilon\oslash \subseteq \varepsilon\pounds = P(\mathcal{B})$ but Δ is an absorber of $\underline{B}$ because $\Delta\underline{B} = \varepsilon^2 £ \subset \varepsilon £ = \underline{B}$. So this system satisfies the condition of Part 1 of Theorem 4.4. By applying Cramer's rule we get

$$\xi_1 = \frac{\begin{vmatrix} 1+\oslash & -1+\varepsilon^2\oslash \\ \varepsilon £ & \varepsilon \end{vmatrix}}{\Delta} = \frac{\varepsilon £}{5\varepsilon + \varepsilon^3\oslash} = £$$

$$\xi_2 = \frac{\begin{vmatrix} 3 & 1+\oslash \\ 2\varepsilon & \varepsilon £ \end{vmatrix}}{\Delta} = \frac{\varepsilon £}{5\varepsilon + \varepsilon^3\oslash} = £.$$

These results are clearly not valid, because

$$3\xi_1 + \left(-1+\varepsilon^2\oslash\right)\xi_2 = 3£ + \left(-1+\varepsilon^2\oslash\right)£ = £ \supset 1+\oslash.$$

Observe that the results produced by Cramer's rule are not even zeroless though the determinant is zeroless and the constant term vector is upper zeroless.

If we ignore the uncertainties of the constant term vector and the uncertainty of Δ, by the application of Part 1 of Theorem 4.4, the solution produced by Cramer's rule is now admissible. One has

$$\xi_1 = \frac{\begin{vmatrix} 1 & -1+\varepsilon^2\oslash \\ 0 & \varepsilon \end{vmatrix}}{d} = \frac{\varepsilon}{5\varepsilon} = \frac{1}{5}$$

$$\xi_2 = \frac{\begin{vmatrix} 3 & 1 \\ 2\varepsilon & 0 \end{vmatrix}}{d} = -\frac{2\varepsilon}{5\varepsilon} = -\frac{2}{5}.$$

When testing the validity of this solution, we have indeed that

$$3\xi_1 + \left(-1+\varepsilon^2\oslash\right)\xi_2 = \frac{3}{5} - \frac{2}{5}\left(-1+\varepsilon^2\oslash\right) = 1+\varepsilon^2\oslash \subset 1+\oslash$$

and

$$2\varepsilon\xi_1 + \varepsilon\xi_2 = \frac{2\varepsilon}{5} - \frac{2\varepsilon}{5} = 0 \subset \varepsilon £.$$

4.2 Proof of a Cramer's rule with external numbers

We present now some preliminary results and some Lemmas that will be used in the proof of Theorem 4.4.

Below some useful upper bounds with respect to matrices and determinants will be derived.

Remark 4.10 Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$ and $\Delta = \det \mathcal{A}$. Since Δ is zeroless, one has $\overline{\alpha} \subseteq 1 + \oslash$ by Lemma 2.5. Consequently $A_{ij} \subseteq \oslash$ for all $i, j \in \{1, \ldots, n\}$, hence $\overline{A} \subseteq \oslash$.

Lemma 4.11 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$ and $\Delta = \det \mathcal{A} = d + D$. Then*

$$D = N(\Delta) \subseteq \overline{A}.$$

Proof. Let S_n denote the set of all permutations of the set $\{1, \ldots, n\}$ and $\sigma = (p_1, \ldots, p_n) \in S_n$. Let $\gamma_\sigma = (a_{1p_1} + A_{1p_1}) \cdot \cdots \cdot (a_{np_n} + A_{np_n})$. Because $\overline{a} = 1$, by Remark 4.10, one has $|a_{kp_k}| \leqslant \overline{a} = 1$ and $A_{kp_k} \subseteq \overline{A} \subseteq \oslash$ for all $k \in \{1, \ldots, n\}$. So, by Lemma 2.7, $N(\gamma_\sigma) \subseteq N\left(\left(1 + \overline{A}\right)^n\right) = \overline{A}$.

Now,

$$\begin{aligned} \Delta &= \begin{vmatrix} a_{11} + A_{11} & \cdots & a_{1n} + A_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + A_{n1} & \cdots & a_{nn} + A_{nn} \end{vmatrix} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \gamma_\sigma \\ &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(a_{1p_1} \cdot \ldots \cdot a_{np_n} + N(\gamma_\sigma) \right), \end{aligned}$$

with $\operatorname{sgn}(\sigma) \in \{-1, 1\}$. Then

$$N(\Delta) = \sum_{\sigma \in S_n} N(\gamma_\sigma) \subseteq n!\overline{A} = \overline{A}. \qquad \blacksquare$$

Lemma 4.12 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ be a reduced non-singular matrix with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\mathcal{B} = [\beta_i]_{n \times 1}$ be an upper zeroless vector with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. Then, for all $j \in \{1, \ldots, n\}$*

1. $\det \mathcal{M}_j < 2n!\overline{\beta}$.

2. $N(\det \mathcal{M}_j(b)) \subseteq \overline{b}.\overline{A}$ *and* $N(\det \mathcal{M}_j) \subseteq \overline{b}.\overline{A} + \overline{B}$.

Proof. Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$ and $\sigma = (p_1, \ldots, p_n)$ a permutation of S_n. We have $\overline{\beta}$ zeroless and, for $1 \leqslant j \leqslant n$,

$$\mathcal{M}_j = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1(j-1)} & \beta_1 & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & \beta_n & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}.$$

Let $\gamma_\sigma = \alpha_{1p_1} \cdot \cdots \cdot \alpha_{(j-1)p_{j-1}} \alpha_{(j+1)p_{j+1}} \cdot \cdots \cdot \alpha_{np_n}$ and $i(= i_\sigma)$ be such that $\operatorname{sgn}(\sigma)\gamma_\sigma\beta_i$ is one of the terms of $\det \mathcal{M}_j$. Because $\overline{a} = 1$, by Remark 4.10, it holds that $\overline{\alpha} \subseteq 1 + \oslash$ and $\overline{A} \subseteq \oslash$. So $|\gamma_\sigma| \leqslant \overline{\alpha}^{n-1} \leqslant 1 + \oslash$.

1. One has

$$\det \mathcal{M}_j = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\gamma_\sigma\beta_i \leqslant \sum_{\sigma \in S_n} |\gamma_\sigma\beta_i| \leqslant n!(1 + \oslash)\overline{\beta} < 2n!\overline{\beta}.$$

2. By Lemma 2.7, $N(\gamma_\sigma) \subseteq N\left(\left(1 + \overline{A}\right)^{n-1}\right) = \overline{A}$. Then, for $1 \leqslant j \leqslant n$

$$\begin{aligned} N(\det \mathcal{M}_j(b)) &= N\left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\gamma_\sigma b_i\right) = \sum_{\sigma \in S_n} N(\gamma_\sigma b_i) \\ &= \sum_{\sigma \in S_n} b_i N(\gamma_\sigma) \subseteq n!\overline{b}.\overline{A} = \overline{b}.\overline{A}. \end{aligned}$$

Also, for $1 \leqslant j \leqslant n$

$$\begin{aligned} N(\det \mathcal{M}_j) &= N\left(\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\,\gamma_\sigma\beta_i\right) = \sum_{\sigma\in S_n} N(\gamma_\sigma\beta_i) \\ &= \sum_{\sigma\in S_n} \gamma_\sigma N(\beta_i) + \beta_i N(\gamma_\sigma) \subseteq \sum_{\sigma\in S_n} |\gamma_\sigma|\,\overline{B} + b_i N(\gamma_\sigma) \\ &\subseteq n!\left(\overline{B} + \overline{b}.\overline{A}\right) = \overline{B} + \overline{b}.\overline{A}. \quad \blacksquare \end{aligned}$$

Lemma 4.13 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be a reduced non-singular matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$ and $\Delta = \det \mathcal{A} = d + D$, and let $\mathcal{B} = [\beta_i]$ be an upper zeroless vector, with $\beta_i = b_i + B_i \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. Consider the reduced flexible system $\mathcal{A}\mathcal{X} \subseteq \mathcal{B}$. Assume that $\mathcal{X} = \left[\xi_j\right]$, with $\xi_j = x_j + X_j \in \mathbb{E}$ for all $j \in \{1, \ldots, n\}$, is an admissible solution, and $R(\mathcal{A}) \subseteq P(\mathcal{B})$. Then*

1. $\overline{A}\overline{x} \subseteq \left(\overline{A}\diagup\Delta\right)\overline{\beta} \subseteq \underline{B}$, *with* $\overline{x} = \max\limits_{1\leqslant j\leqslant n} |x_j|$.

2. *If $N\left(\xi_j\right) \subseteq \underline{B}$ for all $j \in \{1, \ldots, n\}$, one has, for all $i \in \{1, \ldots, n\}$,*

$$N\left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right) \subseteq N(\beta_i).$$

Proof. *1.* Because $\mathcal{A}$ is a non-singular matrix, Δ is zeroless. So $d \neq 0$. Moreover, since $\mathcal{A}$ is a reduced matrix, $\overline{a} = 1$ and so $R(\mathcal{A}) = \overline{A}\diagup\Delta$.

By Cramer's rule $\begin{bmatrix} \frac{\det \mathcal{M}_1(a,b)}{d} \\ \vdots \\ \frac{\det \mathcal{M}_n(a,b)}{d} \end{bmatrix}$ is the only solution of the classical linear system $\mathcal{P}\mathcal{Y} = \mathcal{C}$, where $\mathcal{P} = [a_{ij}]_{n\times n}$ is a real matrix and $\mathcal{Y} = [x_i]_{n\times 1}$ and $\mathcal{C} = [b_i]_{n\times 1}$ are real column vectors, with $i, j \in \{1, \ldots, n\}$.

So $\overline{x} = \left|\frac{\det \mathcal{M}_k(a,b)}{d}\right|$ for some $k \in \{1, ..., n\}$. By Part 1 of Lemma 4.12 we have in particular that $\det \mathcal{M}_k(a,b) < 2n!\overline{b} \leqslant 2n!\overline{\beta}$. Then using Lemma 2.6,

$$\begin{aligned}\overline{A}\overline{x} &= \overline{A}\frac{\det \mathcal{M}_k(a,b)}{d} \subseteq \frac{\overline{A}}{d}2n!\overline{\beta} = \frac{\overline{A}}{\Delta}\overline{\beta} \\ &= R(\mathcal{A})\,\overline{\beta} \subseteq P(\mathcal{B})\,\overline{\beta} = \left(\underline{B}\diagup\overline{\beta}\right)\overline{\beta} \subseteq \underline{B}.\end{aligned}$$

Hence $\overline{A}\overline{x} \subseteq \left(\overline{A}\diagup\Delta\right)\overline{\beta} \subseteq \underline{B}$.

2. Suppose that $N(\xi_j) \subseteq \underline{B}$ for all $j \in \{1,\dots,n\}$. Then, using Lemma 2.6 and Part 1, one has for all $i \in \{1,\dots,n\}$

$$\begin{aligned}N\left(\sum_{j=1}^{n}\alpha_{ij}\xi_j\right) &= \sum_{j=1}^{n}N(\alpha_{ij}\xi_j) = \sum_{j=1}^{n}(\alpha_{ij}N(\xi_j) + \xi_j N(\alpha_{ij})) \\ &= \sum_{j=1}^{n}(a_{ij}N(\xi_j) + x_j N(\alpha_{ij})) \subseteq \sum_{j=1}^{n}(\overline{a}\underline{B} + \overline{x}\overline{A}) \\ &= n\left(\underline{B} + \overline{x}\overline{A}\right) \subseteq \underline{B} + \underline{B} = \underline{B} \subseteq N(\beta_i).\end{aligned}$$

Hence $N\left(\sum_{j=1}^{n}\alpha_{ij}\xi_j\right) \subseteq N(\beta_i)$, for all $i \in \{1,\dots,n\}$. ■

We are now able to present the proof of the Theorem 4.4, starting with the case of reduced flexible systems.

Proof of Theorem 4.4. We assume first that $\overline{a} = 1$. Because $\mathcal{A}$ is a non-singular matrix, $\Delta = \det \mathcal{A} = d + D$ is zeroless. So $d \neq 0$ and $\frac{1}{\Delta} = \frac{1}{d+D} = \frac{1}{d} + \frac{D}{d^2}$. Hence, by Lemma 2.6

$$N\left(\frac{1}{\Delta}\right) = \frac{D}{d^2} = \frac{D}{\Delta^2}. \tag{4.1}$$

For all $i,j \in \{1,\dots,n\}$, let $x = [x_j]$ be a solution of the system $\sum_{j=1}^{n} a_{ij}x_j = b_i$. Then by distributivity regarding multiplication by real numbers [6] and Part 1 of Lemma 4.13

$$\begin{aligned}\alpha_{i1}x_1+\cdots+\alpha_{in}x_n &= (a_{i1}+A_{i1})\,x_1+\cdots+(a_{in}+A_{in})\,x_n\\ &= (a_{i1}x_1+\cdots+a_{in}x_n)+(A_{i1}x_1+\cdots+A_{in}x_n)\\ &\subseteq b_i+\overline{A}\overline{x}\subseteq b_i+\underline{B}\subseteq b_i+B_i=\beta_i.\end{aligned}$$

To complete the proof consider now the neutricial part of the system $\mathcal{AX}\subseteq\mathcal{B}$.

1. By Part 2 of Lemma 4.12, Lemma 2.6 and Part 1 of Lemma 4.13, for all $j\in\{1,...,n\}$

$$N\left(\frac{\det\mathcal{M}_j\left(b\right)}{d}\right)=\frac{1}{d}N\left(\det\mathcal{M}_j\left(b\right)\right)\subseteq\frac{\overline{b}.\overline{A}}{d}=\left(\overline{A}\diagup\Delta\right)\overline{\beta}\subseteq\underline{B}. \tag{4.2}$$

So $N\left(\xi_j\right)=N\left(\frac{\det\mathcal{M}_j(b)}{d}\right)\subseteq\underline{B}$ for all $j\in\{1,\ldots,n\}$. Hence $\mathcal{X}=\left[\frac{\det\mathcal{M}_j(b)}{d}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{AX}\subseteq\mathcal{B}$ by Part 2 of Lemma 4.13.

2. Suppose that Δ is not an absorber of $\underline{B}$. So $\underline{B}\subseteq\Delta\underline{B}$ and we have

$$\underline{B}\diagup\Delta\subseteq\underline{B}. \tag{4.3}$$

Then using Lemma 2.6 and formula (4.1), for all $j\in\{1,\ldots,n\}$

$$\begin{aligned}N\left(\xi_j\right) &= N\left(\frac{\det\mathcal{M}_j\left(b\right)}{\Delta}\right)=\frac{1}{\Delta}N\left(\det\mathcal{M}_j\left(b\right)\right)+\det\mathcal{M}_j\left(b\right)\cdot N\left(\frac{1}{\Delta}\right)\\ &= \frac{1}{d}N\left(\det\mathcal{M}_j\left(b\right)\right)+\det\mathcal{M}_j\left(b\right)\cdot\frac{D}{\Delta^2}\\ &= N\left(\frac{\det\mathcal{M}_j\left(b\right)}{d}\right)+\frac{\det\mathcal{M}_j\left(b\right)}{\Delta}\cdot\frac{D}{\Delta}.\end{aligned}$$

Using formula (4.2), Part 1 of Lemma 4.12 and Lemma 4.11 one derives

$$N\left(\frac{\det\mathcal{M}_j\left(b\right)}{d}\right)+\frac{\det\mathcal{M}_j\left(b\right)}{\Delta}\frac{D}{\Delta}\subseteq\underline{B}+\frac{2n!\overline{\beta}}{\Delta}\frac{\overline{A}}{\Delta}=\underline{B}+\frac{\left(\overline{A}\diagup\Delta\right)\overline{\beta}}{\Delta}.$$

Moreover, by Part 1 of Lemma 4.13 and formula (4.3)

$$\frac{\left(\overline{A}\diagup\Delta\right)\overline{\beta}}{\Delta}\subseteq \underline{B}\diagup\Delta\subseteq\underline{B}. \tag{4.4}$$

Hence for all $j\in\{1,\ldots,n\}$

$$N\left(\xi_j\right)\subseteq\underline{B}+\underline{B}=\underline{B}.$$

Therefore Part 2 of Lemma 4.13 implies that $\mathcal{X}=\left[\frac{\det\mathcal{M}_j(b)}{\Delta}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$.

3. Suppose now that Δ is not an absorber of $\underline{B}$ and that $\underline{B}=\overline{B}$. Then using Lemma 4.12 and formula (4.1), for all $j\in\{1,\ldots,n\}$

$$\begin{aligned} N\left(\xi_j\right) &= N\left(\frac{\det\mathcal{M}_j}{\Delta}\right)=\frac{1}{\Delta}N\left(\det\mathcal{M}_j\right)+\det\mathcal{M}_j\cdot N\left(\frac{1}{\Delta}\right)\\ &\subseteq \frac{1}{\Delta}\left(\overline{b}.\overline{A}+\overline{B}\right)+2n!\overline{\beta}N\left(\frac{1}{\Delta}\right)=\frac{1}{\Delta}\left(\overline{b}.\overline{A}+\underline{B}\right)+\overline{\beta}\frac{D}{\Delta^2}. \end{aligned}$$

By Lemmas 2.6 and 4.11 and formula (4.3)

$$\frac{1}{\Delta}\left(\overline{b}.\overline{A}+\underline{B}\right)+\overline{\beta}\frac{D}{\Delta^2}\subseteq\overline{\beta}\left(\overline{A}\diagup\Delta\right)+\underline{B}\diagup\Delta+\frac{\overline{\beta}}{\Delta}\left(\overline{A}\diagup\Delta\right)\subseteq\left(\overline{A}\diagup\Delta\right)\overline{\beta}+\underline{B}+\frac{1}{\Delta}\left(\overline{A}\diagup\Delta\right)\overline{\beta}.$$

It follows from Part 1 of Lemma 4.13 and formula (4.4) that $\left(\overline{A}\diagup\Delta\right)\overline{\beta}\subseteq\underline{B}$ and $\frac{1}{\Delta}\left(\overline{A}\diagup\Delta\right)\overline{\beta}\subseteq\underline{B}$. So

$$N\left(\xi_j\right)\subseteq\underline{B}. \tag{4.5}$$

Hence $\mathcal{X}=\left[\frac{\det\mathcal{M}_j}{\Delta}\right]_{1\leqslant j\leqslant n}$ is a solution of $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$ by Part 2 of Lemma 4.13.

As for the general case, let $\overline{a}$ be arbitrary. Because $\mathcal{A}=[\alpha_{ij}]$ is a non-singular matrix, $\Delta=\det\mathcal{A}$ is zeroless. So $d\neq 0$ and $\overline{a}\neq 0$. Consider the $n\times n$ matrix $\mathcal{A}'=[\alpha_{ij}\diagup\overline{a}]\equiv[c_{ij}+C_{ij}]$ and the column vector $\mathcal{B}'=[\beta_i\diagup\overline{a}]$. Then $\mathcal{A}'$ is a non-singular matrix and $\mathcal{B}'$ is an upper zeroless vector, with $\overline{c}=\max\limits_{1\leqslant i,j\leqslant n}|c_{ij}|=1$. So $\mathcal{A}'\mathcal{X}\subseteq\mathcal{B}'$ is a reduced flexible system with the same solutions as the system $\mathcal{A}\mathcal{X} \subseteq \mathcal{B}$. One has

$$R\left(\mathcal{A}'\right) = \frac{\left(\overline{A}\diagup\overline{a}\right)\overline{c}^{n-1}}{\Delta\diagup\overline{a}^{n}} = \overline{A}\overline{a}^{n-1}\diagup\Delta = R\left(\mathcal{A}\right) \subseteq P\left(\mathcal{B}\right) = \left(\underline{B}\diagup\overline{\beta}\right)\left(\overline{a}\diagup\overline{a}\right) = P\left(\mathcal{B}'\right).$$

Hence $\mathcal{X} = \left[\frac{\det \mathcal{M}_j\diagup\overline{a}^n}{\Delta\diagup\overline{a}^n}\right]_{1\leqslant j\leqslant n} = \left[\frac{\det \mathcal{M}_j}{\Delta}\right]_{1\leqslant j\leqslant n}$ satisfies the equation $\mathcal{A}'\mathcal{X} \subseteq \mathcal{B}'$. Then $\mathcal{X}$ satisfies also the equation $\mathcal{A}\mathcal{X}\subseteq\mathcal{B}$.

Finally we prove that $\mathcal{X}$ is maximal. Indeed, let $\xi_1, ..., \xi_n$ be such that $(\xi_1, ..., \xi_n)^T$ satisfies (3.1), and $x_j \in \xi_j$ for $1 \leqslant j \leqslant n$. Then for every choice of representatives $a_{ij} \in \alpha_{ij}$ with $1 \leqslant i, j \leqslant n$ there exist $b_1 \in \beta_1$,..., $b_n \in \beta_n$ such that

$$\begin{cases} a_{11}x_1 + & \cdots & +a_{1n}x_n & = b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1}x_1 + & \cdots & +a_{nn}x_n & = b_n \end{cases}.$$

Put

$$d = \det \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.$$

Then $x_j = \frac{\mathcal{M}_j(a,b)}{d} \in \frac{\det \mathcal{M}_j}{\Delta}$ for $1 \leqslant j \leqslant n$. Hence $\xi_j \subseteq \frac{\det \mathcal{M}_j}{\Delta}$ for $1 \leqslant j \leqslant n$ and so $\mathcal{X}$ is maximal. ■

Chapter 5

Gauss-Jordan elimination

Theorem 4.4 yields closed form formulae for column vectors of external numbers satisfying the flexible system (3.1) by inclusion. In this chapter we study their relation with solutions obtained by Gauss-Jordan elimination, which are of more practical interest.

The solution of flexible systems by the operations of Gauss-Jordan elimination corresponds to multiplication by certain matrices. Sum and product of matrices will be defined pointwise.

Indeed, let $\mathcal{A} = [\alpha_{ij}]_{m\times n}$, $\mathcal{B} = \left[\beta_{ij}\right]_{m\times n}$ and $\mathcal{C} = \left[\gamma_{jk}\right]_{n\times p}$, where $m, n, p \in \mathbb{N}$, $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant p$ and $\alpha_{ij}, \beta_{ij}, \gamma_{jk}$ are all external numbers. Then

$$\mathcal{A} + \mathcal{B} = \left[\alpha_{ij} + \beta_{ij}\right]_{m\times n}$$

and

$$\mathcal{AC} = \left[\sum_{1\leqslant j\leqslant n} \alpha_{ij}\gamma_{jk}\right]_{m\times p}.$$

One difficulty to overcome is the fact that multiplication of matrices with external numbers is not fully distributive and associative. These are consequences of the fact that multiplication of external numbers is not fully distributive. For an example, let $A \supset \{0\}$

be a neutrix. Then

$$\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right)\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv [0]$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} A & A \\ A & A \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} A & A \\ A & A \end{bmatrix} \neq [0]\,.$$

Still, monotony for inclusion is preserved in the following way. Let $\gamma_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant 2$ and let $U, V, X, Y \in \mathcal{N}$ with $U \subseteq X$ and $V \subseteq Y$. Then

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} \subseteq \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}. \tag{5.1}$$

Indeed

$$\begin{aligned} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} &= \begin{bmatrix} \gamma_{11}U + \gamma_{12}V \\ \gamma_{21}U + \gamma_{22}V \end{bmatrix} \\ &\subseteq \begin{bmatrix} \gamma_{11}X + \gamma_{12}Y \\ \gamma_{21}X + \gamma_{22}Y \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}. \end{aligned}$$

We use the property of subdistributivity of interval calculus in the next proposition on matrix calculation with differences. We consider the general case, for the proof is straightforward.

Proposition 5.1 *Let $n \in \mathbb{N}$ be standard and let $\alpha_{ij}, \beta_i, \xi_j \in \mathbb{E}$ for all $i, j \in \{1, ..., n\}$. Assume*

$$\begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \subseteq \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Let $B_i = N(\beta_i)$ for all $i \in \{1, ..., n\}$. Let $x_i, y_i \in \xi_i$ and $u_i = x_i - y_i$ for $1 \leqslant i \leqslant n$. Then the column vector $(u_1, ..., u_n)^T$ satisfies

$$\begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \dots & \alpha_{nn} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \subseteq \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix}.$$

Proof. It follows from subdistributivity that for $1 \leqslant i \leqslant n$

$$\begin{aligned}
\alpha_{i1}u_1 + \cdots + \alpha_{in}u_n &= \alpha_{i1}(x_1 - y_1) + \cdots + \alpha_{in}(x_n - y_n) \\
&\subseteq \alpha_{i1}x_1 - \alpha_{i1}y_1 + \cdots + \alpha_{in}x_n - \alpha_{in}y_n \\
&= \alpha_{i1}x_1 + \cdots + \alpha_{in}x_n - (\alpha_{i1}y_1 + \cdots + \alpha_{in}y_n) \\
&\subseteq \beta_i - \beta_i = B_i. \qquad \blacksquare
\end{aligned}$$

5.1 Gauss-operations

For the solution of flexible systems by Gauss-Jordan elimination we will consider operations with matrices which contain only real entries. Then, taking profit of (2.2), distributivity holds to a large extent, which leads to some convenient simplifications. In a sense this may be compared with the usual numerical procedure, where matrices with entries, say, in floating-point are nullified using numbers of less complexity, i.e. truncated rational numbers.

It is to be expected that full nullification of a flexible system cannot be realized and that instead of zeros we will obtain neutrices. So instead of nullification we speak about *neutrification*. The Gauss-Jordan operations will be represented by matrices whose entries will neutrify step by step each column of the matrix of coefficients except its diagonal elements; the diagonal elements will be external numbers that may be written as the sum of 1 and a neutrix. This procedure corresponds to the classic Gauss-Jordan elimination method.

First we need to prove some useful properties concerning the minors of the matrix of coefficients of a flexible system. Below we will maintain the notations of Notation 3.1.

Notation 5.2 Let $n \in \mathbb{N}$ be standard and $k \in \{1, \ldots, n-1\}$. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} = a_{ij} + A_{ij} \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. We denote

1. $[\mathcal{A}]_{i_1 \cdots i_k, j_1 \cdots j_k}$ as the $(n-k) \times (n-k)$ matrix formed by removing from $\mathcal{A}$ the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$ and $1 \leqslant j_1 < \cdots < j_k \leqslant n$;
2. $M_{i_1 \cdots i_k, j_1 \cdots j_k} \equiv \det [\mathcal{A}]_{i_1 \cdots i_k, j_1 \cdots j_k}$ as the *$(i_1 \cdots i_k, j_1 \cdots j_k)$ k^{th} minor* of $\mathcal{A}$;
3. $m_{i_1 \cdots i_k, j_1 \cdots j_k}$ as a representative of $M_{i_1 \cdots i_k, j_1 \cdots j_k}$.

For matrices with external numbers the Laplace expansion becomes an inclusion.

Lemma 5.3 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$, and $\Delta = \det \mathcal{A}$. Then, for all $j \in \{1, ..., n\}$,*

$$(-1)^{1+j} \alpha_{1j} M_{1,j} + \cdots + (-1)^{n+j} \alpha_{nj} M_{n,j} \subseteq \Delta.$$

Proof. Let S_n denote the set of all permutations of the set $\{1, ..., n\}$ and $\sigma = (p_1, ..., p_n) \in S_n$. Suppose first that $j = 1$. By subdistributivity, one has

$$\begin{aligned}
& \alpha_{11} M_{1,1} - \alpha_{21} M_{2,1} + \cdots + (-1)^{n+1} \alpha_{n1} M_{n,1} \\
= \; & \alpha_{11} \sum_{\substack{\sigma \in S_n \\ p_1 = 1}} \operatorname{sgn}(\sigma) \alpha_{2p_2} \cdot \cdots \cdot \alpha_{np_n} + \cdots + \alpha_{n1} \sum_{\substack{\sigma \in S_n \\ p_n = 1}} \operatorname{sgn}(\sigma) \alpha_{1p_1} \cdot \cdots \cdot \alpha_{(n-1)p_{n-1}} \\
\subseteq \; & \sum_{\substack{\sigma \in S_n \\ p_1 = 1}} \operatorname{sgn}(\sigma) \alpha_{11} \alpha_{2p_2} \cdot \cdots \cdot \alpha_{np_n} + \cdots + \sum_{\substack{\sigma \in S_n \\ p_n = 1}} \operatorname{sgn}(\sigma) \alpha_{1p_1} \cdot \cdots \cdot \alpha_{(n-1)p_{n-1}} \alpha_{n1} \\
= \; & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \alpha_{1p_1} \cdot \cdots \cdot \alpha_{np_n} = \det \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} = \Delta.
\end{aligned}$$

The proof is the same for $j \in \{2, ..., n\}$. ■

Corollary 5.4 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ matrix, with $\alpha_{ij} \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$, and $\Delta = \det \mathcal{A}$. Any expansion of Δ in cofactors is contained in Δ.*

We now prove some useful properties of the minors of the matrix of coefficients of reduced systems.

The next Lemmas show that, in the case of a reduced matrix of coefficients, its minors have the same order of magnitude as the determinant.

Lemma 5.5 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ be a reduced non-singular matrix with $\alpha_{ij} \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$, and $\Delta = \det \mathcal{A}$. Then, for all $j \in \{1, ..., n\}$,*

$$|M_{i,j}| > \oslash\Delta$$

for some $i \in \{1, ..., n\}$.

Proof. By Lemma 5.3, one has $\alpha_{11}M_{1,1} - \alpha_{21}M_{2,1}... + (-1)^{n+1}\alpha_{n1}M_{n,1} \subseteq \Delta$. Also $|\alpha_{ij}| \leqslant 1 + \oslash$ for all $i, j \in \{1, ..., n\}$.

Suppose that $M_{i,1} \subseteq \oslash\Delta$ for all $i \in \{1, ..., n\}$. Then also $\alpha_{i1}M_{i,1} \subseteq (1 + \oslash)\oslash\Delta = \oslash\Delta$ for all $i \in \{1, ..., n\}$. So

$$\alpha_{11}M_{1,1} - \alpha_{21}M_{2,1} + \cdots + (-1)^{n+1}\alpha_{n1}M_{n,1} \subseteq \oslash\Delta,$$

which is absurd by Lemma 2.8 because Δ is zeroless. Hence $|M_{i,1}| > \oslash\Delta$ for some $i \in \{1, ..., n\}$.

The proof is the same for $j \in \{2, ..., n\}$. ■

Lemma 5.6 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced matrix with $\alpha_{ij} \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$, and $\Delta = \det \mathcal{A}$. Then, $|\Delta| < n! + 1$ and, for all $i, j \in \{1, ..., n\}$,*

$$|M_{i,j}| < (n-1)! + 1.$$

Proof. Let S_n denote the set of all permutations of the set $\{1, ..., n\}$ and $\sigma = (p_1, ..., p_n) \in S_n$. Since $\mathcal{A}$ is a reduced matrix, $|\alpha_{ij}| \leqslant 1 + \oslash$ for all $i, j \in \{1, ..., n\}$. So

$$\begin{aligned} |\Delta| &= \left| \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdot \cdots \cdot \alpha_{np_n} \right| \leqslant \sum_{\sigma \in S_n} |\alpha_{1p_1}| \cdot \cdots \cdot |\alpha_{np_n}| \\ &\leqslant \sum_{\sigma \in S_n} (1+\oslash)^n = n!\,(1+\oslash) = n! + \oslash. \end{aligned}$$

Hence $|\Delta| < n! + 1$. In the same way one proves that for all $i, j \in \{1, ..., n\}$

$$|M_{i,j}| \leqslant (n-1)! + \oslash < (n-1)! + 1. \qquad \blacksquare$$

Corollary 5.7 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced matrix with $\alpha_{ij} \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$, and $\Delta = \det \mathcal{A}$. Then for all $i, j \in \{1, ..., n\}$,*

$$|M_{i,j}| \subseteq [@\,|\Delta|\,, @]\,.$$

In fact, the same lower and upper bounds are hold for the k^{th} minors. The upper bound is obvious and is a consequence of the next Lemma. The proof of the lower bound needs more care due to the specific properties of external Gauss-Jordan elimination and will be postponed to Theorem 5.30.

Lemma 5.8 *Let $n \in \mathbb{N}$ be standard and $k \in \{1, \ldots, n-1\}$. Let $\mathcal{A} = [\alpha_{ij}]_{n \times n}$ be a reduced matrix with $\alpha_{ij} \in \mathbb{E}$, for $1 \leqslant i, j \leqslant n$. Then, for all $i_1, \ldots, i_k, j_1, \ldots, j_k \in \{1, ..., n\}$ such that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$,*

$$|M_{i_1 \cdots i_k, j_1 \cdots j_k}| < (n-k)! + 1.$$

Proof. The proof is similar to the proof of Lemma 5.6. ■

Remark 5.9 Let $n \in \mathbb{N}$ be standard and $\alpha_{ij}, \beta_i, \xi_j \in \mathbb{E}$ for $1 \leqslant i, j \leqslant n$. Let $a_{ij} \in \alpha_{ij}$, for all $i, j \in \{1, \ldots, n\}$. Consider the flexible system (3.1) with matrix representation given by $\mathcal{AX} \subseteq \mathcal{B}$. Unless otherwise said, we will assume that the system is reduced and that all the conditions of Part 3 of Theorem 4.4 are satisfied which here correspond to:

$$\begin{cases} \text{(i) } N(\beta_i) \equiv B, \text{ for all } i \in \{1, \ldots, n\}; \\ \text{(ii) } \Delta = \det \mathcal{A} \text{ is not an absorber of } B; \\ \text{(iii) } \overline{A} \diagup \Delta \subseteq B \diagup \overline{\beta}. \end{cases}$$

Moreover, we will write the first entry α_{11} in the form of $\alpha_{11} = 1 + A_{11} = \overline{\alpha}$.

Notation 5.10 Consider the flexible system (3.1). Let $k \in \{1, \ldots, n-1\}$. We denote

1. $\overline{M_k} \equiv M_{r_1 \cdots r_k, c_1 \cdots c_k}$ for some $r_1, \ldots, r_k, c_1, \ldots, c_k \in \{2, \ldots, n\}$ such that

$$\left|\overline{M_k}\right| = \max_{\substack{2 \leqslant i_1 < \cdots < i_k \leqslant n \\ 2 \leqslant j_1 < \cdots < j_k \leqslant n}} |M_{i_1 \cdots i_k, j_1 \cdots j_k}|;$$

2. $\overline{m_k}$ as a representative of $\overline{M_k}$;

3. $\Delta = \det \mathcal{A} \equiv d + N(\Delta)$, for some $d \in \Delta$.

Remark 5.11 From now on we will assume that $\overline{M_k} = M_{(n-k+1)\cdots n,(n-k+1)\cdots n}$ for all $k \in \{1, \ldots, n-1\}$. This is without loss of generality for we can make row changes and/or column changes in the system (3.1) so that $\overline{M_1} = M_{n,n}$, $\overline{M_2} = M_{(n-1)n,(n-1)n}, \ldots,$ $\overline{M_{n-1}} = M_{2\cdots(n-1)n,2\cdots(n-1)n}$. Indeed, for row and/or column changes, condition (i) stays the same; also conditions (ii) and (iii) remain true for Δ will possibly only change its sign. So we can always obtain an equivalent system of system (3.1) which still verifies conditions (i), (ii) and (iii) and also $\overline{M_k} = M_{(n-k+1)\cdots n,(n-k+1)\cdots n}$ for all $k \in \{1, \ldots, n-1\}$.

Definition 5.12 Consider the flexible system (3.1). Let $\widehat{\mathcal{A}} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be a matrix whose entries a_{ij} are representatives of the entries α_{ij} of the matrix $\mathcal{A}$, where $i, j \in \{1, \ldots, n\}$. For every $p \in \{1, \ldots, 2n-1\}$ we define matrices $\mathcal{G}_p$ such that

$$\mathcal{G}_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1/\overline{m_{n-2}} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_3 = \begin{bmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{24\cdots n,3\cdots n} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{2\cdots(n-1),3\cdots n} & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_4 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \overline{m_{n-2}} \diagup \overline{m_{n-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_5 = \begin{bmatrix} 1 & 0 & m_{3\cdots n,14\cdots n} \diagup \overline{m_{n-2}} & \cdots & 0 \\ 0 & 1 & -m_{3\cdots n,24\cdots n} \diagup \overline{m_{n-2}} & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -m_{35\cdots n,4\cdots n} \diagup \overline{m_{n-2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -m_{3\cdots(n-1),4\cdots n} \diagup \overline{m_{n-2}} & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \overline{m_{n-3}} \diagup \overline{m_{n-4}} & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\mathcal{G}_7 = \begin{bmatrix} 1 & 0 & 0 & -m_{4\cdots n,15\cdots n} \diagup \overline{m_{n-3}} & \cdots & 0 \\ 0 & 1 & 0 & m_{4\cdots n,25\cdots n} \diagup \overline{m_{n-3}} & \cdots & 0 \\ 0 & 0 & 1 & -m_{4\cdots n,35\cdots n} \diagup \overline{m_{n-3}} & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & -m_{46\cdots n,5\cdots n} \diagup \overline{m_{n-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -m_{4\cdots(n-1),5\cdots n} \diagup \overline{m_{n-3}} & \cdots & 1 \end{bmatrix}, \cdots,$$

$$\mathcal{G}_{2n-2} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \overline{m_1} \diagup d \end{bmatrix},$$

$$\mathcal{G}_{2n-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & (-1)^{n+1} m_{n,1} \diagup \overline{m_1} \\ 0 & 1 & \cdots & 0 & (-1)^{n+2} m_{n,2} \diagup \overline{m_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & (-1)^{2n-1} m_{n,n-1} \diagup \overline{m_1} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

We write $\mathcal{G}[.]$ to indicate the repeated multiplication of matrices $\mathcal{G}_{2n-1}\left(\mathcal{G}_{2n-2}\left(\cdots \mathcal{G}_3\left(\mathcal{G}_2\left(\mathcal{G}_1[.]\right)\right)\cdots\right)\right)$.

The above matrices correspond to the Gauss-Jordan elimination operations for the matrix $\widehat{\mathcal{A}}$. Indeed $\mathcal{G}\widehat{\mathcal{A}} = I_n$. For the seek of clarity we present explicit calculations in the special case where $\widehat{\mathcal{A}}$ is an 3×3 matrix.

By Remark 5.11 one has $\overline{m_1} = m_{3,3}$. Also

$$\begin{aligned} m_{3,3}a_{13} - a_{12}m_{3,2} &= (a_{22} - a_{21}a_{12})\, a_{13} - a_{12}\,(a_{23} - a_{21}a_{13}) \\ &= a_{22}a_{13} - a_{12}a_{23} = -m_{3,1} \end{aligned}$$

and clearly in the last step of Gauss-Jordan elimination we obtain the determinant

$$m_{3,3}m_{2,2} - m_{2,3}m_{3,2} = d.$$

So

$$\begin{aligned}
\mathcal{G}\widehat{\mathcal{A}} &= \mathcal{G}_5\left(\mathcal{G}_4\left(\mathcal{G}_3\left(\mathcal{G}_2\left(\mathcal{G}_1\widehat{\mathcal{A}}\right)\right)\right)\right) \\
&= \mathcal{G}_5\left(\mathcal{G}_4\left(\mathcal{G}_3\left(\mathcal{G}_2\left(\begin{bmatrix} 1 & 0 & 0 \\ -a_{21} & 1 & 0 \\ -a_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right)\right)\right)\right) \\
&= \mathcal{G}_5\left(\mathcal{G}_4\left(\mathcal{G}_3\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\overline{m_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & m_{3,3} & m_{3,2} \\ 0 & m_{2,3} & m_{2,2} \end{bmatrix}\right)\right)\right) \\
&= \mathcal{G}_5\left(\mathcal{G}_4\left(\begin{bmatrix} 1 & -a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & -m_{2,3} & 1 \end{bmatrix}\begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & m_{3,2}/\overline{m_1} \\ 0 & m_{2,3} & m_{2,2} \end{bmatrix}\right)\right) \\
&= \mathcal{G}_5\left(\mathcal{G}_4\left(\begin{bmatrix} 1 & 0 & a_{13} - a_{12}m_{3,2}/\overline{m_1} \\ 0 & 1 & m_{3,2}/\overline{m_1} \\ 0 & 0 & m_{2,2} - m_{2,3}m_{3,2}/\overline{m_1} \end{bmatrix}\right)\right) \\
&= \mathcal{G}_5\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{m_1}/d \end{bmatrix}\begin{bmatrix} 1 & 0 & -m_{3,1}/\overline{m_1} \\ 0 & 1 & m_{3,2}/\overline{m_1} \\ 0 & 0 & d/\overline{m_1} \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & 0 & m_{3,1}/\overline{m_1} \\ 0 & 1 & -m_{3,2}/\overline{m_1} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -m_{3,1}/\overline{m_1} \\ 0 & 1 & m_{3,2}/\overline{m_1} \\ 0 & 0 & 1 \end{bmatrix} = I_3.
\end{aligned}$$

In general, with $\mathcal{A} = \begin{bmatrix} 1 & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$ a reduced matrix, the matrix $\mathcal{G}_1$ corresponds to the neutrification of the first column of $\mathcal{A}$ except its first position, the matrix $\mathcal{G}_2$ places a nearly unit entry in the second position of the second column of $\mathcal{A}$, the matrix $\mathcal{G}_3$ corresponds to the neutrification of the second column of $\mathcal{A}$ except its second position, the matrix $\mathcal{G}_4$ places a nearly unit entry in the third position of the third column of $\mathcal{A}$, the matrix $\mathcal{G}_5$ corresponds to the neutrification of the third column of $\mathcal{A}$ except its third position, and so on until the matrix $\mathcal{G}_{2n-2}$ places a nearly unit entry in the last position of the last column of $\mathcal{A}$ and the matrix $\mathcal{G}_{2n-1}$ neutrifies the last column of $\mathcal{A}$ except its last position. So the even matrices reduce the rows and the odd matrices neutrify the corresponding columns. Observe that if $p = 2j - 1$, the entries of the column j of the matrix $\mathcal{G}_p$ are of alternate sign above the principal diagonal and of negative sign bellow the principal diagonal. Working with a matrix of representatives $\widehat{\mathcal{A}}$, we illustrate this phenomenon with the matrix $\mathcal{G}_7$. We start with the minors below the principal diagonal. The minor $m_{46\cdots n,5\cdots n}$ equals the determinant of

$$\mathcal{T} \equiv \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{51} & a_{52} & a_{53} & a_{54} \end{bmatrix}.$$

The operations of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$ and $\mathcal{G}_6$ transform the matrix $\mathcal{T}$ into the matrix which is of the form

$$\mathcal{T}' \equiv \begin{bmatrix} 1 & 0 & 0 & t_{14} \\ 0 & 1 & 0 & t_{24} \\ 0 & 0 & 1 & t_{34} \\ 0 & 0 & 0 & t_{44} \end{bmatrix}.$$

The determinant of $\mathcal{T}$ is modified by the matrices $\mathcal{G}_2$ and $\mathcal{G}_4$. Indeed, $\mathcal{G}_2$ corresponds to a multiplication of $m_{46\cdots n,5\cdots n}$ by $1/\overline{m_{n-2}}$. Then $\mathcal{G}_4$ corresponds to a multiplication of $m_{46\cdots n,5\cdots n}$ by $(1/\overline{m_{n-2}})\,(\overline{m_{n-2}}/\overline{m_{n-3}}) = 1/\overline{m_{n-3}}$. Observe that $\mathcal{G}_6$ does not have an impact on the matrix $\mathcal{T}$ and consequently not on its determinant. Hence

$$t_{44} = \det \mathcal{T}' = (1/\overline{m_{n-3}}) \det \mathcal{T} = m_{46\cdots n,5\cdots n}/\overline{m_{n-3}}.$$

The matrix $\mathcal{G}_7$ nullifies t_{44} with the pivot-entry at the position $4,4$ which, due to the previous operations, has been turned into 1. For this reason the entry $5,4$ of the matrix $\mathcal{G}_7$ must be equal to $-m_{46\cdots n,5\cdots n}/\overline{m_{n-3}}$.

In the same manner we obtain that, for $i \in \{6,\ldots,n\}$, the entries $i,4$ of the matrix $\mathcal{G}_7$ must be equal to $-m_{45\cdots(i-1)(i+1)\cdots n,5\cdots n}/\overline{m_{n-3}}$, in particular, always have a negative sign.

Now we consider the minors above the principal diagonal. The minor $m_{4\cdots n,35\cdots n}$ equals the determinant of

$$\mathcal{U} \equiv \begin{bmatrix} 1 & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{bmatrix}.$$

The operations of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$ and $\mathcal{G}_6$ transform the matrix $\mathcal{U}$ into the matrix which is of the form

$$\mathcal{U}' \equiv \begin{bmatrix} 1 & 0 & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

The determinant of $\mathcal{U}$ is modified by the matrices $\mathcal{G}_2$ and $\mathcal{G}_4$. As above, $\mathcal{G}_2$ corresponds to a multiplication of $m_{4\cdots n,35\cdots n}$ by $1/\overline{m_{n-2}}$, $\mathcal{G}_4$ corresponds to a multiplication of $m_{4\cdots n,35\cdots n}$ by $(1/\overline{m_{n-2}})\,(\overline{m_{n-2}}/\overline{m_{n-3}}) = 1/\overline{m_{n-3}}$ and $\mathcal{G}_6$ does not have an impact. Hence

$$u_{33} = \det \mathcal{U}' = (1/\overline{m_{n-3}}) \det \mathcal{U} = m_{4\cdots n,35\cdots n}/\overline{m_{n-3}}.$$

The matrix $\mathcal{G}_7$ nullifies u_{33} with the pivot-entry 1 at the position $4,4$, so the entry $3,4$ of the matrix $\mathcal{G}_7$ must be equal to $-m_{4\cdots n,35\cdots n}/\overline{m_{n-3}}$.

To explain the change of sign for the entry $2,4$ we note that we have to deal with the matrix

$$\mathcal{V} \equiv \begin{bmatrix} 1 & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \end{bmatrix},$$

which is transformed into

$$\mathcal{V}' \equiv \begin{bmatrix} 1 & 0 & v_{13} \\ 0 & 0 & v_{23} \\ 0 & 1 & v_{33} \end{bmatrix}.$$

Observe that

$$\det \mathcal{V}' = -\det \begin{bmatrix} 1 & 0 & v_{13} \\ 0 & 1 & v_{33} \\ 0 & 0 & v_{23} \end{bmatrix} = -v_{23}.$$

As above, we argue that we have to nullify with $m_{4\cdots n,25\cdots n}/\overline{m_{n-3}}$ but now with opposite sign.

For the entry $1,4$ we have two row changes and so we do not have a change of sign. Thus the parity of such row changes explains the change of signs in the entries of column j above the principal diagonal of the odd matrices $\mathcal{G}_{2j-1}$.

Notice that to the lack of associativity, in general, $\mathcal{G}$ does not correspond to the multiplication of matrices and so it should be treated as an operator. Also by (2.3) distributivity holds with respect to expressions of the form $a + A$, with $a \in \mathbb{R}$ and $A \in \mathcal{N}$. Hence the operator $\mathcal{G}$ is distributive in the following sense:

$$\begin{aligned} &\mathcal{G} \begin{bmatrix} 1 + A_{11} & \cdots & a_{1n} + A_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} + A_{n1} & \cdots & a_{nn} + A_{nn} \end{bmatrix} \\ &= \mathcal{G} \begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} + \mathcal{G} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}. \end{aligned} \tag{5.2}$$

With the operator $\mathcal{G}$ we do not achieve a complete inverse for the matrix $\mathcal{A}$ but still we obtain an approximate inverse admitting at most infinitesimal errors. Indeed, as it will be shown in the next section (Proposition 5.32),

$$\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \subseteq \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash \end{bmatrix}. \tag{5.3}$$

Hence, by (5.2), one has

$$\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \subseteq \begin{bmatrix} 1+\oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & 1+\oslash \end{bmatrix}. \tag{5.4}$$

5.2 Gauss-solution

In this section we present a general theorem that guarantees that the maximal solution produced by Cramer´s rule applied to a n by n flexible system satisfying the conditions of Part 3 of Theorem 4.4 is the same solution produced by Gauss-Jordan elimination.

Definition 5.13 Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$. We call $(x_1, \ldots, x_n)^T$ a *Gauss-solution* of the flexible system (3.1), with matrix representation given by $\mathcal{AX} \subseteq \mathcal{B}$, if for all choices of representatives of α_{ij}, for $1 \leqslant i, j \leqslant n$, and corresponding matrices one has

$$(\mathcal{GA})\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \subseteq \mathcal{GB}.$$

Theorem 5.14 *The Cramer-solution of the flexible system* (3.1) *equals the external set of all Gauss-solutions.*

The proof of this theorem will be given for a reduced system. If the system is not reduced, we can always divide all coefficients of matrix $\mathcal{A}$ by its *pivot* and obtain thereby an equivalent system of (3.1) which is reduced and still verifies all the conditions of Part 3 of Theorem 4.4.

We recall that, although the condition that $N(\beta_i) = B$, for all $i \in \{1, \ldots, n\}$, is not many times satisfied by a flexible system, one may solve the flexible system, now with $\underline{B} = \min_{1 \leqslant i \leqslant n} B_i$ instead of the $N(\beta_i)$. If for this new system we have $R(\mathcal{A}) \subseteq P(\mathcal{B})$ and also that Δ is not an absorber of $\underline{B}$, both by Cramer's rule and Gauss-Jordan elimination, one obtains the maximal solution of the modified flexible system which is an admissible solution of the original system.

We start with an example given by a 3×3 system.

Example 5.15 Consider the flexible system of Example 4.7 of previous Chapter 4:

$$\begin{cases} \left(1+\varepsilon^2\oslash\right)\xi_1 + \xi_2 + \left(1+\varepsilon^3\pounds\right)\xi_3 \subseteq \frac{1}{\varepsilon} + \varepsilon\oslash \\ \left(2+\varepsilon^3\pounds\right)\xi_1 + \left(-1+\varepsilon^2\oslash\right)\xi_2 - \xi_3 \subseteq \varepsilon\oslash \\ \left(\varepsilon+\varepsilon^3\oslash\right)\xi_1 + \xi_2 + \left(2+\varepsilon^2\oslash\right)\xi_3 \subseteq 1 + \varepsilon\oslash, \end{cases}$$

where ε is a positive infinitesimal. Let $\mathcal{A}$ be its matrix of coefficients and $\mathcal{B}$ the constant term vector. One has already seen that $\Delta = \det \mathcal{A} = -3 + \varepsilon^2\oslash$ is zeroless and that this system satisfies all of the conditions of Part 3 of Theorem 4.4. When applying Gauss-Jordan elimination, we get

$$\begin{aligned}
\mathcal{A}|\mathcal{B} &= \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & \frac{1}{\varepsilon}+\varepsilon\oslash \\ 2+\varepsilon^3\pounds & -1+\varepsilon^2\oslash & -1 & \varepsilon\oslash \\ \varepsilon+\varepsilon^3\oslash & 1 & 2+\varepsilon^2\oslash & 1+\varepsilon\oslash \end{array}\right] \\
&\begin{matrix} \longrightarrow \\ L_2-2L_1 \\ L_3-\varepsilon L_1 \end{matrix} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & \frac{1}{\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & -3+\varepsilon^2\oslash & -3+\varepsilon^3\pounds & -\frac{2}{\varepsilon}+\varepsilon\oslash \\ \varepsilon^3\oslash & 1-\varepsilon & 2-\varepsilon+\varepsilon^2\oslash & \varepsilon\oslash \end{array}\right] \\
&\begin{matrix} \longrightarrow \\ -\frac{1}{3}L_2 \end{matrix} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & 1 & 1+\varepsilon^3\pounds & \frac{1}{\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & 1+\varepsilon^3\pounds & \frac{2}{3\varepsilon}+\varepsilon\oslash \\ \varepsilon^3\oslash & 1-\varepsilon & 2-\varepsilon+\varepsilon^2\oslash & \varepsilon\oslash \end{array}\right] \\
&\begin{matrix} L_1-L_2 \\ \longrightarrow \\ L_3-(1-\varepsilon)\,L_2 \end{matrix} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & \varepsilon^2\oslash & \varepsilon^3\pounds & \frac{1}{3\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & 1+\varepsilon^3\pounds & \frac{2}{3\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & \varepsilon^2\oslash & 1+\varepsilon^2\oslash & \frac{2}{3}-\frac{2}{3\varepsilon}+\varepsilon\oslash \end{array}\right] \\
&\begin{matrix} \longrightarrow \\ L_2-L_3 \end{matrix} \left[\begin{array}{ccc|c} 1+\varepsilon^2\oslash & \varepsilon^2\oslash & \varepsilon^3\pounds & \frac{1}{3\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & 1+\varepsilon^2\oslash & \varepsilon^2\oslash & -\frac{2}{3}+\frac{4}{3\varepsilon}+\varepsilon\oslash \\ \varepsilon^2\oslash & \varepsilon^2\oslash & 1+\varepsilon^2\oslash & \frac{2}{3}-\frac{2}{3\varepsilon}+\varepsilon\oslash \end{array}\right] \\
&\equiv \mathcal{A}'|\mathcal{B}',
\end{aligned}$$

with $\mathcal{A}' \subseteq I_3 + [\oslash]_{3\times 3}$. So the solution produced by Gauss-Jordan elimination is

$$\mathcal{X} \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3\varepsilon}+\varepsilon\oslash \\ -\frac{2}{3}+\frac{4}{3\varepsilon}+\varepsilon\oslash \\ \frac{2}{3}-\frac{2}{3\varepsilon}+\varepsilon\oslash \end{bmatrix}.$$

As shown on Example 4.7, this solution is exactly the same one obtained when applying Cramer's rule.

The next example, which is a 4×4 system, illustrates how the higher order minors of the matrix of coefficients intervene in the Gauss-Jordan elimination process.

Example 5.16 Let ε be a positive infinitesimal. Consider the following flexible system:

$$\begin{cases} 2\xi_1 + \left(2+\varepsilon^2\oslash\right)\xi_2 + \xi_3 + \xi_4 = -2+\varepsilon\pounds \\ -\xi_1 + 2\xi_2 + (1+\varepsilon\oslash)\,\xi_3 + \xi_4 = \varepsilon\pounds \\ \xi_1 - \xi_3 + \varepsilon\oslash\xi_4 = -1+\varepsilon\pounds \\ (1+\varepsilon\pounds)\,\xi_1 + \xi_2 = 4+\varepsilon\pounds. \end{cases}$$

First we reduce the flexible system dividing all the coefficients of the system by its pivot which is 2. We then obtain the equivalent reduced flexible system

$$\begin{cases} \xi_1 + \left(1+\varepsilon^2\oslash\right)\xi_2 + \frac{1}{2}\xi_3 + \frac{1}{2}\xi_4 = -1 + \varepsilon\pounds \\ -\frac{1}{2}\xi_1 + \xi_2 + \left(\frac{1}{2}+\varepsilon\oslash\right)\xi_3 + \frac{1}{2}\xi_4 = \varepsilon\pounds \\ \frac{1}{2}\xi_1 - \frac{1}{2}\xi_3 + \varepsilon\oslash\xi_4 = -\frac{1}{2} + \varepsilon\pounds \\ \left(\frac{1}{2}+\varepsilon\pounds\right)\xi_1 + \frac{1}{2}\xi_2 = 2 + \varepsilon\pounds. \end{cases}$$

Let

$$\mathcal{A} = \begin{bmatrix} 1 & 1+\varepsilon^2\oslash & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2}+\varepsilon\oslash & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & \varepsilon\oslash \\ \frac{1}{2}+\varepsilon\pounds & \frac{1}{2} & 0 & 0 \end{bmatrix},$$

$$\mathcal{X} = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} -1+\varepsilon\pounds \\ \varepsilon\pounds \\ -\frac{1}{2}+\varepsilon\pounds \\ 2+\varepsilon\pounds \end{bmatrix}.$$

Then the determinant $\Delta = \det\mathcal{A}$ is zeroless. Indeed, one verifies that $\Delta = \frac{3}{16} + \varepsilon\pounds$. Also $R(\mathcal{A}) = \overline{A}\diagup\Delta = \varepsilon\pounds$, $P(\mathcal{B}) = \underline{B}\diagup\overline{\beta} = \varepsilon\pounds$ and $\Delta\underline{B} = \varepsilon\pounds = \underline{B}$. Hence $R(\mathcal{A}) \subseteq P(\mathcal{B})$, Δ is not an absorber of $\underline{B}$ and $\underline{B} = \overline{B} = \varepsilon\pounds$, so all the conditions of Part 3 of Theorem 4.4 are satisfied.

One has

$$M_{23,34} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash \\ \frac{1}{2}+\varepsilon\pounds & \frac{1}{2} \end{vmatrix} = \varepsilon\pounds,$$

$$M_{24,34} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{2} + \varepsilon^2\oslash,$$

$$M_{34,34} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash \\ -\frac{1}{2} & 1 \end{vmatrix} = \frac{3}{2} + \varepsilon^2\oslash,$$

$$M_{34,14} = \begin{vmatrix} 1+\varepsilon^2\oslash & \frac{1}{2} \\ 1 & \frac{1}{2}+\varepsilon\oslash \end{vmatrix} = \varepsilon\oslash,$$

$$M_{34,24} = \begin{vmatrix} 1 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}+\varepsilon\oslash \end{vmatrix} = \frac{3}{4} + \varepsilon\oslash,$$

$$M_{3,4} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2}+\varepsilon\oslash \\ \frac{1}{2}+\varepsilon\pounds & \frac{1}{2} & 0 \end{vmatrix} = -\frac{3}{8}+\varepsilon\pounds,$$

$$M_{4,1} = \begin{vmatrix} 1+\varepsilon^2\oslash & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2}+\varepsilon\oslash & \frac{1}{2} \\ 0 & -\frac{1}{2} & \varepsilon\oslash \end{vmatrix} = \varepsilon\oslash,$$

$$M_{4,2} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}+\varepsilon\oslash & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \varepsilon\oslash \end{vmatrix} = \frac{3}{8}+\varepsilon\oslash,$$

$$M_{4,3} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \varepsilon\oslash \end{vmatrix} = \varepsilon\oslash,$$

$$M_{4,4} = \begin{vmatrix} 1 & 1+\varepsilon^2\oslash & \frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2}+\varepsilon\oslash \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{vmatrix} = -\frac{3}{4}+\varepsilon\oslash.$$

We may choose

$$\begin{aligned} \overline{m_2} &= m_{34,34} = \frac{3}{2}, \\ \overline{m_1} &= m_{4,4} = -\frac{3}{4}, \\ d &= \frac{3}{16}. \end{aligned}$$

Using the Gauss-operations defined in 5.12, one has $\mathcal{G}\mathcal{A} = \mathcal{G}_7\left(\mathcal{G}_6\left(\ldots\left(\mathcal{G}_1\mathcal{A}\right)\cdots\right)\right)$, with

$$\mathcal{G}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a_{21} & 1 & 0 & 0 \\ -a_{31} & 0 & 0 & 0 \\ -a_{41} & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\overline{m_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{G}_3 = \begin{bmatrix} 1 & -a_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m_{24,34} & 1 & 0 \\ 0 & -m_{23,34} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{G}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{m_2} \diagup \overline{m_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathcal{G}_5 = \begin{bmatrix} 1 & 0 & m_{34,14} \diagup \overline{m_2} & 0 \\ 0 & 1 & -m_{34,24} \diagup \overline{m_2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -m_{3,4} \diagup \overline{m_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 1 \end{bmatrix},$$

$$\mathcal{G}_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overline{m_1} \diagup d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix},$$

$$\mathcal{G}_7 = \begin{bmatrix} 1 & 0 & 0 & -m_{4,1} \diagup \overline{m_1} \\ 0 & 1 & 0 & m_{4,2} \diagup \overline{m_1} \\ 0 & 0 & 1 & -m_{4,3} \diagup \overline{m_1} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So

$$\mathcal{G}_1 \mathcal{A} = \begin{bmatrix} 1 & 1 + \varepsilon^2 \oslash & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} + \varepsilon^2 \oslash & \frac{3}{4} + \varepsilon \oslash & \frac{3}{4} \\ 0 & -\frac{1}{2} + \varepsilon^2 \oslash & -\frac{3}{4} & -\frac{1}{4} + \varepsilon \oslash \\ \varepsilon \pounds & \varepsilon^2 \oslash & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$\mathcal{G}_2 \left(\mathcal{G}_1 \mathcal{A} \right) = \begin{bmatrix} 1 & 1 + \varepsilon^2 \oslash & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 + \varepsilon^2 \oslash & \frac{1}{2} + \varepsilon \oslash & \frac{1}{2} \\ 0 & -\frac{1}{2} + \varepsilon^2 \oslash & -\frac{3}{4} & -\frac{1}{4} + \varepsilon \oslash \\ \varepsilon \pounds & \varepsilon^2 \oslash & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$\mathcal{G}_3 \left(\mathcal{G}_2 \left(\mathcal{G}_1 \mathcal{A} \right) \right) = \begin{bmatrix} 1 & \varepsilon^2 \oslash & \varepsilon \oslash & 0 \\ 0 & 1 + \varepsilon^2 \oslash & \frac{1}{2} + \varepsilon \oslash & \frac{1}{2} \\ 0 & \varepsilon^2 \oslash & -\frac{1}{2} + \varepsilon \oslash & \varepsilon \oslash \\ \varepsilon \pounds & \varepsilon^2 \oslash & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$\mathcal{G}_4 \left(\mathcal{G}_3 \left(\mathcal{G}_2 \left(\mathcal{G}_1 \mathcal{A} \right) \right) \right) = \begin{bmatrix} 1 & \varepsilon^2 \oslash & \varepsilon \oslash & 0 \\ 0 & 1 + \varepsilon^2 \oslash & \frac{1}{2} + \varepsilon \oslash & \frac{1}{2} \\ 0 & \varepsilon^2 \oslash & 1 + \varepsilon \oslash & \varepsilon \oslash \\ \varepsilon \pounds & \varepsilon^2 \oslash & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix},$$

$$\mathcal{G}_5 \left(\mathcal{G}_4 \left(\ldots \left(\mathcal{G}_1 \mathcal{A} \right) \cdots \right) \right) = \begin{bmatrix} 1 & \varepsilon^2 \oslash & \varepsilon \oslash & 0 \\ 0 & 1 + \varepsilon^2 \oslash & \varepsilon \oslash & \frac{1}{2} + \varepsilon \oslash \\ 0 & \varepsilon^2 \oslash & 1 + \varepsilon \oslash & \varepsilon \oslash \\ \varepsilon \pounds & \varepsilon^2 \oslash & \varepsilon \oslash & -\frac{1}{4} + \varepsilon \oslash \end{bmatrix},$$

$$\mathcal{G}_6\left(\mathcal{G}_5\left(\ldots\left(\mathcal{G}_1\mathcal{A}\right)\cdots\right)\right)=\begin{bmatrix}1 & \varepsilon^2\oslash & \varepsilon\oslash & 0\\ 0 & 1+\varepsilon^2\oslash & \varepsilon\oslash & \frac{1}{2}+\varepsilon\oslash\\ 0 & \varepsilon^2\oslash & 1+\varepsilon\oslash & \varepsilon\oslash\\ \varepsilon\pounds & \varepsilon^2\oslash & \varepsilon\oslash & 1+\varepsilon\oslash\end{bmatrix},$$

$$\mathcal{G}\mathcal{A}=\begin{bmatrix}1 & \varepsilon^2\oslash & \varepsilon\oslash & 0\\ \varepsilon\pounds & 1+\varepsilon^2\oslash & \varepsilon\oslash & \varepsilon\oslash\\ 0 & \varepsilon^2\oslash & 1+\varepsilon\oslash & \varepsilon\oslash\\ \varepsilon\pounds & \varepsilon^2\oslash & \varepsilon\oslash & 1+\varepsilon\oslash\end{bmatrix}\subseteq I_4+[\oslash]_{4\times 4}\,.$$

On the other hand,

$$\begin{aligned}\mathcal{G}\mathcal{B} &= \mathcal{G}_7\left(\ldots\left(\mathcal{G}_1\begin{bmatrix}-1+\varepsilon\pounds\\ \varepsilon\pounds\\ -\frac{1}{2}+\varepsilon\pounds\\ 2+\varepsilon\pounds\end{bmatrix}\right)\right)=\mathcal{G}_7\left(\ldots\left(\mathcal{G}_2\begin{bmatrix}-1+\varepsilon\pounds\\ -\frac{1}{2}+\varepsilon\pounds\\ \varepsilon\pounds\\ \frac{5}{2}+\varepsilon\pounds\end{bmatrix}\right)\right)\\ &= \mathcal{G}_7\left(\ldots\left(\mathcal{G}_3\begin{bmatrix}-1+\varepsilon\pounds\\ -\frac{1}{3}+\varepsilon\pounds\\ \varepsilon\pounds\\ \frac{5}{2}+\varepsilon\pounds\end{bmatrix}\right)\right)=\mathcal{G}_7\left(\ldots\left(\mathcal{G}_4\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ -\frac{1}{3}+\varepsilon\pounds\\ -\frac{1}{6}+\varepsilon\pounds\\ \frac{5}{2}+\varepsilon\pounds\end{bmatrix}\right)\right)\\ &= \mathcal{G}_7\left(\mathcal{G}_6\left(\mathcal{G}_5\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ -\frac{1}{3}+\varepsilon\pounds\\ \frac{1}{3}+\varepsilon\pounds\\ \frac{5}{2}+\varepsilon\pounds\end{bmatrix}\right)\right)=\mathcal{G}_7\left(\mathcal{G}_6\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ -\frac{1}{2}+\varepsilon\pounds\\ \frac{1}{3}+\varepsilon\pounds\\ \frac{31}{12}+\varepsilon\pounds\end{bmatrix}\right)\\ &= \mathcal{G}_7\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ -\frac{1}{2}+\varepsilon\pounds\\ \frac{1}{3}+\varepsilon\pounds\\ -\frac{31}{3}+\varepsilon\pounds\end{bmatrix}=\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ \frac{14}{3}+\varepsilon\pounds\\ \frac{1}{3}+\varepsilon\pounds\\ -\frac{31}{3}+\varepsilon\pounds\end{bmatrix}.\end{aligned}$$

So

$$\mathcal{X}=\begin{bmatrix}-\frac{2}{3}+\varepsilon\pounds\\ \frac{14}{3}+\varepsilon\pounds\\ \frac{1}{3}+\varepsilon\pounds\\ -\frac{31}{3}+\varepsilon\pounds\end{bmatrix}$$

represents the external set of all Gauss-solutions. By Theorem 5.14, it matches the Cramer-solution.

5.3 Proof of a Gauss-Jordan elimination theorem with external numbers

First we prove Theorem 5.14 in the case of a 2 by 2 reduced system. This case serves as a guide for the general case for it avoids some of its complications due to the presence of minors of higher order.

5.3.1 The case of 2 by 2 matrices

Definition 5.17 Let $\alpha_{12}, \alpha_{21}, \alpha_{22}, \beta_1, \beta_2, \xi_1, \xi_2 \in \mathbb{E}$. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Consider the reduced non-singular non-homogeneous flexible system of linear equations

$$\left\{ \begin{array}{cccc} (1 + A_{11})\,\xi_1 & + & \alpha_{12}\xi_2 & \subseteq \beta_1 \\ \alpha_{21}\xi_1 & + & \alpha_{22}\xi_2 & \subseteq \beta_2. \end{array} \right. \tag{5.5}$$

Let $d = a_{22} - a_{21}a_{12}$, then $d \neq 0$. The matrices $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ take the form

$$\mathcal{G}_1 = \begin{bmatrix} 1 & 0 \\ -a_{21} & 1 \end{bmatrix}, \mathcal{G}_2 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix}, \mathcal{G}_3 = \begin{bmatrix} 1 & -a_{12} \\ 0 & 1 \end{bmatrix},$$

with $\mathcal{G}\,[.]$ equal to the repeated multiplication of matrices $\mathcal{G}_3(\mathcal{G}_2(\mathcal{G}_1 \cdot [.]))$.

Observe that, with $\mathcal{A} = \begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the matrix $\mathcal{G}_1$ corresponds to the subtraction of a_{21} times the first row of the second row of $\mathcal{A}$, the matrix $\mathcal{G}_2$ divides the second row of $\mathcal{G}_1\mathcal{A}$ by d and the matrix $\mathcal{G}_3$ subtracts the second row a_{12} times of the first row of $\mathcal{G}_2(\mathcal{G}_1\mathcal{A})$. These are the appropriate Gauss-Jordan elimination operations for the matrix $\mathcal{A}$, indeed $\mathcal{G}\mathcal{A} = I_2$ with $\mathcal{G}_3\,(\mathcal{G}_2 \cdot \mathcal{G}_1) = \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & 1 \end{bmatrix}$.

Let $(x, y) \in \mathbb{R}^2$. We recall that $(x, y)^T$ is a *Gauss-solution* of (5.5) if for all choices of representatives of $\alpha_{12}, \alpha_{21}, \alpha_{22}$ and corresponding matrices one has

$$\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.$$

We will assume that $N(\beta_1) = N(\beta_2) \equiv B$. In case Δ is not an absorber of B and $\overline{A}/\Delta \subseteq B/\overline{\beta}$, every element of the solution given by Cramer's rule is a Gauss-solution and vice-versa. This will be shown in the remaining part of this section. We start with some useful properties of multiplication of matrices.

As already observed, because the matrices $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ contain only real numbers, by (2.3) distributivity holds with respect to expressions of the form $a + A$, with $a \in \mathbb{R}$ and $A \in \mathcal{N}$. Hence

$$\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{5.6}$$

Lemma 5.18 *Consider the reduced non-singular non-homogeneous flexible system (5.5). Assume that Δ is not an absorber of B. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Then*

1. $B = B\Delta = B/\Delta$.

2. $\mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}$.

3. *If* $\overline{A}/\Delta \subseteq B/\overline{\beta}$ *one has*

$$\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \subseteq \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}$$

and

$$\left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} B \\ B \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.$$

Proof. *1.* Because (5.5) is a reduced non-singular flexible system,

$$0 < |\Delta| \leqslant 2 + \oslash \leqslant 3.$$

Moreover, Δ is not an absorber of B. So

$$B \subseteq \Delta B \subseteq 3B = B.$$

Hence $B = B\Delta$. Moreover $B \diagup \Delta = (B\Delta)/\Delta = B(\Delta/\Delta) = B$, since $\Delta/\Delta \subseteq 1 + \oslash$.

2. Firstly, since $|a_{21}| \leqslant 1$, one has

$$\begin{aligned} \mathcal{G}_1 \begin{bmatrix} B \\ B \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -a_{21} & 1 \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \\ &= \begin{bmatrix} B \\ a_{21}B + B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}. \end{aligned}$$

Secondly, by Part 1,

$$\begin{aligned} \mathcal{G}_2 \begin{bmatrix} B \\ B \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \\ &= \begin{bmatrix} B \\ \frac{B}{d} \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}. \end{aligned}$$

Thirdly, since $|a_{12}| \leqslant 1$,

$$\begin{aligned} \mathcal{G}_3 \begin{bmatrix} B \\ B \end{bmatrix} &= \begin{bmatrix} 1 & -a_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B \\ B \end{bmatrix} \\ &= \begin{bmatrix} B + a_{12}B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}. \end{aligned}$$

Hence

$$\mathcal{G} \begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}_3 \left(\mathcal{G}_2 \left(\mathcal{G}_1 \cdot \begin{bmatrix} B \\ B \end{bmatrix} \right) \right) = \begin{bmatrix} B \\ B \end{bmatrix}.$$

3. If $\overline{A} \diagup \Delta \subseteq B \diagup \overline{\beta}$, by Part 1 one has $\overline{A} \subseteq B \diagup \overline{\beta}$. Then, because for all $i, j \in \{1, 2\}$, $A_{ij} \subseteq \overline{A} \subseteq B \diagup \overline{\beta}$, using formula (5.1) and Part 2, one obtains, whenever

b is a representative of $\overline{\beta}$

$$\begin{aligned}
\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &\subseteq \mathcal{G}\begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix} \\
&= \mathcal{G}\begin{bmatrix} B/b & B/b \\ B/b & B/b \end{bmatrix} = \frac{1}{b}\mathcal{G}\begin{bmatrix} B & B \\ B & B \end{bmatrix} \\
&= \frac{1}{b}\begin{bmatrix} B & B \\ B & B \end{bmatrix} = \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}.
\end{aligned}$$

Moreover, also using Lemma 2.5 and Part 2,

$$\begin{aligned}
\left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} B \\ B \end{bmatrix} &\subseteq \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \\
&\subseteq \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} B \\ B \end{bmatrix} \\
&\subseteq \begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}. \qquad \blacksquare
\end{aligned}$$

We also need a property on the order of magnitude of the entries of a matrix with respect to its determinant.

Lemma 5.19 *Let* $\mathcal{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ *be the matrix of coefficients of the reduced non-singular flexible system (5.5) and* $\Delta = \det \mathcal{A}$. *Then* $|\alpha_{12}| > \oslash\Delta$ *or* $|\alpha_{22}| > \oslash\Delta$.

Proof. One has $\Delta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$, with $|\alpha_{ij}| \leqslant 1 + \oslash$ for all $i, j \in \{1, 2\}$. Suppose that $\alpha_{12} \subseteq \oslash\Delta$ and $\alpha_{22} \subseteq \oslash\Delta$. Then

$$\alpha_{11}\alpha_{22} \subseteq (1 + \oslash) \oslash \Delta = \oslash\Delta$$

and

$$\alpha_{12}\alpha_{21} \subseteq \oslash (1 + \oslash) \Delta = \oslash\Delta.$$

So $\Delta \subseteq \oslash\Delta$, which is absurd because Δ is zeroless. Hence $|\alpha_{12}| > \oslash\Delta$ or $|\alpha_{22}| > \oslash\Delta$. $\blacksquare$

The next two propositions yield a lower bound on the uncertainty of Cramer-solutions and an upper bound on the uncertainty of Gauss-solutions.

Proposition 5.20 *Consider the reduced flexible system* (5.5). *Assume that* Δ *is not an absorber of* B *and that* $\overline{A}/\Delta \subseteq B/\overline{\beta}$. *Then*

$$N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B.$$

Proof. By formula (4.5), $N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) \subseteq B$ and $N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) \subseteq B$. On the other hand one has

$$\begin{aligned} a_{22}B + a_{12}B &\subseteq (a_{22}B + b_1A_{22} + BA_{22}) + (a_{12}B + b_2A_{12} + BA_{12}) \\ &= N\left(\det \begin{bmatrix} b_1 + B & a_{12} + A_{12} \\ b_2 + B & a_{22} + A_{22} \end{bmatrix}\right) = N\left(\det \mathcal{M}_1\right). \end{aligned}$$

By Lemma 5.19, $|\alpha_{12}| > \oslash\Delta$ or $|\alpha_{21}| > \oslash\Delta$. So $a_{22} = c_1 d$, with $|c_1| > \oslash$, or $a_{12} = c_2 d$, with $|c_2| > \oslash$. Using Part 1 of Lemma 5.18, we find $a_{22}B = c_1 dB = c_1 B \supseteq B$ or $a_{12}B = c_2 dB = c_2 B \supseteq B$. Therefore $B \subseteq a_{22}B + a_{12}B \subseteq N\left(\det \mathcal{M}_1\right)$. Hence

$$\frac{B}{\Delta} \subseteq \frac{N\left(\det \mathcal{M}_1\right)}{\Delta} \subseteq N\left(\frac{\det \mathcal{M}_1}{\Delta}\right).$$

Again by Part 1 of Lemma 5.18 one has $B = \frac{B}{\Delta}$. So $B \subseteq N\left(\frac{\det \mathcal{M}_1}{\Delta}\right)$ and we conclude that $N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = B$.

The proof is the same for $N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B$. ∎

Proposition 5.21 *Consider the reduced non-singular non-homogeneous flexible system of linear equations* (5.5). *Assume that* Δ *is not an absorber of* B *and that* $\overline{A}/\Delta \subseteq B/\overline{\beta}$. *Let* $x_1,, x_2, y_1, y_2 \in \mathbb{R}$ *such that* $(x_1, x_2)^T$ *and* $(y_1, y_2)^T$ *are Gauss-solutions of* (5.5). *Let* $u_1 = x_1 - y_1$ *and* $u_2 = x_2 - y_2$. *Then* $u_1 \in B$ *and* $u_2 \in B$.

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Then

$$\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} B \\ B \end{bmatrix}, \tag{5.7}$$

for, using Part 2 of Lemma 5.18,

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\subseteq\ &\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
\subseteq\ &\mathcal{G}\begin{bmatrix} b_1+B \\ b_2+B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1+B \\ b_2+B \end{bmatrix} \\
=\ &\mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} \\
=\ &\begin{bmatrix} B \\ B \end{bmatrix} - \begin{bmatrix} B \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix}.
\end{aligned}$$

Also

$$\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{5.8}$$

Indeed, by distributivity, Part 3 of Lemma 5.18 and Lemma 2.5

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
=\ &\left(\mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\subseteq\ &\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} B/\overline{\beta} & B/\overline{\beta} \\ B/\overline{\beta} & B/\overline{\beta} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
\subseteq\ &\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\end{aligned}$$

Assume $(u_1, u_2) \in \mathbb{R}^2$ such that $(u_1, u_2)^T$ satisfies

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \oslash & \oslash \\ \oslash & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \subseteq \begin{bmatrix} B \\ B \end{bmatrix}. \tag{5.9}$$

Then

$$\begin{cases} u_1 + \oslash u_1 + \oslash u_2 \subseteq B \\ u_2 + \oslash u_1 + \oslash u_2 \subseteq B. \end{cases} \tag{5.10}$$

Suppose first that $\max(|u_1|, |u_2|) = |u_1|$. So $u_1 + \oslash u_1 + \oslash u_2 = u_1 + \oslash u_1 = (1 + \oslash) u_1$. If $u_1 \notin B$, also $u_1/2 \notin B$. Hence $|u_1 + \oslash u_1 + \oslash u_2| > |u_1|/2 \notin B$, which contradicts the first equation of system (5.10). Therefore $u_1 \in B$ and also $u_2 \in B$. The case that $\max(|u_1|, |u_2|) = |u_2|$ is analogous. Hence all solutions $(u_1, u_2)^T$ of (5.9) satisfy $u_1 \in B$ and $u_2 \in B$. By (5.8) all solutions of (5.7) satisfy (5.9). Hence all solutions of (5.7) satisfy $u_1 \in B$ and $u_2 \in B$. ■

By Part 3 of Theorem 4.4, if $\triangle$ is not an absorber of B and $\overline{A}/\Delta \subseteq B/\overline{\beta}$, a Cramer-solution of the system (5.5) is a maximal solution. We show now that under these conditions any element of this solution is a Gauss-solution.

Theorem 5.22 *Assume that $\triangle$ is not an absorber of B and that $\overline{A}/\Delta \subseteq B/\overline{\beta}$. Let $(x,y)^T \in \left(\frac{\det \mathcal{M}_1}{\Delta}, \frac{\det \mathcal{M}_2}{\Delta}\right)^T$. Then $(x,y)^T$ is a Gauss-solution of* (5.5)*.*

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Choose $b_1 \in \beta_1$ and $b_2 \in \beta_2$ and let $b = \max(|b_1|, |b_2|)$. Put $d_1 = b_1 a_{22} - b_2 a_{12}$, $d_2 = b_2 - b_1 a_{21}$ and $d = a_{22} - a_{12}a_{21}$. One has $|d_1| \leqslant 3b$ and $|d_2| \leqslant 3b$.

We assume first that $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \frac{d_2}{d} \end{bmatrix}$. Then

$$\left(\mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \frac{d_2}{d} \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Now we prove that

$$\left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.$$

Indeed, using Lemma 5.18, one obtains that

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} \\
\subseteq\ & \begin{bmatrix} B/b & B/b \\ B/b & B/b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{B}{b}x + \frac{B}{b}y \\ \frac{B}{b}x + \frac{B}{b}y \end{bmatrix} \\
=\ & \begin{bmatrix} \frac{B}{b}\frac{d_1}{d} + \frac{B}{b}\frac{d_2}{d} \\ \frac{B}{b}\frac{d_1}{d} + \frac{B}{b}\frac{d_2}{d} \end{bmatrix} \subseteq \begin{bmatrix} \frac{B}{b}\frac{b}{d} + \frac{B}{b}\frac{b}{d} \\ \frac{B}{b}\frac{b}{d} + \frac{B}{b}\frac{b}{d} \end{bmatrix} \\
=\ & \begin{bmatrix} \frac{B}{\Delta} \\ \frac{B}{\Delta} \end{bmatrix} = \begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix}.
\end{aligned}$$

Then it follows by distributivity that

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} \\
=\ & \left(\mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1 + B \\ b_2 + B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.
\end{aligned}$$

Hence $(x, y)^T$ is an admissable solution of (5.5).

Finally, let $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \begin{bmatrix} \frac{\det \mathcal{M}_1}{\triangle} \\ \frac{\det \mathcal{M}_2}{\triangle} \end{bmatrix}$ be arbitrary. By Proposition 5.20 one has

$$N\left(\frac{\det \mathcal{M}_1}{\Delta}\right) = N\left(\frac{\det \mathcal{M}_2}{\Delta}\right) = B.$$

So $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}$. Then by distributivity and Lemma 5.18

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} x' \\ y' \end{bmatrix} \\
\subseteq\ & \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\right)\begin{bmatrix} B \\ B \end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)\begin{bmatrix} B \\ B \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right)\begin{bmatrix} B \\ B \end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} \\
=\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}.
\end{aligned}$$

Hence $(x', y')^T$ is also an admissable solution of (5.5). To complete the proof that $(x', y')^T$ is a Gauss-solution, we observe that the previous calculations do not depend on the choice of representatives, except for the entry $1, 1$, where we made the particular choice of the a representative 1. Here the 2 by 2 case does not lead to a particular simplification, and we refer to the final part of the proof of the general case (Theorem 5.36) ■

Next theorem is a converse to Theorem 5.22. Under the usual conditions, a Gauss-solution must be an element of the Cramer-solution.

Theorem 5.23 *Assume that* $\triangle$ *is not an absorber of* B *and that* $\overline{A}/\Delta \subseteq B/\overline{\beta}$. *Let* $(x, y)^T$ *be a Gauss-solution of* (5.5). *Then* $(x, y)^T$ *satisfies* (5.5), *in fact*

$$(x, y)^T \in \left(\frac{\det \mathcal{M}_1}{\Delta}, \frac{\det \mathcal{M}_2}{\Delta} \right)^T .$$

Proof. Let $a_{12} \in \alpha_{12}, a_{21} \in \alpha_{21}$ and $a_{22} \in \alpha_{22}$. Choose $b_1 \in \beta_1$ and $b_2 \in \beta_2$ and let $b = \max(|b_1|, |b_2|)$. Put $d_1 = b_1 a_{22} - b_2 a_{12}$, $d_2 = b_2 - b_1 a_{21}$ and $d = a_{22} - a_{12} a_{21}$. It follows from Theorem 5.22 that $(x, y)^T = \left(\frac{d_1}{d}, \frac{d_2}{d} \right)^T$ is a Gauss-solution, and it clearly satisfies (5.5). Let $(x', y')^T$ be an arbitrary Gauss-solution of (5.5). By Propositions 5.21 and 5.20 it holds that $x' \in \frac{d_1}{d} + B = \frac{\det \mathcal{M}_1}{\triangle}$ and $y' \in \frac{d_2}{d} + B = \frac{\det \mathcal{M}_2}{\triangle}$. Then it follows from Part 3 of Theorem 4.4 that $(x, y)^T$ satisfies (5.5). ■

We end the 2 by 2 case with the next theorem.

Theorem 5.24 *Assume that* $\triangle$ *is not an absorber of* B *and that* $\overline{A}/\Delta \subseteq B/\overline{\beta}$. *Then the Cramer-solution of the reduced flexible system* (5.5) *equals the external set of all Gauss-solutions.*

Proof. By Theorem 5.22 and 5.23 it holds that $\left(\frac{\det \mathcal{M}_1}{\Delta}, \frac{\det \mathcal{M}_2}{\Delta}\right)^T$ is equal to the external set of all Gauss-solutions. ■

5.3.2 The case of n by n matrices

We will assume that system (3.1) is reduced. If the system is not reduced, we start by dividing all coefficients of matrix $\mathcal{A}$ by some "largest" representative $\overline{a}$ of the entries of $\mathcal{A}$. We obtain thereby an equivalent system of (3.1) which is reduced and still verifies all the conditions of Part 3 of Theorem 4.4, as it was shown in the proof of Theorem 4.4.

Remark 5.25 Since system (3.1) is non-homogeneous, one has $B/\overline{\beta} \subseteq \oslash$ by Lemma 2.5. So condition (iii) implies that $\overline{A}/\Delta \subseteq \oslash$. Hence $\overline{A} \subseteq \oslash\Delta$.

The general case needs some estimations about the order of magnitude of the determinant of the matrix of coefficients and its minors. We start by showing a useful relation between Δ and the determinant of the matrix obtained by adding c times the t^{th} row to the k^{th} row of $\mathcal{A}$, where $k \neq t$.

Proposition 5.26 *Let $n \in \mathbb{N}$ be standard, $c \in \mathbb{R}$, with $|c| \leqslant 1$, and $k, t \in \{1, ..., n\}$ with $k \neq t$. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced non-singular matrix, with $\alpha_{ij} \in \mathbb{E}$ and $\Delta = \det \mathcal{A}$, and let $\mathcal{A}' = \left[\alpha'_{ij}\right]_{n\times n}$ such that*

$$\alpha'_{ij} = \begin{cases} \alpha_{ij} & , i \neq k \\ \alpha_{ij} + c\alpha_{tj} & , i = k \end{cases}$$

and $\Delta' = \det \mathcal{A}'$, for $1 \leqslant i, j \leqslant n$. Then

$$\Delta' \subseteq (1 + \oslash)\,\Delta.$$

Proof. One has

$$\mathcal{A}' = \begin{bmatrix} \alpha_{11} & ... & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{t1} & ... & \alpha_{tn} \\ \vdots & & \vdots \\ \alpha_{k1} + c\alpha_{t1} & ... & \alpha_{kn} + c\alpha_{tn} \\ \vdots & & \vdots \\ \alpha_{n1} & ... & \alpha_{nn} \end{bmatrix}.$$

Let S_n denote the set of all permutations of the set $\{1, ..., n\}$ and $\sigma = (p_1, ..., p_n) \in S_n$. So, using subdistributivity,

$$\begin{aligned} \Delta' &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot (\alpha_{kp_k} + c\alpha_{tp_k}) \cdot ... \cdot \alpha_{np_n} \\ &\subseteq \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{kp_k} \cdot ... \cdot \alpha_{np_n} + \\ &\quad c \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n}. \end{aligned}$$

Now $\alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n} = \alpha_{1p_1} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{np_n}$ and they appear with opposite signs in the sum of permutations. So

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n} = N(\alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n}).$$

Since $|\alpha_{ij}| \leqslant 1 + \oslash$ and $N(\alpha_{ij}) \subseteq \overline{A}$, by Lemma 2.7,

$$N(\alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n}) \subseteq N\left(\left(1 + \overline{A}\right)^n\right) = \overline{A}.$$

Hence, by Remark 5.25,

$$\begin{aligned} \Delta' &\subseteq \Delta + cN(\alpha_{1p_1} \cdot ... \cdot \alpha_{tp_t} \cdot ... \cdot \alpha_{tp_k} \cdot ... \cdot \alpha_{np_n}) \\ &\subseteq \Delta + c\overline{A} \subseteq \Delta + \oslash\Delta = (1 + \oslash)\,\Delta. \qquad \blacksquare \end{aligned}$$

Proposition 5.27 *Let $n \in \mathbb{N}$ be standard. Let $\mathcal{A} = [\alpha_{ij}]_{n\times n}$ be a reduced non-singular matrix, with $\alpha_{ij} \in \mathbb{E}$, $\alpha_{11} = 1 + A_{11}$ and $\Delta = \det \mathcal{A}$. Let $\mathcal{A}' = \left[\alpha'_{ij}\right]_{n\times n}$ be such that*

$$\alpha'_{ij} = \begin{cases} \alpha_{ij} & , i = 1 \\ \alpha_{ij} - \alpha_{i1}\alpha_{1j} & , i \neq 1 \end{cases}$$

and $\Delta' = \det \mathcal{A}'$, for $1 \leqslant i, j \leqslant n$. Then

$$\Delta_1 \equiv \det \left[\mathcal{A}'\right]_{1,1} \subseteq \Delta'.$$

Proof. Put

$$\mathcal{A}' = \begin{bmatrix} 1 + A_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ A'_{21} & M_{3\cdots n,3\cdots n} & \cdots & M_{3\cdots n,2\cdots(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ A'_{n1} & M_{2\cdots(n-1),3\cdots n} & \cdots & M_{2\cdots(n-1),2\cdots(n-1)} \end{bmatrix}$$

and

$$\left[\mathcal{A}'\right]_{1,1} = \begin{bmatrix} M_{3\cdots n,3\cdots n} & \cdots & M_{3\cdots n,2\cdots(n-1)} \\ \vdots & \ddots & \vdots \\ M_{2\cdots(n-1),3\cdots n} & \cdots & M_{2\cdots(n-1),2\cdots(n-1)} \end{bmatrix}.$$

By Remark 5.25, for all $i \in \{2, \ldots, n\}$,

$$A'_{i1} = \max\left(A_{i1}, \alpha_{i1}A_{11}\right) \subseteq \overline{A} \subseteq \oslash\Delta \subseteq \oslash.$$

By Lemma 5.8, for $2 \leqslant i_1 < \cdots < i_{n-2} \leqslant n$, $2 \leqslant j_1 < \cdots < j_{n-2} \leqslant n$,

$$M_{i_1\cdots i_{n-2},j_1\cdots j_{n-2}} \leqslant \left|\overline{M_{n-2}}\right| < 2! + 1 = 3.$$

Let $\Delta' = \det \mathcal{A}'$. Since any expansion of Δ' in cofactors is contained in Δ', one has

$$(1 + A_{11}) \det \left[\mathcal{A}'\right]_{1,1} + A'_{21} \det \left[\mathcal{A}'\right]_{2,1} + \cdots + A'_{n1} \det \left[\mathcal{A}'\right]_{n,1} \subseteq \Delta',$$

where, for all $i \in \{2, \ldots, n\}$,

$$\left|\det \left[\mathcal{A}'\right]_{i,1}\right| \leqslant (n-1)!\,(1 + \oslash)\left|\overline{M_{n-2}}\right|^{n-2} < (n-1)!3^{n-2} \in @.$$

Hence

$$\begin{aligned}\Delta_1 &= \det\left[\mathcal{A}'\right]_{1,1} \\ &\subseteq (1+A_{11})\det\left[\mathcal{A}'\right]_{1,1} + A'_{21}\det\left[\mathcal{A}'\right]_{2,1} + \cdots + A'_{n1}\det\left[\mathcal{A}'\right]_{n,1} \\ &\subseteq \Delta'. \qquad \blacksquare\end{aligned}$$

Lemma 5.28 *Consider the flexible system* (3.1)*. Then for all* $k \in \{1, \ldots, n-1\}$,

$$N\left(\overline{M_k}\right) \subseteq \overline{A}.$$

Proof. Let $k \in \{1, \ldots, n-1\}$ be arbitrary. One has, for $1 \leqslant i, j \leqslant n-k$,

$$[\mathcal{A}]_{(n-k+1)\cdots n,(n-k+1)\cdots n} = [\alpha_{ij}]_{(n-k)\times(n-k)}\,.$$

Also $|\alpha_{ij}| \leqslant 1 + \oslash$ and $N(\alpha_{ij}) \subseteq \overline{A}$ for all $i, j \in \{1, \ldots, n\}$. Let S_{n-k} denote the set of all permutations of the set $\{1, \ldots, n-k\}$ and $\sigma = (p_1, \ldots, p_{n-k}) \in S_{n-k}$. Then, using Lemma 2.7,

$$\begin{aligned}N\left(\overline{M_k}\right) &= N\left(M_{(n-k+1)\cdots n,(n-k+1)\cdots n}\right) \\ &= N\left(\det[\mathcal{A}]_{(n-k+1)\cdots n,(n-k+1)\cdots n}\right) \\ &= N\left(\sum_{\sigma\in S_{n-k}} \operatorname{sgn}(\sigma)\,\alpha_{1p_1}\cdots\alpha_{(n-k)p_{n-k}}\right) \\ &= \sum_{\sigma\in S_{n-k}} N\left(\alpha_{1p_1}\cdots\alpha_{(n-k)p_{n-k}}\right) \subseteq \sum_{\sigma\in S_{n-k}} N\left(\left(1+\overline{A}\right)^{n-k}\right) \\ &= \sum_{\sigma\in S_{n-k}} \overline{A} = (n-k)!\overline{A} = \overline{A}. \qquad \blacksquare\end{aligned}$$

Since system (3.1) is reduced and non-singular, the next estimation on the order of magnitude of real part of the determinant of the matrix of coefficients is straightforward.

Lemma 5.29 *Consider the flexible system* (3.1). *Then*

$$\oslash\Delta < |d| \leqslant \pounds.$$

Proof. Since $\Delta = \det \mathcal{A} = d + N(\Delta)$ is zeroless, one has $|d| > \oslash\Delta$.

On the other hand, because $\mathcal{A}$ is a reduced matrix, $\overline{\alpha} = \max_{1\leqslant i,j\leqslant n} |\alpha_{ij}| \leqslant 1 + \oslash$. Let S_n denote the set of all permutations of the set $\{1, \ldots, n\}$ and $\sigma = (p_1, \ldots, p_n) \in S_n$. Hence

$$\begin{aligned} \Delta &= \det \mathcal{A} = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdots \alpha_{np_n} \leqslant \left| \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1} \cdots \alpha_{np_n} \right| \\ &\leqslant \sum_{\sigma\in S_n} |\alpha_{1p_1}| \cdots |\alpha_{np_n}| \leqslant \sum_{\sigma\in S_n} \overline{\alpha}^n \leqslant n!\,(1+\oslash) < 2n! \in @. \end{aligned}$$

So $|d| \leqslant \pounds$. ■

In fact, the maximum of all minors of the matrix of coefficients have the same upper and lower bound.

Theorem 5.30 *Consider the flexible system* (3.1). *Then, for all* $k \in \{1, \ldots, n-1\}$,

$$\oslash\Delta < |\overline{m_k}| \leqslant \pounds.$$

Proof. We will use external induction, starting with the smallest minors. By Remark 5.11 one has $\overline{M_{n-1}} = M_{2\cdots n, 2\cdots n} = \alpha_{11} = 1 + A_{11}$, where $A_{11} \subseteq \overline{A} \subseteq \oslash\Delta$ by Remark 5.25. So $|\overline{m_{n-1}}| = 1$ and therefore

$$\oslash\Delta < |\overline{m_{n-1}}| \leqslant \pounds.$$

We treat separately the cases of the $(n-2)^{th}$ and $(n-3)^{th}$ minors. The case of the $(n-3)^{th}$ minor suggest how to treat the general case.

Let

$$\mathcal{A}' = \mathcal{G}_1\mathcal{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix} \mathcal{A}.$$

Hence $\mathcal{A}'$ is of the form

$$\mathcal{A}' = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha'_{21} & \alpha'_{22} & \cdots & \alpha'_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha'_{n1} & \alpha'_{n2} & \cdots & \alpha'_{nn} \end{bmatrix},$$

where, for all $i \in \{2, \ldots, n\}$,

$$\alpha'_{i1} = \max\left(A_{i1}, \alpha_{i1}A_{11}\right) \subseteq \overline{A} \subseteq \oslash\Delta$$

and, for all $i, j \in \{2, \ldots, n\}$,

$$\left|\alpha'_{ij}\right| \leqslant \left|\overline{M_{n-2}}\right|.$$

Suppose that $|\overline{m_{n-2}}| \in \oslash\Delta$. One has $N\left(\overline{M_{n-2}}\right) \subseteq \overline{A} \subseteq \oslash\Delta$ by Lemma 5.28. So $\overline{M_{n-2}} = \overline{m_{n-2}} + N\left(\overline{M_{n-2}}\right) \subseteq \oslash\Delta$ which implies that, for all $i, j \in \{2, \ldots, n\}$,

$$\alpha'_{ij} \subseteq \oslash\Delta.$$

Let S_n be the set of all permutations of $\{1, \ldots, n\}$ and $\sigma = (p_1, \ldots, p_n)$ a permutation of S_n. Then

$$\Delta' = \det \mathcal{A}' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \alpha_{1p_1}\alpha'_{2p_2} \cdots \alpha'_{np_n} \subseteq \oslash\Delta.$$

By Proposition 5.26,

$$\Delta' \subseteq (1 + \oslash)\,\Delta.$$

So $\Delta' \subseteq \Delta \cap \oslash\Delta$, which is absurd by Lemma 2.8 because Δ is zeroless. Therefore $|\overline{m_{n-2}}| > \oslash\Delta$.

Moreover, $\left|\overline{M_{n-2}}\right| < 2! + 1 = 3 \in @$ by Lemma 5.8. Hence $|\overline{m_{n-2}}| \leqslant \pounds$ and one concludes that

$$\oslash\Delta < |\overline{m_{n-2}}| \leqslant \pounds. \tag{5.11}$$

Now let

$$\begin{aligned}
\mathcal{A}'' &= [\mathcal{G}_3]_{1,1}\left([\mathcal{G}_2]_{1,1}\left[\mathcal{A}'\right]_{1,1}\right)\\
&= \begin{bmatrix} 1 & 0 & \cdots & 0\\ -m_{24\cdots n,3\cdots n} & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -m_{2\cdots(n-1),3\cdots n} & 0 & \cdots & 1 \end{bmatrix}\left(\begin{bmatrix} 1/\overline{m_{n-2}} & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1 \end{bmatrix}\left[\mathcal{A}'\right]_{1,1}\right).
\end{aligned}$$

By Remark 5.11, one has $\overline{M_{n-2}} = M_{3\cdots n,3\cdots n} = \alpha'_{22}$. By (5.11) one has $\frac{A'_{22}}{\alpha'_{22}} = A'_{22}$. Then we obtain that $\mathcal{A}''$ is a $(n-1)\times(n-1)$ matrix of the form

$$\mathcal{A}'' = \begin{bmatrix} 1 + A'_{22} & \alpha'_{23}/\overline{m_{n-2}} & \cdots & \alpha'_{2n}/\overline{m_{n-2}}\\ \alpha''_{32} & \alpha''_{33} & \cdots & \alpha''_{3n}\\ \vdots & \vdots & \ddots & \vdots\\ \alpha''_{n2} & \alpha''_{n3} & \cdots & \alpha''_{nn} \end{bmatrix};$$

here, for all $i \in \{3, \ldots, n\}$,

$$\alpha''_{i2} = \max\left(A'_{i2}, \alpha'_{i2}A'_{22}\right) \subseteq \overline{A} \subseteq \oslash\Delta$$

and, for all $i, j \in \{3, \ldots, n\}$,

$$\left|\alpha''_{ij}\right| \leqslant \left|\overline{M_{n-3}}\right|.$$

Suppose that $|\overline{m_{n-3}}| \in \oslash\Delta$. By Lemma 5.28 one has $N\left(\overline{M_{n-3}}\right) \subseteq \overline{A} \subseteq \oslash\Delta$. Then $\overline{M_{n-3}} = \overline{m_{n-3}} + N\left(\overline{M_{n-3}}\right) \subseteq \oslash\Delta$ and one concludes that, for all $i, j \in \{3, \ldots, n\}$,

$$\alpha''_{ij} \subseteq \oslash\Delta.$$

Let $\Delta_1 = \det[\mathcal{A}']_{1,1}$ and $\Delta'' = \det\mathcal{A}''$. One has $\Delta_1 \subseteq \Delta'$ by Proposition 5.27, $\det[\mathcal{G}_3]_{1,1} = 1$ and $\det[\mathcal{G}_2]_{1,1} = 1\diagup\overline{m_{n-2}}$. Let S_{n-1} be the set of all permutations of $\{2,\ldots,n\}$ and $\sigma = (p_2,\ldots,p_n) \in S_{n-1}$. So, by (5.11),

$$\begin{aligned}\Delta'' &= (1\diagup\overline{m_{n-2}}) \sum_{\sigma\in S_{n-1}} \operatorname{sgn}(\sigma)\, \alpha'_{2p_2}\alpha''_{3p_3}\cdots\alpha''_{np_n} \\ &\subseteq \frac{@}{\Delta}(\oslash\Delta)^{n-2} \subseteq \oslash.\end{aligned}$$

Also, using Proposition 5.26 and (5.11),

$$\begin{aligned}\Delta'' &= (1\diagup\overline{m_{n-2}})\,\Delta_1 \subseteq (1\diagup\overline{m_{n-2}})\,\Delta' \\ &\subseteq (1+\oslash)\,(1\diagup\overline{m_{n-2}})\,\Delta \subseteq @.\end{aligned}$$

So $\Delta'' \subseteq \oslash \cap @ = \emptyset$ which is absurd. Therefore $|\overline{m_{n-3}}| > \oslash\Delta$.

On the other hand, $\left|\overline{M_{n-3}}\right| < 3!+1 = 7 \in @$ by Lemma 5.8. Hence $|\overline{m_{n-3}}| \leqslant \pounds$ which implies that

$$\oslash\Delta < |\overline{m_{n-3}}| \leqslant \pounds.$$

Finally, let $k \in \{1,\ldots,n-1\}$ be arbitrary. Assume that $\mathcal{A}^{(k-1)}$, $\Delta^{(k-1)} = \det\mathcal{A}^{(k-1)}$ and $\Delta_{k-1} = \det\left[\mathcal{A}^{(k-1)}\right]_{1,1}$ are defined. By Proposition 5.27 one has $\Delta_{k-1} \subseteq \Delta^{(k-1)}$. Also by the induction hypothesis

$$\oslash\Delta < |\overline{m_{n-k}}| \leqslant \pounds. \tag{5.12}$$

Let

$$\mathcal{A}^{(k)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -m_{k(k+2)\cdots n,(k+1)\cdots n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{k\cdots(n-1),(k+1)\cdots n} & 0 & \cdots & 1 \end{bmatrix} \left(\begin{bmatrix} 1\diagup\overline{m_{n-k}} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \left[\mathcal{A}^{(k-1)}\right]_{1,1} \right).$$

By Remark 5.11, one has $\overline{M_{n-k}} = M_{(k+1)\cdots n,(k+1)\cdots n} = \alpha_{kk}^{(k-1)}$. By (5.12) one has $\frac{A_{kk}^{(k-1)}}{a_{kk}^{(k-1)}} = A_{kk}^{(k-1)}$. By analogy, $\mathcal{A}^{(k)}$ is a $(n-k+1)\times(n-k+1)$ matrix of the form

$$\mathcal{A}^{(k)} = \begin{bmatrix} 1 + A_{kk}^{(k-1)} & \alpha_{k(k+1)}^{(k-1)} \diagup \overline{m_{n-k}} & \cdots & \alpha_{kn}^{(k-1)} \diagup \overline{m_{n-k}} \\ \alpha_{(k+1)k}^{(k)} & \alpha_{(k+1)(k+1)}^{(k)} & \cdots & \alpha_{(k+1)n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{nk}^{(k)} & \alpha_{n(k+1)}^{(k)} & \cdots & \alpha_{nn}^{(k)} \end{bmatrix};$$

here, for all $i \in \{k+1, \ldots, n\}$,

$$\alpha_{ik}^{(k)} = \max\left(A_{ik}^{(k-1)}, \alpha_{ik}^{(k-1)} A_{kk}^{(k-1)}\right) \subseteq \overline{A} \subseteq \oslash\Delta$$

and, for all $i, j \in \{k+1, \ldots, n\}$

$$\left|\alpha_{ij}^{(k)}\right| \leqslant \left|\overline{M_{n-k-1}}\right|.$$

Suppose that $|\overline{m_{n-k-1}}| \in \oslash\Delta$. One has $N\left(\overline{M_{n-k-1}}\right) \subseteq \overline{A} \subseteq \oslash\Delta$ by Lemma 5.28 and so $\overline{M_{n-k-1}} = \overline{m_{n-k-1}} + N\left(\overline{M_{n-k-1}}\right) \subseteq \oslash\Delta$. Hence, for all $i, j \in \{k+1, \ldots, n\}$,

$$\alpha_{ij}^{(k)} \subseteq \oslash\Delta.$$

Let $\Delta^{(k)} = \det \mathcal{A}^{(k)}$. Let S_{n-k} be the set of all permutations of $\{k, \ldots, n\}$ and $\sigma = (p_k, \ldots, p_n)$ a permutation S_{n-k}. Then

$$\begin{aligned} \Delta^{(k)} &= (1 \diagup \overline{m_{n-k}}) \sum_{\sigma \in S_{n-k}} \operatorname{sgn}(\sigma)\, \alpha_{kp_k}^{(k-1)} \alpha_{(k+1)p_{k+1}}^{(k)} \cdots \alpha_{np_n}^{(k)} \\ &\subseteq \frac{@}{\Delta} (\oslash\Delta)^{n-k} \subseteq \oslash. \end{aligned}$$

Also, by Proposition 5.26 and (5.12),

$$\begin{aligned}\Delta^{(k)} &= (1/\overline{m_{n-k}})\,\Delta_{k-1} \subseteq (1/\overline{m_{n-k}})\,\Delta^{(k-1)} \\ &\subseteq (1/\overline{m_{n-k}})\,(1/\overline{m_{n-k-1}})\,\Delta^{(k-2)} \\ &\subseteq (1/\overline{m_{n-k}})\cdots(1/\overline{m_{n-2}})\,\Delta_1 \\ &\subseteq \left[@, (1+\oslash)\,\frac{@}{\Delta^{k-2}}\right].\end{aligned}$$

So $\Delta^{(k)} \subseteq \oslash \cap \left[@, (1+\oslash)\,\frac{@}{\Delta^{k-2}}\right] = \emptyset$, which is absurd. Hence $|\overline{m_{n-k-1}}| > \oslash\Delta$.

Moreover, by Lemma 5.8 one has $\left|\overline{M_{n-k-1}}\right| < (k+1)! + 1 \in @$. Therefore $|\overline{m_{n-k-1}}| \leqslant £$ and so

$$\oslash\Delta < |\overline{m_{n-k-1}}| \leqslant £.$$

Using external induction one concludes that, for all $k \in \{1, \dots, n-1\}$,

$$\oslash\Delta < |\overline{m_k}| \leqslant £. \qquad \blacksquare$$

Lemma 5.31 *Consider the flexible system* (3.1). *Then*

1. $B = B\Delta = B/\Delta$.

2. $\left(\frac{\overline{m_1}}{d}\right) B = B$ *and* $\left(\frac{\overline{m_{k+1}}}{\overline{m_k}}\right) B = B$ *for* $1 \leqslant k \leqslant n-2$.

Proof. *1.* By Lemma 5.29

$$0 < |\Delta| \leqslant £.$$

Moreover, Δ is not an absorber of B. So

$$B \subseteq \Delta B \subseteq £B = B.$$

Hence

$$B = B\Delta.$$

On the other hand, since $\Delta/\Delta \subseteq 1 + \oslash$,

$$B \diagup \Delta = (B\Delta)/\Delta = B(\Delta/\Delta) = B.$$

2. By the previous Part 1 and Theorem 5.30 one has

$$\left(\frac{\overline{m_1}}{d}\right) B = |\overline{m_1}| \frac{B}{\Delta} = |\overline{m_1}| B \subseteq \pounds B \subseteq B.$$

Also, $|\overline{m_1}|, |d| \in [@\Delta, @]$ by Lemma 5.29 and Theorem 5.30. So $\left|\frac{\overline{m_1}}{d}\right| \geqslant @\Delta$ and, as a consequence of Part 1,

$$B = @B = @\Delta B \subseteq \left(\frac{\overline{m_1}}{d}\right) B.$$

Hence

$$\frac{\overline{m_1}}{d} B = B.$$

Moreover, again by Theorem 5.30, one has $|\overline{m_{k+1}}|, |\overline{m_k}| \in [@\Delta, @]$ for $1 \leqslant k \leqslant n-2$. So

$$@\,|\Delta| \leqslant \left|\frac{\overline{m_{k+1}}}{\overline{m_k}}\right| \leqslant \frac{@}{@\,|\Delta|} = \frac{@}{|\Delta|},$$

which implies

$$@\Delta B \subseteq \left(\frac{\overline{m_{k+1}}}{\overline{m_k}}\right) B \subseteq @\frac{B}{\Delta}.$$

By Part 1, one has $B = B\Delta = B \diagup \Delta$. Also $@B = B$. Hence

$$B \subseteq \left(\frac{\overline{m_{k+1}}}{\overline{m_k}}\right) B \subseteq B$$

and one concludes that, for all $k \in \{1, \ldots, n-2\}$,

$$\left(\frac{\overline{m_{k+1}}}{\overline{m_k}} \right) B = B \qquad \blacksquare$$

In the remaining part of this section it will be shown that every element of the solution given by Cramer's rule is a Gauss-solution and vice-versa. We start with some useful properties of multiplication of matrices.

Proposition 5.32 *Consider the flexible system* (3.1). *Let* $a_{ij} \in \alpha_{ij}$ *for all* $i, j \in \{1, \ldots, n\}$. *Then*

1. $\mathcal{G} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$.

2. $\mathcal{G} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \subseteq \left[B / \overline{\beta} \right]_{n \times n}$.

3. $\left(\mathcal{G} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} \right) \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \subseteq \mathcal{G} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$.

Proof. *1.* Firstly, since $|a_{i1}| \leqslant 1$ for all $i \in \{2, \ldots, n\}$, one has

$$\mathcal{G}_1 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -a_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ a_{21}B + B \\ \vdots \\ a_{n1}B + B \end{bmatrix} = \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix}.$$

Also, by Theorem 5.30 and Lemma 5.31,

$$\mathcal{G}_2 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1/\overline{m_{n-2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ B/\overline{m_{n-2}} \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix}.$$

For $2 \leqslant i_1 < \cdots < i_{n-2} \leqslant n$ and $2 \leqslant j_1 < \cdots < j_{n-2} \leqslant n$ one has $\left|m_{i_1 \cdots i_{n-2}, j_1 \cdots j_{n-2}}\right| \leqslant |\overline{m_{n-2}}| \leqslant \pounds$ by Theorem 5.30. Also $|a_{12}| \leqslant 1$ and so

$$\mathcal{G}_3 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -m_{24\cdots n,3\cdots n} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -m_{2\cdots(n-1),3\cdots n} & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ B \\ \vdots \\ B \end{bmatrix}$$
$$= \begin{bmatrix} B + a_{12}B \\ B \\ B + m_{24\cdots n,3\cdots n}B \\ \vdots \\ B + m_{2\cdots(n-1),3\cdots n}B \end{bmatrix} = \begin{bmatrix} B \\ B \\ B \\ \vdots \\ B \end{bmatrix}.$$

Using Part 2 of Lemma 5.31 one has

$$\mathcal{G}_4 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \overline{m_{n-2}} \diagup \overline{m_{n-3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ B \\ \vdots \\ B \end{bmatrix}$$
$$= \begin{bmatrix} B \\ B \\ \left(\overline{m_{n-2}} \diagup \overline{m_{n-3}}\right) B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ B \\ B \\ \vdots \\ B \end{bmatrix}.$$

Again by Theorem 5.30, for $2 \leqslant i_1 < \cdots < i_{n-2} \leqslant n$ and $1 \leqslant j_1 < \cdots < j_{n-2} \leqslant n$,

$$\left|m_{i_1 \cdots i_{n-2}, j_1 \cdots j_{n-2}}\right| \leqslant |\overline{m_{n-2}}| \leqslant \pounds,$$

$$\left|m_{i_1 \cdots i_{n-3}, j_1 \cdots j_{n-3}}\right| \leqslant |\overline{m_{n-3}}| \leqslant \pounds.$$

and

$$|\overline{m_{n-2}}| > \oslash \Delta.$$

Also $\frac{B}{\Delta} = B$ by Part 1 of Lemma 5.31. So

$$\begin{aligned}
\mathcal{G}_5 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} &= \begin{bmatrix} 1 & 0 & m_{3\cdots n,14\cdots n} / \overline{m_{n-2}} & \cdots & 0 \\ 0 & 1 & -m_{3\cdots n,24\cdots n} / \overline{m_{n-2}} & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & -m_{35\cdots n,4\cdots n} / \overline{m_{n-2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -m_{3\cdots (n-1),4\cdots n} / \overline{m_{n-2}} & \cdots & 1 \end{bmatrix} \begin{bmatrix} B \\ B \\ B \\ B \\ \vdots \\ B \end{bmatrix} \\
&= \begin{bmatrix} B + \left(m_{3\cdots n,14\cdots n} / \overline{m_{n-2}}\right) B \\ B + \left(m_{3\cdots n,24\cdots n} / \overline{m_{n-2}}\right) B \\ B \\ B + \left(m_{35\cdots n,4\cdots n} / \overline{m_{n-2}}\right) B \\ \vdots \\ B + \left(m_{3\cdots (n-1),4\cdots n} / \overline{m_{n-2}}\right) B \end{bmatrix} = \begin{bmatrix} B \\ B \\ B \\ B \\ \vdots \\ B \end{bmatrix}.
\end{aligned}$$

Applying the same techniques to the subsequent matrices one concludes that

$$\mathcal{G} \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \mathcal{G}_{2n-1} \left(\cdots \mathcal{G}_3 \left(\mathcal{G}_2 \left(\mathcal{G}_1 \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \right) \right) \right) = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

2. This part is a clear generalization of the 2 by 2 case. Since $\overline{A} / \Delta \subseteq B / \overline{\beta}$, by Part 1 of Lemma 5.31 one has $\overline{A} \subseteq B / \overline{\beta}$. So, for all $i, j \in \{1, \ldots, n\}$,

$$A_{ij} \subseteq \overline{A} \subseteq B / \overline{\beta}.$$

Using formula (5.1) and Part 1, one obtains, whenever b is a representative of $\overline{\beta}$,

$$\begin{aligned}
\mathcal{G} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} &\subseteq \mathcal{G} \begin{bmatrix} B / \overline{\beta} & \cdots & B / \overline{\beta} \\ \vdots & \ddots & \vdots \\ B / \overline{\beta} & \cdots & B / \overline{\beta} \end{bmatrix} = \mathcal{G} \begin{bmatrix} B / b & \cdots & B / b \\ \vdots & \ddots & \vdots \\ B / b & \cdots & B / b \end{bmatrix} \\
&= \frac{1}{b} \mathcal{G} \begin{bmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} = \frac{1}{b} \begin{bmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix} \\
&= \begin{bmatrix} B / b & \cdots & B / b \\ \vdots & \ddots & \vdots \\ B / b & \cdots & B / b \end{bmatrix} = \begin{bmatrix} B / \overline{\beta} & \cdots & B / \overline{\beta} \\ \vdots & \ddots & \vdots \\ B / \overline{\beta} & \cdots & B / \overline{\beta} \end{bmatrix}.
\end{aligned}$$

3. Using Part 2, Lemma 2.5 and Part 1 we obtain

$$\begin{aligned}\left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} &\subseteq \begin{bmatrix} B/b & \cdots & B/b \\ \vdots & \ddots & \vdots \\ B/b & \cdots & B/b\end{bmatrix}\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} \\ &\subseteq \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash\end{bmatrix}\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} \\ &\subseteq \begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix}. \qquad \blacksquare\end{aligned}$$

We are now able to justify formula (5.3). Notice that $B/b \subseteq \oslash$ by Lemma *2.5*. So Part 2 of Lemma 5.32 implies that

$$\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn}\end{bmatrix} \subseteq \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash\end{bmatrix}.$$

The next two propositions yield a lower bound on the uncertainty of Cramer-solutions and an upper bound on the uncertainty of Gauss-solutions.

Proposition 5.33 *Consider the flexible system* (3.1). *Then, for all* $j \in \{1, \ldots, n\}$,

$$N\left(\frac{\det \mathcal{M}_j}{\Delta}\right) = B.$$

Proof. Let $j \in \{1, \ldots, n\}$ be arbitrary. By formula (4.5)

$$N\left(\frac{\det \mathcal{M}_j}{\Delta}\right) \subseteq B.$$

On the other hand, by Lemma 5.5 one has $|M_{ij}| > \oslash\Delta$ for some $i \in \{1,\ldots,n\}$, with Δ zeroless . So $B \subseteq M_{ij}B$. Therefore

$$\begin{aligned} B &\subseteq M_{1j}B + \cdots + M_{ij}B + \cdots + M_{nj}B \\ &\subseteq N\left(\det\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1(j-1)} & b_1+B & \alpha_{1(j+1)} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{n(j-1)} & b_n+B & \alpha_{n(j+1)} & \cdots & \alpha_{nn} \end{bmatrix}\right) \\ &= N\left(\det \mathcal{M}_j\right). \end{aligned}$$

By Part 1 of Lemma 5.31 one has $B = \frac{B}{\Delta}$. So

$$\begin{aligned} B &= \frac{B}{\Delta} \subseteq \frac{N\left(\det \mathcal{M}_j\right)}{\Delta} \subseteq \frac{N\left(\det \mathcal{M}_j\right)}{\Delta} + \det \mathcal{M}_j \cdot N\left(\frac{1}{\Delta}\right) \\ &= N\left(\frac{\det \mathcal{M}_j}{\Delta}\right). \end{aligned}$$

Combining, we conclude that $N\left(\frac{\det \mathcal{M}_j}{\Delta}\right) = B$. ■

Clearly, any number $u \in \mathbb{R}$ that verifies $u + \oslash u \subseteq B$ should satisfy $u \in B$. The next lemma generalizes this property to higher dimensions.

Lemma 5.34 *Let B be a neutrix and $(u_1,\ldots,u_n) \in \mathbb{R}^n$ such that $(u_1,\ldots,u_n)^T$ satisfies*

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \subseteq \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}. \tag{5.13}$$

Then, for all $i \in \{1,\ldots,n\}$

$$u_i \in B.$$

Proof. Let $(u_1,\ldots,u_n) \in \mathbb{R}^n$ such that $(u_1,\ldots,u_n)^T$ satisfies (5.13). Then

$$\begin{cases} u_1 + \oslash u_1 + \cdots + \oslash u_n \subseteq B \\ \vdots \\ u_n + \oslash u_1 + \cdots + \oslash u_n \subseteq B. \end{cases} \tag{5.14}$$

Suppose first that $\max_{1 \leqslant i \leqslant n} |u_i| = |u_1|$. So

$$u_1 + \oslash u_1 + \cdots + \oslash u_n = u_1 + \oslash u_1 = (1 + \oslash)\, u_1.$$

Suppose that $u_1 \notin B$. Then also $\frac{u_1}{2} \notin B$. Hence

$$|u_1 + \oslash u_1 \cdots + \oslash u_n| > \frac{|u_1|}{2} \notin B,$$

which contradicts the first equation of system (5.14). Therefore $u_1 \in B$ which implies that also $u_i \in B$ for $2 \leqslant i \leqslant n$.

The cases where $\max_{1 \leqslant i \leqslant n} |u_i| = |u_p|$ for $2 \leqslant p \leqslant n$ are analogous. Hence all solutions $(u_1, \ldots, u_n)^T$ of (5.13) satisfy $u_i \in B$ for all $i \in \{1, \ldots, n\}$. ■

Proposition 5.35 *Consider the flexible system* (3.1)*. Let $x_{i,}, y_i \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$ such that $(x_1, \ldots, x_n)^T$ and $(y_1, \ldots, y_n)^T$ are Gauss-solutions of* (3.1)*. Let $u_i = x_i - y_i$ for all $i \in \{1, \ldots, n\}$. Then, for $1 \leqslant i \leqslant n$,*

$$u_i \in B.$$

Proof. Again we follow the steps of the 2 by 2 case as a generalization. Let $a_{ij} \in \alpha_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Then

$$\left(\mathcal{G} \begin{bmatrix} 1 + A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix} \right) \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \subseteq \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}, \tag{5.15}$$

for $(x_1,\ldots,x_n)^T$ and $(y_1,\ldots,y_n)^T$ are Gauss-solutions and, using Part 1 of Proposition 5.32,

$$
\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\
\subseteq\ &\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
\subseteq\ &\mathcal{G}\begin{bmatrix} b_1+B \\ \vdots \\ b_n+B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1+B \\ \vdots \\ b_n+B \end{bmatrix} \\
=\ &\mathcal{G}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} - \mathcal{G}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} - \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \\
=\ &\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} - \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.
\end{aligned}
$$

Also

$$
\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \subseteq \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}. \tag{5.16}
$$

Indeed, by distributivity, Part 2 of Lemma 5.32 and Lemma 2.5

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\
= &\left(\mathcal{G}\begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\right)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\
\subseteq &\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} B/\overline{\beta} & \cdots & B/\overline{\beta} \\ \vdots & \ddots & \vdots \\ B/\overline{\beta} & \cdots & B/\overline{\beta} \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\
\subseteq &\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} \oslash & \cdots & \oslash \\ \vdots & \ddots & \vdots \\ \oslash & \cdots & \oslash \end{bmatrix}\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.
\end{aligned}$$

By (5.16) all solutions of (5.15) satisfy (5.13). So, by Lemma 5.34, all solutions of (5.15) satisfy $u_i \in B$ for all $i \in \{1, \ldots, n\}$. ■

By Part 3 of Theorem 4.4, if $\triangle$ is not an absorber of B and $\overline{A}/\Delta \subseteq B/\overline{\beta}$, a Cramer-solution of the system (3.1) is a maximal solution. We show now that under these conditions any element of this solution is a Gauss-solution.

Theorem 5.36 *Let* $(x_1, \ldots, x_n)^T \in \left(\frac{\det \mathcal{M}_1}{\Delta}, \ldots, \frac{\det \mathcal{M}_n}{\Delta}\right)^T$. *Then* $(x_1, \ldots, x_n)^T$ *is a Gauss-solution of* (3.1).

Proof. Let $a_{ij} \in \alpha_{ij}$ for all $i, j \in \{1, \ldots, n\}$. Choose $b_i \in \beta_i$ for all $i \in \{1, \ldots, n\}$ and let $b = \max\limits_{1 \leqslant i \leqslant n}(|b_i|)$. Put, for all $j \in \{1, \ldots, n\}$,

$$d_j = \det \begin{bmatrix} 1 & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{3n} \end{bmatrix}.$$

Because $|a_{ij}| \leqslant 1 + \oslash$ for all $i, j \in \{1, \ldots, n\}$, one has $|d_j| \leqslant ((n-1)! + \oslash)\, b$ for all $j \in \{1, \ldots, n\}$. So, for all $j \in \{1, \ldots, n\}$,

$$d_j B \subseteq bB. \tag{5.17}$$

We assume first that $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \vdots \\ \frac{d_n}{d} \end{bmatrix}$, where $d = \det \begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$. Then

$$\left(\mathcal{G}\begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = I_n \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d_1}{d} \\ \vdots \\ \frac{d_n}{d} \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

By Part 2 of Proposition 5.32 and formula (5.17), one has

$$\begin{aligned}
\left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} &\subseteq \begin{bmatrix} B/b & \cdots & B/b \\ \vdots & \ddots & \vdots \\ B/b & \cdots & B/b \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} \frac{B}{b}x_1 + \cdots + \frac{B}{b}x_n \\ \vdots \\ \frac{B}{b}x_1 + \cdots + \frac{B}{b}x_n \end{bmatrix} \\
&= \begin{bmatrix} \frac{B}{b}\frac{d_1}{d} + \cdots + \frac{B}{b}\frac{d_n}{d} \\ \vdots \\ \frac{B}{b}\frac{d_1}{d} + \cdots + \frac{B}{b}\frac{d_n}{d} \end{bmatrix} \\
&\subseteq \begin{bmatrix} \frac{B}{b}\frac{b}{d} + \cdots + \frac{B}{b}\frac{b}{d} \\ \vdots \\ \frac{B}{b}\frac{b}{d} + \cdots + \frac{B}{b}\frac{b}{d} \end{bmatrix} = \begin{bmatrix} \frac{B}{d} \\ \vdots \\ \frac{B}{d} \end{bmatrix}.
\end{aligned}$$

Also by Part 1 of Lemma 5.31 and Part 1 of Proposition 5.32,

$$\begin{bmatrix} \frac{B}{\Delta} \\ \vdots \\ \frac{B}{\Delta} \end{bmatrix} = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

Hence

$$\left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \subseteq \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}. \tag{5.18}$$

Then it follows by distributivity that

$$\begin{aligned} & \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ = & \left(\mathcal{G}\begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ \subseteq & \mathcal{G}\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} b_1+B \\ \vdots \\ b_n+B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}. \end{aligned}$$

Hence $(x_1, \ldots, x_n)^T$ is an admissible solution of (3.1).

Let $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \begin{bmatrix} \frac{\det \mathcal{M}_1}{\triangle} \\ \vdots \\ \frac{\det \mathcal{M}_n}{\Delta} \end{bmatrix}$ be arbitrary.

By Proposition 5.33 one has $N\left(\frac{\det \mathcal{M}_p}{\Delta}\right) = B$ for all $p \in \{1, \ldots, n\}$. So

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}.$$

Then, by distributivity and Proposition 5.32,

$$\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
\subseteq\ & \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{bmatrix}\right)\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right)\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \\
&\quad + \left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}\right)\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \\
=\ & \mathcal{G}\begin{bmatrix} \beta_1 + B \\ \vdots \\ \beta_n + B \end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.
\end{aligned}$$

Hence $(y_1, \ldots, y_n)^T$ is an admissible solution of (3.1).

Finally, we prove that $(y_1, \ldots, y_n)^T$ is a Gauss-solution. For this must now choose an arbitrary $a_{11} \in \alpha_{11}$. Then $a_{11} = 1 + \varepsilon$ with $\varepsilon \in A_{11}$. So, by distributivity, formula (5.18) and Part 3 of Lemma 5.32,

$$
\begin{aligned}
&\left(\mathcal{G}\begin{bmatrix} 1+\varepsilon+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n\end{bmatrix} \\
=\ & \left(\mathcal{G}\begin{bmatrix} 1+A_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn}\end{bmatrix}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n\end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} \varepsilon & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0\end{bmatrix}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n\end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n\end{bmatrix} + \left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix} x_1 \\ \vdots \\ x_n\end{bmatrix} \\
&+ \left(\mathcal{G}\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn}\end{bmatrix}\right)\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} \\
\subseteq\ & \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n\end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} + \mathcal{G}\begin{bmatrix} B \\ \vdots \\ B\end{bmatrix} \\
=\ & \mathcal{G}\begin{bmatrix} \beta_1 + B \\ \vdots \\ \beta_n + B\end{bmatrix} = \mathcal{G}\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n\end{bmatrix}. \qquad \blacksquare
\end{aligned}
$$

Next theorem is a converse to Theorem 5.36. Under the usual conditions, a Gauss-solution must be an element of the Cramer-solution.

Theorem 5.37 *Let* $(x_1,\ldots,x_n)^T$ *be a Gauss-solution of* (3.1)*. Then* $(x_1,\ldots,x_n)^T$ *satisfies* (3.1)*, in fact* $(x_1,\ldots,x_n)^T \in \left(\frac{\det\mathcal{M}_1}{\Delta},\ldots,\frac{\det\mathcal{M}_n}{\Delta}\right)^T$.

Proof. Let $a_{ij}\in\alpha_{ij}$ for all $i,j\in\{1,\ldots,n\}$. Let $b_i\in\beta_i$ for all $i\in\{1,\ldots,n\}$ and let $b=\max\limits_{1\leqslant i\leqslant n}(|b_i|)$.

Put, for all $j \in \{1, \ldots, n\}$,

$$
\begin{aligned}
d_j &= \det \begin{bmatrix} 1 & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{3n} \end{bmatrix}, \\
d &= \det \begin{bmatrix} 1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}.
\end{aligned}
$$

It follows from Theorem 5.36 that $(x_1, \ldots, x_n)^T = \left(\frac{d_1}{d}, \ldots, \frac{d_n}{d}\right)^T$ is a Gauss-solution, and it clearly satisfies (3.1).

Let $(y_1, \ldots, y_n)^T$ be an arbitrary Gauss-solution of (3.1). By Propositions 5.35 and 5.33 it holds that, for all $i \in \{1, \ldots, n\}$,

$$
y_i \in \frac{d_i}{d} + B = \frac{\det \mathcal{M}_i}{\triangle}.
$$

Then it follows from Part 3 of Theorem 4.4 that $(y_1, \ldots, y_n)^T$ also satisfies (3.1). ■

Proof of Theorem 5.14 By Theorem 5.36 and 5.37 it holds that $\left(\frac{\det \mathcal{M}_1}{\triangle}, \ldots, \frac{\det \mathcal{M}_n}{\triangle}\right)^T$ is equal to the external set of all Gauss-solutions. ■

This final theorem implies that the external set of all Gauss-solutions, being equal to the Cramer-solution, by Part 3 of Theorem 4.4, also constitutes an admissible and maximal solution of the reduced flexible system (3.1).

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