Existência e Regularidade de Minimizantes para Integrais em Dimensão 1 do Cálculo das Variações com Lagrangiano Não-convexo Autónomo

Dissertação apresentada

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Esta dissertação não inclui as críticas e as sugestões feitas pelo júri

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Existence and regularity of minimizers for 1-dim integrals of the calculus of variations with nonconvex autonomous lagrangian

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Existência e Regularidade de Minimizantes

para Integrais em Dimensão 1 do Cálculo das Variações

com Lagrangiano Não-convexo Autónomo

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Resumo

Nesta dissertação apresentamos novas condições que provamos serem suficientes para garantir a existência, e alguma regularidade, de minimizantes para integrais simples autónomos não-convexos

$$\int_a^b \ell\left(x(t),\, x'(t)\right)\, dt, \qquad \int_a^b L\left(x(t),\, x'(t)\right)\, dt,$$

com $\ell : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$ (resp. $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$), na classe das funções absolutamente contínuas $x : [a,b] \to \mathbb{R}$ (resp. $x : [a,b] \to \mathbb{R}^n$) com $x(a) = A$, $x(b) = B$.

O lagrangiano $\ell : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$ pode: ter $\ell(s,\cdot)$ não-convexa (mesmo em $\xi = 0$), assumir o valor $+\infty$ livremente, ou ser não-boreliano. De facto, impomos apenas as seguintes hipóteses básicas: $\ell(\cdot)$ é $\mathcal{L} \otimes \mathcal{B}$–mensurável com $\ell(s,\cdot)$ semicontínua inferiormente e superlinear. Para um tal lagrangiano $\ell(\cdot)$, mostramos existência e regularidade sob mais uma hipótese extra a ser escolhida entre várias possibilidades.

Relativamente ao lagrangiano $L(\cdot)$, este pode ser e.g. semicontínuo inferiormente e superlinear; e substituimos a hipótese usual de convexidade pela hipótese mais geral de "almost convexity", que no caso superlinear radial $L(s,|v|)$ é automaticamente satisfeita quando $L(s,\cdot)$ é convexa no zero.

No que diz respeito ao caso escalar, os resultados foram obtidos generalizando os resultados de A. Ornelas para o caso 0–convexo, $\ell^{**}(\cdot,0) = \ell(\cdot,0)$; nomeadamente usando a bimonotonia. Esta propriedade de regularidade foi a base de partida para conseguirmos provar os resultados aqui apresentados.

Relativamente ao caso vectorial, os resultados foram obtidos usando reparametrizações, e aplicando os referidos resultados de bimonotonia a tais reparametrizações.

Apresentamos também algumas aplicações destes resultados para provar a existência de minimizantes em exemplos concretos onde não se podem aplicar resultados já conhecidos.

Palavras chave: cálculo das variações, integrais não-lineares não-convexos, propriedades de regularidade.

Existence and regularity of minimizers for 1-dim integrals of the calculus of variations with nonconvex autonomous lagrangian

Maria Clara da Palma Carlota

Abstract

In this thesis we present new conditions which we prove to be sufficient to guarantee existence, and some regularity, of minimizers for nonconvex autonomous 1-dim integrals

$$\int_a^b \ell\left(x(t),\, x'(t)\right)\, dt\,, \quad \int_a^b L\left(x(t),\, x'(t)\right)\, dt$$

with $\ell : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$ (resp. $L : \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$) among the absolutely continuous functions $x : [a,b] \to \mathbb{R}$ (resp. $x : [a,b] \to \mathbb{R}^n$) with $x(a) = A\,,\ x(b) = B$.

The lagrangian $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ may: have $\ell(s,\cdot)$ nonconvex (even at $\xi = 0$), assume $+\infty$ values freely, or be non-Borel. Indeed, our only basic hypotheses are: $\ell(\cdot)$ is $\mathcal{L}\otimes\mathcal{B}-$measurable and has $\ell(s,\cdot)$ lower semicontinuous and superlinear. For such $\ell(\cdot)$, we prove existence and regularity under an extra hypothesis to be chosen among several possibilities.

As to $L(\cdot)$, it may be e.g. superlinear lower semicontinuous; and we replace convexity by almost convexity, an hypothesis which in the radial superlinear case $L(s,|v|)$ is automatically satisfied when $L(s,\cdot)$ is convex at zero.

Concerning the scalar case, our results have been obtained by generalizing the results of A. Ornelas for the $0-$convex case, $\ell^{**}(\cdot,0) = \ell(\cdot,0)$; namely using bimonotonicity. This regularity property has been the starting basis to reach the results here presented.

As to the vector case, our results have been obtained by using reparametrizations, and by applying the bimonotonicity results to such reparametrizations.

We also present some applications of these results to show existence of minimizers in concrete examples, not covered by previous results.

Key words: calculus of variations, nonconvex nonlinear integrals, regularity properties

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Chapter 1

Introduction

Consider the classical problem of the calculus of variations: minimize the integral

$$\int_a^b L\left(t\,,\, x(t)\,,\, x'(t)\right)\, dt \qquad\qquad on \quad \mathcal{X}^n_{A,B}\,, \tag{1.1}$$

where $\mathcal{X}^n_{A,B}$, $n \geq 1$, is the class of AC (*absolutely continuous*) functions $x : [a,b] \to \mathbb{R}^n$ satisfying boundary conditions $x(a) = A$, $x(b) = B$. (In the *scalar* case $n = 1$ we will use $\mathcal{X}_{A,B}$ instead of $\mathcal{X}^1_{A,B}$.)

The aim of this thesis is to prove new *sufficient conditions* guaranteeing existence and regularity of minimizers for the problem (1.1) when the *lagrangian* $L(\cdot)$ is *autonomous*, i.e., does not depend on the variable t (*time*):

$$L : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]\,.$$

(In the scalar case we will use $\ell(\cdot)$ instead of $L(\cdot)$.)

A classical example is the brachistochrone (or path of quickest descent) problem which consists in the following: given two points in the vertical plane $\alpha = (a, B)$ and $\beta = (b, B)$ $(A < B)$, find the curve C joining α and β such that a material point, starting at α, will glide from α to β along C, under the force of gravity only, in least time (neglecting frictional effects). Labelling the horizontal axis with t and the vertical axis with s, and assuming s to be positive downwards, well known mechanical arguments lead to the conclusion that the curve C should be the graph of a minimizer of the integral (1.1) with the lagrangian

$$\ell(s, \xi) = \frac{\sqrt{1+\xi^2}}{\sqrt{s - A}}.$$

The solution is the (unique) cycloid with a vertical cusp at α and passing through β. One could argue that the calculus of variations was born in June 1696, when Johann Bernoulli presented this problem to the international mathematical community, as a challenge.

The first approach to solve problem (1.1) was what we can call an *indirect method*, based on the naive idea they had that (1.1) always has a solution $y(\cdot)$: to determine it, one looks for conditions that minimizers must satisfy (*necessary conditions*) and then eliminate possible candidates until eventually they become reduced to a unique $y(\cdot)$. But this method can fail due to several reasons, one of them being because problem (1.1) may have no solutions.

In 1915 Leonida Tonelli ([T]) presented a direct approach to problem (1.1), the so called *direct method of the calculus of variations*. Indeed, he realized that by transferring to the integrals of the calculus of variations the Ascoli-Arzelà compactness and Baire semicontinuity theorems for real functions, he could obtain an existence theory. The reasoning to prove existence of minimizers for the integral (1.1) is the following: prove that it is *lsc* (*lower semicontinuous*) with respect to the weak topology of $\mathcal{X}^n_{A,B}$ and then show that there exists a minimizing sequence $(y_n(\cdot))$ which converges weakly to some $y(\cdot) \in \mathcal{X}^n_{A,B}$. Indeed, in such case we have:

$$\inf_{x(\cdot)\in\mathcal{X}^n_{A,B}} \int_a^b L(x(t), x'(t))\, dt \le \int_a^b L(y(t), y'(t))\, dt \le$$

$$\le \liminf_{n\to\infty} \int_a^b L(y_n(t), y_n'(t))\, dt := \inf_{x(\cdot)\in\mathcal{X}^n_{A,B}} \int_a^b L(x(t), x'(t))\, dt.$$

If $L(\cdot)$ is lsc with *superlinear growth*, i.e.

$$L(s,\xi) \ge \theta(|\xi|) \quad \forall (s,\xi) \qquad \text{with} \qquad \theta(r)/r \to +\infty \;\; as \;\; r \to +\infty,$$

and $\xi \mapsto L(s,\xi)$ is convex then these conditions are satisfied. Indeed, lower semicontinuity and convexity ensure the weak sequential lower semicontinuity of the integral (see e.g. [De G], [Ol 1], [Io]), while the superlinear growth ensures the relative compactness of minimizing sequences (see e.g. [Bu Gi Hi], [Ek Te], [C]).

Though these hypotheses are sufficient to get existence of minimizers, they are not necessary, as simple examples show. For instance, if $\ell : \mathbb{R} \to [0,+\infty]$ is a lsc function, not necessarily convex or with superlinear growth, for which $\overline{co}\, epi\, \ell(\cdot)$ (i.e. the closed convex hull of the epigraph of $\ell(\cdot)$) has only bounded faces, then it is easy to see that for any A, $B \in \mathbb{R}$ there exists a minimizer for the integral

$$\int_a^b \ell\left(x'(t)\right)\, dt \qquad\qquad on \;\; \mathcal{X}_{A,B},$$

which, moreover, is Lipschitz continuous.

Therefore, in the last years several authors investigated the possibility of eliminating the hypotheses of convexity and / or superlinearity and of weakening the regularity required on $L(\cdot)$.

Concerning the scalar convex case, the regularity hypotheses on the lagrangian have been much weakened by [De G Bu Dal M] and [Amb]: they proved that to obtain weak sequential lower semicontinuity of the integral one does not need to impose $\ell(\cdot)$ to be lsc, it suffices to ask $\ell(\cdot)$ to be only $\mathcal{L} \otimes \mathcal{B}$ *– measurable* (as in [Roc We, 14.34]) with $\ell(\cdot, 0)$ lsc, provided *the slope at zero is integrable*, i.e. there exists $m(\cdot)$ in $L^1_{loc}(\mathbb{R})$ for which

$$\ell(s,\xi) \geq \ell(s,0) + m(s)\,\xi \quad \forall s,\ \xi. \tag{1.2}$$

Therefore for such a convex $\ell(\cdot)$ with superlinear growth there exist minimizers for the integral

$$\int_a^b \ell\,(x(t),\,x'(t))\;dt \qquad\qquad on \quad \mathcal{X}_{A,B}.$$

Even though this hypothesis is quite weak, in [Or 3] it was proved that (1.2) may still be weakened, namely replaced by any one of the following hypotheses: either $A = B$; or $\ell(\cdot)$ is lsc at $(s,0)$ $\forall s$; or $\ell(\cdot)$ is *integrably bounded near zero*, i.e.

$$\exists\, l, M : \mathbb{R} \to (0,+\infty),\ l \times M \in L^1(A,B):\ \ \ell(s,\xi) \leq l(s) \quad \forall\, |\xi| \leq \frac{1}{M(s)} \quad \forall s; \tag{1.3}$$

or else, more generally, $\ell(\cdot)$ is *approximable with integrable slopes at zero*, i.e.:

$$\forall n \in \mathbb{N} \quad \exists\, \varphi_n : \mathbb{R} \to [0,n] \quad lsc \quad with \quad (\varphi_n(s)) \nearrow \ell(s,0) \qquad \forall s, \tag{1.4}$$

$$\exists\, m_n(\cdot) \in L^1(A,B): \qquad \ell(s,\xi) \;\geq\; \varphi_n(s) \;+\; m_n(s)\;\xi \qquad \forall s, \xi. \tag{1.5}$$

This hypothesis (1.4)+(1.5) is weaker than (1.2) even in the case of a constant sequence – i.e. $\varphi_n(s) = \ell(s,0)$ and $m_n(s) = m(s)\ \forall n$ – because $m(\cdot)$ needs to be integrable only along $co\,\{A,B\}$, instead of on any bounded interval (even if the minimizer may have values outside of $co\,\{A,B\}$). Moreover, (1.4) + (1.5) is also weaker than either (1.3) or the hypothesis of $\ell(\cdot)$ being lsc at $(s,0)$ $\forall s$.

The above considerations concern the convex case, i.e., $\ell(s,\cdot)$ convex. But also this hypothesis may be weakened. Indeed, a series of papers have been devoted to the progressive weakening of the hypotheses allowing existence of minimizers to be proved in the nonconvex case (see e.g. [Ol 2], [Au Ta], [Marc], [Ray], [Ce Co], [Ce Mari], [Am Ce], [Am Mari], [Marq Or]). In all of these papers (except for [Ray], see (3.11)), existence has been proved for $\ell(\cdot)$ of *sum – type*, i.e.

$$\ell(s,\xi) = \psi(s) + \rho(s)\,h(\xi) \tag{1.6}$$

with $\rho(\cdot) \equiv 1$, imposing on the lsc $\psi(\cdot)$ different geometric restrictions. In the last one ([Marq Or]) $\psi(\cdot)$ is assumed *concave – monotone*, namely: $\psi(\cdot)$ is *concave* (resp. *monotone*) along each interval of an open set $\mathcal{C}$ (resp. $\mathcal{M}$) with $\mathcal{C} \bigcup \mathcal{M} = \mathbb{R}$; while in [Am Ce], $\psi(\cdot)$ should satisfy the following: $\forall s \in \mathbb{R}$ $\exists$ open nonempty interval, with an extremity at s, where $\psi(\cdot)$ decreases strictly (in case $h^{**}(0) < h(0)$) as the distance from s increases.

Afterwards it was shown that these geometric restrictions on the lagrangian may be weakened under the extra hypothesis of 0 *– convexity* :

$$\ell(\cdot,0) = \ell^{**}(\cdot,0), \tag{1.7}$$

where $\ell^{**}(\cdot)$ is the bipolar, defined by $epi\ \ell^{**}(s,\cdot) := \overline{co}\ epi\ \ell(s,\cdot)\ \ \forall s$. Indeed, several papers ([Am Ce], [Fu Marc Or], [Or 1], [Or 2]) have weakened progressively the hypotheses imposed to prove existence of *minimizers* under (1.7). The first 3 of these papers dealt mainly with the *sum* case cited above, while [Or 2] treated completely the "*affine*" case (1.6).

Finally, the general nonlinear case $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ was treated in [Or 4] (see also [Or 5]) under extremely weak hypotheses. Namely assuming $\ell(\cdot)$ $\mathcal{L}\otimes\mathcal{B}-$*measurable* with $\ell(s,\cdot)$ *lsc*, [Or 4] proved existence of a $0-$*relaxed* minimizer $y_0(\cdot)$ (i.e. a minimizer for an integral whose lagrangian is, in a sense, the convexification of $\ell(s,\cdot)$ only at the velocity zero $\forall s$ (see (2.11))) which is a true minimizer whenever

$$\varphi(\cdot) = \ell^{**}(\cdot,0) = \ell(\cdot,0),$$

where $\varphi(\cdot) := \ell^c(\cdot,0)$, with $\ell^c(\cdot)$ the largest of the $0-lsc-convex$ *functions* $\leq \ell(\cdot)$ (see definitions 2.1 and 2.2), so that, in particular $\ell^c(\cdot) = \ell^{**}(\cdot)$ whenever $\ell^{**}(\cdot)$ is lsc at $\xi = 0$, i.e. at $(s,0)\ \forall s$. Moreover, it suffices to $0-$*convexify* the lagrangian $\ell(s,\cdot)$ only at one special point s', because such $y_0(\cdot)$ is known to satisfy a special regularity property, it is *bimonotone* (namely: apart from a (*possibly empty*) interval (a',b') where it *remains stopped* at the mentioned point s', $y_0(\cdot)$ is *strictly monotone* (with $y_0'(t) \neq 0$ a.e.) along each one of the remaining intervals, $[a,a']$ and $[b',b]$). More precisely, once a *bimonotone* minimizer $y_0(\cdot)$ is known, then either the stopping interval is empty and no convexity condition is needed, even at zero; or else $a' < b'$ and – without affecting the property of $y_0(\cdot)$ being a minimizer – one may redefine $\ell(s,0)$ to become $+\infty$ at every $s \neq s'$. This is the main basis on which this thesis stands.

In chapter 2 we present the results obtained in [Or 4] (see also [Or 5]), concerning the existence of $0-$*relaxed* minimizers, in the superlinear case, for the 1-dim integral

$$\int_a^b \ell\left(x(t),\, x'(t)\right)\, dt \qquad\qquad on\ \ \mathcal{X}_{A,B}\,.$$

Chapter 3 is devoted to prove existence of *true* minimizers, in the superlinear case, even with $\varphi(s) < \ell^{**}(s,0) < \ell(s,0)\ \ \forall s$, provided an adequate extra hypothesis is satisfied, which needs a previously known $0-$*relaxed minimizer* $y_0(\cdot)$, to be stated precisely. Indeed, defining $S_0 := y_0([a,b])$ and $\varphi_0(\cdot) := \varphi_{|S_0}(\cdot)$, then such extra hypothesis consists in imposing: there must exist some minimizer s' of $\varphi_0(\cdot)$ which is not a *mean – strict minimizer* of $\varphi(\cdot)$. This hypothesis generalizes directly the hypotheses of [Am Ce] and [Marq Or], and also the preceding ones (of [Au Ta], [Marc], [Ray], and [Ce Co]). Notice that one can also give very general hypotheses, namely e.g. (3.10) and (3.12) (as in corollary 3.7), under which existence of minimizers may be guaranteed without needing to know $y_0(\cdot)$.

On the contrary, in chapter 4 we prove a new *SC* (*sufficient condition*) on the boundary data (a,A,b,B), which is applicable even when $\ell(\cdot)$ does not satisfy the extra hypothesis of chapter 3. Indeed, such SC guarantees existence of a *bimonotone* $0-relaxed$ minimizer $y_0(\cdot)$ which *does not stop* so that it is a *true* minimizer. Therefore this SC replaces completely the condition of $0-convexity$; under it, existence of minimizers is obtained *without any convexity hypothesis*, and with *almost no regularity hypotheses* on the lagrangian $\ell(\cdot)$. Indeed, besides the Basic Hypotheses of chapter 2 and superlinearity, we only need the validity of the $DuBois-Reymond\ differential\ inclusion$ (2.15) for the relaxed minimizer.

The bimonotonicity results of [Or 4] are also used in chapter 5, where we prove new sufficient conditions for existence and regularity of minimizers for the 1-dim integral

$$\int_a^b L(x(t), x'(t))\, dt \qquad\qquad on\ \ \mathcal{X}^n_{A,B}.$$

Since $L(s,\cdot)$ is allowed to be nonconvex, we consider the bipolar $L^{**}(s,\cdot)$ of $L(s,\cdot)$, and the corresponding convexified integral

$$\int_a^b L^{**}(x(t), x'(t))\, dt \qquad\qquad on\ \ \mathcal{X}^n_{A,B}.$$

We call $y_c(\cdot)$ a relaxed minimizer provided $y_c(\cdot)$ minimizes this integral.

Unlike the scalar case, in this vector case the hypothesis of 0−convexity does not suffice to guarantee existence of minimizers (see section 5.5). Indeed, one needs to impose more than just 0−convexity in order to obtain, for general dimension, the same operational possibilities; namely *almost convexity*. This concept was born, for multifunctions, in the paper [Ce Or], to prove existence of solutions to nonconvex differential inclusions and to time-optimal control problems, using reparametrizations. The technique of reparametrizations has been used by A. Cellina and collaborators, during the last decade, to prove also: Lipschitz properties for minimizers (see e.g. [Ce], [Ce Fe]) and existence results for convex noncoercive lagrangians (see e.g. [Ce Tr Za], [Ce Fe]).

In the first result we present, $L(\cdot)$ is assumed to be lsc with superlinear growth at infinity. This ensures existence of a relaxed minimizer $y_c(\cdot)$, which is then changed to become a new relaxed minimizer $y(\cdot)$ for which $L^{**}(y(t), y'(t)) = L(y(t), y'(t))$ a.e. on $[a,b]$; so that $y(\cdot)$ is a true minimizer. In the second result, existence of $y_c(\cdot)$ is used as one hypothesis. We need no growth assumption to turn $y_c(\cdot)$ into $y(\cdot)$, in particular we do not need to impose $co\,epi\,L(s,\cdot)$ closed.

We also present applications of these results to show existence of true minimizers in concrete examples, not covered by previous results.

As said above, two techniques have been combined to prove these results. The first one is the above cited reparametrizations, while the second technique is bimonotonicity. Here, for the first time, bimonotonicity is applied to the reparametrizations.

Concerning the convex noncoercive vector case, there are available results (see e.g. [Cl], [Ce Tr Za], [Ce Fe]), which ensure existence of relaxed minimizers. However, as far as we know, corollary 5.5 is the first existence result obtained in the general vector nonconvex noncoercive autonomous case. Indeed, nonconvex, noncoercive vector results have been obtained but just for lagrangians of sum-type (see e.g. [Cr], [Cr Mal 1]), or for radial lagrangians (see e.g. [Cr Mal 2]).

Chapter 2

Preliminaries for the scalar case

2.1 Introduction

This chapter contains preliminaries for the scalar case treated in chapters 3 and 4, where the problem considered is to prove existence, and regularity, of minimizers for the 1-dim integral

$$\int_a^b \ell\left(x(t), x'(t)\right) dt \qquad\qquad on \quad \mathcal{X}_{A,B}, \tag{2.1}$$

where $\mathcal{X}_{A,B}$ is the class of AC (*absolutely continuous*) functions $x : [a,b] \to \mathbb{R}$ with $x(a) = A$ and $x(b) = B$. The lagrangian $\ell(\cdot)$ will always satisfy (in chapters 2, 3, 4) the following extremely weak

Basic Hypotheses : $\ell : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$

is $\mathcal{L} \otimes \mathcal{B} - measurable$ (as in [Roc We]) with $\ell(s, \cdot)$ *lsc* (*lower semicontinuous*) $\forall s$.

Any function as this $\ell(\cdot)$ will be called a $BH - function$, for easier reference.

Since we allow $\ell(s, \cdot)$ to be nonconvex, we also use its bipolar $\ell^{**}(\cdot)$ (defined by $epi\ \ell^{**}(s, \cdot) := \overline{co}\ epi\ \ell(s, \cdot)\ \ \forall s$, namely the closed convex hull of the epigraph of $\ell(s, \cdot)$) together with its $0 - lsc - convexified\ \ lagrangian\ \ \ell^c(\cdot)$.

We call $y(\cdot)$ a *relaxed minimizer* of (2.1) provided $y(\cdot)$ minimizes the integral obtained by replacing $\ell(\cdot)$ in (2.1) by $\ell^c(\cdot)$; and the *superlinear growth hypothesis* (2.2) is used to obtain (via *direct method*) existence of *relaxed minimizers*, from which $0 - relaxed\ \ minimizers$ of (2.1) are built. These are minimizers of the *nonconvexified* integral, which is obtained from (2.1) by replacing $\ell(\cdot)$ with the *nonconvexified lagrangian* $\ell^0(\cdot)$.

The aim of this chapter is to define precisely all these concepts, and to review the preceding results (of [Or 4] or [Or 5]), concerning existence and regularity properties of $0-$ *relaxed minimizers*. This is important for chapters 3 and 4, since $0-$ *relaxed minimizers* are the starting basis for those chapters.

2.2 Existence of 0−relaxed minimizers

Definition 2.1 *A $BH-$ function is called superlinear provided*

$$\frac{\inf \ell(\mathbb{R},\xi)}{|\xi|} \to +\infty \qquad as \qquad |\xi| \to \infty\,. \tag{2.2}$$

A $BH-$ function $\ell(\cdot)$ having $\ell(s,\cdot)$ convex $\forall s$ is called $0-lsc-$ convex provided $\ell(\cdot,0)$ is lsc and $\ell(\cdot)$ is approximable by integrable slopes at zero, i.e.

$$\forall n \in \mathbb{N} \qquad \exists\, \varphi_n : \mathbb{R} \to [0,n] \quad lsc \quad with \quad (\varphi_n(s)) \nearrow \ell(s,0) \qquad \forall s \tag{2.3}$$

$$\exists\, m_n(\cdot) \in L^1_{loc}(\mathbb{R}) : \qquad \ell(s,\xi) \geq \varphi_n(s) + m_n(s)\,\xi \qquad \forall s,\xi\,. \tag{2.4}$$

To see an example, consider the $BH-$ *function*

$$\ell(s,\xi) := \begin{cases} \left(2+\xi\, s\, |s|^{-\delta}\right)^+ & for \quad s \neq 0 \quad and \quad (|\xi|=1 \quad or \quad \xi=0) \\ 1 & for \quad s=0 \quad and \quad (|\xi|=1 \quad or \quad \xi=0) \\ +\infty & \forall s \quad for \quad |\xi|>1\,. \end{cases}$$

Then $\ell^{**}(\cdot)$ is $0-lsc-convex$ for $\delta<2$, but $\ell^{**}(\cdot)$ is not $0-lsc-convex$ for $\delta>2$.

Notice: any superlinear function $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ which is lsc has $\ell^{**}(\cdot)$ $0-lsc-convex$, since $\ell^{**}(\cdot)$ is then lsc; indeed, more generally, any *superlinear* $BH-function$ with $\ell(s,\cdot)$ *convex* and $\ell(\cdot)$ lsc at $(s,0)$ $\forall s$ is $0-lsc-convex$ (see [Or 3, th. 1]). In particular, a $0-lsc-convex$ function $\ell(\cdot)$ may have $\ell(s,0)=+\infty$, or $\partial\ell(s,0)=\emptyset$ with finite $\ell(s,0)$; indeed, what matters is, somehow, integrability in s of the slope of $\ell(s,\cdot)$ near zero. There are $0-lsc-convex$ functions which are not lsc at $\xi=0$; see e.g. the example above with $\delta<2$.

Definition 2.2 *Given any $BH-function$ $\ell(\cdot)$, define the $0-lsc-convexified$ lagrangian $\ell^c(\cdot)$ to be the largest of the $0-lsc-convex$ $BH-functions$ $\leq \ell(\cdot)$. Define also*

$$\varphi(\cdot) := \ell^c(\cdot,0)\,. \tag{2.5}$$

One easily checks that such $\ell^c(\cdot)$ always exists, hence we may consider the corresponding $0-lsc-convexified\ integral$

$$\int_a^b \ell^c\left(x(t),\, x'(t)\right)\, dt \qquad\qquad on \quad \mathcal{X}_{A,B}\,, \tag{2.6}$$

for any $BH-function$ $\ell(\cdot)$. In concrete applications, with an explicitly given superlinear $\ell(\cdot)$, one may not know exactly what $\ell^c(\cdot)$ is; however, in such cases $\ell^c(\cdot)$ may be replaced by the following easily computable function $\overline{\ell}(\cdot)$, in all that follows. (But notice: $\ell^c(\cdot)$ is in some cases, as in the example above, better than $\overline{\ell}(\cdot)$.) To compute $\overline{\ell}(\cdot)$, define $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ by: $epi\, f(\cdot) := \overline{epi}\, \ell(\cdot)$; $epi\, g(s,\cdot) := \overline{co}\, epi\, f(s,\cdot)$ $\forall s$; $h(\cdot,\xi) := \ell(\cdot,\xi)$ for $\xi \neq 0$ and $h(\cdot,0) := g(\cdot,0)$; $epi\, \overline{\ell}(s,\cdot) := \overline{co}\, epi\, h(s,\cdot)$. Then $\overline{\ell}(\cdot)$ is convex and lsc at $\xi = 0$ (see [Or 3, part (j) of proof, p. 10]). Therefore $\overline{\ell}(\cdot)$ is $0 - lsc - convex$.

Proposition 2.3 *(See [Or 3, th. 1])*

Let $\ell(\cdot)$ be a superlinear $BH - function$. Then for any A, B there exists a relaxed minimizer $y_c(\cdot)$ (i.e. a minimizer of (2.6)). Moreover, we may impose $y_c(\cdot)$ to be bimonotone (i.e. properties (ii) and (iii), of definition 2.4 below, hold with $y_0(\cdot)$ replaced by $y_c(\cdot)$).

In the remaining of chapter 2 (and chapters 3, 4) we restrict our attention mostly to the set $S_0 := y_c([a,b])$ with $y_c(\cdot)$ as given by proposition 2.3, we define

$$\varphi_0(\cdot) := \varphi_{|S_0}(\cdot) \tag{2.7}$$

and consider the sets

$$S_{A,B} := \{s' \in S_0 : \varphi_0(s') = \min \varphi_0(\cdot)\} \tag{2.8}$$

$$S^{=}_{A,B} := \{s' \in S_{A,B} : \varphi_0(s') = \ell(s',0)\}, \quad S^{<}_{A,B} := \{s' \in S_{A,B} : \varphi_0(s') < \ell(s',0)\}.$$

Consider now the *nonzero extremities* $\alpha(s)$, $\beta(s)$ of the intervals of affinity of $\ell^c(s,\cdot)$ which have the other extremity at $\xi = 0$. Or, more precisely, consider the subdifferential $\partial\ell^c(s,\cdot)$ of $\ell^c(s,\cdot)$ (see [Ek Te, p. 20]), and define the set

$$F_0(s) := (\partial\ell^c(s,\cdot))^{-1}(\partial\ell^c(s,0)) = \{\xi \in \mathbb{R} : \partial\ell^c(s,\xi) \cap \partial\ell^c(s,0) \neq \emptyset\}.$$

Then, assuming $\ell(\cdot)$ to be a *superlinear* $BH - function$, as we always do here, we have: the set $\{0\} \cup F_0$ must be an interval

$$[\alpha(s), \beta(s)] \quad with \quad \alpha(s) \leq 0 \leq \beta(s); \tag{2.9}$$

$\ell^c(s,\cdot)$ is affine along $[\alpha(s),0]$ and along $[0,\beta(s)]$ – and even affine along $[\alpha(s),\beta(s)]$ whenever $s \in S_{afz}$, see (3.1), in particular for those $s \in S^{<}_{A,B}$ for which $\ell^{**}(s,0) = \varphi(s)$ – and the following equalities always hold true at those s where $\alpha(s) \neq 0$ (resp. $\beta(s) \neq 0$) : $\ell^c(s,\alpha(s)) = \ell(s,\alpha(s))$ (*resp.* $\ell^c(s,\beta(s)) = \ell(s,\beta(s))$).

For each $s \in S_0$, consider the set $V(s)$ of those $\xi \neq 0$ for which (see [Roc])

$$\{\ (\xi, \ell^c(s,\xi))\ \} \quad is \quad a \quad 0-dimensional \quad face \quad of \quad epi\, \ell^c(s,\cdot); \tag{2.10}$$

and define the *nonconvexified lagrangian*

$$\ell^0(s,\xi) := \begin{cases} \ell^c(s,\xi) & \text{for} \quad \xi \in V(s) \text{ and } s \in S_0 \\ \varphi(s) & \text{for} \quad \xi = 0 \text{ and } s \in S_{A,B} \\ +\infty & \text{elsewhere}, \end{cases}$$

together with the *nonconvexified integral*

$$\int_a^b \ell^0(x(t), x'(t))\, dt \qquad \text{on} \quad \mathcal{X}_{A,B}, \tag{2.11}$$

for any $BH-function$ $\ell(\cdot)$. (Notice: in concrete examples in which $y_c(\cdot)$, hence S_0 and $S_{A,B}$, may be not explicitly known, one may define $\ell^0(\cdot)$ using $\mathbb{R}$ instead of S_0 and $S_{A,B}$, but just for the definition of $\ell^0(\cdot)$. Notice also: the idea of $\ell^0(\cdot)$ is, somehow, to obtain the largest function having bipolar $\ell^{0**}(\cdot) = \ell^c(\cdot)$ where it matters; and $\ell^c(s,\xi) = \ell(s,\xi)\ \forall \xi \in V(s)\ \forall s \in S_0$.)

Definition 2.4 *We call* $y_0(\cdot)$ *a* $0-relaxed$ *minimizer of* (2.1) *on* $\mathcal{X}_{A,B}$ *provided* $y_0(\cdot)$ *has values* $y_0([a,b]) = S_0$ *and is a bimonotone minimizer of the nonconvexified integral, in the sense that* $y_0(\cdot) \in \mathcal{X}_{A,B}$ *minimizes the* $0-lsc-convexified$ *integral* (2.6) *on this class and (using* $\ell^c(\cdot)$, $\varphi(\cdot)$, $\ell^0(\cdot)$ *as defined in* (2.5) *and* (2.11)) $y_0(\cdot)$ *satisfies the following regularity:*

(i) $\ell^c(y_0(t), y_0'(t)) = \ell^0(y_0(t), y_0'(t))$ *a.e. in* $[a,b]$;

(ii) $y_0(\cdot)$ *remains a constant* s' *along some subinterval* (a',b'), *with* $a' \le b'$;

(iii) $y_0(\cdot)$ *is monotone along each one of the remaining subintervals,* $[a,a']$ *and* $[b',b]$, *with derivative "bounded away" from zero, in the sense that*

$$y_0'(t) \notin \{0\} \cup (\alpha(y_0(t)), \beta(y_0(t))) \qquad \text{a.e. in} \quad [a,a'] \cup [b',b]; \tag{2.12}$$

(iv)

$$\ell^c(y_0(t), y_0'(t)) = \ell(y_0(t), y_0'(t)) \qquad \text{a.e. in} \quad [a,a'] \cup [b',b]; \tag{2.13}$$

(v)

$$s' \in S_{A,B}, \tag{2.14}$$

whenever $\int_a^b \ell^c(y_c(t), y_c'(t))\, dt < +\infty$.

We say that $y_0(\cdot)$ *stops (resp. does not stop) in case* $a' < b'$ *(resp.* $a' = b'$*). One may always choose the stopping point* $s' \in S_{A,B}$.

Proposition 2.5 *(See [Or 4, th. 1] or [Or 5, th. 1])*

Let $\ell(\cdot)$ *be a superlinear* $BH-$ *function. Then for any* A, B *the nonconvex integral* (2.1), *defined on the class* $\mathcal{X}_{A,B}$, *has a* $0-$ *relaxed minimizer* $y_0(\cdot)$.

Remark 2.6 *According to [Amb As Bu, th. 4.1] (see also [Dal M Fr, th. 3.10]), any minimizer, as* $y_c(\cdot)$, *of the convexified integral* (2.6) *satisfies the DuBois – Reymond differential inclusion (i.e. there exists a constant* q *for which*

$$\ell^c(y_c(t), y_c'(t)) \in q + y_c'(t)\,\partial\ell^c(y_c(t), y_c'(t)) \quad a.e. \quad in \quad [a,b]) \tag{2.15}$$

provided the minimum value is finite and

$$y_c'(t) \in interior\,(\ell^c(y_c(t), \cdot))^{-1}(\mathbb{R}) \quad a.e.. \tag{2.16}$$

Clearly a simple way of guaranteeing this is by asking the domain $(\ell^c(y_c(t), \cdot))^{-1}(\mathbb{R})$ *to be open for a.e.* t; *and this happens automatically in case*

$$\ell^c(y_c(t), \mathbb{R}) \subset \mathbb{R} \quad for \quad a.e. \quad t \in [a,b]. \tag{2.17}$$

Given a relaxed minimizer $y_c(\cdot)$ *(guaranteed to exist by proposition* 2.3*), the proof of proposition* 2.5 *consists in changing it so as to obtain a* $0-$*relaxed minimizer* $y_0(\cdot)$. *This proof shows that if* $y_c(\cdot)$ *satisfies the DuBois-Reymond differential inclusion* (2.15) *for* $\ell^c(\cdot)$, *independently of* (2.16) *holding true or not, either for* $y_c(\cdot)$ *or for* $y_0(\cdot)$, *then also* $y_0(\cdot)$ *satisfies it for* $\ell^0(\cdot)$, *with the same constant* q.

Remark 2.7 *For a function* $\ell : \mathbb{R} \times \mathbb{R} \to [0, +\infty]$, *a simple way to obtain* $\mathcal{L} \otimes \mathcal{B}-$ *measurability is to ask that* $\ell(\cdot, \xi)$ *be measurable* $\forall \xi$ *and* $\ell(s, \cdot)$ *be either continuous (e.g. convex with finite values) or else convex lsc with domain never a singleton* $\forall s$ *(see [Roc We, 14.34, 14.39, 14.42]). Notice also: definition* 2.2 *implies the measurability of* $\ell^c(x(\cdot), x'(\cdot))$ $\forall x(\cdot) \in \mathcal{X}_{A,B}$, *see [Or 3, prop. 2], hence definitions* 2.4, 3.1 *make sense.*

Chapter 3

Existence in the scalar case without mean-strict minimizers

3.1 Introduction

This chapter is devoted to prove existence of *true* minimizers, for *superlinear* $BH-functions$ $\ell(\cdot)$, even with $\varphi(s) < \ell^{**}(s,0) < \ell(s,0)$ $\forall s$, provided an adequate extra hypothesis is satisfied, which is stated by using a $0-relaxed$ *minimizer* $y_0(\cdot)$, already known to exist. Indeed, defining $S_0 := y_0([a,b])$ and $\varphi_0(\cdot) := \varphi_{|S_0}(\cdot)$, then such extra hypothesis consists in imposing: there must exist some minimizer s' of $\varphi_0(\cdot)$ which is not a *mean−strict minimizer* of $\varphi(\cdot)$. This generalizes directly the hypotheses of [Am Ce] and [Marq Or], and also the preceding ones (of [Au Ta], [Marc], [Ray], and [Ce Co]); and allows to show existence of minimizers in many cases even without knowing $y_0(\cdot)$, as e.g. in corollary 3.7.

Given the extreme weakness of our Basic Hypotheses, we should clarify precisely what we mean by a solution of (2.1).

Definition 3.1 *A function* $y(\cdot) \in \mathcal{X}_{A,B}$ *is called a minimizer of the integral* (2.1) *provided* $y(\cdot)$ *minimizes the* $0-lsc-convexified$ *integral* (2.6); *and* $\ell(y(\cdot), y'(\cdot)) = \ell^c(y(\cdot), y'(\cdot))$ *a.e.* (*unless* $\int_a^b \ell^c(y(t), y'(t))\, dt = +\infty$).

This definition makes sense: $y(\cdot)$ will give to the integral (2.1) a value $\leq$ than any other $x(\cdot) \in \mathcal{X}_{A,B}$ for which the integral (2.1) is defined.

3.2 Existence and regularity of true minimizers

From now on we will consider a $0-relaxed\ minimizer$ $y_0(\cdot)$ (guaranteed to exist by proposition 2.5), hence $S_0 = y_0([a,b])$ and $\varphi_0(\cdot) := \varphi_{|S_0}(\cdot)$. In what follows, when we mention the nonempty interval (s', s) we mean, more precisely, *interior* $(co\ \{s', s\})$, regardless of having $s > s'$ or $s < s'$.

Using $\alpha(\cdot)$, $\beta(\cdot)$ as in (2.9), define the set S_{afz} of those $s \in \mathbb{R}$ at which $\ell^c(s, \cdot)$ is *affine at zero*, i.e.

$$\alpha(s) < 0 < \beta(s) \qquad and \qquad \ell^c(s,0) = (1-\lambda)\,\ell^c(s,\alpha(s)) + \lambda\,\ell^c(s,\beta(s)), \tag{3.1}$$

for an adequate λ to have $0 = (1-\lambda)\,\alpha(s) + \lambda\,\beta(s)$. By (2.9), $\ell^c(\cdot)$ may be replaced by $\ell(\cdot)$ in the *rhs* of equality (3.1). Consider now the function

$$\mu : \mathbb{R} \to (0, +\infty], \qquad \mu(s) := \begin{cases} \frac{1}{|\alpha(s)|} + \frac{1}{\beta(s)} & for \quad s \in S_{afz} \\ +\infty & for \quad s \notin S_{afz}; \end{cases} \tag{3.2}$$

and, for each bounded open interval $S \neq \emptyset$ satisfying the integrability condition

$$\mu(\cdot) \in L^1(S), \tag{3.3}$$

define the $\mu - mean\ integral$ of $\varphi(\cdot)$ over S by:

$$\oint_S \varphi(s)\,ds := \frac{1}{\int_S \mu(s)\,ds} \int_S \varphi(s)\,\mu(s)\,ds. \tag{3.4}$$

For each $s' \in S_0$ consider $S_{afz}(s')$, namely the set of those $s \neq s'$ for which the interval $S := (s', s)$ satisfies (3.3). Clearly $S_{afz}(s')$ is always the union of two intervals, each one of them possibly empty or bounded (maybe including the extremity away from s') or unbounded. Define also the set $S^{-}_{afz}(s')$ consisting of the points $s \in S_{afz}(s')$ for which $\varphi(s) = \ell(s,0) \leq \min \varphi_0(\cdot)$.

Definition 3.2 *We say that* $s' \in S_{A,B}$ *is not a* $mean-strict\ minimizer$ *of* $\varphi(\cdot)$ *provided: either*

$$s' \in S^{=}_{A,B}; \tag{3.5}$$

or

$$s' \ belongs\ to\ an\ open\ \neq \emptyset\ interval\ S \subset S_{A,B}\ having\ \mu(\cdot) \in L^1(S); \tag{3.6}$$

or else $\exists\, s'_1 \in S_0$ *for which: either*

$$\exists\, s'' \in S_{afz}(s'_1) \setminus S_0 : \quad \frac{b'-a'}{\int_{s'_1}^{s''} \mu(s)\,ds} \in \mathbb{Z} \quad and \quad \oint_{s'_1}^{s''} \varphi(s)\,ds \leq \min \varphi_0(\cdot) \tag{3.7}$$

or

$$\exists\, s'' \in S_{afz}\left(s_1'\right) \setminus S_0 : \quad \oint_{s_1'}^{s} \varphi(\sigma)\, d\sigma \le \min \varphi_0(\cdot) \quad \forall s \in \left(s_1', s''\right); \tag{3.8}$$

or else

$$\exists\, s'' \in S_{afz}^{-}\left(s_1'\right) \setminus S_0 : \quad \left| \int_{s_1'}^{s''} \mu(s)\, ds \right| \le b' - a' \quad \textit{and} \quad \oint_{s_1'}^{s''} \varphi(s)\, ds \le \min \varphi_0(\cdot)\,. \tag{3.9}$$

Remark 3.3 *In definition 3.2, it appears easier to grasp its meaning by naming it, as we did, in the negative way. In concrete applications, to prove existence of minimizers for (2.1) with explicitly given $\ell(\cdot)$, it may turn out more convenient to select $s_1' \ne s'$. Moreover, in the rest of this remark we will assume, to simplify, $s_1' = s'$.*

In the special "affine" case (3.18) in which $\rho(\cdot)$ is constant and $\psi(\cdot) = \varphi(\cdot)$ is lsc and is concave − monotone at s', in the sense of s' belonging to an open interval I where $\varphi(\cdot)$ is either concave or monotone, then clearly $\varphi(\cdot)$ satisfies (3.6) or (3.8) for some s'' with either $s'' < s'$ or $s' < s''$; indeed, such $\varphi(\cdot)$ must satisfy

$$\forall s' \in \mathbb{R} \quad \exists\, s'' \ne s' : \quad \begin{array}{l} \varphi(\cdot) \textit{ decreases along } co\,\{s', s''\} \\ \textit{as the distance from } s' \textit{ increases.} \end{array} \tag{3.10}$$

(Notice: in the reality we need (3.10) to be satisfied only for those $s' \in S_{A,B}$, see (2.7); and by $\varphi(\cdot)$ decreasing we mean either non-strictly or strictly.) Since this is the hypothesis used (with strict decreasing) in [Am Ce], in particular definition 3.2 generalizes both the hypotheses of [Marq Or] and of [Am Ce]. Of course definition 3.2 also generalizes the one of [Ray], imposing

$$\frac{\partial}{\partial s}\left[\ell^c(s,\xi) - \xi\, \frac{\partial}{\partial \xi}\ell^c(s,\xi)\right] \ne 0 \quad \forall\, s\,, \xi\,; \tag{3.11}$$

indeed, if one asks this for $\xi = 0$ only, then it means this C^2 $\varphi(\cdot) = \ell^c(\cdot\,,0)$ cannot have local minimum points, hence has to be strictly monotone. As to [Au Ta], they ask (3.11) to hold $\forall \xi \in [\alpha(s), \beta(s)]$ $\forall s$.

Let us comment now on definition 3.2 itself. As to the integrability condition $\mu(\cdot) \in L^1(S)$, is easily imposed, e.g. by asking:

$$\varphi(\cdot) = \ell^{**}(\cdot\,,0) < \ell(\cdot\,,0)\,, \quad \mu(\cdot) := \frac{1}{|\alpha(\cdot)|} + \frac{1}{\beta(\cdot)} \in L^1_{loc}(\mathbb{R})\,. \tag{3.12}$$

(This holds e.g. in the "affine" case (3.18), with $\rho(\cdot) \ge 1$, $h(\cdot)$, $\psi(\cdot)$ and $\rho(\cdot)$ lsc, and $h^{**}(\cdot) < h(\cdot)$.) Once such integrability is guaranteed, then $S_{afz}(s') = \mathbb{R} \setminus \{s'\}$.

Therefore what really matters, in general, to be able to apply definition 3.2 in concrete examples, is to know whether it is possible to find such a point s'' satisfying the inequality (3.7) or (3.8) or (3.9). And, of course, this is possible only in case $\varphi(\cdot)$ has some one-sided mean, as in the lhs of (3.8) with s near s', which is $\le \varphi(s')$. One possibility for this to happen is e.g. in case $\varphi(\cdot)$ increases (resp. decreases) or *remains constant along a small open* $\neq \emptyset$ *interval to the* $left$ *(resp. right) of* s'. *But clearly* (3.7) *or* (3.8) *or* (3.9), *can never be satisfied at a minimizer* s' *of* $\varphi_0(\cdot)$ *if, say,* s' *is also the unique global minimizer of* $\varphi(\cdot)$; *or if* $\ell(\cdot)$ *is so wild that* $|(s',s)\setminus S_{afz}|>0$ *or* $\mu(\cdot)\notin L^1(s',s)$ *or* $\varphi(\cdot)\,\mu(\cdot)\notin L^1(s',s)$, $\forall s\neq s'$.

Another possibility to get (3.7) *or* (3.8) *or* (3.9), *is:* $\varphi(\cdot)$ *might, say, decrease strictly as one approaches* s' *from the left; while* $\varphi(\cdot)$ *might oscillate wildly to the right of* s', *in such a way as to yield a right-sided mean* $s\mapsto ⨏_{s'}^{s}\varphi(\sigma)\,d\sigma$ *decreasing as* s *increases (at least for small enough* $s-s'$*), i.e. in some sense having* $\varphi(\cdot)$ *to decrease more than increase, in each oscillation.*

Definition 3.4 *We call* $y(\cdot)$ *a* $finitely-monotone$ $minimizer$ *of the integral* (2.1) *on* $\mathcal{X}_{A,B}$ *provided:* $y(\cdot)$ *minimizes the* $0-lsc-convexified$ *integral* (2.6); $\ell^c(y(\cdot),y'(\cdot))=\ell(y(\cdot),y'(\cdot))$ *a.e. in case the corresponding minimum value is finite; and, for some* $N\in\mathbb{N}$, $[a,b]$ *may be partitioned into* N *subintervals* $[a_i,b_i]$, *along each of which* $y(\cdot)$ *is strictly monotone and satisfies* (2.12), *except possibly along one subinterval* $[a_{i_0},b_{i_0}]$ *where* $y(\cdot)$ *is constant. In such case we also call* $y(\cdot)$ *a* $N-monotone$ $minimizer$ *of* (2.1). *We say that* $y(\cdot)$ *stops (resp. does not stop) in case* $a_{i_0}<b_{i_0}$ *(resp.* $a_{i_0}=b_{i_0}$*).*

Theorem 3.5 *(Sufficient condition for the existence of a true minimizer)*

Let $\ell(\cdot)$ *be a superlinear* $BH-function$, *so that proposition* 2.5 *may be applied to reach* $y_0(\cdot)$ *and* $\varphi_0(\cdot)$, S_0 *as in* (2.7).

Then there exists a true minimizer $y(\cdot)$, *for the fully nonconvex integral* (2.1) *on* $\mathcal{X}_{A,B}$, *which is* $finitely-monotone$, *provided: either* $a'=b'$; *or* $S^{=}_{A,B}\neq\emptyset$; *or else* $\exists\, s'\in S_{A,B}$ *which is not a* $mean-strict$ $minimizer$ *of* $\varphi(\cdot)$.

In case $y_0(\cdot)$ *is Lipschitz continuous (see* [Or 6]*) and* $\alpha(\cdot)$, $\beta(\cdot)$ *are locally bounded then also* $y(\cdot)$ *is Lipschitz.*

Proof : Denote by N the positive integer in (3.7). (As to (3.8), one easily checks that such N must always exist; while in case (3.9) one takes $N=1$.) Let us assume, say, $s'<s''$ just to fix ideas. Define

$$\theta := \frac{N}{b'-a'}\int_{s'}^{s''}\frac{1}{|\alpha(s)|}\,ds\,, \tag{3.13}$$

so that

$$1-\theta = \frac{N}{b'-a'}\int_{s'}^{s''}\frac{1}{\beta(s)}\,ds\,; \tag{3.14}$$

and define

$$\tau_+ : [s', s''] \longrightarrow \left[a', a' + (1-\theta)\, \tfrac{b'-a'}{N}\right], \qquad \tau_+(s) := a' + \int_{s'}^{s} \tfrac{1}{\beta(\sigma)}\, d\sigma,$$

$$\tau_- : [s', s''] \longrightarrow \left[a' + (1-\theta)\, \tfrac{b'-a'}{N}, a' + \tfrac{b'-a'}{N}\right], \quad \tau_-(s) := a' + \tfrac{b'-a'}{N} + \int_{s'}^{s} \tfrac{1}{\alpha(\sigma)}\, d\sigma$$

Since we are assuming $1/\alpha(\cdot)$, $1/\beta(\cdot) \in L^1(s', s'')$, these functions $\tau_+(\cdot)$, $\tau_-(\cdot)$ will be AC and monotone with derivative $\neq 0$ a.e.:

$$\tau_+(\cdot) \quad \textit{increases}, \quad \textit{with} \quad \tau_+'(s) = 1/\beta(s) > 0 \quad \textit{a.e.};$$

while

$$\tau_-(\cdot) \quad \textit{decreases}, \quad \textit{with} \qquad \tau_-'(s) = 1/\alpha(s) < 0 \quad \textit{a.e.}.$$

Moreover $\tau_+(s') = a'$, $\tau_-(s') = a' + \frac{b'-a'}{N}$, $\tau_-(s'') = a' + (1-\theta)\, \frac{b'-a'}{N} = \tau_+(s'')$, by (3.13), (3.14). The inverse functions of $\tau_+(\cdot)$, $\tau_-(\cdot)$, respectively

$$x_+ : \left[a', a' + (1-\theta)\, \frac{b'-a'}{N}\right] \to [s', s''],$$

$$x_- : \left[a' + (1-\theta)\, \frac{b'-a'}{N}, a' + \frac{b'-a'}{N}\right] \to [s', s''],$$

are well-defined and are AC (see [Or 1, remark 4]), $x_+(\cdot)$ increases and $x_-(\cdot)$ decreases (both with derivative $\neq 0$ a.e.); and

$$x_+(a') = s' = x_-\left(a' + \frac{b'-a'}{N}\right),$$

$$x_+\left(a' + (1-\theta)\, \frac{b'-a'}{N}\right) = s'' = x_-\left(a' + (1-\theta)\, \frac{b'-a'}{N}\right).$$

We may therefore define the function

$$x_1 : \left[a', a' + \tfrac{b'-a'}{N}\right] \longrightarrow [s', s''],$$

$$x_1(t) := \begin{cases} x_+(t) & \textit{for } t \textit{ in } \left[a', a' + (1-\theta)\, \tfrac{b'-a'}{N}\right] \\ x_-(t) & \textit{for } t \textit{ in } \left[a' + (1-\theta)\, \tfrac{b'-a'}{N}, a' + \tfrac{b'-a'}{N}\right]. \end{cases}$$

Clearly $x_1(a') = s' = x_1\left(a' + \frac{b'-a'}{N}\right)$, $x_1\left(a' + (1-\theta)\, \frac{b'-a'}{N}\right) = s''$,

$$x_1'(t) = \begin{cases} \beta(x_1(t)) & \textit{for a.e. } t \textit{ in } \left[a', a' + (1-\theta)\, \tfrac{b'-a'}{N}\right] \\ \alpha(x_1(t)) & \textit{for a.e. } t \textit{ in } \left[a' + (1-\theta)\, \tfrac{b'-a'}{N}, a' + \tfrac{b'-a'}{N}\right]. \end{cases}$$

Therefore,

$$+\infty > \int_{a'}^{a'+\frac{b'-a'}{N}} \ell^0 \left(y_0(t), y_0'(t)\right) dt = \varphi(s') \frac{b'-a'}{N} \geq$$

$$\geq \int_{s'}^{s''} \varphi(s)\, \mu(s)\, ds = \int_{s'}^{s''} \left[\ell(s, \alpha(s)) \frac{1}{|\alpha(s)|} + \ell(s, \beta(s)) \frac{1}{\beta(s)}\right] ds =$$

$$= \int_{s'}^{s''} \ell(s, \beta(s)) \frac{1}{\beta(s)}\, ds + \int_{s'}^{s''} \ell(s, \alpha(s)) \frac{1}{|\alpha(s)|}\, ds =$$

$$= \int_{a'}^{a'+(1-\theta)\frac{b'-a'}{N}} \ell\left(x_+(t), x_+'(t)\right) dt + \int_{a'+(1-\theta)\frac{b'-a'}{N}}^{a'+\frac{b'-a'}{N}} \ell\left(x_-(t), x_-'(t)\right) dt =$$

$$= \int_{a'}^{a'+\frac{b'-a'}{N}} \ell\left(x_1(t), x_1'(t)\right) dt,$$

using [Or 3, prop. 3 (b)]. But then the inequality $\geq$ will have to be an equality. (Otherwise a contradiction would be reached, since $y_0(\cdot)$ has to be already a $0-$*relaxed minimizer* hence $x_1(\cdot)$ cannot yield a smaller value to this integral.) Therefore

$$\int_{a'}^{a'+\frac{b'-a'}{N}} \ell\left(x_1(t), x_1'(t)\right) dt = \int_{a'}^{a'+\frac{b'-a'}{N}} \ell^0\left(y_0(t), y_0'(t)\right) dt. \tag{3.15}$$

Let us repeat now N times this construction. Namely: we construct $x_2(\cdot)$ in $\left[a' + \frac{b'-a'}{N}, a' + 2\frac{b'-a'}{N}\right]$, ..., $x_N(\cdot)$ in $\left[a' + (N-1)\frac{b'-a'}{N}, b'\right]$ in the same way as $x_1(\cdot)$ was constructed above in $\left[a', a' + \frac{b'-a'}{N}\right]$. By gluing together these N patches, we end up with an AC function

$$y : [a,b] \longrightarrow [s', s''],$$

$$y(t) := \begin{cases} y_0(t) & for\ t\ in\ [a, a'] \\ x_1(t) & for\ t\ in\ \left[a', a' + \frac{b'-a'}{N}\right] \\ \vdots & \vdots \\ x_N(t) & for\ t\ in\ \left[a' + (N-1)\frac{b'-a'}{N}, b'\right] \\ y_0(t) & for\ t\ in\ [b', b] \end{cases}$$

with $y'(t) \neq 0$ a.e., $y(a') = s' = y(b')$, and, by repeating the equality (3.15) N times and adding,

$$\int_{a'}^{b'} \ell\left(y(t), y'(t)\right) dt = \int_{a'}^{b'} \ell^0\left(y_0(t), y_0'(t)\right) dt.$$

Therefore $y(\cdot)$ is the desired $2(N+1)-$*monotone true minimizer* of the fully nonconvex integral (2.1) on the class $\mathcal{X}_{A,B}$. The proof is complete.

Notice that this $y(\cdot)$ also minimizes the integral obtained from (2.1) by replacing $\ell(\cdot)$ with a new lagrangian $\ell^1(\cdot)$ defined to be equal to $\ell(\cdot)$ on R and $+\infty$ on $\mathbb{R}^2 \setminus R$, where R is the set of those (s,ξ) for which: either $\xi \in V(s)$ and $s \in S_0$; or $\xi = 0$ and $\ell(s,0) = \varphi(s)$; or else $\xi \in \{\alpha(s), \beta(s)\}$ and $s \notin S_0$. (Taking $V(s)$ as in (2.10).)

Let us present now a simpler version of theorem 3.5, conceived for the special "*affine*" case of definition 3.2, as in (3.18). In this case we need to assume $h(\cdot)$ lsc; and let us assume also, for simplicity, $\psi(\cdot)$ and $\rho(\cdot)$ lsc, so that $\varphi(\cdot) := \psi(\cdot)+\rho(\cdot)\, h^{**}(0)$ and there exists a $0-$*relaxed minimizer* $y_0(\cdot)$ as in definition 2.4, which gives $S_0 := y_0([a,b])$ and $\varphi_0(\cdot) := \varphi_{|S_0}(\cdot)$. Then $s' \in S_{A,B}$ will not be a *mean – strict minimizer* of $\varphi(\cdot)$ provided either $h^{**}(0) = h(0)$ or else, either $s' \in \textit{interior}\,(S_{A,B}) \neq \emptyset$ or else, considering the maximal open interval (α,β) containing 0 along which $h^{**}(\cdot)$ is affine and setting $\mu := \frac{1}{|\alpha|} + \frac{1}{\beta}$, we have: $\exists\, s_1' \in S_0$ for which:
either

$$\exists\, s'' \notin S_0 : \quad \frac{b'-a'}{(s''-s_1')\,\mu} \in \mathbb{Z} \quad \text{and} \quad \frac{1}{s''-s_1'} \int_{s_1'}^{s''} \varphi(s)\; ds \leq \min \varphi_0(\cdot) \tag{3.16}$$

or

$$\exists\, s'' \notin S_0 : \quad \frac{1}{s-s_1'} \int_{s_1'}^{s} \varphi(\sigma)\; d\sigma \leq \min \varphi_0(\cdot) \quad \forall s \in (s_1', s''). \tag{3.17}$$

Corollary 3.6 *Let $h : \mathbb{R} \to [0,+\infty]$ be a lsc function having $h(\xi)\,/\,|\xi| \to +\infty$ as $|\xi| \to \infty$, and let $\psi : \mathbb{R} \to [0,+\infty)$ and $\rho : \mathbb{R} \to [1,+\infty)$ be lsc functions, so that proposition* 2.5 *may be applied to reach a $0-$relaxed minimizer $y_0(\cdot)$ of*

$$\int_a^b \psi(x(t)) + \rho(x(t))\, h(x'(t))\; dt \qquad \text{on } \mathcal{X}_{A,B}. \tag{3.18}$$

Then there exists a true minimizer $y(\cdot)$, for the nonconvex integral (3.18), *which is finitely – monotone, provided either $a' = b'$ or $h^{**}(0) = h(0)$ or $\varphi(\cdot)$ satisfies* (3.10) *or else $\exists\, s' \in S_{A,B}$ that is not a mean – strict minimizer of $\varphi(\cdot)$ (i.e. as in* (3.16) *or* (3.17)).

In case $y_0(\cdot)$ is Lipschitz continuous (see e.g. [Or 6]) then also $y(\cdot)$ is.

Notice, however: $\psi(\cdot)$ and $\rho(\cdot)$ could be taken just Lebesgue–measurable and apply theorem 3.5.

Corollary 3.7 *Let $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ be lsc and superlinear, with: either $\varphi(\cdot) := \ell^{**}(\cdot,0) = \ell(\cdot,0)$; or else* (3.10) *and* (3.12).

Then there exists a minimizer for the integral (2.1) *which is finitely – monotone.*

Chapter 4

Existence in the scalar case under appropriate mean speeds

4.1 Introduction

In this chapter we consider the problem of existence of minimizers for the integral (2.1) defined in the more restricted class

$$\mathcal{Z}_{A,B} := \{x(\cdot) \in \mathcal{X}_{A,B} : \quad x'(t) \neq 0 \quad for \quad a.e. \quad t \quad in \quad [a,b]\}.$$

Indeed, we present a SC (*sufficient condition*) on the boundary data (a, A, b, B) guaranteeing existence of a *bimonotone* $0-$*relaxed* minimizer $y_0(\cdot)$, for the integral (2.1) on $\mathcal{X}_{A,B}$, which *does not stop* (so that $y_0(\cdot)$ is actually in $\mathcal{Z}_{A,B}$). Another way of seeing this result is the following. If the boundary data satisfies such SC then it ceases to matter whether $\ell(s', \cdot)$ is $0-$*convex* or not; so that, in particular, $y_0(\cdot)$ is not only a $0-$*relaxed* minimizer, it is indeed a true minimizer of this integral (2.1). That is: this SC replaces completely the condition of $0-$*convexity*; under it, existence of minimizers is obtained *without any convexity hypothesis*, and with *almost no regularity hypotheses* on the lagrangian $\ell(\cdot)$. Indeed, besides $\ell: \mathbb{R} \times \mathbb{R} \to [0, +\infty]$ being $\mathcal{L} \otimes \mathcal{B}-$*measurable* with $\ell(s, \cdot)$ *lsc* $\forall s$, having *superlinear growth* (as in (2.2)), we only need the validity of the *DuBois – Reymond differential inclusion* (2.15) for the relaxed minimizer (in particular imposing e.g. the restriction $x'(\cdot) \geq 0$ causes no problem). Notice that we use in this chapter the same definition 3.1 of minimizer.

We start by obtaining an inequality which is a NC (*necessary condition*) for a *bimonotone* $0-$*relaxed* minimizer $y_0(\cdot)$ *to stop* (i.e. to have $a' < b'$); and the opposite inequality yields immediately the above SC for $y_0(\cdot)$ *not to stop* (i.e. to have $a' = b'$, so that $y_0(\cdot) \in \mathcal{Z}_{A,B}$). Then a similar reasoning gives another inequality which is a NC for $y_0(\cdot)$ *not to stop*; and, again, the opposite inequality yields a SC for $y_0(\cdot)$ *to stop*, so that $y_0(\cdot) \in \mathcal{X}_{A,B} \setminus \mathcal{Z}_{A,B}$.

This research was suggested by the paper [Mu Pe], in which interesting numerical examples have been reported.

4.2 Existence and regularity of true minimizers

From now on we will consider a $0-$*relaxed minimizer* $y_0(\cdot)$ (guaranteed to exist by proposition 2.5); will assume (2.15) (assuming e.g. (2.16) or (2.17)); and will use

$$\varphi(\cdot)=\ell^c(\cdot,0),\quad S_0=y_0([a,b]),\quad \varphi_0(\cdot)=\varphi_{|S_0}(\cdot), \tag{4.1}$$

as given by proposition 2.5 and (2.7), (2.8). Before stating our first result, let us introduce useful notations. In what follows, s' is always a parameter in $S_{A,B}$. Define: $V(s',s'):=\{0\}$,

$$V(s,s'):=\{\xi\notin(\alpha(s),\beta(s)):\ \varphi(s')\in\ell^c(s,\xi)-\xi\,\partial\ell^c(s,\xi)\}\quad for\ \ s\neq s'; \tag{4.2}$$

$$\begin{gathered} V_-(s,s'):=V(s,s')\cap(-\infty,0),\\ \beta_-(s,s'):=\min V_-(s,s'),\quad \alpha_-(s,s'):=\max V_-(s,s'), \end{gathered} \tag{4.3}$$

$$\begin{gathered} V_+(s,s'):=V(s,s')\cap(0,+\infty),\\ \alpha_+(s,s'):=\min V_+(s,s'),\quad \beta_+(s,s'):=\max V_+(s,s'). \end{gathered} \tag{4.4}$$

4.2.1 A necessary condition in case the 0−relaxed minimizer stops

Define

$$\beta_A(\cdot,s'):co\{A,s'\}\to\mathbb{R},\qquad \beta_B(\cdot,s'):co\{s',B\}\to\mathbb{R},$$

$$\beta_A(s,s'):=\begin{cases}\beta_-(s,s') & for\ \ s\ \ in\ \ [s',A]\ \ if\ \ s'<A\\ \beta_+(s,s') & for\ \ s\ \ in\ \ [A,s']\ \ if\ \ A\leq s',\end{cases}$$

$$\beta_B(s,s'):=\begin{cases}\beta_-(s,s') & for\ \ s\ \ in\ \ [B,s']\ \ if\ \ B<s'\\ \beta_+(s,s') & for\ \ s\ \ in\ \ [s',B]\ \ if\ \ s'\leq B;\end{cases}$$

and, to use in case $y_0(\cdot)$ is monotone, $\beta_{A,B}:\ co\{A,B\}\longrightarrow\mathbb{R}$,

$$\beta_{A,B}(s):=\beta_A(s,s')=\beta_B(s,s')\quad\forall s'\in S_{A,B}.$$

(Notice: at points where $V_-(\cdot) = \emptyset$ (resp. $V_+(\cdot) = \emptyset$) we assume $\beta_-(\cdot) = 0 = \alpha_-(\cdot)$ (resp. $\alpha_+(\cdot) = 0 = \beta_+(\cdot)$).)

In the next theorem we use the definitions of sections 2.1 and 2.2.

Theorem 4.1 (*Necessary condition, in case the* $0-$*relaxed minimizer stops*)

Let $\ell(\cdot)$ *be a superlinear* $BH-$*function.*

Let $y_0(\cdot)$ *be as in* (4.1). *In case* $y_0(\cdot)$ *does not minimize the integral* (2.1) *we must have:*

(*a*) $y_0(\cdot) \equiv s'$ *along some nonempty subinterval* $(a', b') \subset [a, b]$, *for some* $s' \in S^{<}_{A,B}$;

(*b*) *if, moreover,* $S^{=}_{A,B} \neq \emptyset$, *then* $y_0(\cdot)$ *may be modified so as to become a true minimizer* $y(\cdot)$ *of the integral* (2.1);

(*c*) *otherwise* $S^{=}_{A,B}$ *is empty and,* (*under* (2.15) *or* (2.16) *or* (2.17))

$$\int_A^{s'} \frac{1}{\beta_A(s, s')}\, ds + \int_{s'}^{B} \frac{1}{\beta_B(s, s')}\, ds < b - a. \tag{4.5}$$

In case $y_0(\cdot)$ *is monotone and satisfies inclusion* (2.15) *then*

$$\int_A^B \frac{1}{\beta_{A,B}(s)}\, ds < b - a. \tag{4.6}$$

Proof: (*a*) By (2.13), $y_0(\cdot)$ has to minimize also the integrals

$$\begin{aligned} &\textstyle\int_a^{a'} \ell(x(t), x'(t))\, dt, \qquad x(a) = A, \quad x(a') = s', \\ &\textstyle\int_{b'}^{b} \ell(x(t), x'(t))\, dt, \qquad x(b') = s', \quad x(b) = B. \end{aligned} \tag{4.7}$$

Therefore, if $y_0(\cdot)$ does not minimize the original integral (2.1) then

$$\int_{a'}^{b'} \ell^c(y_0(t), y_0'(t))\, dt = (b' - a')\varphi(s') < (b' - a')\ell(s', 0) = \int_{a'}^{b'} \ell(y_0(t), y_0'(t))\, dt,$$

hence $a' < b'$ and $\varphi(s') < \ell(s', 0)$, so that $s' \in S^{<}_{A,B}$. Indeed, otherwise $y_0(\cdot)$ would minimize the integral (2.1):

$$\int_a^b \ell(y_0(t), y_0'(t))\, dt = \int_a^b \ell^c(y_0(t), y_0'(t))\, dt.$$

(*b*) This case is obvious.

(*c*) But even when $S^{=}_{A,B}$ is empty, since $y_0(\cdot)$ does not minimize the integral (2.1), by (*a*), (2.15) and (2.14) we have, along the nonempty interval (a', b'),

$$q = \varphi(s') = \min \varphi_0(\cdot).$$

Therefore $y_0(\cdot)$ satisfies also the explicit differential inclusion (see (4.2))

$$y_0'(t) \in V(y_0(t), s') \qquad a.e. \qquad in \qquad [a,b]. \tag{4.8}$$

Let us consider an interval, say $[a, a']$, where $y_0'(t) > 0$ a.e.; then (4.8) and (4.4) yield

$$\alpha_+(y_0(t), s') \leq y_0'(t) \leq \beta_+(y_0(t), s') \qquad a.e. \qquad in \qquad [a, a'].$$

Since the function $t \mapsto s = y_0(t)$ restricted to $[a, a']$ has an inverse $\tau_A(\cdot)$ which is AC with derivative $\tau_A'(\cdot) > 0$ a.e. on $[A, s']$, we may consider the function $v_A(\cdot) := 1/\tau_A'(\cdot)$, obtaining

$$y_0'(t) = v_A(y_0(\cdot)) = v_A(s) \qquad for \ \ a.e. \ \ t \in [a, a'] \ \ and \ \ a.e. \ \ s \in [A, s'].$$

Therefore

$$\alpha_+(s, s') \leq v_A(s) \leq \beta_+(s, s') \qquad a.e. \ in \ \ [A, s'].$$

We may define a new lagrangian $\ell_A(s, \xi) := \xi / v_A(s)$, thus obtaining a $\mathcal{L} \otimes \mathcal{B}$–*measurable* function $\ell_A(\cdot)$ with $\ell_A(\cdot, 0) \equiv 0$, to which [Or 3, prop. 2] is applicable, yielding the measurability of the function $y_0'(\cdot) / v_A(y_0(\cdot))$; and since $y_0'(\cdot) / v_A(y_0(\cdot)) = 1$ a.e., in particular $y_0'(\cdot) / v_A(y_0(\cdot)) \in L^\infty(a, a')$. Therefore [Or 3, prop. 3 (a)] may be applied to justify the change of variable in the integral

$$a' - a = \int_a^{a'} 1 \, dt = \int_a^{a'} \frac{y_0'(t)}{v_A(y_0(t))} \, dt = \int_A^{s'} \frac{1}{v_A(s)} \, ds \geq \int_A^{s'} \frac{1}{\beta_+(s, s')} \, ds.$$

In case we also have $y_0'(t) > 0$ a.e. in $[b', b]$ then, similarly, since $y_0(\cdot)$ restricted to $[b', b]$ has an inverse $\tau_B(\cdot)$ which is AC, we may consider the function $v_B(\cdot) := 1 / \tau_B'(\cdot)$, obtaining

$$b - b' = \int_{b'}^{b} 1 \, dt = \int_{b'}^{b} \frac{y_0'(t)}{v_B(y_0(t))} \, dt = \int_{s'}^{B} \frac{1}{v_B(s)} \, ds \geq \int_{s'}^{B} \frac{1}{\beta_+(s, s')} \, ds.$$

In particular $1 / \beta_+(\cdot, s') \in L^1(A, B)$, and adding these two inequalities one gets

$$(b - a) - (b' - a') \geq \int_A^B \frac{1}{\beta_+(s, s')} \, ds.$$

Since $a' < b'$ we finally reach

$$\int_A^B \frac{1}{\beta_+(s, s')} \, ds < b - a. \tag{4.9}$$

In particular, in this case $y_0(\cdot)$ is monotone, hence there could exist plenty of possible stopping points $s' \in S_{A,B}$ for $y_0(\cdot)$. However they all yield the same $\beta_{A,B}(\cdot)$. This proves (c) of the statement in case $y_0(\cdot)$ always increases.

The other cases may be treated similarly. The proof of theorem 4.1 is complete.

Theorem 4.2 (*Sufficient condition for the existence of a true minimizer*)

Let $\ell(\cdot)$ be a superlinear BH – function.

Then there exists a bimonotone minimizer of the nonconvex integral (2.1) *on* $\mathcal{X}_{A,B}$ (*namely: as in definition* 2.4 *with* $a' = b'$) *provided* $\exists y_0(\cdot)$ *as in* (4.1), *satisfying* (2.15), (2.16) *or* (2.17), *and: either* $y_0(\cdot)$ *is* (*bimonotone but*) *non-monotone and*

$$b - a \leq \int_A^{s'} \frac{1}{\beta_A(s, s')}\, ds + \int_{s'}^{B} \frac{1}{\beta_B(s, s')}\, ds\,, \tag{4.10}$$

where s' *is the point in* $S_0 \setminus co\,\{A, B\}$ *at maximum distance from* $co\,\{A, B\}$; *or else* $y_0(\cdot)$ *is monotone and*

$$b - a \leq \int_A^{B} \frac{1}{\beta_{A,B}(s)}\, ds\,. \tag{4.11}$$

Proof : It suffices to notice that if the inequality (4.10), opposite to the inequality (4.5) of theorem 4.1, holds true then $y_0(\cdot)$ cannot stop because: if it stopped then the inequality (4.5) would be true (by theorem 4.1); and since also the opposite inequality (4.10) holds true (by the hypotheses of theorem 4.2), we would reach a contradiction.

The proof is complete.

4.2.2 A sufficient condition for 0−relaxed minimizer to stop

Let us introduce further useful notation. Define:

$$\alpha_A : co\left\{A, s'\right\} \to \mathbb{R}, \qquad \alpha_B : co\left\{s', B\right\} \to \mathbb{R},$$

$$\alpha_A(s,s') := \begin{cases} \alpha_-(s,s') & for\ \ s\ \ in\ \ [s',A]\ \ if\ \ s' < A \\ \alpha_+(s,s') & for\ \ s\ \ in\ \ [A,s']\ \ if\ \ A \leq s', \end{cases}$$

$$\alpha_B(s,s') := \begin{cases} \alpha_-(s,s') & for\ \ s\ \ in\ \ [B,s']\ \ if\ \ B < s' \\ \alpha_+(s,s') & for\ \ s\ \ in\ \ [s',B]\ \ if\ \ s' \leq B; \end{cases}$$

and, to use in case $y_0(\cdot)$ is monotone, $\alpha_{A,B} : co\{A,B\} \longrightarrow \mathbb{R}$,

$$\alpha_{A,B}(s) := \alpha_A(s,s') = \alpha_B(s,s') \quad \forall s' \in S_{A,B}.$$

Using similar proofs as above, one obtains the following results.

Theorem 4.3 *(Necessary condition, in case the* $0-relaxed$ *minimizer does not stop)*

Let $\ell(\cdot)$ *be a superlinear* $BH-function$.

Let $y_0(\cdot)$, *a* $0-relaxed$ *minimizer of the nonconvex integral* (2.1) *on* $\mathcal{X}_{A,B}$ *given by proposition* 1, *satisfy the DuBois – Reymond differential inclusion* (2.15). *In case* $y_0(\cdot)$ *does not stop then we must have : either* $y_0(\cdot)$ *is non-monotone and*

$$b - a \leq \int_A^{s'} \frac{1}{\alpha_A(s,s')}\, ds + \int_{s'}^{B} \frac{1}{\alpha_B(s,s')}\, ds, \tag{4.12}$$

where s' *is as after* (4.10); *or else* $y_0(\cdot)$ *is monotone and*

$$b - a \leq \int_A^{B} \frac{1}{\alpha_{A,B}(s)}\, ds. \tag{4.13}$$

Theorem 4.4 *(Sufficient condition for* $0-relaxed$ *minimizers to stop)*

Let $\ell(\cdot)$ *be a superlinear* $BH-function$.

Let $y_0(\cdot)$, *a* $0-relaxed$ *minimizer of the nonconvex integral* (2.1) *on* $\mathcal{X}_{A,B}$ *given by proposition* 2.5, *satisfy the DuBois – Reymond differential inclusion* (2.15).

Then $y_0(\cdot)$ *stops at some minimizer* s' *of* $\varphi_0(\cdot)$ *provided : either* $y_0(\cdot)$ *is non-monotone and*

$$\int_A^{s'} \frac{1}{\alpha_A(s,s')}\, ds + \int_{s'}^{B} \frac{1}{\alpha_B(s,s')}\, ds < b - a, \tag{4.14}$$

where s' *is as after* (4.10); *or else* $y_0(\cdot)$ *is monotone and*

$$\int_A^B \frac{1}{\alpha_{A,B}(s)}\, d\, s < b - a\,. \tag{4.15}$$

Remark 4.5 *The case* $\alpha_+(\cdot, s') \equiv \beta_+(\cdot, s')$ *a.e. is specially interesting, in the sense that the above gap disappears: for* $b-a$ *small enough* $y_0(\cdot)$ *does not stop; while from* $\int_a^b 1/\beta_+(\cdot, s')$ *on,* $y_0(\cdot)$ *stops. Therefore one is never in doubt about what happens, in this case. We get thus a* NSC *for* $y_0(\cdot)$ *to stop. This is the case namely whenever* $\ell^{**}(s,\cdot)$ *is strictly convex outside of the interval* $(\alpha(s), \beta(s))$ $\forall s \in y_0([a,b])$. *(An example is presented in the next section.)*

4.3 Determination of boundary data for which minimizers exist in a specific example

Set $\ell(s,\xi) := \varphi(s) + h(\xi)^{1+\delta}$ with $\varphi(s) := |s - s'|^{\varepsilon}$ and $h(\xi) := \big|\, |\xi|^{1+\delta} - \beta^{1+\delta}\,\big|$, $\varepsilon > 0$, $\delta > 0$, and $\beta > 0$. Then $\alpha(s) \equiv -\beta$, $\beta(s) \equiv \beta$, and one easily checks, using the results of chapter 3, that s' is the unique possible stopping point. Hence

$$V(s,s') = \begin{cases} \left\{\xi \notin [-\beta,\beta] :\ h(\xi)^{\delta}\left[\beta^{1+\delta} + \delta\,(1+\delta)\,|\xi|^{1+\delta}\right] = \varphi(s)\right\} & \text{for} \quad s \neq s' \\ \{0\} & \text{for} \quad s = s'. \end{cases}$$

Let us consider, for simplicity, the special case $\delta = 1$. Defining, for $s \neq s'$,

$$v(s) := \frac{1}{\sqrt{3}}\,\beta\,\sqrt{1 + \sqrt{4 + 3\,\varphi(s)\,\beta^{-4}}} > \beta > 0\,,$$

we get, for $s \neq s'$,

$$V(s,s') = \{-v(s), v(s)\}\,,$$

$$\frac{1}{\beta_-(s,s')} = \frac{1}{\alpha_-(s,s')} = -\frac{1}{v(s)}\,, \quad \frac{1}{\alpha_+(s,s')} = \frac{1}{\beta_+(s,s')} = \frac{1}{v(s)}\,,$$

and one obtains, as NSC for the *existence* of a *minimizer* for the integral (2.1):

$$b - a \leq \left| \int_A^{s'} \frac{1}{v(s)}\, d\, s \right| + \left| \int_{s'}^B \frac{1}{v(s)}\, d\, s \right|. \tag{4.16}$$

(Notice: the hypotheses of the relaxation result of [Ek Te, th. IX.4.1, p. 287] are fulfilled.)

To consider still a more specific example, fix $\beta = 1$, $\varepsilon = 2$, $s' = 0$, $B = 1$; then a SC for the existence of minimizers for the fully nonconvex integral (2.1), for any $A \leq 0$, is the inequality

$$b - a \leq R(A) := \sqrt{3} \int_A^1 \frac{1}{\sqrt{1 + \sqrt{4 + 3\, s^2}}}\, ds\,.$$

One easily checks that $R(A) \geq R(0) = 0.953\ldots$ $\forall A \leq 0$. Therefore it suffices to choose $b - a \leq 0.953$ to be sure of the existence of a minimizer for the fully nonconvex integral (2.1) with $A \leq 0$, $B = 1$. In particular, there always exists a true minimizer for the integral

$$\int_0^{1/2} x(t)^2 + \left(x'(t)^2 - 1\right)^2 dt \qquad with \qquad x(0) = A \leq 0\,,\ x(1/2) = 1\,.$$

On the other hand, if we fix the interval $[a, b]$ (or, more precisely, its length: e.g. we may set $a = 0$ and fix b), then the integral (2.1) will have minimizers whenever A (is negative and) has modulus large enough.

However, for $\varepsilon = 5$ existence of minimizers would demand

$$b \leq \sqrt{3} \int_{-\infty}^1 \frac{1}{\sqrt{1 + \sqrt{4 + 3\,|s|^5}}}\, ds = 7.07404\ldots\,.$$

Therefore: in case $a = 0$, $B = 1$ and $b \geq 7.075$, there exists no minimizer, for any $A < 0$. In particular, there exists no true minimizer for the integral

$$\int_0^8 |x(t)|^5 + \left(x'(t)^2 - 1\right)^2 dt \qquad with \qquad x(0) = A \leq 0\,,\ x(8) = 1\,.$$

We can state the morale synthetically as: by imposing a mean speed $|B - A| \,/\, (b - a)$ large enough then the $0-$ *relaxed minimizer* cannot afford to stop, hence is a true minimizer.

Chapter 5

Existence in the vector case under almost convexity

5.1 Introduction

The aim of this chapter is to prove new sufficient conditions for existence and regularity of minimizers for the 1-dim integral

$$\int_a^b L\left(x(t), x'(t)\right)\, dt \qquad\qquad on \quad \mathcal{X}^n_{A,B}, \tag{5.1}$$

where $\mathcal{X}^n_{A,B}$ is the class of AC (absolutely continuous) functions $x : [a,b] \to \mathbb{R}^n$ satisfying boundary conditions $x(a) = A$, $x(b) = B$, and $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$.

Since $L(s, \cdot)$ is allowed to be nonconvex, we consider the bipolar $L^{**}(s, \cdot)$ of $L(s, \cdot)$, so that $epi\, L^{**}(s, \cdot) = \overline{co}\ epi\ L(s, \cdot)$, and the corresponding convexified integral

$$\int_a^b L^{**}\left(x(t), x'(t)\right)\, dt \qquad\qquad on \quad \mathcal{X}^n_{A,B}. \tag{5.2}$$

We call $y_c(\cdot)$ a relaxed minimizer provided $y_c(\cdot)$ minimizes the integral (5.2).

Since in the vector case the hypothesis of 0−convexity does not suffice to guarantee existence of minimizers (see (5.16)), we have used instead almost convexity, a concept that was born, for multifunctions, in the paper [Ce Or].

In the first result we present, $L(\cdot)$ is assumed to be lsc with superlinear growth at infinity, i.e.

$$L(s,\xi) \geq \theta(|\xi|) \quad \forall (s,\xi) \qquad \text{with} \qquad \theta(r)/r \to +\infty \ \ as \ \ r \to +\infty, \tag{5.3}$$

so that there exists a relaxed minimizer $y_c(\cdot)$. Changing $y_c(\cdot)$, by applying to reparametrizations the bimonotonicity results of A. Ornelas, we obtain a new relaxed minimizer $y(\cdot)$ which is a true minimizer: it also minimizes the original, nonconvex, integral (5.1).

In the second result, existence of $y_c(\cdot)$ is used as one hypothesis, and we need no growth assumption to turn $y_c(\cdot)$ into $y(\cdot)$.

We also present some concrete examples of application of these results to prove existence of true minimizers.

5.2 Almost convexity

Definition 5.1 *For a function* $L : \mathbb{R}^m \times \mathbb{R}^n \to (-\infty, +\infty]$, *we call* $L(s,\cdot)$ *almost convex provided*

$$\forall\, \xi \ \ with \ \ L^{**}(s,\xi) < L(s,\xi) \tag{5.4}$$

$$\exists\, \lambda \in [0,1] \quad \exists\, \Lambda \in [1,+\infty) \quad \exists\, \alpha \in [0,1] \quad for\ which \tag{5.5}$$

$$L^{**}(s,\xi) = (1-\alpha)\, L(s,\lambda\xi) + \alpha\, L(s,\Lambda\xi) \tag{5.6}$$

$$\xi = (1-\alpha)\,(\lambda\xi) + \alpha\,(\Lambda\xi)\,. \tag{5.7}$$

For completeness, we also set $\lambda = 1 = \Lambda = \alpha$ *at those* ξ *where* $L^{**}(s,\xi) = L(s,\xi)$, *in particular at* $\xi = 0$. *(The convention* $0 \cdot (+\infty) = 0$ *is used.) We will denote by* $\lambda_L(\cdot)$ *the function* $(s,\xi) \mapsto \lambda$ *for* $L(\cdot)$. *Similarly for* Λ, α.

Clearly $L(s,\cdot)$ convex lsc implies $L(s,\cdot)$ almost convex. Moreover $L(s,\cdot)$ almost convex implies $L^{**}(s,0) = L(s,0)$. But the opposite implication does not hold, even for simple 2−dim superlinear polynomials. Indeed, e.g.

$$L(s,\xi) := h(\xi) := \left(\xi_1^2 + \xi_2^2\right)\,\left(\xi_1^2 - 1\right)^2 + \xi_2^2$$

satisfies $h^{**}(0) = h(0) = 0$ but:

$$\exists\, \xi = (1/2\,,1) \quad \exists\, \lambda = 0 \quad \exists\, \Lambda = 2 \quad \exists\, \alpha = 1/2 \quad with$$

$$\xi = (1-\alpha)\,(\lambda\xi) + \alpha\,(\Lambda\xi)\,, \qquad h^{**}(\lambda\xi) = h(\lambda\xi) = 0\,, \qquad h^{**}(\Lambda\xi) = h(\Lambda\xi) = 4$$

$$h^{**}(\xi) = 1 < h(\xi) < (1-\alpha)\ h(\lambda\xi) + \alpha\, h(\Lambda\xi) = 2 \tag{5.8}$$

(and: λ must be zero, while 2 is the best value of Λ, i.e. the one yielding the smallest *rhs* in (5.8)); moreover, even though $\ell(\cdot)$ is superlinear,

$$\exists\, \xi = (0,1)\ :\quad h^{**}(\Lambda\xi) < h(\Lambda\xi) \quad \forall\, \Lambda \geq 1\,. \tag{5.9}$$

Indeed, $h^{**}(\xi_1,\xi_2) = \xi_2^2 \quad \forall\, |\xi_1| \leq 1 \quad \forall \xi_2$.

Typical examples of almost convex functions may be obtained by increasing arbitrarily (e.g. to become $= +\infty$) the values of any given $L(\cdot)$ as follows. Denote by $\widehat{F}(s)$ the vertical projection into $\mathbb{R}^n$ of any (relatively open) face $F(s)$ of $epi\, L^{**}(s,\cdot)$. Then one may change $L(s,\xi)$ by increasing it, starting from the value $L^{**}(s,\xi)$, at those $\xi \neq 0$ contained in any bounded subset of any $k-$dim $\widehat{F}(s)$ which is contained in a $k-$dim *linear* subspace of $\mathbb{R}^n$, $1 \leq k \leq n$. Or, as a simpler alternative, increase arbitrarily $L(s,\xi)$ only at those $\xi \in \widehat{F}(s)$, for each bounded $n-$dim face $F(s)$.

Notice also that for $L(s,\cdot):\mathbb{R}^n \to [0,+\infty]$ *lsc* superlinear we do have

$$L^{**}(s,0) = L(s,0) \qquad \Rightarrow \qquad L(s,\cdot) \;\; \textit{almost convex} \tag{5.10}$$

whenever the faces of *epi* $L^{**}(s,\cdot)$ have all dimension ≤ 1, namely in the scalar $n=1$ or radial $L(s,\xi) = f(s,|\xi|)$ case. Here superlinearity is really not needed: it suffices to have boundedness of the nonconvexity faces (i.e. of the subset of each $\widehat{F}(s)$ where $f^{**}(s,\cdot) < f(s,\cdot)$).

5.3 Existence of minimizers

Theorem 5.2 (*Existence superlinear*)

Let $L:\mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ *be a lsc function with superlinear growth* (5.3) *having* $L(s,\cdot)$ *almost convex* $\forall s$.

Then for any A, B *the nonconvex integral* (5.1) *has minimizers.*

(*Notice: the regularity of theorem* 5.4 *applies here too.*)

Corollary 5.3 (*Existence radial*)

Let $f:\mathbb{R}^n \times [0,+\infty) \to [0,+\infty]$ *be a lsc function with* $f(s,\cdot) \geq \theta(\cdot)$ $\forall s$, $\theta(\cdot)$ *as in* (5.3).

Then for any A, B *the nonconvex integral*

$$\int_a^b f\left(x(t), |x'(t)|\right) dt \qquad \textit{on} \quad \mathcal{X}^n_{A,B}$$

has minimizers provided $f^{**}(\cdot,0) = f(\cdot,0)$ (*using* $f(\cdot,-r) := f(\cdot,r)$ $\forall r > 0$).

Theorem 5.4 (*Regularity in all cases*)

Let $L:\mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty]$ *be a Borel function with* $L^{**}(\cdot)$ *Borel. Fix* A, $B \in \mathbb{R}^n$, $y_c(\cdot) \in \mathcal{X}^n_{A,B}$ *and assume* $L(\cdot,0)$ *to be lsc on* $y_c([a,b])$ *and* $L(y_c(t),\cdot)$ *to be almost convex lsc* $\forall\, t \in [a,b]$.

Then there exists $y(\cdot) \in \mathcal{X}^n_{A,B}$ *for which*

(*i*) $\int_a^b L(y(t), y'(t))\,dt \leq \int_a^b L^{**}(y_c(t), y_c'(t))\,dt$;

(*ii*) $\exists\, a' \leq b'$: $\quad y'(t) \neq 0$ *a.e. in* $[a,a'] \cup [b',b]$;

(*iii*) $\exists\, s'$ *minimizer of* $L^{**}(\cdot,0)$ *on* $y([a,b]) = y_c([a,b])$: $\quad y(\cdot) \equiv s'$ *on* $[a',b']$;

(*iv*) $L^{**}(y(\cdot), y'(\cdot)) = L(y(\cdot), y'(\cdot))$ *a.e.*.

Corollary 5.5 (*Existence, given relaxed minimizer*)

Under the same hypotheses of theorem 5.4, *the existence of a relaxed minimizer* (*i.e. a minimizer of* (5.2)) *implies the existence of a true minimizer* (*of* (5.1)).

Proposition 5.6 (*See [Cl]*) *Let* $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ *be a Borel function with* $L^{**}(\cdot)$ *Borel.*

Fix $A, B \in \mathbb{R}^n$ *and a closed subset* $\Omega \subset \mathbb{R}^n$ *for which* $L^{**}(\cdot)$ *is lsc on* $\Omega \times \mathbb{R}^n$, $(L^{**}(s, \cdot))^{-1}(\mathbb{R})$ *is open* $\neq \emptyset$ $\forall s \in \Omega$. *Assume there exists* $m_\Omega > 0$ *for which the class* $\mathcal{X}_\Omega$, *of those* $x(\cdot)$ *in* $\mathcal{X}^n_{A,B}$ *having* $x([a,b]) \subset \Omega$ *and giving a value* $\leq m_\Omega$ *to the convexified integral* (5.2), *has a Lipschitz continuous element.*

Define

$$Q(\xi) := \{L^{**}(s,\xi) - \langle \xi, \partial L^{**}(s,\xi)\rangle : s \in \Omega\}$$

$$q^- := \lim_{R \to +\infty} \sup \{q \in Q(\xi) : |\xi| > R\}$$

and, for $K > 0$,

$$q^+(K) := \inf \{q \in Q(\xi) : |\xi| < K\}.$$

Assume, moreover:

(*a*) $\exists K_\Omega > 0 : \quad |\{t \in [a,b] : |x'(t)| < K_\Omega\}| > 0 \quad \forall x(\cdot) \in \mathcal{X}_\Omega$,

(*b*) $q^- < q^+(K_\Omega)$.

Then there exists a relaxed minimizer $y_c(\cdot)$ (*i.e. a minimizer of* (5.2)) *which is Lipschitz continuous.*

Corollary 5.7 *Let* $L(\cdot)$ *and* $y_c(\cdot)$ *be as in proposition* 5.6.

Then the nonconvex integral (5.1) *has minimizers provided* $L(y_c(t), \cdot)$ *is almost convex lsc* $\forall t \in [a,b]$.

If, in addition,

$$\forall M > 0 \quad \exists M_1 : \qquad \Lambda_L(s,\xi)\,|\xi| \leq M_1 \quad \forall |\xi| \leq M \quad \forall s \in y_c([a,b])$$

then $y(\cdot)$ *is Lipschitz continuous.*

Notice: in theorem 5.2 and corollaries 5.3, 5.5, 5.7, clearly theorem 5.4 may be applied to obtain a minimizer $y(\cdot)$ satisfying regularity properties (*i*), (*ii*), (*iii*), (*iv*).

Proof : (*a*) We will consider the following class of reparametrizations of the interval $[a,b]$: $\mathcal{R}_{a,b}$ is the class of all AC maps $\tau:[a,b]\to[a,b]$ having $\tau(a)=a$, $\tau(b)=b$ and $\tau'(\cdot)\ge 0$ a.e..

For each nonconstant $y(\cdot)$ in $\mathcal{X}^n_{A,B}$ (see e.g. [C]), there exists $Y(\cdot)$ in $\mathcal{X}^n_{A,B}$ characterized by having constant speed, i.e. $|Y'(t)|=m$ $\forall t\in[a,b]\setminus\mathcal{N}$, where $\mathcal{N}$ is a null set and m is the mean speed of $y(\cdot)$:

$$m:=\frac{1}{b-a}\int_a^b |y'(t)|\ dt.$$

Defining

$$\tau(t):=a+\frac{1}{m}\int_a^t |y'(r)|\ dr,$$

we have $\tau'(t)=\frac{|y'(t)|}{m}\ge 0$, $\tau(a)=a$ and $\tau(b)=b$, hence $\tau(\cdot)\in\mathcal{R}_{a,b}$. Moreover

$$y(t)=Y(\tau(t))\quad \forall t\in[a,b].$$

Clearly $Y(\cdot)$ is Lipschitz continuous.

(*b*) As is well know (see [Ol 1], [Io]), the convexified integral (5.2) has a minimizer $y_c(\cdot)$. We may assume the minimizer $y_c(\cdot)$ to be nonconstant. Let us consider the corresponding special function $Y(\cdot)$ having constant speed m, as in (a). Let $\mathcal{N}$ be the set of those τ in $[a,b]$ where the derivative $Y'(\tau)$ does not exist or $|Y'(\tau)|\ne m$. Let $\tau_c\in\mathcal{R}_{a,b}$ be such that (as in (a)) $Y(\tau_c(t))=y_c(t)$ $\forall t\in[a,b]$.

Define the function $\ell_0:\mathbb{R}\times\mathbb{R}\to[0,+\infty]$,

$$\ell_0(\tau,r):=\begin{cases} L^{**}(Y(\tau),Y'(\tau)r) & \textit{for } \tau\in[a,b]\setminus\mathcal{N} \textit{ and } r\in[0,+\infty) \\ L^{**}(Y(\tau),0) & \textit{for } (\tau\in[a,b] \textit{ and } r=0) \\ & \textit{or } (\tau\in\mathcal{N} \textit{ and } r=1) \\ +\infty & \textit{for other } (\tau,r)\in\mathbb{R}\times\mathbb{R}. \end{cases}$$

Clearly $\ell_0(\cdot)$ is $\mathcal{L}\otimes\mathcal{B}$–measurable and $\ell_0(\cdot,0)$ is lsc. Therefore the integrand $\ell_0(\tau(\cdot),\tau'(\cdot))$ is measurable (by [Or 3, propos. 2]) for any reparametrization $\tau(\cdot)$ in $\mathcal{R}_{a,b}$. By [Or 3, propos. 1 with $\alpha\equiv 0$], there exists a reparametrization $\tau_1(\cdot)\in\mathcal{R}_{a,b}$ for which: $\exists a'\le b'$ such that $\tau_1'(t)\ne 0$ a.e. in $[a,a']\cup[b',b]$, $\exists\tau'$ minimizer of $\ell_0(\cdot,0)$ on $[a,b]$ such that $\tau_1(\cdot)\equiv\tau'$ on $[a',b']$, and

$$\int_a^b \ell_0(\tau_1(t),\tau_1'(t))\ dt\le\int_a^b \ell_0(\tau_c(t),\tau_c'(t))\ dt.$$

In particular, setting $y_1(t):=Y(\tau_1(t))$ it follows that $y_1(\cdot)$ is a new minimizer for the convexified integral (5.2), since $\tau_1'(\cdot)$ (resp. $\tau_c'(\cdot)$), hence $y_1'(\cdot)$ (resp. $y_c'(\cdot)$), is zero a.e. on $\tau_1^{-1}(\mathcal{N})$ (resp. $\tau_c^{-1}(\mathcal{N})$).

The functions $\overline{\lambda}(\cdot) := \lambda_L(y_1(\cdot), y_1'(\cdot))$, $\overline{\Lambda}(\cdot) := \Lambda_L(y_1(\cdot), y_1'(\cdot))$ are measurable. Therefore there exist Borel functions $\lambda,\ \Lambda : [a,b] \to \mathbb{R}$ such that $\lambda(\cdot) = \overline{\lambda}(\cdot)$ and $\Lambda(\cdot) = \overline{\Lambda}(\cdot)$ a.e. in $[a,b]$. Let $T := \{t \in [a,b] : \lambda(\cdot) \neq \overline{\lambda}(\cdot) \ \text{or} \ \Lambda(\cdot) \neq \overline{\Lambda}(\cdot)\}$,

$$\mathcal{N}_0 := \tau_1^{-1}(\mathcal{N}) \cup \{t \in [a,b] : \nexists \tau_1'(t) \ \text{or} \ \exists \tau_1'(t) = 0\} \cup$$
$$\cup \{t \in [a,b] : \nexists y_1'(t) \ \text{or} \ y_1'(t) \neq Y'(\tau_1(t))\tau_1'(t)\} \cup T,$$

and $\mathcal{N}_1 := \tau_1(\mathcal{N}_0)$, hence $\mathcal{N}_1$ is a null set. Notice that $\tau_1|_{[a,a']\cup(b',b]}(\cdot)$ has an inverse $\tau_1^{-1} : [a,b] \to [a,a'] \cup (b',b]$ which is a measurable function with measurable derivative $\tau_1^{-1\prime}(\tau_1(t)) = 1/\tau_1'(t) > 0$ a.e. in $[a,a'] \cup (b',b]$. Define the measurable functions $\lambda_1,\ \Lambda_1 : \mathbb{R} \to \mathbb{R}$ setting

$$\lambda_1(\tau) := \begin{cases} \frac{\lambda(\tau_1^{-1}(\tau))}{\tau_1^{-1\prime}(\tau)} & \text{for } \tau \in [a,b] \setminus \mathcal{N}_1 \\ 1 & \text{otherwise} \end{cases}$$

$$\Lambda_1(\tau) := \begin{cases} \frac{\Lambda(\tau_1^{-1}(\tau))}{\tau_1^{-1\prime}(\tau)} & \text{for } \tau \in [a,b] \setminus \mathcal{N}_1 \\ 1 & \text{otherwise}. \end{cases}$$

Define the new function $\ell_1 : \mathbb{R} \times \mathbb{R} \to [0,+\infty]$,

$$\ell_1(\tau,r) := \begin{cases} L(Y(\tau), Y'(\tau)r) & \text{for } \tau \in [a,b] \setminus \mathcal{N}_1 \text{ and } r \in \{\lambda_1(\tau), \Lambda_1(\tau)\} \\ L(Y(\tau), 0) & \text{for } (\tau \in [a,b] \text{ and } r = 0) \\ & \text{or } (\tau \in \mathcal{N}_1 \text{ and } r = 1) \\ +\infty & \text{for other } (\tau, r) \in \mathbb{R} \times \mathbb{R}. \end{cases}$$

Then $\ell_1(\cdot)$ is $\mathcal{L}\otimes\mathcal{B}$–measurable with $\ell_1(\tau,\cdot)$ lsc; and one easily checks the following: also $\ell_1^{**}(\cdot)$ is $\mathcal{L}\otimes\mathcal{B}$–measurable with $\ell_1^{**}(\cdot,0) = \ell_1(\cdot,0)$ lsc; $\ell_1^{**}(\tau,r) = +\infty$ whenever $\tau \notin [a,b]$ or $r \notin [0,\Lambda_1(\tau)]$; and

$$L^{**}(Y(\tau), Y'(\tau)r) \leq \ell_1^{**}(\tau,r) \leq \ell_1(\tau,r) \quad \forall r \in \mathbb{R} \quad \forall \tau \in [a,b] \setminus \mathcal{N}_1 \tag{5.11}$$

with equality at $r \in \{0, \lambda_1(\tau), \Lambda_1(\tau)\}$ in both inequalities, and at $r \in \{0\} \cup [\lambda_1(\tau), \Lambda_1(\tau)]$ (in particular at $r = 1/\tau_1^{-1\prime}(\tau)$) in the first one. Indeed, the bipolar of $\ell_1(\tau,\cdot)$ is:

$$\ell_1^{**}(\tau,r) := \begin{cases} L^{**}(Y(\tau),Y'(\tau)r) & \textit{for } \tau\in[a,b]\setminus\mathcal{N}_1 \textit{ and } r\in[\lambda_1(\tau),\Lambda_1(\tau)] \\ \left(1-\frac{r}{\lambda_1(\tau)}\right)L(Y(\tau),0)+ & \\ \frac{r}{\lambda_1(\tau)}L(Y(\tau),Y'(\tau)\lambda_1(\tau)) & \textit{for } r\in(0,\lambda_1(\tau)) \textit{ and } \tau\in[a,b]\setminus\mathcal{N}_1 \\ L(Y(\tau),0) & \textit{for } (\tau\in[a,b] \textit{ and } r=0) \\ & \textit{or } (\tau\in\mathcal{N}_1 \textit{ and } r=1) \\ +\infty & \textit{for other } (\tau,r)\in\mathbb{R}\times\mathbb{R}, \end{cases}$$

since

$$L^{**}\left(Y(\tau),Y'(\tau)r\right)=L\left(Y(\tau),Y'(\tau)r\right) \quad \forall r\in\{0,\lambda_1(\tau),\Lambda_1(\tau)\} \quad \forall\tau\in[a,b]\setminus\mathcal{N}_1.$$

(c) Now we claim that the reparametrization $\tau_1(\cdot)$ (yielding $Y(\tau_1(\cdot))=y_1(\cdot)$) is a minimizer for the convexified integral

$$\int_a^b \ell_1^{**}\left(\tau(t),\tau'(t)\right)\,dt, \tag{5.12}$$

defined on the class $\mathcal{R}_{a,b}$. To prove this claim, notice that for each $\tau(\cdot)$ in $\mathcal{R}_{a,b}$ we have, setting $x(t):=Y(\tau(t))$, an AC map:

$$\int_a^b x'(t)\,dt=\int_a^b Y'(\tau(t))\tau'(t)\,dt=\int_a^b Y'(\tau)\,d\tau,$$

since the last integral exists and $\tau(\cdot)$ is monotone. Moreover,

$$L^{**}\left(x(t),x'(t)\right)=L^{**}\left(Y(\tau(t)),Y'(\tau(t))\tau'(t)\right)\le\ell_1^{**}\left(\tau(t),\tau'(t)\right) \tag{5.13}$$

for a.e. t in $[a,b]$; with equality at a.e. t in $[a,b]$ where $\tau'(t)\in\{0\}\cup[\lambda_1(\tau(t)),\Lambda_1\tau(t)]$, in particular in case $\tau(\cdot)=\tau_1(\cdot)$.

To see this, notice that this follows from (5.11) for those t where $\tau'(t)$ exists, $x'(t)=Y'(\tau(t))\tau'(t)$ and $\tau(t)\in[a,b]\setminus\mathcal{N}_1$ (i.e. for almost $t\in\tau^{-1}([a,b]\setminus\mathcal{N}_1)$), while, on the other hand, since $\mathcal{N}_1$ is a null set, we have $\tau'(t)=0$ for a.e. $t\in\tau^{-1}(\mathcal{N}_1)$, hence the *lhs* of (5.13) is $L^{**}(Y(\tau(t)),0)$ and the *rhs* is $\ell_1^{**}(\tau(t),0)$ for a.e. $t\in\tau^{-1}(\mathcal{N}_1)$ (and these two are equal at any $\tau(t)\in[a,b]$, by the definition of $\ell_1^{**}(\cdot)$ and the almost convexity of $L(\cdot)$). (In particular: equality in (5.13) holds for a.e. $t\in\tau^{-1}(\mathcal{N}_1)$.)

In the special case $\tau(\cdot)=\tau_1(\cdot)$, as one easily checks,

$$\lambda_1(\tau_1(t))=\lambda(t)\tau_1'(t)\le\tau_1'(t)$$

$$\Lambda_1(\tau_1(t)) = \Lambda(t)\,\tau_1'(t) \geq \tau_1'(t)\,,$$

i.e. $\tau_1'(t) \in [\lambda_1(\tau_1(t)), \Lambda_1(\tau_1(t))]$ for a.e. $t \notin \mathcal{N}_0$; $\tau_1'(t) = 0$ for a.e. $t \in \mathcal{N}_0$. Hence equality holds in (5.13).

Using (5.13) we may now complete the proof of the claim stated at the beginning of (c):

$$\int_a^b \ell_1^{**}(\tau(t), \tau'(t))\, dt \geq \int_a^b L^{**}(x(t), x'(t))\, dt \geq$$

$$\geq \int_a^b L^{**}(y_1(t), y_1'(t))\, dt = \int_a^b \ell_1^{**}(\tau_1(t), \tau_1'(t))\, dt.$$

(d) Define $\alpha : [a,b] \to \mathbb{R}$

$$\alpha(\tau) := \begin{cases} \lambda_1(\tau) & for \quad \tau \in \left\{ \begin{array}{l} \tau \in [a,b] :\ \lambda_1(\tau) > 0 \ \ and \ \ \ell^{**}(\tau, 0) < +\infty \\ \qquad\qquad and \ \ \ell^{**}(\tau, \lambda_1(\tau)) < +\infty \end{array} \right\} \\ \\ 0 & for \quad \tau \in \{\tau \in [a,b] : \ell^{**}(\tau, 0) = +\infty \ \ or \ \ \ell^{**}(\tau, \lambda_1(\tau)) = +\infty\} \\ \\ \Lambda_1(\tau) & for \quad \tau \in \{\tau \in [a,b]\ : \lambda_1(\tau) = 0\}\,. \end{cases}$$

By [Or 3, propos. 1 with this $\alpha(\cdot)$], there exists a reparametrization $\tau_2(\cdot) \in \mathcal{R}_{a,b}$ for which: $\exists a'' \leq b''$, with $a'' \leq a'$ and $b' \leq b''$, such that $\tau_2'(t) \neq 0$ a.e. in $[a, a''] \cup [b'', b]$, $\tau_2(\cdot) \equiv \tau'$ on $[a'', b'']$, $\tau_2'(t) \notin \{0\} \cup (0, \alpha(\tau_2(t)))$ a.e. in $[a, a''] \cup [b'', b]$, $\tau_2'(t) \in [\lambda_1(\tau_2(t)), \Lambda_1(\tau_2(t))]$ a.e. in $[a, a''] \cup [b'', b]$, and

$$\int_a^b \ell_1^{**}(\tau_2(t), \tau_2'(t))\, dt \leq \int_a^b \ell_1^{**}(\tau_1(t), \tau_1'(t))\, dt.$$

Therefore the reparametrization $\tau_2(\cdot)$ is also a minimizer for the convexified integral (5.12).

(e) By [Or 4, th. 1], or [Or 5, th. 1], there exists a reparametrization $\tau_3(\cdot)$ in the class $\mathcal{R}_{a,b}$ having $\tau_3'(t) \neq 0$ a.e. in $[a, a''] \cup [b'', b]$, $\tau_3(\cdot) \equiv \tau'$ on $[a'', b'']$, $\tau_3'(t) \notin \{0\} \cup (0, \alpha(\tau_3(t)))$ a.e. in $[a, a''] \cup [b'', b]$, $\tau_3'(t) \in \{\lambda_1(\tau_2(t)), \Lambda_1(\tau_2(t))\}$ a.e. in $[a, a''] \cup [b'', b]$,

$$\ell_1^{**}(\tau_3(t), \tau_3'(t)) = \ell_1(\tau_3(t), \tau_3'(t))$$

a.e. in $[a,b]$ (hence, in particular, the rhs is measurable in t), and

$$\int_a^b \ell_1(\tau_3(t), \tau_3'(t))\, dt \leq \int_a^b \ell_1^{**}(\tau_2(t), \tau_2'(t))\, dt. \tag{5.14}$$

Let us define a new function $y(t) := Y(\tau_3(t))$, obtaining: $y(a) = A$, $y(b) = B$, $y(\cdot)$ is AC with $y'(t) = Y'(\tau_3(t))\tau_3'(t)$ for a.e. t in $[a,b]$. Since $\tau_3'(t) \in \{0, \lambda_1(\tau_3(t)), \Lambda_1(\tau_3(t))\}$ for a.e. t on $[a,b]$, by a reasoning similar to the one used to prove (5.13) (but with $\ell(\cdot)$, $L(\cdot)$, $\tau_3^{-1}(\mathcal{N}_1)$ instead) we get

$$\begin{aligned} \ell_1(\tau_3(t), \tau_3'(t)) &= L\big(Y(\tau_3(t)), Y(\tau_3(t))\tau_3'(t)\big) = \\ &= L(y(t), y'(t)) \quad for \quad a.e. \quad t \in [a,b]\,. \end{aligned} \tag{5.15}$$

To complete this proof it only lacks to show that $y(\cdot)$ minimizes indeed the integral (5.1) on the class $\mathcal{X}^n_{A,B}$. But, for any $x(\cdot)$ in this class we have, by (5.15) and (5.14),

$$\int_a^b L(y(t), y'(t))\, dt = \int_a^b \ell_1(\tau_3(t), \tau_3'(t))\, dt \leq$$

$$\leq \int_a^b \ell_1^{**}(\tau_1(t), \tau_1'(t))\, dt \leq \int_a^b L^{**}(y_c(t), y_c'(t))\, dt \leq$$

$$\leq \int_a^b L^{**}(x(t), x'(t))\, dt \leq \int_a^b L(x(t), x'(t))\, dt.$$

5.4 Special regularity for n=1

Corollary 5.8 (*Regularity*)

Let $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ be a Borel function with $\ell^{**}(\cdot,0) = \ell(\cdot,0)$ lsc and $\ell(s,\cdot)$ lsc $\forall s$.

Fix A, $B \in \mathbb{R}^n$ and $y_c(\cdot) \in \mathcal{X}_{A,B}$ for which the faces of epi $\ell^{**}(y_c(t),\cdot)$ are bounded $\forall\, t \in [a,b]$.

Then there exists $y(\cdot) \in \mathcal{X}^n_{A,B}$ satisfying the properties (i), (ii), (iii), (iv) of theorem 5.4 plus:

- *(v) $y'(t) \notin (\alpha(y(t)), \beta(y(t)))$ a.e. in $[a,a'] \cup [b',b]$, with $\alpha(\cdot)$, $\beta(\cdot)$ as in remark 5.9;*
- *(vi) $y(\cdot)$ is monotone on $[a,a']$ and on $[b',b]$ provided $\int_a^b \ell^{**}(y_c(t), y_c'(t))\, dt < +\infty$ and $\partial\ell^{**}(y_c(t),0) \neq \emptyset$ $\forall t \in [a,b]$.*

Remark 5.9 *For each s, $\alpha(s)$, $\beta(s)$ are the nonzero extremities of the intervals of affinity of $\ell^{**}(s,\cdot)$ which have the other extremity at $\xi = 0$. More precisely, consider the subdifferential $\partial\ell^{**}(s,\cdot)$ of $\ell^{**}(s,\cdot)$ (see [Ek Te, p. 20]), and define the set*

$$F_0(s) := (\partial\ell^{**}(s,\cdot))^{-1}(\partial\ell^{**}(s,0)) = \{\xi \in \mathbb{R} : \partial\ell^{**}(s,\xi) \cap \partial\ell^{**}(s,0) \neq \emptyset\}.$$

Then, under the hypotheses of corollary 5.8, we have: the set $\{0\} \cup F_0$ is an interval

$$[\alpha(s), \beta(s)] \quad \text{with} \quad \alpha(s) \leq 0 \leq \beta(s),$$

and $\ell^{**}(s,\cdot)$ is affine along $[\alpha(s), 0]$ and along $[0, \beta(s)]$.

Proposition 5.10 (*See [Cl]*) *Let $\ell : \mathbb{R}\times\mathbb{R} \to [0,+\infty]$ be a Borel function with $\ell^{**}(\cdot,0) = \ell(\cdot,0)$ lsc and $\ell(s,\cdot)$ lsc $\forall s$.*

Fix A, $B \in \mathbb{R}$ and a closed subset $\Omega \subset \mathbb{R}$ for which $\ell^{**}(\cdot)$ is lsc on $\Omega \times \mathbb{R}^n$, $(\ell^{**}(s,\cdot))^{-1}(\mathbb{R})$ is open $\neq \emptyset$ $\forall s \in \Omega$. Assume there exists $m_\Omega > 0$ for which the *class* $\mathcal{X}_\Omega$, *of those* $x(\cdot)$ *in* $\mathcal{X}_{A,B}$ *having* $x([a,b]) \subset \Omega$ *and giving a value* $\leq m_\Omega$ *to the convexified integral* (5.2), *has a Lipschitz continuous element.*

Define

$$Q(\xi) := \{\ell^{**}(s,\xi) - \xi\,\partial\,\ell^{**}(s,\xi) : s \in \Omega\}$$

$$q^- := \lim_{R\to+\infty} \sup\{q \in Q(\xi) : |\xi| > R\}$$

and, for $K > 0$,

$$q^+(K) := \inf\{q \in Q(\xi) : |\xi| < K\}.$$

Assume, moreover:

(*a*) *every* $x(\cdot) \in \mathcal{X}_\Omega$ *is such that* $x([a,b]) \subset$ *interior* (Ω),

(*b*) $\ell^{**}(\cdot)$ *is locally Lipschitz in* (s,ξ) *and satisfies, for constants* k_0 *and* c_0,

$$|\partial_s \ell^{**}(s,\xi)| \leq k_0\,|\ell^{**}(s,\xi)| + c_0 \qquad \forall\,(s,\xi) \in \Omega \times \mathbb{R},$$

where

$$|\partial_s \ell^{**}(s_0,\xi)| := \sup\{|v| : v \text{ is in the Clarke's differential of } \ell^{**}(\cdot,\xi) \text{ at } s = s_0\},$$

(*c*) $q^- < q^+(K)$ *for some* $K > \frac{|B-A|}{b-a}$.

Then there exists a relaxed minimizer $y_c(\cdot)$ (*i.e. a minimizer of* (5.2)) *which is Lipschitz continuous.*

Corollary 5.11 (*Existence noncoercive*)

Let $\ell(\cdot)$ *and* $y_c(\cdot)$ *be as in proposition* 5.10.

Then the nonconvex integral (5.1) *has minimizers provided the faces of epi* $\ell^{**}(y_c(t),\cdot)$ *are bounded* $\forall\, t \in [a,b]$.

Moreover, there exist a minimizer $y(\cdot)$ *which satisfies the regularity properties* (*i*), (*ii*), (*iii*), (*iv*) *of theorem* 5.4 *plus properties* (*v*) *and* (*vi*) *of corollary* 5.8.

If, in addition,

$$\forall\, M > 0 \quad \exists\, M_1 : \qquad \Lambda_L(s,\xi)\,|\xi| \leq M_1 \quad \forall\,|\xi| \leq M \quad \forall\, s \in y_c([a,b])$$

then $y(\cdot)$ *is Lipschitz continuous.*

5.5 Examples of application

Theorem 5.2 ensures the existence of a minimizer for the nonconvex integral (5.1) when $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ is e.g.

$$L(s,\xi) = \begin{cases} |s - s_0|^2 + \left(|\xi|^2 - \gamma^2\right)^2 & \text{for } \xi \neq 0 \\ |s - s_0|^2 & \text{for } \xi = 0 . \end{cases}$$

As to corollary 5.7, it ensures existence and Lipschitz continuity of a minimizer e.g. in case $L : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ has the form

$$L(s,\xi) = g(s)\, f(\xi)$$

with $f : \mathbb{R}^n \to [0, +\infty]$,

$$f(\xi) = \begin{cases} \left(1 + |\xi|^2\right)^{\frac{1}{2}} & \text{for } |\xi| \geq 1 \\ +\infty & \text{for } 0 < |\xi| < 1 \\ \sqrt{2} & \text{for } \xi = 0 , \end{cases}$$

and $g : \mathbb{R}^n \to [1, +\infty)$ is a lsc function, locally bounded.

Finally, to see a simple 2−dim example where convexity at zero does not imply existence, let

$$h(\xi) = \left(\xi_1^2 + \xi_2^2\right)\left(\xi_1^2 - 1\right)^2 + \xi_2^2 , \tag{5.16}$$

and

$$L(s,\xi) = s_1^2 + h(\xi) , \tag{5.17}$$

$s = (s_1, s_2)$, $a = 0$, $A = (0,0)$, $b = 1$, $B = (0,1)$. Clearly $y_c(t) = (0,t)$ is a relaxed minimizer, giving the value 1 to the integral (5.2). However, as one easily checks, to satisfy the boundary conditions the value of the nonconvex integral (5.1) must always be > 1 (while the *inf* is, clearly, $= 1$).

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