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RESEARCH ARTICLE

Profit optimization for cattle growing in a randomly fluctuating environment

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A class of stochastic differential equations models was applied to describe the evolution of the weight of mertolengo cattle. We have determined the optimal mean profit obtained by selling an animal at the cattle market, using two approaches. One consists in determining the optimal selling age (independently of the weight) and the other consists in selling the animal when a fixed optimal weight is achieved for the first time (independently of the age). The profit probability distribution can be computed for such optimal age/weight. For typical market values, we observed that the second approach achieves a higher optimal mean profit compared with the first one, and, in most cases, even provides a lower standard deviation.

Keywords: stochastic differential equations; cattle growth; profit optimization; probability density functions

AMS Subject Classification: 60H10; 60E05; 62G07; 91B70; 92D99

1. Introduction

In earlier work we have studied a class of stochastic differential equation (SDE) models for individual growth in randomly fluctuating environments and we have applied such models using real data on the evolution of bovine weight (see, for instance, [8–11]). The Gompertz and the Bertalanffy-Richards stochastic models, are particular cases of this more general class of SDE models. This type of models might be useful in cattle breeding or forestry in order to optimize the exploitation of such resources. The work we present here is focused on a new application of these models, namely the optimization of the mean profit obtained by raising and selling an animal and the computation of probabilities involving the selling profit.

Consider a farmer raising an animal. On one hand, based on our models, we can compute the mean profit obtained by selling the animal at different ages and, in particular, we can determine the optimal age at which the farmer should sell the animal in order to maximize the mean profit. We can also obtain the probability distribution of the selling profit and then compute probabilities involving that profit. On the other hand, knowing which animal weight is demanded by the market, we can study the properties of the time required for an animal to reach such weight for the first time. We present expressions for the mean and variance

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of these times, known from first passage time theory (see, for instance, [3]), and use these results to determine the optimal weight in order to obtain the maximum mean profit. A comparison between these two approaches is presented.
The cattle data we work with was collected by technicians of the Mertolengo's Cattle Breeders Association (ACBM) and provided by Professor Carlos Roquete from the Institute of Mediterranean Agricultural and Environmental Sciences of University of Évora (ICAAM-UE). The animals were raised in "Herdade da Aboóboda" in the Serpa region at the left margin of the Guadiana river, together with their mothers during nursing and later supplemented with silage when pasture is in shortage (from August till January). Mertolengo cattle breed is, at the moment, considered by many as the Portuguese cattle breed with higher progression in terms of population increment and market potential.
The computations were performed using Software packages MAPLE and R.

2. Individual growth models

Individual growth models represent individual size changes over time. Although our study is exclusively dedicated to individual growth, many of the models applied to population growth are also used to model individual growth.

Several patterns are often observed in the growth rates of organisms. The so-called exponential growth is typical in certain short periods, particularly soon after birth. The asymptotic growth pattern applies to the length of some organisms, the size of the skull and of the brain. It is characterized by a steady decrease in the growth rate, thus has no inflection point. The weight and volume of the body and of most organs follow a sigmoid or S-shaped growth. Initially, the growth rate is low but increasing, until it reaches a maximum, corresponding to the curve inflection point, and then decreases to zero as the animal reaches its weight at maturity. The sigmoid curve is prevalent among animals with determined growth, and this led to the emergence of a specific class of "sigmoid functions" to describe growth.

The most common deterministic models used to describe the individual growth of an animal (plant or other organism) in terms of its size (weight, volume, height, length, etc.) X_t at age (instant) t, can be written in the form of an autonomous differential equation

$$dY_t = \beta(\alpha - Y_t)dt,\ Y_{t_0} = y_0, \tag{1}$$

where Y_t can be seen as a modified size, i.e., $Y_t = g(X_t)$ where g a is strictly increasing continuously differentiable function (which we assume known), and $y_0 = g(x_0)$, x_0 being the size at age t_0 (first observation). With A denoting the asymptotic size or size at maturity of the animal, we put $\alpha = g(A)$. The parameter β, positive, is the growth coefficient and represents the rate of approach to maturity. The choice of g leads to some well known classic models. For instance, when $g(x) = x$ we get the monomolecular model; the Bertalanffy Richards model [6, 19] corresponds to the case $g(x) = x^c (c > 0)$, the Gompertz model to the case $g(x) = \ln x$ (can be considered the limiting case of BertalanffyRichards model when $c \to 0$), and the Logistic model [23] to the case $g(x) = -x^{-1}$.

In the determinist individual growth models, random variations in data have been treated by classical regression models. The traditional assumption of regression models that observed deviations from the regression curve are independent at different times would be realistic if the deviations were due to measurement errors. It is totally unrealistic when the deviations are due to random changes on growth rates induced by environmental random fluctuations. For instance, in such regression models, a delay in growth at a certain age has no repercussions on future sizes. Stochastic differential equation models do not have such shortcomings, since they are built precisely to incorporate the dynamics of the growth process and the effect environmental random fluctuations have on such dynamics. Thus, we have considered that individual growth in a random environment can be described by stochastic differential equations of the form

$$dY_t = \beta(\alpha - Y_t)dt + \sigma dW_t,\ Y_{t_0} = y_0, \tag{2}$$

where σ is an environmental noise intensity parameter and W_t is the standard Wiener process. This model is a variant of the Ornstein Uhlenbeck model, usually called the Vasicek model, and was introduced to study interest rate dynamics [22]. We can see applications of this type of models, for instance, in [13, 17] for tree growth and in [16] for fish growth.

The solution of (2), Y_t, is a homogeneous diffusion process with drift coefficient $a(y) = \beta\,(\alpha - y)$ and diffusion coefficient $b(y) = \sigma^2$. The drift coefficient is the mean speed of growth described by Y_t and the diffusion coefficient gives a measure of the local magnitude of the fluctuations. It can be seen, for instance in [4], that the explicit solution of (2) is given by

$$Y_t = \alpha - (\alpha - y_0)\exp\left(-\beta(t-t_0)\right) + \sigma \exp\left(-\beta t\right)\int_{t_0}^{t} \exp\left(\beta u\right) dW_u, \tag{3}$$

and follows a Gaussian distribution with mean $\alpha - (\alpha - y_0)\exp\left(-\beta(t-t_0)\right)$ and variance $\frac{\sigma^2}{2\beta}\left(1 - \exp\left(-2\beta(t-t_0)\right)\right)$.

We have used maximum likelihood estimation theory to estimate the parameter vector $\mathbf{p} = (\alpha, \beta, \sigma)$. We describe details on this procedure, for instance, in [9, 11] and [10]. In [11], we have seen that the best models for our data (using an AIC criterion), were the stochastic Bertalanffy-Richards model with $c = 1/3$, (SBRM) and the stochastic Gompertz model (SGM).

3. Profit optimization and related probability distribution

The SDE models presented can be useful in financial context. In our application, by having more information on the growth of animals, growers can, for instance, optimize the average profit obtained from selling an animal. Applying these models to the weight of mertolengo cattle, we compute the mean and standard deviation of the profit obtained from selling an animal to the meat market at different ages and, in particular, determine which is the optimal selling age. In this case, the profit is defined as a function of age. Another approach is also presented, defining the profit as a function of the animals weight. Using first passage time theory, we can characterize the time taken for an animal to achieve a certain weight of market interest for the first time. In particular, expressions for the mean and standard deviation of these times are presented and applied to our data. These last results can be used to determine the optimal selling weight in terms of mean profit. Probability distributions of the selling profit can be obtained, allowing to compute probabilities involving this profit.

The profit obtained from selling an animal can be defined as $L = V - C$, where

V represents the selling price and C the acquisition (if it is the case) and animal raising costs.

3.1. *Profit as a function of age*

Let x_0 be the weight of the animal at age t_0 (the age, assumed known, when it is bought) and $t > t_0$ the selling age. The profit at age t is given by $L_t = V_t - C_t$, with $V_t = PRX_t$ (R being the dressing proportion and P the selling price per unit weight) and $C_t = C_1 - c_2(t - t_0)$ (C_1 being the fixed costs and c_2 the variable raising costs per unit time). Since $X_t = g^{-1}(Y_t)$, we can write the profit as a function of Y_t. For the SGM and the SBRM, we obtain

$$L_t = l_t(Y_t) = \begin{cases} PRe^{Y_t} - C_1 - c_2(t - t_0), & \text{for SGM} \\ PRY_t^3 - C_1 - c_2(t - t_0), & \text{for SBRM.} \end{cases} \quad (4)$$

Considering the Gaussian probability distribution of Y_t, we can determine the probability density function of L_t using $f_{L_t}(u) = f_{Y_t}(l_t^{-1}(u)) \left| \frac{dl_t^{-1}(u)}{du} \right|$, where

$$f_{Y_t}(y) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t-t_0)})}} \exp\left(-\frac{\left(y - \alpha - (y_0 - \alpha)\, e^{-\beta(t-t_0)}\right)^2}{2\frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t-t_0)})} \right),$$

and

$$l_t^{-1}(u) = \begin{cases} \ln\left(\frac{u + C_1 + c_2(t-t_0)}{PR}\right), & \text{for SGM} \\ \left(\frac{u + C_1 + c_2(t-t_0)}{PR}\right)^{1/3}, & \text{for SBRM} \end{cases} \quad (5)$$

$$\frac{dl_t^{-1}(u)}{du} = \begin{cases} \left(u + C_1 + c_2(t - t_0)\right)^{-1}, & \text{for SGM} \\ \frac{(u + C_1 + c_2(t-t_0))^{-2/3}}{3(PR)^{1/3}}, & \text{for SBRM.} \end{cases} \quad (6)$$

The expressions for the mean and variance of L_t are, respectively, given by

$$\mathrm{E}\left[L_t\right] = PR\mathrm{E}\left[X_t\right] - C_1 - c_2(t - t_0) \quad (7)$$

and

$$\mathrm{Var}\left[L_t\right] = P^2R^2\mathrm{Var}\left[X_t\right], \quad (8)$$

where the expressions for $\mathrm{E}[X_t]$ and $\mathrm{Var}[X_t]$ are determined, according to the model used, as follows. In the SGM case, X_t follows a log-normal distribution, and consequently

$$\mathrm{E}\left[X_t\right] = \exp\left(\alpha + e^{-\beta(t-t_0)}(\ln x_0 - \alpha) + \frac{\sigma^2}{4\beta}(1 - e^{-2\beta(t-t_0)})\right)$$

and

$$\mathrm{Var}\left[X_t\right] = \exp\left(2\alpha + 2e^{-\beta(t-t_0)}(\ln x_0 - \alpha) + \frac{\sigma^2}{\beta}(1 - e^{-2\beta(t-t_0)})\right) + \\ - \exp\left(2\alpha + 2e^{-\beta(t-t_0)}(\ln x_0 - \alpha) + \frac{\sigma^2}{2\beta}(1 - e^{-2\beta(t-t_0)})\right).$$

For the SBRM case, using Stein's Lemma, we get

$$\mathrm{E}\left[X_t\right] = \frac{3\sigma^2}{2\beta}\left(\alpha + e^{-\beta(t-t_0)}(x_0^{1/3} - \alpha)\right)(1 - e^{-2\beta(t-t_0)}) \\ + \left(\alpha + e^{-\beta(t-t_0)}(x_0^{1/3} - \alpha)\right)^3$$

and

$$\mathrm{Var}\left[X_t\right] = \frac{9\sigma^4}{\beta^2}\left(\alpha + e^{-\beta(t-t_0)}(x_0^{1/3} - \alpha)\right)^2(1 - e^{-2\beta(t-t_0)})^2 + \\ + \frac{9\sigma^2}{2\beta}\left(\alpha + e^{-\beta(t-t_0)}(x_0^{1/3} - \alpha)\right)^4(1 - e^{-2\beta(t-t_0)}) + \frac{15\sigma^6}{8\beta^3}(1 - e^{-2\beta(t-t_0)})^3.$$

For both models, SGM and SBRM, by maximizing expression (7) with respect to age t, we can obtain the optimal age for selling an animal in order to reach a maximum mean profit.

The computation of probabilities involving the selling profit can be obtained. For instance, we can determine the probability of achieving a profit higher than a certain value v by selling the animal at a certain age t (and, in particular, at the optimal selling age). This probability can be obtained based on the following

$$\mathrm{P}\left[L_t > v\right] = \mathrm{P}\left[X_t > \frac{v + C_1 + C_2(t)}{PR}\right] = \begin{cases} \mathrm{P}\left[Y_t > \ln\left(\frac{v+C_1+C_2(t)}{PR}\right)\right], & \text{for SGM} \\ \mathrm{P}\left[Y_t > \left(\frac{v+C_1+C_2(t)}{PR}\right)^{1/3}\right], & \text{for SBRM} \end{cases}$$

$$= \begin{cases} 1 - \Phi\left(\frac{\ln\left(\frac{v+C_1+C_2(t)}{PR}\right) - \alpha - (\ln x_0 - \alpha)e^{-\beta(t-t_0)}}{\sqrt{\frac{\sigma^2}{2\beta}(1-e^{-2\beta(t-t_0)})}}\right), & \text{for SGM} \\ 1 - \Phi\left(\frac{\left(\frac{v+C_1+C_2(t)}{PR}\right)^{1/3} - \alpha - (x_0^{1/3} - \alpha)e^{-\beta(t-t_0)}}{\sqrt{\frac{\sigma^2}{2\beta}(1-e^{-2\beta(t-t_0)})}}\right), & \text{for SBRM} \end{cases}. \quad (9)$$

3.2. *Profit as a function of weight*

For the case in which we want to compute the profit obtained from selling the animal when a certain weight Q^* is achieved for the first time, we can use the expression $L_{Q^*} = PRQ^* - C_1 - c_2 T_Q$, where $T_Q = \inf\{t > 0 : Y_t = Q\}$ represents the time required for an animal to reach a specific size $Q^* = g^{-1}(Q)$ for the first time.

The expressions for the mean and variance of the profit are now given by

$$\mathrm{E}[L_{Q^*}] = PRQ^* - C_1 - c_2\mathrm{E}[T_Q|Y(0) = y_0] \tag{10}$$

and

$$\mathrm{Var}[L_{Q^*}] = c_2^2\mathrm{Var}[T_Q|Y(0) = y_0], \tag{11}$$

where $\mathrm{E}[T_Q|Y(0) = y_0]$ and $\mathrm{Var}[T_Q|Y(0) = y_0]$ represent the mean and variance of the first passage time by $Q = g(Q^*)$. In [3, 5] we have determined explicit expressions (in the form of simple integrals that can be numerically computed) for the mean and variance of the time required for an animal to reach a given size for the first time. For our model (2), we obtain the following expressions

$$\mathrm{E}[T_Q|Y(0) = y_0] = \frac{1}{\beta}\int_{\sqrt{2\beta}(y_0-\alpha)/\sigma}^{\sqrt{2\beta}(Q-\alpha)/\sigma} \frac{\Phi(y)}{\phi(y)}dy \tag{12}$$

and

$$\mathrm{Var}[T_Q|Y(0) = y_0] = \frac{2}{\beta^2}\int_{\sqrt{2\beta}(y_0-\alpha)/\sigma}^{\sqrt{2\beta}(Q-\alpha)/\sigma}\int_{-\infty}^{z} \frac{\Phi^2(y)}{\phi(y)\phi(z)}dydz, \tag{13}$$

where Φ and ϕ are the distribution function and the probability density function of a standard normal random variable. To obtain the mean and variance of T_Q, one needs to numerically integrate in (12) and (13).

We now present some results related to the probability density function of T_Q. More details on the study of the first passage time density function can be found, for instance, in [1, 7, 18, 20, 24]. For our model (2), in [14] the probability density function of the first passage time by $Q = g(Q^*)$ is described as

$$f_{T_Q}(t) = \sum_{n=1}^{\infty} c_n\lambda_n \exp(-\lambda_n t),\ t > 0, \tag{14}$$

where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$ when $n \to \infty$. For all $t_0 > 0$, this series converges uniformly in $[t_0, \infty)$. Considering the notation

$$Q_l = \frac{\sqrt{2\beta}}{\sigma}(\alpha - y_0),\ Q_L = \frac{\sqrt{2\beta}}{\sigma}(\alpha - Q),\ \nu_n = \frac{\lambda_n}{\beta},$$

where $0 < \nu_1 < \nu_2 < \ldots < \nu_n$ ($\to \infty$ when $n \to \infty$) are the positive roots of the equation

$$H_\nu\left(\frac{Q_L}{\sqrt{2}}\right) = 0$$

and

$$c_n = -\frac{H_{\nu_n}\left(\frac{Q_l}{\sqrt{2}}\right)}{\nu_n \left.\frac{\partial H_\nu\left(\frac{Q_L}{\sqrt{2}}\right)}{\partial \nu}\right|_{\nu=\nu_n}},$$

where $H_\nu(.)$ is the Hermite function. The distribution function of T_Q is given by

$$F_{T_Q}(t) = \sum_{n=1}^{\infty} c_n(1 - \exp(-\lambda_n t)). \tag{15}$$

In [14], an application to interest rates in Japan is shown and asymptotic expressions for λ_n and c_n are also presented. This method is applied in [16] to evaluate probability density functions for hitting times in the context of fish growth and mortality.

For model (2), an expression for the probability density function of the first passage time by the asymptotic value α can be found, for instance, in [14] and [18]:

$$f_{T_\alpha}(t) = \frac{|y_0 - \alpha|}{\sigma\sqrt{2\pi}} \left(\frac{\beta}{\sinh(\beta t)}\right)^{3/2} \exp\left(\frac{\beta t}{2} - \frac{\beta(y_0-\alpha)^2 e^{-\beta t}}{2\sigma^2 \sinh(\beta t)}\right). \tag{16}$$

More recently, in [24], considering the Ornstein-Uhlenbeck model, an interesting result relates the probability of the process being above the asymptotic value α with the distribution of the first passage time by α. For model (2), for a future instant $T > 0$,

$$F_{T_\alpha}(T) = P[T_\alpha \leq T] = 2P[Y_T \geq \alpha].$$

Since Y_t follows a gaussian distribution with mean and variance as described above, we obtain

$$F_{T_\alpha}(T) = 2\left(1 - \Phi\left(-\frac{(y_0-\alpha)e^{-\beta T}}{\sigma\sqrt{1-e^{-2\beta T}}}\right)\right) = 2\Phi\left(-\frac{|y_0-\alpha|e^{-\beta T}}{\sigma\sqrt{1-e^{-2\beta T}}}\right), \tag{17}$$

where Φ denotes the distribution function of a standard normal random variable. Expressions (16) and (17) are easy to apply, but they can only be used when considering the first passage time by the asymptotic value, i.e., when $Q = \alpha$. To apply, as we did, to cattle data, we need the distribution of the first passage time through other values and so we need to use expression (15), although it requires more extensive computational work.
These methods were applied to our cattle data, allowing to obtain the distribution curves of the time required for an animal to reach a specific weight for the first time.

The probability distribution function for the profit, L_{Q^*}, obtained from selling the animal when a certain weight Q^* is achieved for the first time, can be written as

$$\begin{aligned} F_{L_{Q^*}}(v) &= \mathrm{P}\left[L_{Q^*} < v\right] = \mathrm{P}\left[PRQ^* - C_1 - c_2 T_Q \leq v\right] \\ &= \mathrm{P}\left[T_Q > \frac{PRQ^* - C_1 - v}{c_2}\right] = 1 - F_{T_Q}\left(\frac{PRQ^* - C_1 - v}{c_2}\right). \end{aligned} \tag{18}$$

4. Application to cattle data

The data set we worked with contains information on the weight, in kilograms, of 97 females of Mertolengo cattle, with several observations *per* animal corresponding to ages varying from birth until a maximum age between 0.6 and 18 years, making a total of 2129 observations.

Table 1 shows the maximum likelihood estimates, for both the SGM and the SBRM. We have also obtained the asymptotic confidence bands based on the results of the empirical Fisher information matrix.

Let us now give an example and consider a typical cattle market situation, where a mertolengo cow with 160 kg, previously raised with the mother up to 7 months (0.58 years) of age (approximate weaning age), is bought by a producer for 200 euros to be raised for market sale up to age 16 months (1.33 years). We must consider, in the case of mertolengo cattle breed, that the dressing proportion is 50% of live weight ($R = 0.5$); the usual raising costs (in euros) for an animal from the age of 7 months to age t are: 18.85 for commercialization and transportation, 26.68 for feeding/month, 7.25 for sanitation costs and 1.55 for other costs. We can then establish the fixed costs as $C_1 = 200 + 18.85 + 7.25 + 1.55 = 227.45$ and the cost of animal raising per unit time as $c_2 = 26.68/\text{month} \times 12\text{months/year} = 320.16/\text{year}$. We consider typical selling prices P (euros/kg) of the animal to be 3.00, 3.25, 3.50 or 3.75. Subsidies for slaughter or RPU (unit income) were not considered.

What is the expected profit of this producer when the animal is marketed at age t? Considering the maximum likelihood estimates of the model parameters given in Table 1 and the typical market values described above we can obtain the expected profit curve according to age. This is shown in Figure 1.(a), for both SGM and SBRM, for a selling price of 3.25 euros/kg. Maximizing expression (7) with respect to age t, we have obtained the optimal age for selling the animal in order to reach a maximum mean profit. Table 2 presents, for both SGM and SBRM, the optimal age t (A_{opt}) and correspondent expected weight ($\text{E}[X_{A_{opt}}]$), maximum mean profit ($\text{E}[L_{A_{opt}}]$) and standard deviation of the profit ($sd[L_{A_{opt}}]$).

For the approach in which we consider the profit as a function of the animals weight, we have started by using expression (12) to compute the expected times to achieve weights from 200 to 400 kg. These results were then used in (10) and, trough the maximization with respect to Q^*, we have obtained the maximum mean profit ($\text{E}[L_{Q^*_{opt}}]$), and corresponding optimal selling weight (Q^*_{opt}). Figure 1.(b) illustrates the case for which $P = 3.25$ euros/kg and Table 2 shows these results for both models (SGM and SBRM), as well as the standard deviation of the profit ($sd[L_{Q^*_{opt}}]$) and expected age of the animal when the optimal weight is achieved ($\text{E}[A_{opt}]$). Since the animal was bought at 7 months of age (0.583 years), the expected age when Q^*_{opt} is achieved for the first time can be computed as $\text{E}[A_{opt}] = 0.583 + \text{E}[T_{Q_{opt}}]$.

Considering a selling price of $P = 3.25$ euros/kg, figure 2 shows the selling profit standard deviation for both approaches, obtained by using the square root of expressions (8) and (11).

By the analysis of the results on Table 2, we can conclude that the approach based on the profit as a function of the animal weight lead us to higher values of the optimal mean profit than the first approach of optimization by the animal age. The optimal mean profit values are higher in the case of SGM than in the SBRM case. In terms of the profit standard deviation, we observe that the second approach provides considerably lower values than the first approach in the case of SGM and slightly higher values than the first approach in the case of SBRM.

We have seen that it is possible to obtain probabilities involving the selling profit. In particular, based on (9) and (18) when $v = 0$ we can compute the probability of having a positive profit by selling the animal at a certain age or a certain weight. For instance, for a selling price of 3.25 euros/kg, considering the SGM, the probability of having a positive profit by selling the animal at the optimal age (1.05 years) is 0.715, while the probability of a positive profit is only 0.632 if one sells the animal at the usual (non optimal) age of 1.33 years.

Looking at the second approach, the probability of having a positive profit when the animal is sold after reaching the optimal weight (292 kg) is 0.807. As one can see in this case, the second approach provides a higher probability of having a positive profit then the first approach. This is also true for the other values of P we have considered.

Considering $P = 3.25$ euros/kg, figure 3 illustrates the probability of having a selling profit higher than a certain value v, varying from 0 to 100 euros, when the animal is sold at the optimal age (Figure 3.(a)), and when the animal is sold at the usual age of 16 months (Figure 3.(b)). We can see that these probabilities are higher in the case of SGM, with the only exception of low values of v when the animal is sold at the optimal age.

Based on the probability density function and distribution function of T_Q, the first time an animal achieves a certain weight $Q^* = g^{-1}(Q)$, the plots for the probability of having a selling profit higher than a certain value v can be shown. As an example we show the probability density function (Figure 4.(a)) and the distribution function (Figure 4.(b)) of the first time an animal reaches $Q^* = 300$ kg. Figure 5, illustrates the probability that the selling profit exceeds a certain value v, varying from 0 to 100 euros, when the animal is sold at $P = 3.25$ euros/kg after reaching 300 kg. Higher values of these probabilities are obtained in the case of the SGM.

5. Conclusions

Based on SDE individual growth models applied to the evolution of the weight of Mertolengo cattle, we have studied two different approaches for the optimization of the mean profit obtained by raising and selling an animal in the cattle market. The first approach consists in selling the animal at a fixed age, independently of the animal's weight. We have determined the most adequate age to sell the animal in order to obtain a maximum mean profit. The other approach consists in selling the animal when a fixed weight is achieved for the first time, independently of the animal's age. The cattle market demands may be, for example, the search for an animal with a certain specific weight. In this case, it is important to be able to determine the average time required for the animal to achieve the desired weight. To this end, the theory of first passage times was applied.

We have observed that, for typical market values, the second methodology achieves a higher optimal mean profit compared with the first methodology, and even provides a lower standard deviation for this optimal profit in the SGM case, showing a higher, but only slightly, standard deviation for the SBRM case.

Probability distribution functions for the selling profit were developed for both approaches. Probabilities of obtaining a profit higher than a certain value were computed. The second methodology revealed higher probabilities of exceeding a certain value of the profit when compared with the first one.

Optimization problems concerning the average profit of the sale of a single animal were studied. In practice, a producer usually sells several animals at once, possibly with different weights and ages, so they do not reach a certain age or a certain weight all at the same time. We can put up the question of what is the optimal time to sell a heterogenous group of animals in order to optimize profit concerning some shared costs, such as transport of animals. This would be a situation that leads to a more complex optimization problem which would be interesting to study in future work.

In our case, animals feed freely in the field and we have no records of food intake. However, there are cases in which we can control food intake and it would be interesting to determine the optimal control (for stochastic optimal control see, for instance, [12, 15, 25]). Other interesting developments would be to consider jump diffusion models (see, for instance, [15]), in which environmental fluctuations may have jumps, or even stochastic hybrid systems (see [2, 21]), in which there are different growth regimes with random switching among them.

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Table 1. Maximum likelihood estimates and approximate half-width of the 95% confidence bands

	SGM	SBRM
A (kg)	411.2 ± 8.1	425.7 ± 9.5
β (year^{-1})	1.676 ± 0.057	1.181 ± 0.057
σ (year$^{-1/2}$)	0.302 ± 0.009	0.597 ± 0.019

Table 2. Mean profit optimization results.

		Optimization by age				Optimization by weight			
P (€/kg)		A_{opt}	$E[X_{A_{opt}}]$	$E[L_{A_{opt}}]$	$sd[L_{A_{opt}}]$	$E[A_{opt}]$	Q^*_{opt}	$E[L_{Q^*_{opt}}]$	$sd[L_{Q^*_{opt}}]$
SGM	3.00	0.97	253	29.41	91.11	1.04	271	33.14	57.03
	3.25	1.05	271	62.19	109.60	1.14	292	68.38	68.79
	3.50	1.13	285	96.94	127.00	1.23	309	105.95	79.85
	3.75	1.19	296	133.26	143.72	1.31	324	145.43	91.13
SBRM	3.00	0.71	189	14.14	29.53	0.77	200	14.71	34.37
	3.25	0.86	219	39.73	47.70	0.92	232	41.74	51.22
	3.50	0.97	240	68.49	61.21	1.05	256	72.29	64.62
	3.75	1.07	257	99.63	72.99	1.16	275	105.50	76.37

Figure 1. Expected profit, for SGM (solid line) and SBRM (dashed line), considering $P = 3.25$ euros/kg. Maximum expected profit is signaled "o". (a) for profit as a function of age (expected profit from selling the animal at the usual age of 16 months is signaled with "$*$"); (b) for profit as a function of weight.

Figure 2. Profit standard deviation, for SGM (solid line) and SBRM (dashed line), considering $P = 3.25$ euros/kg. Profit standard deviation corresponding to the optimal weight is signaled "o". (a) for profit as a function of age; (b) for profit as a function of weight.

Figure 3. $P[L_t > v]$ for $P = 3.25$ euros/kg at optimal age (a) and usual age (b), for SGM (solid line) and SBRM (dashed line).

Figure 4. Probability density function (a) and distribution function (b) of the time required for an animal to reach for the first time 300kg, for SGM (solid line) and SBRM (dashed line).

Figure 5. $P[L_{Q^*} > v]$ for $P = 3.25$ euros/kg when the animal reaches 300kg, for SGM (solid line) and SBRM (dashed line).

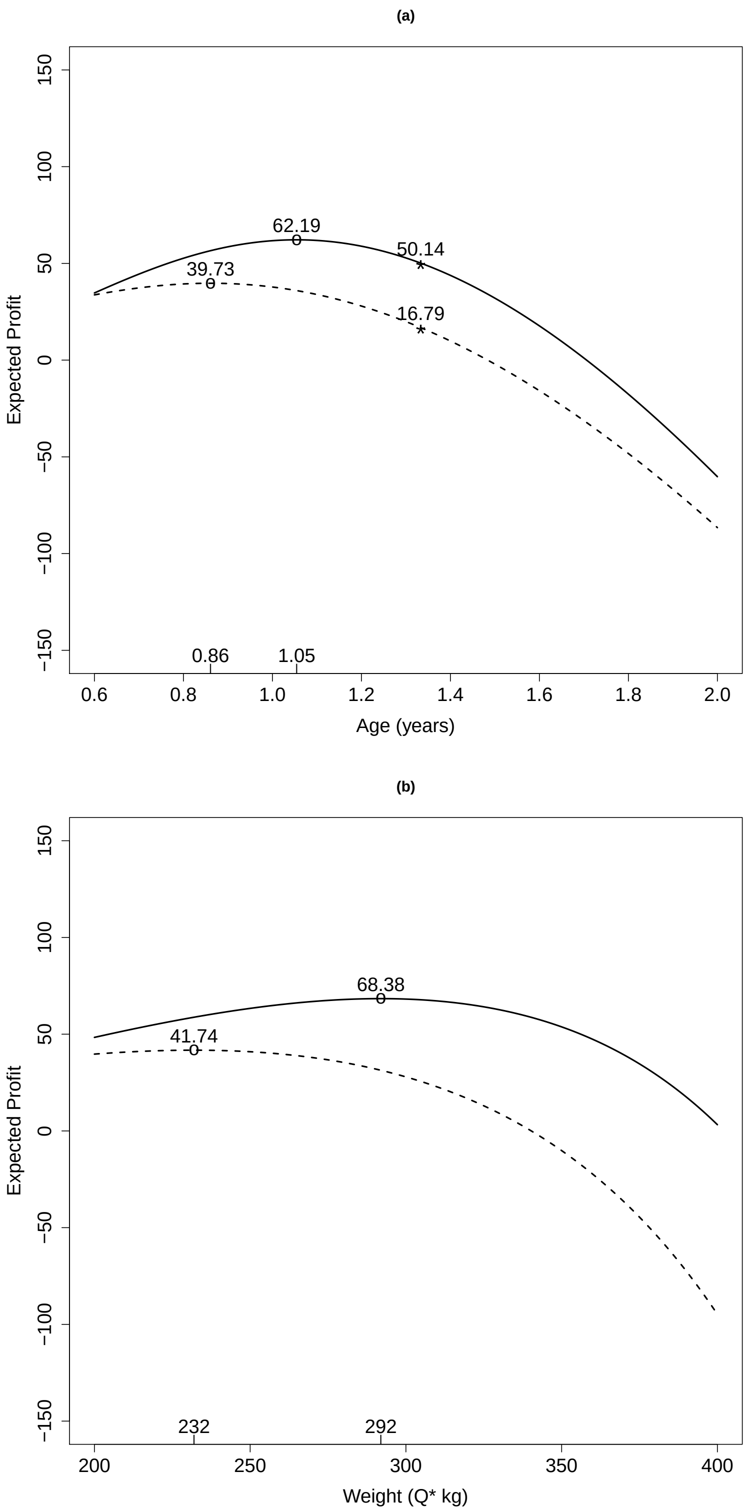
(a)
Expected Profit
150
100
50
0
−50
−100
−150
62.19
50.14
39.73
16.79
0.86
1.05
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Age (years)
(b)
Expected Profit
150
100
50
0
−50
−100
−150
68.38
41.74
232
292
200
250
300
350
400
Weight (Q* kg)

Figure 1.

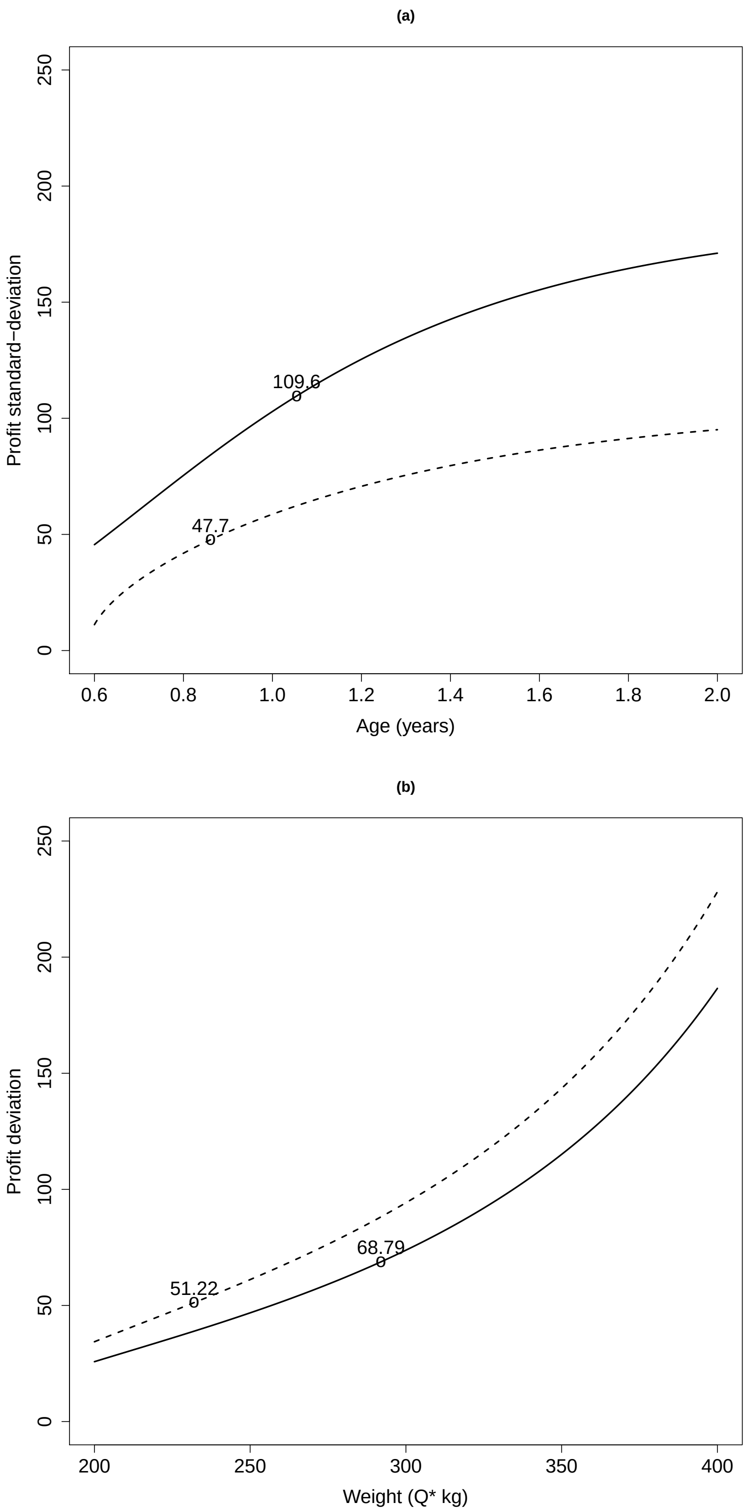
(a)
Profit standard−deviation
Age (years)
109.6
47.7
0
50
100
150
200
250
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
(b)
Profit deviation
Weight (Q* kg)
68.79
51.22
0
50
100
150
200
250
200
250
300
350
400

Figure 2.

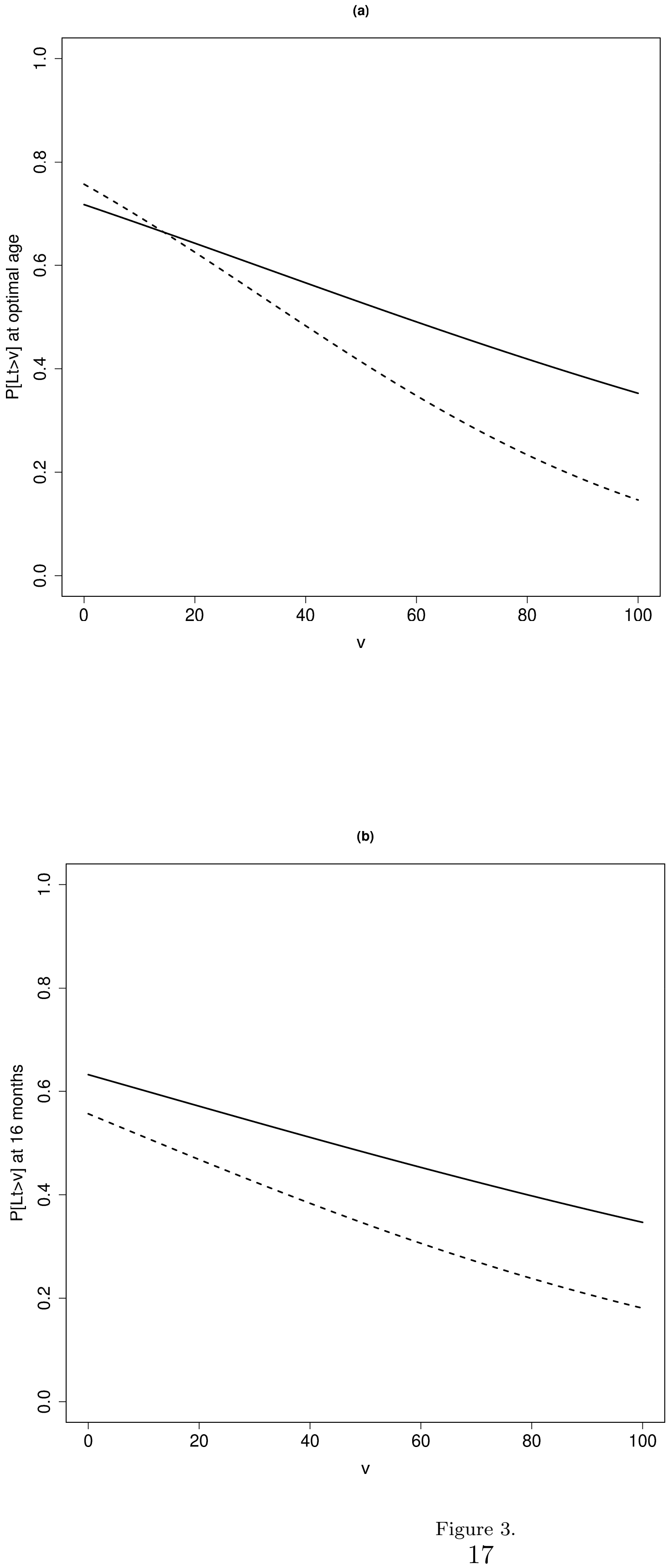
(a)
P[Lt>v] at optimal age
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
v
(b)
P[Lt>v] at 16 months
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
v

Figure 3.

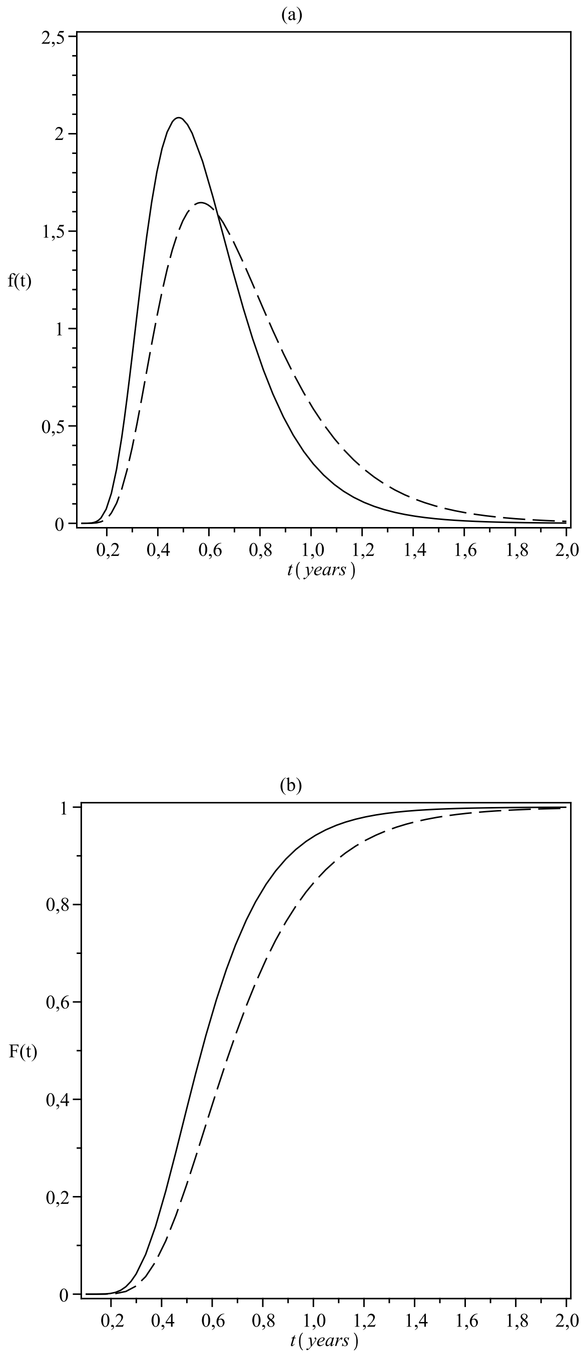
(a)
f(t)
2,5
2
1,5
1
0,5
0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
t (years)
(b)
F(t)
1
0,8
0,6
0,4
0,2
0
0,2
0,4
0,6
0,8
1,0
1,2
1,4
1,6
1,8
2,0
t (years)

Figure 4.

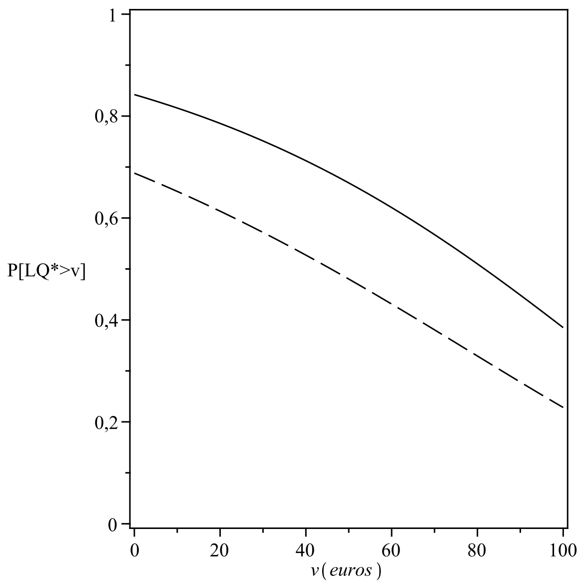

Figure 5.