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New Trends on Nonlocal and Functional Boundary Value Problems

Guest Editors: Gennaro Infante, To Fu Ma, and Feliz Manuel Minhós

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Editorial Board

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Editorial

New Trends on Nonlocal and Functional Boundary Value Problems

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In the last decades, boundary value problems with nonlocal and functional boundary conditions have become a rapidly growing area of research. The study of this type of problems not only has a theoretical interest that includes a huge variety of differential, integrodifferential, and abstract equations, but also is motivated by the fact that these problems can be used as a model for several phenomena in engineering, physics, and life sciences that standard boundary conditions cannot describe. In this framework, fall problems with feedback controls, such as the steady states of a thermostat, where a controller at one of its ends adds or removes heat depending upon the temperature registered in another point, or phenomena with functional dependence in the equation and/or in the boundary conditions, with delays or advances, maximum or minimum arguments, such as beams where the maximum (minimum) of the deflection is attained in some interior or endpoint of the beam. Topological and functional analysis tools, for example, degree theory, fixed point theorems, or variational principles, have played a key role in the developing of this subject.

This volume contains a variety of contributions within this area of research. The articles deal with second and higher order boundary value problems with nonlocal and functional conditions for ordinary, impulsive, partial, and fractional differential equations on bounded and unbounded domains. In the contributions, existence, uniqueness, and asymptotic behaviour of solutions are considered by using several methods as fixed point theorems, spectral analysis, and oscillation theory.

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Research Article

Existence Results for a Fully Fourth-Order Boundary Value Problem

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We discuss the existence of solution for the fully fourth-order boundary value problem $u^{(4)} = f(t, u, u', u'', u''')$, $0 \le t \le 1$, $u(0) = u(1) = u''(0) = u''(1) = 0$. A growth condition on f guaranteeing the existence of solution is presented. The discussion is based on the Fourier analysis method and Leray-Schauder fixed point theorem.

1. Introduction and Main Results

In this paper we deal with the existence of solution for the fully fourth-order ordinary differential equation boundary value problem (BVP)

$$\begin{gathered} u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{1}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. This problem models deformations of an elastic beam whose two ends are simply supported in equilibrium state, and its research has important significance in mechanics.

For the special case of BVP(1) that f does not contain derivative terms u' and u''', namely, simply fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) = f\left(t, u(t), u''(t)\right), \quad 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{2}$$

the existence of solution has been studied by many authors; see [1–8]. In [1], Aftabizadeh showed the existence of a solution to PBV(2) under the restriction that f is a bounded function. In [2, Theorem 1], Yang extended Aftabizadeh's result and showed the existence for BVP(2) under the growth condition of the form

$$\left| f(t, u, v) \right| \le a|u| + b|v| + c, \tag{3}$$

where a, b, and c are positive constants such that

$$\frac{a}{\pi^4} + \frac{b}{\pi^2} < 1. \tag{4}$$

In [3], under a more general linear growth condition of two-parameter nonresonance, del Pino and Manásevich also discussed the existence of BVP(2) and the result of Yang was further extended. For more results involving two-parameter nonresonance condition see [4, 7]. All these works are based on Leray-Schauder degree theory. In [5, 6], the upper and lower solutions method is applied to discuss the existence of BVP(2). Recently, in [8] the fixed point index theory in cones is employed to BVP(2) and some existence results of positive are obtained, where f may be super-linear growth.

For the more simple case of BVP(1) that f does not contain any derivative terms, the following fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) = f(t, u(t)), \quad 0 \le t \le 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{5}$$

has been studied by more researchers, and various theorems and methods of nonlinear analysis have been applied; see [9–13] and reference therein.

However, few researchers consider the fully fourth-order boundary value problem BVP(1). The purpose of this paper is to discuss the existence of solution of BVP(1). We will extend the Yang's result previously mentioned from BVP(2) to the general BVP(1). Our results are as follows.

Theorem 1. *Assume that $f \in C([0,1]\times\mathbb{R}^4,\mathbb{R})$ and it satisfies the growth condition*

$$\begin{aligned} &\left|f(t,x_0,x_1,x_2,x_3)\right| \\ &\quad \le c_0\left|x_0\right| + c_1\left|x_1\right| + c_2\left|x_2\right| + c_3\left|x_3\right| + M, \end{aligned} \tag{6}$$

for all $t \in [0,1]$ and $(x_0,x_1,x_2,x_3)\in\mathbb{R}^4$, where $c_0,c_1,c_2,c_3 \ge 0$ and $M > 0$ are constants and c_0,c_1,c_2,c_3 satisfy the restriction

$$\frac{c_0}{\pi^4} + \frac{c_1}{\pi^3} + \frac{c_2}{\pi^2} + \frac{c_3}{\pi} < 1. \tag{7}$$

Then the BVP(1) *possesses at least one solution.*

Theorem 1 is a directly extension of Yang's result previously mentioned. In Theorem 1, the condition (7) is optimal. If the condition (7) does not hold, the existence of solution of BVP(1) cannot be guaranteed. Strengthening the condition (6) of Theorem 1, we can obtain the following uniqueness result.

Theorem 2. *Assume that $f \in C([0,1]\times\mathbb{R}^4,\mathbb{R})$ and it satisfies the Lipschitz-type condition*

$$\left|f(t,x_0,x_1,x_2,x_3) - f(t,y_0,y_1,y_2,y_3)\right| \le \sum_{i=0}^{3} c_i\left|x_i - y_i\right|, \tag{8}$$

for any (t,x_0,x_1,x_2,x_3) and $(t,y_0,y_1,y_2,y_3) \in [0,1]\times\mathbb{R}^4$, where c_0, c_1, c_2, $c_3 \ge 0$ are constants and satisfy (7)*. Then BVP*(1) *has a unique solution.*

If the partial derivatives $f_{x_0}, f_{x_1}, f_{x_2}$, and f_{x_3} exist, then from Theorem 2 and the theorem of differential mean value, we have the following.

Corollary 3. *Let $f \in C([0,1]\times\mathbb{R}^4,\mathbb{R})$ and the partial derivatives f_{x_0}, f_{x_1}, f_{x_2}, and f_{x_3} exist. If there exist positive constants c_0,c_1,c_2,c_3 such that*

$$\left|f_{x_i}(t,x_0,x_1,x_2,x_3)\right| \le c_i, \quad i = 0,1,2,3, \tag{9}$$

and the constants c_0,c_1,c_2,c_3 satisfy (7)*, then BVP*(1) *has one unique solution.*

The proofs of Theorems 1 and 2 are based on the Fourier analysis method and Leray-Schauder fixed point theorem, which will be given in Section 2.

2. Proof of the Main Results

Let $I = [0,1]$ and $H = L^2(I)$ be the usual Hilbert space with the interior product $(u,v) = \int_0^1 u(t)v(t)dt$ and the norm $\|u\|_2 = (\int_0^1 |u(t)|^2dt)^{1/2}$. For $m \in \mathbb{N}$, let $W^{m,2}(I)$ be the usual Sobolev space with the norm $\|u\|_{m,2} = \sqrt{\sum_{i=0}^m \|u^{(i)}\|_2^2}$. $u \in W^{m,2}(I)$ means that $u \in C^{m-1}(I)$, $u^{(m-1)}(t)$ is absolutely continuous on I and $u^{(m)} \in L^2(I)$.

Given $h \in L^2(I)$, we consider the linear fourth-order boundary value problem (LBVP)

$$\begin{gathered} u^{(4)}(t) = h(t), \quad t \in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{gathered} \tag{10}$$

Let $G(t,s)$ be the Green's function to the second-order linear boundary value problem

$$-u'' = 0, \qquad u(0) = u(1) = 0, \tag{11}$$

which is explicitly expressed by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{12}$$

For every given $h \in L^2(I)$, it is easy to verify that the LBVP(10) has a unique solution $u \in W^{4,2}(I)$ in Carathéodory sense, which is given by

$$u(t) = \iint_0^1 G(t,\tau)G(\tau,s)h(s)\,ds\,d\tau := Sh(t). \tag{13}$$

If $h \in C(I)$, the solution is in $C^4(I)$ and is a classical solution. Moreover, the solution operator of LBVP(10), $S : L^2(I) \to W^{4,2}(I)$ is a linearly bounded operator. By the compactness of the Sobolev embedding $W^{4,2}(I) \hookrightarrow C^3(I)$ and the continuity of embedding $C^3(I) \hookrightarrow W^{3,2}(I)$, we see that S maps $L^2(I)$ into $W^{3,2}(I)$ and $S : L^2(I) \to W^{3,2}(I)$ is a completely continuous operator.

Choose a subspace of $W^{3,2}(I)$ by

$$D = \left\{u \in W^{3,2}(I) \mid u(0) = u(1) = 0,\ u''(0) = u''(1) = 0\right\}. \tag{14}$$

Clearly, D is a closed subspace, and hence D is a Banach space by the norm $\|u\|_{3,2}$ of $W^{3,2}(I)$. Define another norm on D by

$$\|u\|_X = \left\|u'''\right\|_2, \quad u \in D. \tag{15}$$

One easily verifies that $\|u\|_X$ is equivalent to $\|u\|_{3,2}$. Hereafter, we use X to denote the Banach space D endowed the norm $\|u\|_X$, namely,

$$X = \left(D, \|\cdot\|_X\right). \tag{16}$$

By the boundary condition of LBVP(10), the solution operator S maps H into D. Hence $S : H \to X$ is completely continuous.

Lemma 4. *For LBVP*(10)*, the following two conclusions hold.*

(a) *The norm of the solution operator of LBVP*(10) *$S : H \to X$ satisfies $\|S\|_{\mathscr{L}(H,X)} \le 1/\pi$.*

(b) *For every $h \in H$, the unique solution of LBVP(10) $u \in W^{4,2}(I)$ satisfies the inequalities*

$$\begin{aligned} \|u\|_2 &\le \frac{1}{\pi^3}\left\|u'''\right\|_2, \\ \left\|u'\right\|_2 &\le \frac{1}{\pi^2}\left\|u'''\right\|_2, \\ \left\|u''\right\|_2 &\le \frac{1}{\pi}\left\|u'''\right\|_2. \end{aligned} \tag{17}$$

Proof. Since sine system $\{\sin k\pi t \mid k \in \mathbb{N}\}$ is a complete orthogonal system of $L^2(I)$, every $h \in L^2(I)$ can be expressed by the Fourier series expansion

$$h(t) = \sum_{k=1}^{\infty} h_k \sin k\pi t, \tag{18}$$

where $h_k = 2\int_0^1 h(s)\sin k\pi s\, ds$, $k = 1, 2, \ldots$, and the Parseval equality

$$\|h\|_2^2 = \frac{1}{2}\sum_{k=1}^{\infty} \left|h_k\right|^2, \tag{19}$$

holds. Let $u = Sh$; then $u \in W^{4,2}(I)$ is the unique solution of LBVP(10), and u, u'', and $u^{(4)}$ can be expressed by the Fourier series expansion of the sine system. Since $u^{(4)} = h$, by the integral formula of Fourier coefficient, we obtain that

$$\begin{aligned} u(t) &= \sum_{k=1}^{\infty} \frac{h_k}{k^4\pi^4}\sin k\pi t, \\ u''(t) &= -\sum_{k=1}^{\infty} \frac{h_k}{k^2\pi^2}\sin k\pi t. \end{aligned} \tag{20}$$

On the other hand, since cosine system $\{\cos k\pi t \mid k = 0, 1, 2, \ldots\}$ is another complete orthogonal system of $L^2(I)$, every $v \in L^2(I)$ can be expressed by the cosine series expansion

$$v(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\pi t, \tag{21}$$

where $a_k = 2\int_0^1 h(s)\cos k\pi s\, ds$, $k = 0, 1, 2, \ldots$. For the above $u = Sh$, by the integral formula of the coefficient of cosine series, we obtain the cosine series expansions of u' and u''':

$$u'(t) = \sum_{k=1}^{\infty} \frac{h_k}{k^3\pi^3}\cos k\pi t, \tag{22}$$

$$u'''(t) = -\sum_{k=1}^{\infty} \frac{h_k}{k\pi}\cos k\pi t. \tag{23}$$

Now from (23), (19), and Parseval equality, it follows that

$$\begin{aligned} \|Sh\|_X^2 = \left\|u'''\right\|_2^2 &= \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 \\ &\le \frac{1}{2\pi^2}\sum_{k=-\infty}^{\infty}\left|h_k\right|^2 = \frac{1}{\pi^2}\|h\|_2^2. \end{aligned} \tag{24}$$

This means that $\|S\|_{\mathscr{L}(H,X)} \le 1/\pi$, namely, (a) holds.

By (20)–(22) and Paserval equality, we have that

$$\begin{aligned} \|u\|_2^2 &= \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^4\pi^4}\right|^2 \\ &\le \frac{1}{2\pi^6}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 = \frac{1}{\pi^6}\left\|u'''\right\|_2^2, \end{aligned} \tag{25}$$

$$\begin{aligned} \left\|u'\right\|_2^2 &= \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^3\pi^3}\right|^2 \\ &\le \frac{1}{2\pi^4}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 = \frac{1}{\pi^4}\left\|u'''\right\|_2^2, \end{aligned} \tag{26}$$

$$\begin{aligned} \left\|u''\right\|_2^2 &= \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k^2\pi^2}\right|^2 \\ &\le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{h_k}{k\pi}\right|^2 = \frac{1}{\pi^2}\left\|u'''\right\|_2^2. \end{aligned} \tag{27}$$

□

This shows that the conclusion (b) holds.

Proof of Theorem 1. We define a mapping $F : X \to H$ by

$$F(u)(t) := f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad u \in X. \tag{28}$$

From the assumption (6) and the property of Carathéodory mapping it follows that $F : X \to H$ is continuous and it maps every bounded set of X into a bounded set of H. Hence, the composite mapping $S \circ F : X \to X$ is completely continuous. We use the Leray-Schauder fixed-point theorem to show that $S \circ F$ has at least one fixed-point. For this, we consider the homotopic family of the operator equations:

$$u = \lambda(S \circ F)(u), \quad 0 < \lambda < 1. \tag{29}$$

We need to prove that the set of the solutions of (29) is bounded in X. See [14].

Let $u \in X$ be a solution of an equation of (29) for $\lambda \in (0, 1)$. Set $h = \lambda F(u)$; then by the definition of S, $u = Sh \in W^{4,2}(I)$ is the unique solution of LBVP(10). By (a) of Lemma 4, we have

$$\begin{aligned} \|u\|_X = \|Sh\|_X &\le \|S\|_{\mathscr{L}(X,H)}\|h\|_2 \\ &\le \frac{1}{\pi}\|h\|_2 \le \frac{1}{\pi}\|F(u)\|_2. \end{aligned} \tag{30}$$

From (28), (6), and (b) of Lemma 4, it follows that

$$\begin{aligned} \|F(u)\|_2 &\le c_0\|u\|_2 + c_1\left\|u'\right\|_2 + c_2\left\|u''\right\|_2 + c_3\left\|u'''\right\|_2 + M \\ &\le \left(\frac{c_0}{\pi^3} + \frac{c_1}{\pi^2} + \frac{c_2}{\pi} + c_3\right)\left\|u'''\right\|_2 + M \\ &= \pi\left(\frac{c_0}{\pi^4} + \frac{c_1}{\pi^3} + \frac{c_2}{\pi^2} + \frac{c_3}{\pi}\right)\|u\|_X + M. \end{aligned} \tag{31}$$

Combining this inequality with (30), we obtain that

$$\|u\|_X \le \frac{M}{1-(c_0/\pi^4+c_1/\pi^3+c_2/\pi^2+c_3/\pi)} := C_0. \quad (32)$$

This means that the set of the solutions for (29) is bounded in X. Therefore, by the Leray-Schauder fixed-point theorem [14], $S\circ F$ has a fixed-point $u_0 \in X$. Let $h_0 = F(u_0)$. By the definition of S, $u_0 = Sh_0 \in W^{4,2}(I)$ is a solution of LBVP(10) for $h = h_0$. Since $W^{4,2}(I) \hookrightarrow C^3(I)$, from (28) it follows that $h_0 \in C(I)$. Hence $u_0 \in C^4(I)$ is a classical solution of LBVP(10), and by (28) u_0 is also a solution of BVP(1).

The proof of Theorem 1 is completed. □

Proof of Theorem 2. Let $M = \max\{|f(t,0,0,0,0)| : t\in I\}+1$. From condition (8) of Theorem 2 we easily see that Condition (6) of Theorem 1 holds. By Theorem 1, the BVP(1) has at least one solution.

Now, let $u_1, u_2 \in C^4(I)$ be two solutions of BVP(1); then $u_i = S(F(u_i))$, $i = 1, 2$. From (8) and (28), we obtain that

$$\left|F(u_2)(t) - F(u_1)(t)\right| \le +\sum_{i=0}^{3} c_i \left|u_2^{(i)}(t) - u_1^{(i)}(t)\right|, \quad (33)$$

for $t \in I$. Since $u_2 - u_1$ is the solution of LBVP(10) for $h = F(u_2) - F(u_1)$, by (33) and (b) of Lemma 4, we have

$$\begin{aligned}\left\|F(u_2) - F(u_1)\right\|_2 &\le \sum_{i=0}^{3} c_i \left\|u_2^{(i)} - u_1^{(i)}\right\|_2 \\ &\le \left(\frac{c_0}{\pi^3}+\frac{c_1}{\pi^2}+\frac{c_2}{\pi}+c_3\right)\left\|u_2''' - u_1'''\right\|_2 \\ &= \pi\left(\frac{c_0}{\pi^4}+\frac{c_1}{\pi^3}+\frac{c_2}{\pi^2}+\frac{c_3}{\pi}\right)\left\|u_2 - u_1\right\|_X.\end{aligned} \quad (34)$$

From this and (a) of Lemma 4, it follows that

$$\begin{aligned}\left\|u_2 - u_1\right\|_X &= \left\|S(F(u_2) - F(u_1))\right\|_X \\ &\le \|S\|_{\mathscr{L}(X,H)}\left\|F(u_2) - F(u_1)\right\|_2 \\ &\le \left(\frac{c_0}{\pi^4}+\frac{c_1}{\pi^3}+\frac{c_2}{\pi^2}+\frac{c_3}{\pi}\right)\left\|u_2 - u_1\right\|_X.\end{aligned} \quad (35)$$

Since $c_0/\pi^4 + c_1/\pi^3 + c_2/\pi^2 + c_3/\pi < 1$, from (35) we see that $\|u_2 - u_1\|_X = 0$, that is $u_2 = u_1$. Therefore, BVP(1) has only one solution.

The proof of Theorem 2 is completed. □

Example 5. Consider the following fully linear fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) = a_0(t)u(t) + a_1(t)u'(t) + a_2(t)u''(t) \\ + a_3(t)u'''(t) + h(t), \quad t\in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \quad (36)$$

where the coefficient functions $a_0, a_1, a_2, a_3 \in C(I)$ and the inhomogeneous term $h \in C(I)$. All the known results of [1–13] are not applicable to this equation. Let

$$\begin{gathered} f(t,x_0,x_1,x_2,x_3) = a_0(t)x_0 + a_1(t)x_1 \\ + a_2(t)x_2 + a_3(t)x_3 + h(t), \\ c_i = \max_{t\in I}\left|a_i(t)\right|, \quad i = 0,1,2,3. \end{gathered} \quad (37)$$

It is easy to see that the partial derivatives f_{x_0}, f_{x_1}, f_{x_2}, and f_{x_3} exist and

$$\left|f_{x_i}(t,x_0,x_1,x_2,x_3)\right| = \left|a_i(t)\right| \le c_i, \quad i = 0,1,2,3. \quad (38)$$

Assume that the constants c_0, c_1, c_2, c_3 satisfy (7). Then by Corollary 3, (36) has a unique solution.

Example 6. Consider the following nonlinear fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) = \sum_{i=0}^{3} b_i(t)\left|u^{(i)}(t)\right|^{\alpha_i} + \sin\pi t, \quad t\in I, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \quad (39)$$

where $b_i \in C(I)$, $\alpha_i \in (0,1)$, $i = 0,1,2,3$. Let

$$f(t,x_0,x_1,x_2,x_3) = \sum_{i=0}^{3} b_i(t)\left|x_i\right|^{\alpha_i} + \sin\pi t. \quad (40)$$

Then $f \in C([0,1]\times\mathbb{R}^4, \mathbb{R})$ and it satisfies that

$$\lim_{|x_0|+|x_1|+|x_2|+|x_3|\to\infty} \max_{t\in I} \frac{f(t,x_0,x_1,x_2,x_3)}{|x_0|+|x_1|+|x_2|+|x_3|} = 0. \quad (41)$$

From this one easily proves that there exists a positive constant $M > 0$ such that

$$\left|f(t,x_0,x_1,x_2,x_3)\right| \le |x_0|+|x_1|+|x_2|+|x_3| + M. \quad (42)$$

Since (7) holds for the constants $c_0 = c_1 = c_2 = c_3 = 1$, by (42) f satisfies the conditions of Theorem 1. Hence by Theorem 1, (39) has at least one solution. This conclusion cannot be obtained from the results in [1–13].

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Research Article

On Solutions of Fractional Order Boundary Value Problems with Integral Boundary Conditions in Banach Spaces

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The object of this paper is to investigate the existence of a class of solutions for some boundary value problems of fractional order with integral boundary conditions. The considered problems are very interesting and important from an application point of view. They include two, three, multipoint, and nonlocal boundary value problems as special cases. We stress on single and multivalued problems for which the nonlinear term is assumed only to be Pettis integrable and depends on the fractional derivative of an unknown function. Some investigations on fractional Pettis integrability for functions and multifunctions are also presented. An example illustrating the main result is given.

1. Introduction

The theory of boundary value problems is one of the most important and useful branches of mathematical analysis. Boundary value problems of various types create a significant subject of several mathematical investigations and appear often in many applications, especially in solving numerous problems in physics and engineering. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can be reduced to nonlocal problems with integral boundary conditions. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [1–3] and the references therein.

The class of boundary value problems with integral boundary conditions considered below contains as special cases numerous two, three, multipoint, and nonlocal boundary value problems. Such problems are mainly investigated when considering functions satisfying some conditions expressed in terms of the strong topology of a Banach space E. We will investigate the case, when functions are not strongly continuous and strongly integrable. In this situation we need to introduce more general notion of a solution. We should note that the considered case seems to be a natural case and cover many particular cases considered for both the strong and weak topologies (cf. Lemma 19). A more general notion of solutions allows us to solve the problem under very general assumptions, not so restrictive as before (see our last section).

In contrast to the classical approach for the theory of boundary value problems, the theory for fractional order BVP's is still developing one and not satisfactorily described. It is caused by the fact that it is very difficult to find convenient and handy conditions ensuring the existence of solutions of several nonlinear boundary value problems of fractional order. In the considered case of a weak topology on E our results form a relatively new branch of investigations.

As a pursuit of this, some sufficient conditions for the existence of solutions are presented for the following nonlinear m-point boundary value problem of fractional type:

$$\begin{gathered} D^{\alpha}u(t) + f\left(t, u(t), D^{\beta}u(t)\right) = 0, \\ t \in [0,1], \quad \alpha \in (1,2], \quad \beta \in (0,1), \quad \alpha > 1+\beta, \\ u(1) + \int_0^1 \mathfrak{I}(\tau)\, u(\tau)\, d\tau = l, \quad u(0) = 0, \end{gathered} \tag{1}$$

where x takes values in a Banach space E and $l \in E$. Here $\mathfrak{I} \in L^q[0,1]$ for some $q \in [1,\infty]$ and D^α denotes the pseudo fractional differential operator of order α (to be described later). We will assume that f is a vector-valued Pettis integrable function on $[0,1]$. We remark the following:

(1) for real-valued functions with $\alpha, \beta \in \mathbb{N} := \{1,2,3,4,\ldots\}$, we have problems studied in, for example, [4, 5],

(2) for real-valued functions with $\mathfrak{I} \equiv 0$ and $\alpha, \beta \in \mathbb{N}$, we have problems studied in, for example, [6–10]; see also the references therein,

(3) for real-valued functions $\mathfrak{I} \equiv 0$ and when the function f is independent of the fractional derivatives, then we have problems studied in, for example, [1, 11–13],

(4) in abstract spaces (for vector-valued functions) with $\mathfrak{I} \equiv 0$ and $\alpha, \beta \in \mathbb{N}$, we have problems studied in, for example, [14–19],

(5) in abstract spaces with conditions related to the weak topology on E and when the vector-valued function f is independent of the fractional derivatives, then we have a problems studied in, for example, [20, 21].

In comparison with the existence results in the above list, our assumptions seem to be more natural. In contrast to earlier results, we drop the requirement that f is a real-valued function independent of the fractional derivatives and we consider the case of vector-valued Pettis, but not necessarily Bochner, integrable functions. As we mentioned above, the assumptions in the existence theorem are expressed in terms of the weak topology. Such a result does not appear in the earlier literature and so it seems to be new. We collect all interesting properties for the fractional Pettis integral. Moreover, we are able also to start some studies for multivalued fractional boundary value problems with Pettis-integral boundary conditions and fractionally Pettis integrable multifunctions.

In the paper we stress also on comparison results for Pettis integrals and fractional Pettis integrals. This is also done for the multivalued integrals and seems to be interesting by itself, independently of applicability of our results. The properties of fractional integral operators on the spaces of Pettis integrable functions as well as on some of its subspaces are also investigated.

Finally, we remark that, in the Banach spaces, the existence of solutions of some boundary value problems of fractional orders has been considered in terms of Pettis integrals, for the first time, by Salem [20]. In this paper, for clarity of proofs, we restrict ourselves to the case of reflexive spaces, but it is easy to extend our results for nonreflexive spaces by putting contraction hypothesis with respect to some measure of weak noncompactness and by using appropriate fixed point theorem (cf. [22]). Nevertheless, all auxiliary results in this paper are not restricted to reflexive spaces.

The question of proving the existence of solutions to the problem (1) reduces to proving the existence of solutions of a Fredholm integral equation. Since the space of all Pettis integrable functions is not complete (in general), we restrict our attention to the case of weakly continuous solution of the Fredholm integral equation (modeled off the problem (1)); hence we are ready to find the so-called pseudo-solutions of the problem (1) (cf. [22, 23]).

2. Preliminaries and Auxiliary Results

For the sake of the reader's convenience here we collect a few facts which will be needed further on. Let $I = [0,1]$. According to the custom $L^p(I)$, $1 \le p \le \infty$ will denote the Banach space of real-valued measurable functions x defined on I. Let $L^\infty(I)$ denote the Banach space of real-valued essentially bounded and measurable functions defined on I. Through the paper, E is considered to be a Banach space with norm $\|\cdot\|$ and with its dual space E^*. Moreover, let $E_w = (E,w) = (E,\sigma(E,E^*))$ denote the space E with its weak topology. By $C[I,E]$ we will denote the Banach space of strongly continuous functions $x : I \to E$ endowed with a standard $\|x\|_0 = \sup_{t\in I}\|x(t)\|$, while $P[I,E]$ denotes the space of all E-valued Pettis integrable functions in the interval I (see [24, 25] for the definition). Let us also recall that a function $h : E \to E$ is said to be weakly-weakly sequentially continuous if h takes each weakly convergent sequence in E into weakly convergent sequence in E. We point out that a bounded weakly measurable function $x : I \to E$ need not to be Pettis integrable even if E is reflexive. However, in reflexive Banach spaces, the weakly measurable function $x : I \to E$ is Pettis integrable if and only if $\varphi(x(\cdot))$ is Lebesgue integrable on I for every $\varphi \in E^*$ [26].

Let us recall some basic facts. The following Mazur's lemma can be found in [24, 26].

Lemma 1. *A convex subset of a normed space E is closed if and only if it is weakly closed.*

A simple consequence of the Hahn-Banach theorem is as follows.

Proposition 2. *Let E be a normed space with $x_0 \neq 0$. Then there exits $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\varphi x_0 = \|x_0\|$.*

Now, we are in a position to recall a fixed point theorem being an extension of results from [27].

Theorem 3. *Let E be a Banach space with Q a nonempty, closed, convex, and weakly compact subset of $C[I,E]$. Assume that $T : Q \to Q$ is weakly-weakly sequentially continuous. Then T has a fixed point in Q.*

We need to introduce some subspaces of the space $P[I,E]$ of Pettis integrable functions on I which are important in the sequel.

Definition 4. For $1 \le p \le \infty$, we define the class $\mathscr{H}^p(E)$ to be the class of all functions $x : I \to E$ having $\varphi x \in L^p(I)$ for every $\varphi \in E^*$. If $p = \infty$, the added condition

$$\sup_{\|\varphi\|=1}\left(\operatorname*{ess\,sup}_{t\in I}\left|\varphi x(t)\right|\right) < \infty \tag{2}$$

must be satisfied by each $x \in \mathscr{H}^{\infty}(E)$. The class $\mathscr{H}_0^p(E)$ is defined by

$$\mathscr{H}_0^p(E) := \{x \in P[I,E] : \varphi x \in L^p(I)\}. \quad (3)$$

Remark 5. In a reflexive Banach space E the set $\mathscr{H}_0^1(E)$ coincides with the space $P[I,E]$. This is due to the fact that in reflexive Banach spaces, the weakly measurable function $x : I \to E$ is Pettis integrable if and only if $\varphi(x(\cdot))$ is Lebesgue integrable on I for every $\varphi \in E^*$ [26]. In general, this is the space of Dunford integrable functions.

In the remaining part of this paper we let $p \in [1,\infty]$ be fixed and q is conjugated with p; that is, $1/p + 1/q = 1$. The following results are due to Pettis (see [25, Theorem 3.4 and Corollary 3.41]).

Proposition 6. *In order that $x(\cdot)$ could be in $\mathscr{H}_0^p(E)$, it is necessary and sufficient that $x(\cdot)u(\cdot)$ be Pettis integrable for every $u(\cdot) \in L^q(I)$.*

It is worthwhile to recall the following.

Definition 7. Let $x : I \to E$. The (left-sided) fractional Pettis-integral (shortly LS-FPI) of x of order $\alpha > 0$ is defined by

$$I_+^{\alpha} x(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds, \quad t > 0. \quad (4)$$

In the above definition the sign "$\int$" denotes the Pettis integral. For further purpose, we define the right-sided fractional Pettis-integral (shortly RS-FPI) by

$$I_-^{\alpha} x(t) := \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} x(s)\, ds, \quad t < 1. \quad (5)$$

We will call a function fractionally Pettis integrable provided this integral exists as an element of E (for arbitrary $t < 1$).

We need to clarify the relations between Pettis integrability and fractional Pettis integrability. Similar results will be proved for classes $\mathscr{H}_0^p(E)$. This will be important in our consideration, but it seems to be really interesting in itself. Here we restrict ourselves to the case of left-sided fractional Pettis-integrals.

To make the paper more expository, we will consider fractional Pettis integrability for both cases: $\alpha < 1$ and $\alpha > 1$. The last case is more important in our paper, but the first one is necessary to compare our results with some earlier theorems.

Let us observe that such an integral $I_+^{\alpha}x(t) := \int_0^t ((t-s)^{\alpha-1}/\Gamma(\alpha))x(s)ds$ is a convolution of a function $h(\tau) = \tau^{\alpha-1}/\Gamma(\alpha)$ for $\tau > 0$, $h(\tau) = 0$ for $\tau \le 0$, and the function $(\tilde{x})(t) = x(t)$ for $t \in I$, where $(\tilde{x})(t) = 0$ outside the interval I. Note that Pettis integrability of $x(t)$ implies Pettis integrability of $x(t+h)$ $(h > 0)$ and $x(-t)$, so the convolution of Pettis integrable function with real-valued function h can be properly defined. We start with an obvious observation that for $\varphi \in E^*$

$$\varphi(I_+^{\alpha}x) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi x(s)\, ds. \quad (6)$$

As a consequence of some properties of a convolution for the Pettis integral [28, Proposition 9], for arbitrary α, we have the following.

Theorem 8. *If $x : I \to E$ is Pettis integrable, then*

(a) *$I_+^{\alpha}x$ is defined almost a.e. on I,*

(b) *x is fractionally Pettis integrable on I,*

(c) *if x is Pettis integrable and strongly measurable, then $I_+^{\alpha}x : I \to E$ is bounded, weakly continuous and*

$$\sup_{\|\varphi\|\le 1} \int_0^1 \varphi I_+^{\alpha} x(t)\, dt \le \sup_{\|\varphi\|\le 1} \int_0^1 \varphi x(t)\, dt \cdot \|h\|_1. \quad (7)$$

In the case $E = \mathbb{R}$, it is a well-known consequence of an inequality of Young that the linear fractional integral operators $I_{\pm}^{\alpha}$, send $L^q([0,1])$ continuously into $L^p([0,1])$ if $p \in [1,\infty]$ satisfy $q > 1/(\alpha + (1/p))$ (see [29]) (a deep result from interpolation theory implies that even $q = 1/(\alpha + (1/p))$ is allowed if $1 < p < \infty$). In particular, $I_{\pm}^{\alpha} : L^p([0,1]) \to L^p([0,1])$ is compact for each $p \in [1,\infty]$. Moreover, for $p > \max\{1,(1/\alpha)\}$, the map $I_{\pm}^{\alpha} : L^p([0,1]) \to C([0,1])$ is compact (see, e.g., [20, 30]).

The following results plays a major rule in our analysis.

Lemma 9. *For any $\alpha > 0$ the operator $I_{\pm}^{\alpha}$ takes $C[I,E_w]$ into $C[I,E_w]$ and is well defined.*

Proof. Only the proof in case of the LS-FPI is given since the case of the RS-FPI is very similar.

It can be easily seen that if x is weakly continuous, then $x \in \mathscr{H}_0^{\infty}(E)$. Since $s \to (t-s)^{\alpha-1} \in L^1$, $s < t$, in the view of Proposition 6 and Theorem 8, we have that the function $s \to (t-s)^{\alpha-1}x(s)$, $s < t$, is Pettis integrable. Moreover, I_+^{α} is well defined. To see this, we define $y : I \to E$ by $y(t) := I_+^{\alpha}x(t)$, $t \in [0,1]$. From the definition of fractional Pettis integrals and Theorem 8(c), we have for every $\varphi \in E^*$ that

$$\varphi y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, ds. \quad (8)$$

Since the function φx is continuous, φy is continuous. That is, y is weakly continuous which finishes the proof. ☐

Lemma 10. *Let E be a reflexive Banach space. For any $\alpha \ge 1$ and arbitrary $p \in [1,\infty]$, the operator $I_{\pm}^{\alpha}$ takes $\mathscr{H}_0^p(E)$ into $\mathscr{H}_0^p(E)$ and is well defined.*

Proof. Only the proof in case of the LS-FPI is given since the case of the RS-FPI is very similar.

Note first that, for $x \in \mathscr{H}_0^p(E)$, we have in the view of Proposition 6 that the function $s \to (t-s)^{\alpha-1}x(s)$, $s < t$, is

Pettis integrable. That is, the operator I^α makes sense. Further, I_+^α is well defined. To see this, define $y : I \to E$ by $y(t) := I_+^\alpha x(t)$, $t \in [0,1]$. From the definition of fractional Pettis integrals, we have for every $\varphi \in E^*$ that

$$\varphi(y(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \varphi(x(s))\, ds = I_+^1 I_+^{\alpha-1} \varphi(x(t)). \tag{9}$$

Since $x \in \mathscr{H}_0^p(E)$, one have $\varphi x \in L^p(I)$ for every $\varphi \in E^*$. One could see, by the properties of fractional integral operators on the Banach space $L^p(I)$, that $\varphi y \in L^p(I)$ for every $\varphi \in E^*$. In particular, y is weakly continuous. Since weak continuity implies weak measurability (see [26, page 73]), $y(\cdot)$ is weakly measurable. However, in reflexive Banach spaces, weakly measurable functions $y : I \to E$ are Pettis integrable if and only if y is Dunford integrable; that is, $\varphi(y(\cdot))$ is Lebesgue integrable on I for every $\varphi \in E^*$. Thus $y(\cdot)$ is Pettis integrable. That is, $I_+^\alpha x \in \mathscr{H}_0^p(E)$. □

Remark 11. Let us discuss some properties of $\mathscr{H}^p(E)$ and $\mathscr{H}_0^p(E)$. Recall that $h(\tau) = \tau^{\alpha-1}/\Gamma(\alpha)$ for $\tau > 0$ and $z(\tau) = 0$ for $\tau \le 0$. The case of $\alpha \ge 1$ is trivial; that is, $h \in L^\infty(I)$. When $0 < \alpha < 1$, we see that $\int_0^1 (z(s))^q ds = \int_0^1 (s^{\alpha-1}/\Gamma(\alpha))^q ds < \infty$ whenever $q < 1/(1-\alpha)$. This means that $z \in L^{1/(1-\alpha)}(I)$ and by the converse for the Young inequality I^α takes $\mathscr{H}^{q_1}(E)$ into $\mathscr{H}^s(E)$ whenever $(1/q_1) + (1/q) = 1 + (1/s)$. In particular, the space required for this property depends on α. Let us note that I^α need not be continuous as an operator from $L^1(I)$ into $L^{1/(1-\alpha)}(I)$ (cf. [29, Remark 4.1.1]).

This means that for $\alpha \ge 1$ we have "uniform" estimations for all α, but for $0 < \alpha < 1$ the situation is more complicated (a weakly singular case).

Our consideration as well as Theorem 4.1.1 in [29] gives us a new property.

Lemma 12. *For any $0 < \alpha < 1$ and arbitrary $1 \ge p \ge 1/\alpha$ the operator $I_\pm^\alpha$ takes $\mathscr{H}_0^p(E)$ into $\mathscr{H}_0^s(E)$, where $s = p/(1-p(\alpha-\varepsilon))$ with arbitrary $\varepsilon > 0$.*

As a consequence of Lemma 10, we are able to prove the following.

Lemma 13. *Assume that $\alpha_1, \alpha_2 \ge 1$ and $x \in \mathscr{H}_0^p(E)$. Then,*

$$I_+^{\alpha_1} I_+^{\alpha_2} x = I_+^{\alpha_2} I_+^{\alpha_1} x = I_+^{\alpha_1+\alpha_2} x. \tag{10}$$

Proof. As in the proof of Lemma 10 it follows that $I_+^{\alpha_2} x$ and $I_+^{\alpha_1+\alpha_2} x$ exist. By Lemma 10, $I_+^{\alpha_1} I_+^{\alpha_2} x$ also exists. Therefore, for any $\varphi \in E^*$ we have

$$\begin{aligned} \varphi\left(I_+^{\alpha_1} I_+^{\alpha_2} x(t)\right) &= I_+^{\alpha_1} \varphi\left(I_+^{\alpha_2} x(t)\right) \\ &= I_+^{\alpha_1} I_+^{\alpha_2} \varphi(x(t)) \\ &= I_+^{\alpha_1+\alpha_2} \varphi(x(t)) \\ &= \varphi\left(I_+^{\alpha_1+\alpha_2} x(t)\right), \end{aligned} \tag{11}$$

that is,

$$\varphi\left(I_+^{\alpha_1} I_+^{\alpha_2} x(t) - I_+^{\alpha_1+\alpha_2} x(t)\right) = 0, \quad \text{for every } \varphi \in E^*. \tag{12}$$

Hence $I_+^{\alpha_1} I_+^{\alpha_2} x = I_+^{\alpha_1+\alpha_2} x(t)$. Similarly, we are able to show that $I_+^{\alpha_2} I_+^{\alpha_1} x = I_+^{\alpha_1+\alpha_2} x$. This ends the proof. □

Let us present the case $\alpha \ge 1$.

Lemma 14. *If $\alpha \ge 1$, $\theta \in L^q[0,1]$, and $y \in \mathscr{H}_0^p(E)$, then*

$$\int_0^1 \left[I_-^\alpha \theta(s)\right] y(s)\, ds = \int_0^1 \theta(s) \left[I_+^\alpha y(s)\right] ds. \tag{13}$$

Proof. Define the real-valued function h by

$$h(s) := \frac{1}{\Gamma(\alpha)} \int_s^1 (\tau - s)^{\alpha-1} \theta(\tau)\, d\tau = I_-^\alpha \theta(s). \tag{14}$$

Using the properties of fractional calculus in the Banach space $L^q[0,1]$ (see, e.g., [20, 30]), we deduce that $h \in L^q[0,1]$. Now, for $y \in \mathscr{H}_0^p(E)$ we have, in the view of Lemma 10, that $I_+^\alpha y \in \mathscr{H}_0^p(E)$. Thanks to Proposition 6, the functions $t \to \theta(t) I_+^\alpha y(t)$ and $t \to h(t) y(t)$ are Pettis integrable on $[0,1]$. That is, the integrals in both sides of (13) exist. Then there exists $J \in E$, such that

$$J = \int_0^1 \theta(s) I_+^\alpha y(s)\, ds. \tag{15}$$

By the definition of the Pettis integral, we have

$$\begin{aligned} \varphi J &= \int_0^1 \theta(s) \varphi\left(I_+^\alpha y(s)\right) ds = \int_0^1 \theta(s) I_+^\alpha \varphi y(s)\, ds \\ &= \int_0^1 \theta(s) \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \varphi y(\tau)\, d\tau\, ds, \quad \forall \varphi \in E^*. \end{aligned} \tag{16}$$

By changing the order of integration results in

$$\begin{aligned} \varphi J &= \int_0^1 \left(\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \theta(s)\, ds\right) \varphi y(\tau)\, d\tau \\ &= \int_0^1 h(\tau) \varphi y(\tau)\, d\tau \\ &= \varphi\left(\int_0^1 h(s) y(s)\, ds\right), \quad \forall \varphi \in E^*. \end{aligned} \tag{17}$$

Thus

$$J = \int_0^1 I_-^\alpha \theta(s) y(s)\, ds = \int_0^1 u(s) I_+^\alpha y(s)\, ds. \tag{18}$$

□

Definition 15. Let $x : I \to E$. We define the fractional pseudo-derivative of x of arbitrary order $n + \alpha$, $\alpha \in [0,1)$, where $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ by

$$D^{n+\alpha} x(t) := D^{n+1} I_+^{1-\alpha} x(t), \tag{19}$$

where D denote the pseudo-differential operator (cf. [22, 25]).

The following Lemma is well known in the case $E = \mathbb{R}$, but to see that it also holds in the vector-valued case, we provide a proof.

Lemma 16. *For $0 < \alpha \leq \beta$ we have for every weakly continuous function $x : [0,1] \to E$, $D^0 x = x$, and*

$$D^\beta I_+^\alpha x = D^{\beta-\alpha} x. \tag{20}$$

In particular, when $\alpha = \beta$, (20) means that the operator $D^\alpha I_+^\alpha$ is defined in $C(I, E_w)$ and that D^α is the left-inverse of I_+^α.

Proof. The first claim, that is, $DI_+^1 x = x$, follows from the fact that the integral of weakly continuous function is weakly continuous, then pseudo-differentiable with respect to the right endpoint of the integration interval. Let $\beta = n + \gamma$ and $\beta - \alpha = m + \delta$ with $n, m \in \mathbb{N}_0$ and $\gamma, \delta \in [0, 1)$. Then we have, in view of $DI^1 x = x$ and Lemmas 9 and 10, that

$$\begin{aligned} D^\beta I_+^\alpha x &= D^{n+1} I_+^{1-\gamma} I_+^\alpha x \\ &= D^{n+1} I_+^{1-\gamma+\alpha} x \\ &= D^{n+1} I_+^{1+n-m-\delta} x \\ &= D^{m+1} D^{n-m} I_+^{n-m} I_+^{1-\delta} x \\ &= D^{m+1} I_+^{1-\delta} x \\ &= D^{\beta-\alpha} x. \end{aligned} \tag{21}$$

□

Definition 17. A function $u : I \to E$ is called pseudo-solution of the problem (1) if $u \in C[I, E_w]$ has fractional pseudo-derivative of order $\alpha \in (1,2]$, $u(0) = 0$, $u(1) + \int_0^1 \mathfrak{F}(\tau) u(\tau) d\tau = l$ and satisfies

$$\begin{gathered} D^2 \varphi\left(I_+^{2-\alpha} u(t)\right) + \varphi\left(f\left(t, u(t), D^\beta u(t)\right)\right) = 0 \\ \text{a.e. on } [0,1], \text{ for each } \varphi \in E^*. \end{gathered} \tag{22}$$

The following auxiliary Lemma will be needed in our techniques.

Lemma 18. *If $v \in C[I, E_w]$ is a pseudo-solution to the problem*

$$\begin{gathered} D^{\alpha-\beta} v(t) + f\left(t, I_+^\alpha v(t), v(t)\right) = 0, \\ t \in [0,1], \quad \alpha \in (1,2], \quad \beta \in (0,1], \quad \alpha > 1 + \beta, \\ I_+^\beta v(1) + \int_0^1 \mathfrak{F}(\tau) I_+^\beta v(\tau) d\tau = l, \quad v(0) = 0, \end{gathered} \tag{23}$$

then $u := I_+^\beta v$ is a pseudo-solution for the problem (1).

Proof. Let $v \in C[I, E_w]$ be a pseudo-solution to the problem (23) and $\varphi \in E^*$. As in the proof of Lemma 9 it follows that $I_+^\beta v$ exists and the real function φu is continuous for every $\varphi \in E^*$; moreover

$$\begin{aligned} \lim_{t\to 0^+} \varphi u(t) &= \lim_{t\to 0^+} \left(I_+^\beta \varphi v\right)(t) \\ &= \lim_{t\to 0^+} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \varphi v(s)\, ds = 0. \end{aligned} \tag{24}$$

Thus $\varphi u(0) = 0$ for every $\varphi \in E^*$; that is, $u(0) = 0$. Further

$$u(1) + \int_0^1 \mathfrak{F}(\tau) u(\tau) d\tau = l. \tag{25}$$

In the view of Lemma 16 we also have

$$D^\alpha u(t) = \left(D^\alpha I_+^\beta v\right)(t) = D^{\alpha-\beta} v(t). \tag{26}$$

□

If otherwise is not stated, we will assume from now that $\alpha \in (1,2]$ and $\beta \in (0,1]$.

To obtain the Hammerstein type integral equation modeled off the problem (23), we keep the boundary value problem (23) in mind and we *formally* put (cf. [11, Lemma 2.3])

$$v(t) = -I_+^{\alpha-\beta} f\left(t, I_+^\beta v(t), v(t)\right) + c t^{\alpha-\beta-1}. \tag{27}$$

In the view of Lemma 13, we obtain

$$I_+^\beta v(t) = -I_+^\alpha f\left(t, I_+^\beta v(t), v(t)\right) + c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right] t^{\alpha-1}. \tag{28}$$

To facilitate our discussion, let $q \in [1, \infty]$ be constant with the conjugate exponents p. Suppose $\mathfrak{F} \in L^q[0,1]$ be a non-negative real-valued function and $f : [0,1] \times E \times E \to E$ satisfy the following assumptions:

(1) for each $t \in I = [0,1]$, $f(t, \cdot, \cdot) : (I) \times E \times E \to E$ is weakly-weakly sequentially continuous,

(2) for each $x, y \in C(I, E_w)$, $f(\cdot, x(\cdot), y(\cdot)) \in \mathscr{H}_0^p(E)$,

(3) for any $r > 0$ and $\varphi \in E^*$ there exist a Pettis integrable function $\tilde{f} : I \to E$, function $\Psi \in L^p[I, \mathbb{R}^+]$, and nondecreasing continuous function $\Omega : [0, \infty) \to (0, \infty)$ such that $|\varphi(f(t,x,y))| \leq |\varphi(\tilde{f}(t))| \Omega(r) \leq \|\varphi\| \Psi(t) \Omega(r)$ for a.e. $t \in I$ and all $(x, y) \in B_r \times B_r$.

Let us present two remarks about the above assumptions.

(i) For the interesting discussion about the growth conditions for Pettis integrable functions of the above type see [31]. For differential equations with Caputo fractional integrals (i.e., solutions in the space $C^m(0, 1), \mathbb{R}^d)$, where $m - 1 < \alpha < m$ the problem of dominants for considered functions (Assumption (3)) was considered in [32]. Nevertheless, in the paper by Lin rather strong boundedness conditions are investigated (dominants from $L^2(I)$ or $L^4(I)$ for $\alpha > 1/2$ or bounded functions for $\alpha > 0$) [32, Remark 2.3].

(ii) For convenience of the readers, let us recall the following lemma describing particular sufficient conditions for Pettis integrability of $f(\cdot, x(\cdot), y(\cdot))$ [23, Lemma 15]. It is obvious that additional growth condition for f allows us to characterize the functions from $\mathscr{H}_0^p(E)$.

Lemma 19. *Assume that x is absolutely continuous and $f : I \times E \times E \to E$. Thus $f(\cdot, x(\cdot), y(\cdot))$ is Pettis integrable if at least one of the following cases holds:*

(a) *f satisfies Carathédory conditions; that is, $f(\cdot, x, y)$ is measurable, $f(t, \cdot, \cdot)$ is continuous in $E \times E$, and there exists an integrable function $h : I \to \mathbb{R}$ such that $\|f(t, x, y)\| \leq h(t)$ for all $x, y \in E$ and a.e. $t \in I$,*

(b) *f is weakly-weakly continuous and E is a weakly sequentially complete space,*

(c) *$f(\cdot, x, y)$ is weakly measurable, $f(t, \cdot, \cdot)$ is weakly-weakly continuous in $E \times E$ and E is a WCG-space (weakly compactly generated space),*

(d) *f is strongly measurable and there exists a Young function Γ such that $\lim_{x\to\infty}\Gamma(x)/x = +\infty$ and $\varphi f \in L^{\Gamma}(I)$,*

(e) *f is strongly measurable and there exists $p > 1$ such that $\varphi f \in L^p$ for each $\varphi \in E^*$ (here $f(\cdot, x(\cdot), y(\cdot)) \in \mathscr{H}_0^p(E)$),*

(f) *$f(\cdot, x, y)$ is strongly measurable, $f(t, \cdot, \cdot)$ is weakly-weakly sequentially continuous in $E \times E$, and f is bounded,*

(g) *$f(\cdot, x(\cdot), y(\cdot))$ is strongly measurable, E contains no copy of c_0, and f is bounded.*

Now, we would like to pay our attention to solve (27) for c by

$$I_+^{\beta} v(1) + \int_0^1 \mathfrak{I}(\tau) I_+^{\beta} v(\tau)\, d\tau = l. \tag{29}$$

It follows that

$$\begin{aligned}
&c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right] - \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds = l,\\
&-\int_0^1 \mathfrak{I}(\tau)\left(c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right]\tau^{\alpha-\beta-1}\right.\\
&\qquad\left.-\int_0^{\tau} \frac{(\tau-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds\right) d\tau\\
&= l - c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right]\int_0^1 \mathfrak{I}(\tau)\tau^{\alpha-\beta-1} d\tau\\
&\quad + \int_0^1 \mathfrak{I}(\tau)\left(\int_0^{\tau} \frac{(\tau-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds\right) d\tau.
\end{aligned} \tag{30}$$

Therefore

$$\begin{aligned}
&c\left[\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\right](1+\gamma)\\
&= l + \int_0^1 \mathfrak{I}(\tau)\left(\int_0^{\tau} \frac{(\tau-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds\right) d\tau\\
&\quad + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds,
\end{aligned} \tag{31}$$

where

$$\gamma = \int_0^1 \mathfrak{I}(\tau)\tau^{\alpha-\beta-1} d\tau. \tag{32}$$

Then (in account of Lemma 14), we have

$$\begin{aligned}
c &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)}\\
&\quad\times\left[l + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f\left(s, I_+^{\beta} v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds\right.\\
&\qquad\left. + \int_0^1 h(s) f\left(s, I_+^{\beta} v(s), v(s)\right) ds\right].
\end{aligned} \tag{33}$$

Here $h = I_-^{\alpha-\beta}\mathfrak{I}$. Substituting c into (27), one has

$$\begin{aligned}
v(t) &= -I_+^{\alpha-\beta} f\left(t, I_+^{\beta} v(t), v(t)\right) + \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}\\
&\quad + \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}\\
&\quad \times \int_0^1 \left[\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + h(s)\right] f\left(s, I_+^{\beta} v(s), v(s)\right) ds\\
&= \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}\\
&\quad + \int_0^t \left[\Gamma(\alpha)\frac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)(\Gamma(\alpha-\beta))^2}\right.\\
&\qquad\left. - \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\right] f\left(s, I_+^{\beta} v(s), v(s)\right) ds\\
&\quad + \int_t^1 \Gamma(\alpha)\frac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)(\Gamma(\alpha-\beta))^2}\\
&\qquad \times f\left(s, I_+^{\beta} v(s), v(s)\right) ds\\
&\quad + \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)}\int_0^1 h(s) f\left(s, I_+^{\beta} v(s), v(s)\right) ds.
\end{aligned} \tag{34}$$

Thus

$$v(t) = p(t) + \int_0^1 G(t,s) f\left(s, I_+^{\beta} v(s), v(s)\right) ds, \quad t \in [0,1], \tag{35}$$

where $p(t) = l\Gamma(\alpha)t^{\alpha-\beta-1}/\Gamma(\alpha-\beta)(1+\gamma)$. The Green function G is given by $G(t,s) = G_1(t,s) + G_2(t,s)$ with

$$G_1(t,s) := \begin{cases} \dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\left[\dfrac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)\Gamma(\alpha-\beta)} - \dfrac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}\right], & 0 \le s \le t \le 1, \\ \dfrac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\left[\dfrac{(t(1-s))^{\alpha-\beta-1}}{(1+\gamma)\Gamma(\alpha-\beta)}\right], & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) := \frac{\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} h(s), \quad t,s \in [0,1]. \tag{36}$$

Since $\alpha - \beta > 1$, the following can be easily seen.

Lemma 20. *The map $t \to G(t,\cdot)$ is continuous from $[0,1]$ to $L^q[0,1]$.*

Remark 21. We point out that if E is reflexive, it is not necessary to assume any compactness conditions on the nonlinearity of f. This will be due to [33, Lemma 2] and the fact that a subset of reflexive Banach spaces is weakly compact if and only if it is weakly closed and norm bounded.

3. Weak Solutions of the Hammerstein Integral Equation

In this section, in the light of the Assumptions (1)–(3) imposed on f, we proceed to obtain a result which relies on the fixed point Theorem 3 to ensure the existence of weak solution to the integral equation (35). For the sake of convenience, we introduce the following.

Definition 22. By a solution to (35) we mean a function $v \in C(I,E)$ which satisfies the integral equation (35). This is equivalent to the finding $v \in C(I,E)$ with

$$\varphi(v(t)) = \varphi\left(p(t) + \int_0^1 G(t,s) f\left(s, I_+^{\beta} v(s), v(s)\right) ds\right), \quad t \in I \ \forall \varphi \in E^*. \tag{37}$$

We need to explain why we consider continuous solutions. By the properties of the Pettis integral this should be weakly continuous function. Since p is continuous and we impose (local) boundedness hypothesis for f, our solutions are strongly continuous (cf. [22]). We restrict our attention to the space $C(I,E)$ and then our integral operators will be defined on this space. In contrast to the case of weakly-weakly continuous functions f, we need to replace the space $C(I,E_w)$ endowed with its topology of weak uniform convergence by the space of (strongly) continuous functions $C(I,E)$ with its weak topology. We will utilize in our proofs some characterization of its weak topology.

Now, we are in the position to state and prove the first existence result.

Theorem 23. *Assume that $\alpha \in (1,2]$, $\beta \in (0,1)$ with $\alpha > 1+\beta$ and $\mathfrak{I} \in L^q([0,1])$ be a nonnegative real-valued function. If the Assumptions (1)–(3) hold along with*

$$\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)| \Psi(s)\, ds\right) \limsup_{r\to\infty} \frac{\Omega(r)}{r} < \Gamma(1+\beta), \tag{38}$$

then the integral equation (35) *has at least one solution $v \in C[I,E]$.*

Proof. First of all, observe the expression of G and note that the following implications:

$$\mathfrak{I} \in L^q[0,1] \Longrightarrow h \in L^q[0,1] \Longrightarrow G(t,\cdot) \in L^q[0,1], \quad t \in [0,1], \tag{39}$$

hold. Consequently $G(t,\cdot)\Psi(\cdot) \in L^1[0,1]$ for any $t \in [0,1]$. Let

$$\rho = \left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)| \Psi(s)\, ds\right) \limsup_{r\to\infty} \frac{\Omega(r)}{r}. \tag{40}$$

Consider the set S of real numbers $r \ge 0$ which satisfy the inequality

$$r\Gamma(1+\beta) \le \|p\|_0 + \Omega(r)\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)| \Psi(s)\, ds\right). \tag{41}$$

Then S is bounded above; that is, there exists a constant R_0 with

$$r \le R_0 \quad \forall r \in S. \tag{42}$$

To see this, suppose (42) is false. Then there exists a sequence $0 \ne r_n \in S$ with $r_n \to \infty$ as $n \to \infty$ and

$$\Gamma(1+\beta) \le \frac{\|p\|_0}{r_n} + \frac{\Omega(r_n)}{r_n}\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)| \Psi(s)\, ds\right). \tag{43}$$

Since $\limsup(s_n+t_n) \le \limsup s_n + \limsup t_n$ for any sequences $s_n \ge 0, t_n \ge 0$, we have $\rho \ge \Gamma(1+\beta)$. This contradicts (38). Then, for every $R > R_0$ the inequality

$$\|p\|_0 + \Omega(R)\left(\sup_{t\in[0,1]} \int_0^1 |G(t,s)| \Psi(s)\, ds\right) < \Gamma(1+\beta) R \tag{44}$$

holds, which is in contradiction with $R \in S$ and then contradicts (42).

Now, define the operator $T : C[I, E] \to C[I, E]$ by

$$Tv(t) := p(t) + \int_0^1 G(t,s) f\left(s, I_+^\beta v(s), v(s)\right) ds, \quad t \in [0,1]. \tag{45}$$

We remark that for $v \in C[I, E]$ we have that, by Lemma 9, $I_+^\beta v$ is weakly continuous, and consequently, $f(\cdot, I_+^\beta v(\cdot), v(\cdot)) \in \mathscr{H}_0^p(E)$ (Assumption (2)). Since $s \mapsto G(t,s) \in L^q(I)$, for all $t \in [0,1]$, $G(t,\cdot) f(\cdot, I_-^\beta v(\cdot), v(\cdot))$ is Pettis integrable for all $t \in [0,1]$ (thanks to Proposition 6), and thus the operator T makes sense. Note that T is well defined. To see this, let $t_1, t_2 \in [0,1]$ with $t_2 > t_1$. Since $\beta \in (0,1)$, we deduce that if $\|v\| \le \sigma_1$, then $\|v\| < \sigma$ and $\|I_+^\beta v\| \le \sigma$, where $\sigma = \sigma_1/\Gamma(1+\beta)$. Without loss of generality, assume $Tv(t_2) - Tv(t_1) \neq 0$. Then there exists (as a consequence of Proposition 2) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tv(t_2) - Tv(t_1)\| = \varphi(Tv(t_1) - Tv(t_1))$.

Putting the Assumption (3) in mind, one can write the following chain of inequalities:

$$\begin{aligned}
&\left\|Tv(t_2) - Tv(t_1)\right\| \\
&\quad = \varphi\left(Tv(t_2) - Tv(t_1)\right) \\
&\quad \le \varphi\left(p(t_2) - p(t_1)\right) \\
&\qquad + \Omega(\sigma) \int_0^1 \left|G(t_2,s) - G(t_1,s)\right| \Psi(s)\, ds \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| \\
&\qquad + \Omega(\sigma) \left(\int_0^1 \left|G(t_2,s) - G(t_1,s)\right|^q ds\right)^{1/q} \\
&\qquad \times \left(\int_0^1 |\Psi(s)|^p ds\right)^{1/p} \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| + \Omega(\sigma) \left\|G(t_2,\cdot) - G(t_1,\cdot)\right\|_q \|\Psi\|_p \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| \\
&\qquad + \Omega(\sigma) \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \right. \\
&\qquad\qquad \left. + \left\|G_2(t_2,\cdot) - G_2(t_1,\cdot)\right\|_q\right) \|\Psi\|_p \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| \\
&\qquad + \Omega(\sigma) \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \right. \\
&\qquad\qquad \left. + \frac{\Gamma(\alpha) \left\|(t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}) h(\cdot)\right\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p.
\end{aligned} \tag{46}$$

Then

$$\begin{aligned}
&\left\|Tv(t_2) - Tv(t_1)\right\| \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| \\
&\qquad + \Omega(\sigma) \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \right. \\
&\qquad\qquad \left. + \frac{\Gamma(\alpha) \left|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p.
\end{aligned} \tag{47}$$

Therefore we deduce, in the view of Lemma 20, that T maps $C[I, E]$ into itself.

Let $Q \in C[I, E]$ be the convex, closed and equicontinuous subset (required by Theorem 3). Define this set by

$$\begin{aligned}
Q := \Bigg\{ & v \in C[I, E] : \|v\|_0 \le R_0, \\
& \forall t_1, t_2 \in [0,1] \text{ we have } \left\|v(t_2) - v(t_1)\right\| \\
& \le \left\|p(t_2) - p(t_1)\right\| \\
& \quad + \Omega\left(\frac{R_0}{\Gamma(1+\beta)}\right) \\
& \quad \times \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \right. \\
& \qquad \left. + \frac{\Gamma(\alpha) \left|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p \Bigg\}.
\end{aligned} \tag{48}$$

We claim that T restricted to the set Q maps this set into itself (i.e., $T : Q \to Q$) and is weakly-weakly sequentially continuous. Once the claim is established, Theorem 3 guarantees the existence of a fixed point of T. Hence the integral equation (35) has a solution in $C[I, E]$.

We start by showing that $T : Q \to Q$. To see this, take $v \in Q$, $t \in [0,1]$. Since $\beta \in (0,1)$, we deduce that $\|v\| \le R_0 < R_0/\Gamma(1+\beta)$, $\|I^\beta v\| \le R_0/\Gamma(1+\beta)$. The monotonicity of Ω and the inequality (47) imply that

$$\begin{aligned}
&\left\|Tv(t_2) - Tv(t_1)\right\| \\
&\quad \le \left\|p(t_2) - p(t_1)\right\| \\
&\qquad + \Omega\left(\frac{R_0}{\Gamma(1+\beta)}\right) \\
&\qquad \times \left(\left\|G_1(t_2,\cdot) - G_1(t_1,\cdot)\right\|_q \right. \\
&\qquad\qquad \left. + \frac{\Gamma(\alpha) \left|t_2^{\alpha-\beta-1} - t_1^{\alpha-\beta-1}\right| \|h\|_q}{\Gamma(\alpha-\beta)(1+\gamma)} \right) \|\Psi\|_p.
\end{aligned} \tag{49}$$

Now, without loss of generality, assume $Tv(t) \neq 0$. Then there exists (consequence of Proposition 2) $\varphi \in E^*$ with $\|\varphi\| = 1$ and $\|Tx(t)\| = \varphi(Tv(t))$. By the Assumption (3), we obtain

$$\begin{aligned}
&\|Tv(t)\| \\
&\leq \varphi(p(t)) + \varphi\left(\int_0^1 G(t,s) f\left(s, I_+^{\beta} v(s), v(s)\right) ds\right) \\
&\leq \|p(t)\| + \sup_{t\in[0,1]} \int_0^1 |G(t,s)| \cdot \left|\varphi f\left(s, I_+^{\beta} v(s), v(s)\right)\right| ds \\
&\leq \|p\|_0 + \Omega\left(\frac{R_0}{\Gamma(1+\beta)}\right)\left(\sup_{t\in[0,1]} \int_0^1 G(t,s) \Psi(s) ds\right) \\
&< \frac{R_0}{\Gamma(1+\beta)} \Gamma(1+\beta) = R_0.
\end{aligned} \tag{50}$$

Therefore $\|Tv\|_0 = \sup_{t\in[0,1]}\|Tv(t)\| \leq R_0$. Hence $T : Q \to Q$.

We need to prove now that $T : Q \to Q$ is weakly-weakly sequentially continuous. Let us recall that the weak convergence in $Q \subset C(I,E)$ is exactly the weak pointwise convergence. Let (v_n) be a sequence in Q weakly convergent to v. Then $v_n(t) \to v(t)$ in E_w for each $t \in [0,1]$. Since Q is closed, by Lemma 1 we have $v \in Q$.

Fix $t \in I$ and note, in the view of Lebesgue dominated convergence theorem for the Pettis integral (see [31, 34]), that $I_+^{\beta} v_n(s) \to I_+^{\beta} v(s)$ in E_w. Let us recall that the topology on $C(I, E_w)$ on equicontinuous subsets coincides with the topology of weak pointwise convergence. Since f satisfies Assumption (1), we have $f(t, I_-^{\beta} v_n(t), v_n(t))$ converging weakly to $f(t, I_-^{\beta} v(t), v(t))$; hence again the Lebesgue dominated convergence theorem for Pettis integral yields $Tv_n(t)$ converging weakly to $Tv(t)$ in E, but Q is an equicontinuous subset of $C(I,E)$, and then $T : Q \to Q$ is weakly-weakly sequentially continuous. Applying now Theorem 3, we conclude that T has a fixed point in Q, which completes the proof. □

Let us present a multivalued problem:

$$\begin{gathered}
D^{\alpha-\beta} v(t) \in F\left(t, I_+^{\alpha} v(t), v(t)\right), \\
t \in [0,1], \quad \alpha \in (1,2], \quad \beta \in (0,1], \quad \alpha > 1+\beta, \\
I_+^{\beta} v(1) + \int_0^1 \mathfrak{F}(\tau) I_+^{\beta} v(\tau) d\tau = l, \quad v(0) = 0.
\end{gathered} \tag{51}$$

Some basic results for multivalued boundary value problems with Pettis integrals are due to Maruyama [35], Azzam et al. [36], Azzam-Laouir and Boutana [37], and Satco [38]. However these results are devoted to study the standard case $\alpha - \beta = 2$ and three-point boundary conditions. Our result is an essential extension for the previous ones.

By $ck(E)$ and $cwk(E)$ we denote the family of all nonempty convex compact and nonempty convex weakly compact subsets of E, respectively. For every nonempty convex bounded set $C \subset E$ the support function of C is denoted by $s(\cdot, C)$ and defined on E^* by $s(\varphi, C) = \sup_{x\in C} \varphi x$, for each $\varphi \in E^*$.

Definition 24. A multifunction $F : E \to 2^E$ with nonempty, closed values is weakly sequentially upper hemicontinuous if and only if for each $\varphi \in E^*$ $s(\varphi, G(\cdot)) : E \to \mathbb{R}$ is sequentially upper semicontinuous from (E, w) into $\mathbb{R}$.

In the remaining part of the paper a multifunction F is supposed to be Pettis integrable in the sense of Aumann.

Definition 25. The Aumann-Pettis integral of a multifunction $F : I \to E$ is

$$I_A := (AP) \int_I F(s)\, ds = \left\{(P) \int_I f(s)\, ds : f \in S_F^{Pe}\right\}, \tag{52}$$

where S_F^{Pe} denotes the set of all Pettis integrable selections of F provided that this set is not empty.

Let us note that the multivalued Pettis integral can be defined by other methods. The above definition is the best choice for our consideration. This can be deduced from the following theorem.

Theorem 26 (see [39]). *Let $F : I \to cwk(E)$ $[ck(E)]$ be measurable and scalarly integrable multifunction (i.e., the support functions are real-valued integrable functions). Then the following statements are equivalent:*

(a) *the set $\{\varphi f : \varphi \in B(E^*),\ f \in S_F^{Pe}\}$ is uniformly integrable,*

(b) *every measurable selection of F is Pettis integrable,*

(c) *for every measurable subset A of I the Aumann-Pettis integral I_A belongs to $cwk(E)$ $[ck(E)]$ and, for every $\varphi \in E^*$, one has*

$$s(\varphi, I_A) = \int_A s(\varphi, F(s))\, ds. \tag{53}$$

Taking into account Theorem 8, we are able to add one more condition to the above theorem, which seems to be important in our consideration. Since $\alpha - \beta > 1$, G is continuous, and by taking arbitrary Pettis integrable selection we obtain Pettis integrability of $G(t,\cdot) f(\cdot)$.

Theorem 27. *Each of the conditions from Theorem 26 implies the following:*

(d) *for every t the multifunction F is fractionally Aumann-Pettis integrable; that is, $I_+^{\alpha} F(t) := \{\int_0^t ((t-s)^{\alpha-1}/\Gamma(\alpha)) f(s) ds,\ f \in S_F^{Pe}\}$ belongs to $cwk(E)$ $[ck(E)]$.*

Let us recall that we restrict ourselves to the case of the (left-sided) fractional Pettis-integral.

Note that for multivalued mappings we will utilize Kakutani's fixed point theorem (for continuity concepts see [27]).

Theorem 28 (see [27]). *If C is a nonempty weakly compact convex subset of E and $F : C \to 2^C$ is sequentially weakly upper semi-continuous, then there exists a fixed point of F; that is, $x \in C$ with $x \in F(x)$.*

An immediate consequence of the above theorems as well as our main theorems is the following result.

Theorem 29. *Assume that E is separable. Let $F : I \times E \times E \to 2^E$ with nonempty convex and weakly compact values satisfy the following:*

(a) *$F(t, \cdot, \cdot) : E \times E \to E$ is weakly sequentially upper hemicontinuous for each $t \in I$,*

(b) *$F(\cdot, x, y)$ has a weakly measurable selection for each $(x, y) \in E \times E$,*

(c) *$F(t, x, y) \subset H(t)$ a.e. for some cwk-(E-) valued Pettis integrable multifunction G.*

Then there exists at least one pseudo-solution of the Cauchy problem (51) *on I.*

We will follow the idea of the proof for the single-valued problem. Let us only sketch the main steps of the proof.

Note that the Assumption (c) implies weak compactness of S_H^{Pe} and separability of the space E are sufficient to replace reflexivity of E (as announced in the preliminary part).

In this proof we need to define the multifunction $T : C[I, E] \to 2^{C[I,E]}$ by

$$T v(t) := p(t) + \int_0^1 G(t, s) F\left(s, I_+^\beta v(s), v(s)\right) ds, \quad t \in (I). \tag{54}$$

By [40, Lemma 3.2] our assumptions (a)–(c) imply that the set $S_F^{Pe}(\cdot, v(\cdot))$ is nonempty for arbitrary $v \in C(I, E)$ and the multivalued Nemytskii operator is well defined, so by Theorem 27 the Aumann-Pettis integral of F is nonempty too.

Let $W = \{f \in \mathscr{H}_0^1(E) : f(t) \in H(t) \text{ a.e. on } I\}$ and $U = \{x_f \in C(I, E) : x_f(t) = p(t) + \int_0^t G(t, s) f(s) ds,\ t \in I,\ f \in W\}$.

For $f \in W$ and $\varphi \in E^*$ we have $\varphi f \le s(\varphi, H)$. Then, by our assumptions, W is Pettis uniformly integrable. Thus for arbitrary $x \in U$ and $t, \tau \in I$ there exists an appropriate $f \in W$ and

$$\begin{aligned} \|x(t) - x(\tau)\| &= \sup_{\|\varphi\| \le 1} \varphi(x(t) - x(\tau)) \\ &= \sup_{\|\varphi\| \le 1} \left(\int_0^t \varphi f(s) ds - \int_0^\tau \varphi f(s) ds\right) \\ &= \sup_{\|\varphi\| \le 1} \int_\tau^t \varphi f(s) ds \\ &\le \sup_{\|\varphi\| \le 1} \int_\tau^t s(\varphi, H(s)) ds. \end{aligned} \tag{55}$$

By uniform Pettis integrability of W it follows that U is an equicontinuous subset of $C(I, E)$. The property of the multi-valued Pettis integral gives us the convexity of U. Then U is nonempty, convex, bounded, and equicontinuous in $C(I, E)$.

As S_H^{Pe} is sequentially compact for the topology induced by the tensor product $L^\infty \otimes E^*$, the $\overline{U}$ is closed. Since U is convex, by Mazur's lemma (Lemma 1) $\overline{U}$ is weakly closed. Thus by a weak version of Ascoli's theorem $\overline{U}$ is weakly compact in $C(I, E)$.

As the set U is strongly equicontinuous, then for each $M > 0$ there exists $\alpha \in I$ such that for each $t \in I$ and $f \in S_F^{Pe}$ we have $\| \int_0^t f(s) ds \| \le M$.

Then T restricted to a ball with radius $R_0 = M + \|p\|$ (as in the previous proof) has nonempty, closed, convex and weakly compact values. As a domain Q for T we put $\overline{U} \cap B_{R_0}$. By repeating the proof from [40, Theorem 3.3] we are able to show that T has weakly-weakly sequentially closed graph. Restricted to a weakly compact set Q an operatot T is sequentially weakly upper semi-continuous. This means, that the Kakutani fixed point theorem (Theorem 28) gives us a fixed point of T. The proof is complete.

4. Pseudo-Solutions to Fractional Order Boundary Value Problem

In this section, we are looking for sufficient conditions to ensure the existence of pseudo-solution to the boundary value problem (1) under the Pettis integrability assumption imposed on f. In order to obtain the existence of solutions of the problem (1), we can make use of Theorem 23.

Theorem 30. *Let the assumptions of Theorem 23 be satisfied. Then the boundary value problem* (1) *has at least one pseudo-solution $u \in C(I, E_w)$.*

Proof. Firstly, we remark that, for any $v \in C[I, E]$, we have (according to Proposition 6) that $v(\cdot)\mathfrak{I}(\cdot) \in P[I, E]$ for $\mathfrak{I} \in L^q(I)$, $q \in [1, \infty]$. Thus the integral boundary condition makes sense.

In account of Theorem 23 it can be easily seen that the integral equation (35) has a solution $v \in C[I, E]$. Let v be a weak solution of (35). Then

$$\begin{aligned} v(t) &= \frac{t^{\alpha-\beta-1} l \Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} + \int_0^1 \left[G_1(t, s) + G_2(t, s)\right] \times f\left(s, I_+^\beta v(s), v(s)\right) ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f\left(s, I_+^\beta v(s), v(s)\right) ds + c t^{\alpha-\beta-1}, \\ c &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} \times \left[l + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} f\left(s, I_+^\beta v(s), v(s)\right)}{\Gamma(\alpha-\beta)} ds + \int_0^1 h(s) f\left(s, I_+^\beta v(s), v(s)\right) ds \right]. \end{aligned} \tag{56}$$

By Lemma 14 and using $G_1(0,s) = G_2(0,s) = 0$, a straightforward estimates show that

$$v(0) = 0, \qquad I_+^{\beta} v(1) + \int_0^1 \mathfrak{J}(\tau) I_+^{\beta} v(\tau)\, d\tau = l. \tag{57}$$

Furthermore, we have

$$v(t) = -I_+^{\alpha-\beta} f\left(t, I_+^{\beta} v(t), v(t)\right) + ct^{\alpha-\beta-1}. \tag{58}$$

Thus for any $\varphi \in E^*$ we have

$$\begin{aligned}\varphi v(t) &= -\varphi\left(I_+^{\alpha-\beta} f\left(t, I_+^{\beta} v(t), v(t)\right)\right) + \varphi c t^{\alpha-\beta-1} \\ &= -I_+^{\alpha-\beta}\varphi\left(f\left(t, I_+^{\beta} v(t), v(t)\right)\right) + \varphi c t^{\alpha-\beta-1}.\end{aligned} \tag{59}$$

Operating by $I_+^{2-(\alpha-\beta)}$ on both sides of (59) and using the properties of fractional calculus in the space $L^1[0,1]$ (see, e.g., [20, 30]) result in

$$\begin{aligned}I_+^{2-(\alpha-\beta)}\varphi v(t) &= -I_+^{2}\varphi\left(f\left(t, I_+^{\beta} v(t), v(t)\right)\right) \\ &\quad + \varphi c\frac{\Gamma(\alpha-\beta)}{\Gamma(2)} t.\end{aligned} \tag{60}$$

Therefore

$$\begin{aligned}\varphi\left(I_+^{2-(\alpha-\beta)} v(t)\right) &= -I_+^{2}\varphi\left(f\left(t, I_+^{\beta} v(t), v(t)\right)\right) \\ &\quad + \varphi c\frac{\Gamma(\alpha-\beta)}{\Gamma(2)} t.\end{aligned} \tag{61}$$

Thus

$$\begin{aligned}&\frac{d^2}{dt^2}\varphi\left(I_+^{2-(\alpha-\beta)} v(t)\right) \\ &\quad = -\varphi\left(f\left(t, I_+^{\beta} v(t), v(t)\right)\right) \quad \text{a.e. on } [0,1].\end{aligned} \tag{62}$$

That is, v has the fractional pseudo-derivative of order $\alpha-\beta \in (1,2)$ and satisfies

$$D^{\alpha-\beta} v(t) = -f\left(t, I_+^{\beta} v(t), v(t)\right) \quad \text{on } [0,1]. \tag{63}$$

Therefore v is a pseudo-solution to the problem (23). This together with Lemma 18 implies that the problem (1) has a pseudo-solution $u \in C(I, E_w)$ which completes the proof. □

Now, we consider an example to illustrate our result.

Example 31. Let $\mathfrak{J} \in L^{\infty}(I)$ and assume that $E := \ell^2(I)$ be the space of countably nonzero functions on $I := [0,1]$ that are square-summable, under the ℓ^2-norm. This is even a reflexive space. For each $t \in I$ we define $g(t) := e_t \in \ell^2(I)$ by

$$e_t(s) := \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{if } s \neq t. \end{cases} \tag{64}$$

This function is Pettis, but not Bochner, integrable (it is not even strongly measurable [22, 41]), and for any $\varphi \in \ell^2(I)$ we have [22, 41]

$$\varphi e_t = \sum_{\tau\in[0,1]} \varphi(\tau) e_t(\tau) = \begin{cases} 0, & \text{if } \tau \neq t, \\ \varphi(\tau), & \text{if } \tau = t. \end{cases} \tag{65}$$

Since $(\ell^2(I))^* = \ell^2(I)$, the function φe_t is only countably nonzero for each $\varphi \in (\ell^2(I))^*$. Hence $\varphi g = 0$ a.e. (with respect to the Lebesgue measure).

Now, we investigate the existence of pseudo-solutions for the problem

$$\begin{gathered} D^{\alpha} u(t) + \mu D^{\beta} u(t) + \delta u(t) = g(t), \\ t \in [0,1], \quad \delta, \mu \in \mathbb{R}, \quad \alpha \in (1,2], \quad \beta \in (0,1), \quad \alpha > 1 + \beta, \\ u(1) + \int_0^1 \mathfrak{J}(\tau) u(\tau)\, d\tau = l, \quad u(0) = 0. \end{gathered} \tag{66}$$

Let us define the function $f : I \times E \times E \to E$ by $f(t,x,y) = \mu x + \delta y - g(t)$. Remark that for any $x, y \in C[I, E]$, f is Pettis, but not Bochner, integrable and satisfies, by suitable choice of δ, μ, all requirements of Theorem 30 with $\psi \equiv 1$, $\Omega(r) = (\delta + \mu) r$, $p = 1$, and $q = \infty$.

Therefore one cannot expect the existence of weak or strong solutions to the problem (66). In view of Theorem 30 the existence of pseudo-solutions is guaranteed. Since f satisfies the requirements of Theorem 30 with $p = 1$, $q = \infty$, the problem (66) has a pseudo-solution $u \in C(I, E_w)$ given by $u = I_+^{\beta} v$, where v denotes the weak solution to the problem

$$\begin{aligned} v(t) = {} & \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \\ & + \int_0^1 G(t,s)\left[\mu v(s) + \delta I_+^{\beta} v(s) - g(s)\right] ds, \\ & \qquad t \in (I). \end{aligned} \tag{67}$$

That is, v satisfies the problem

$$\begin{aligned} v(t) = {} & \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} \\ & + \int_0^1 G(t,s)\left[\mu v(s) + \delta I_+^{\beta} v(s)\right] ds \\ & - \int_0^1 G(t,s) g(s)\, ds, \quad t \in (I). \end{aligned} \tag{68}$$

Since $\int_0^1 G(t,s)\varphi g(s) ds = 0$, for each $\varphi \in \ell^2(I)$, we obtain $\int_0^1 G(t,s) g(s) ds = 0$. Hence

$$\begin{aligned} v(t) &= \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} + \int_0^1 G(t,s)\left[\mu v(s) + \delta I_+^{\beta} v(s)\right] ds \\ &= \frac{l\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)(1+\gamma)} - I_+^{\alpha-\beta}\left[\mu v(s) + \delta I_+^{\beta} v(s)\right] + ct^{\alpha-\beta-1}, \end{aligned} \tag{69}$$

where

$$c = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} \times \left[l + \int_0^1 \frac{(1-s)^{\alpha-\beta-1}\left[\mu v(s) + \delta I_+^{\beta} v(s)\right]}{\Gamma(\alpha-\beta)} ds + \int_0^1 h(s)\left[\mu v(s) + \delta I_+^{\beta} v(s)\right] ds \right]. \tag{70}$$

Therefore

$$v(t) + \mu I_+^{\alpha-\beta} v(t) + \delta I_+^{\alpha} v(t) = c^* t^{\alpha-\beta-1}, \tag{71}$$

where

$$c^* = \left(c + \frac{l\Gamma(\alpha)}{\Gamma(\alpha-\beta)(1+\gamma)} \right). \tag{72}$$

Further

$$I_+^{\beta} v(t) + \mu I_+^{\alpha-\beta} I_+^{\beta} v(t) + \delta I_+^{\alpha+\beta} v(t) = c^* \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}. \tag{73}$$

Now if $u = I_+^{\beta} v$, we obtain

$$u(t) + \mu I_+^{\alpha-\beta} u(t) + \delta I_+^{\alpha} u(t) = c^* \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}. \tag{74}$$

Thus for every $\varphi \in (\ell^2(I))^*$, we have

$$\varphi u(t) + \mu I_+^{\alpha-\beta} \varphi u(t) + \delta I_+^{\alpha} \varphi u(t) = \varphi c^* \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} t^{\alpha-1}. \tag{75}$$

Whence

$$I_+^{2-\alpha} \varphi u(t) + \mu I_+^{2-\beta} \varphi u(t) + \delta I_+^{2} \varphi u(t) = \varphi c^* \frac{\Gamma(\alpha-\beta)}{\Gamma(2)} t. \tag{76}$$

Moreover,

$$\frac{d^2}{dt^2} \varphi\left(I_+^{2-\alpha} u(t)\right) + \frac{d^2}{dt^2} \varphi I_+^{2-\beta} u(t) + \delta \varphi u(t) = 0. \tag{77}$$

Since

$$\begin{aligned} \frac{d^2}{dt^2} \varphi I_+^{2-\beta} u(t) &= \frac{d^2}{dt^2} I_+^{2-\beta} \varphi u(t) \\ &= \frac{d}{dt}\left(\frac{d}{dt} I_+^{1} I_+^{1-\beta} \varphi u(t) \right) \\ &= \frac{d}{dt} I_+^{1-\beta} \varphi u(t) \\ &= \frac{d}{dt} \varphi I_+^{1-\beta} u(t), \end{aligned} \tag{78}$$

we arrive, for every $\varphi \in E^*$, at $D^{\alpha}\varphi u(t) + \mu D^{\beta}\varphi u(t) + \delta\varphi u(t) = 0$ for all $t \in [0,1]$ while $\varphi g = 0$ a.e.

Finally

$$D^{\alpha} \varphi u(t) + \mu D^{\beta} \varphi u(t) + \delta u(t) = \varphi e_t, \quad \text{a.e. } [0,1]. \tag{79}$$

Then u is a pseudo- (but not a weak or strong) solution to the problem (66).

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Research Article

Second-Order Impulsive Differential Equations with Functional Initial Conditions on Unbounded Intervals

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We present some results on the existence of solutions for second-order impulsive differential equations with deviating argument subject to functional initial conditions. Our results are based on Schaefer's fixed point theorem for completely continuous operators.

1. Introduction

Differential equations with impulses arise quite often in the study of different problems in particular are used as a model for evolutionary processes subject to a sudden rapid change of their state at certain moments. The theory of impulsive differential equations has become recently a quite active area of research. For an introduction to this theory we refer to the books [1–4], which also contain a variety of interesting examples and applications.

In this paper, we establish new results for the existence of solutions for the second-order impulsive differential equation with deviating argument subject to initial conditions:

$$\begin{gathered} x''(t) = Ax(t) + f(t, x(t), x(\alpha(t))), \\ \qquad t \in [0, +\infty) \setminus \{t_1, \dots, t_l\} \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1, \dots, l \\ x'(t_k^+) - x'(t_k^-) = J_k(x(t_k)), \quad k = 1, \dots, l \\ x(t) = \phi(t), \quad t \in [-r, 0], \\ x'(0) = \phi'(0) := \eta, \end{gathered} \tag{P}$$

where $x(t)$ is in the space $(\mathbb{R}^n, |\cdot|)$ ($|\cdot|$ not necessarily the Euclidean norm), $\Theta := \{0 < t_1 < \cdots < t_l\} \subseteq [0, +\infty)$, A is a real $n \times n$ matrix, $x(\tau^-)$, and $x(\tau^+)$ are the left and right limits of x in $t = \tau$.

Such type of problem arises from a few of scientific applications as, for example, the problem of impulsive maneuver of a spacecraft (see [5] in finite-dimentional setting and [6] in infinite-dimentional setting). More recently, impulsive second-order differential equations or inclusions on compact intervals subject to nonhomogeneous conditions have been studied by several authors (see for example [7–14] and the references therein). The study of differential problems on unbounded interval has been done, for example, in [14–18].

Differential equations with deviating argument are investigated, for example, in [6, 7, 9, 12, 14, 16, 17, 19, 20].

Various techniques are utilized in the above papers: Schauder's fixed point Theorem [6, 9, 16, 17, 19, 20], Leray-Schauder's nonlinear alternative [14, 18], Contractions Principle [8, 13], fixed point index theory [12], and Sadovskii's fixed point Theorem [7]. Moreover, in order to prove the compactness of involved operator, the Ascoli-Arzelà Theorem is often used (see [6, 9, 12, 14, 18]).

The methodology here is to write the problem (P) as a *perturbed* integral equation, and we look for fixed points of an operator T in a suitable functions space. For this purpose we want to utilize the Schaefer's fixed point theorem for completely continuous operators. In order to prove that T is completely continuous, we make use of a variant of the compactness result in the Banach space $BC(Q, \mathbb{R}^n)$ of continuous bounded functions from a topological space Q into $\mathbb{R}^n$ due to De Pascale, Lewicki and G. Marino. We have used similar techniques and tools in [16, 17, 20] to study first-order, impulsive, or nonimpulsive, differential or integrodifferential equations on unbounded intervals. In this paper we extend these ideas to second-order impulsive equations.

Moreover we prove that our method can be easily used also in the case of functional conditions that, to the best of the our knowledge, are not studied for this class of problems. In fact, we discuss in details the Problem (P) in the case of the initial conditions

$$x(t) = \phi(t) \quad t \in [-r, 0], \quad x'(0) = \phi'(0) \tag{1}$$

because it sheds light on the techniques used, but the same approach may be applied to impulsive equations subject to a more general functional initial condition that covers a large number of cases, namely

$$Lx = H(x), \qquad x'(0) = \eta, \tag{2}$$

where L is a bounded linear operator and H is a completely continuous operator. This is done in the last section.

2. Notations and Preliminaries

Firstly, we recall definitions, notations, and useful facts regarding the *cosine families* (see [21–23] for the detailed study of cosine and sine families).

Definition 1. A one-parameter family $(C(t))_{t\in\mathbb{R}}$ of bounded linear mappings on $\mathbb{R}^n$ into itself is called a strongly continuous *cosine family* if and only if

(1) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$,
(2) $C(0) = I$, where I is the identity map,
(3) $C(\cdot)x \in C(\mathbb{R}, \mathbb{R}^n)$, for all $x \in \mathbb{R}^n$.

Moreover, the *sine family* $(S(t))_{t\in\mathbb{R}}$ is defined as

$$S(t) := \int_0^t C(s)\, ds. \tag{3}$$

By definition one obtains that:

(u1) $S(t)x$ is continuous in $t \in \mathbb{R}$, $S(0) = 0$ and $S(-t) = -S(t)$, for all $t \in \mathbb{R}$,
(u2) $C(t) = C(-t)$, for all $t \in \mathbb{R}$,
(u3) $C(t), S(t), S(s), C(s)$ commute, for all $t, s \in \mathbb{R}$,
(u4) $S(t+s) + S(t-s) = 2S(t)C(s)$, for all $t, s \in \mathbb{R}$,
(u5) $S(t+s) = S(t)C(s) + C(t)S(s)$, for all $t, s \in \mathbb{R}$.

Definition 2. The infinitesimal generator of a strongly continuous cosine family is the operator $A : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$A := \left.\frac{d^2}{dt^2} C(t)\right|_{t=0} \tag{4}$$

and $\mathrm{Dom}(A) := \{x \in \mathbb{R}^n\ C(t)x$ is continuously differentiable in $t = 0\}$.

One obtains that, for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$\frac{d^2}{dt^2} C(t)\, x = AC(t)\, x = C(t)\, Ax \tag{5}$$

and then

$$\frac{d}{dt} C(t)\, x = \int_0^t AC(s)\, x ds = A \int_0^t C(s)\, x ds = AS(t)\, x. \tag{6}$$

So, by (u5) one has

(u6) $C(t+s) = C(t)C(s) + AS(t)S(s)$, for all $t, s \in \mathbb{R}$.

Definition 3. A cosine family $(C(t))_{t\in\mathbb{R}}$ (resp., a sine family $(S(t))_{t\in\mathbb{R}}$) is uniformly bounded if there exists $M_C > 0$ (resp., $M_S > 0$) such that

$$|C(t)| \le M_C \quad \left(\text{resp. } |S(t)| \le M_S\right) \quad \forall t \in \mathbb{R}, \tag{7}$$

where $|\cdot|$ denotes the norm in the space of the $n \times n$ matrices.

Example 4. In $\mathbb{R}^n$, let $(C(t))_{t\in\mathbb{R}}$ be the family of bounded linear operators

$$C(t) = \begin{pmatrix} \cos(a_1 t) & 0 & \cdots & 0 \\ 0 & \cos(a_2 t) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cos(a_n t) \end{pmatrix}. \tag{8}$$

Let us observe that $(C(t))_{t\in\mathbb{R}}$ is a uniformly bounded cosine family and its infinitesimal generator is

$$A = \left.\frac{d^2}{dt^2} C(t)\right|_{t=0} \begin{pmatrix} -a_1^2 & 0 & \dots & 0 \\ 0 & -a_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -a_n^2 \end{pmatrix}. \tag{9}$$

We will utilize the Schaefer's fixed point theorem for completely continuous operators.

Theorem 5 (see [24]). *Let E be a normed linear space. Let $S : E \to E$ be a completely continuous operator. Let*

$$\zeta(S) := \{x \in E : x = \lambda Sx \text{ for some } 0 < \lambda < 1\}. \tag{10}$$

Then either $\zeta(S)$ in unbounded or S has a fixed point.

We use the condition for compactness in the Banach space $BC(Q, \mathbb{R}^n)$ of continuous bounded functions y from a topological space Q into $\mathbb{R}^n$, endowed with the norm $\|y\|_\infty := \sup\{|y(t)|,\ t \in Q\}$, due to De Pascale, Lewicki and G. Marino.

Theorem 6 (see [25]). *Let $T : BC(Q, \mathbb{R}^n) \to BC(Q, \mathbb{R}^n)$ be a continuous operator. Suppose that, for any bounded set $F \subset BC(Q, \mathbb{R}^n)$, $T(F)$ is a bounded set and there exist ν bounded functions $\varphi_j : Q \to \mathbb{R}^n$, $j = 1, 2, \ldots, \nu$, such that, for all $t, s \in Q$ and for all $y \in F$,*

$$\left|(Ty)(t) - (Ty)(s)\right| \le \sum_{j=1}^{\nu} \left|\varphi_j(t) - \varphi_j(s)\right|. \tag{11}$$

Then T is a compact operator.

3. The Integral Problem

From now on, we assume that the functions $f, \phi, \alpha, I_k, J_k$ and the matrix A have the following properties.

(h_0) The matrix A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$, uniformly bounded by $M_C > 0$. Suppose moreover that the corresponding sine family $(S(t))_{t\in\mathbb{R}}$ is uniformly bounded by a constant $M_S > 0$.

(h_1) $f : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that there exist a continuous integrable function $p : [0,+\infty) \to [0,+\infty)$ and a continuous nondecreasing function $\Psi : [0,+\infty) \to [1,+\infty)$ for which

$$\left|f(t,x,y)\right| \le p(t)\Psi(|x|+|y|), \qquad t \ge 0,\ x,y \in \mathbb{R}^n, \tag{12}$$

$$\int_0^\infty \frac{ds}{\Psi(s)} = +\infty. \tag{13}$$

(h_2) $\alpha : [0,+\infty) \to [-r,+\infty)$ is a continuous increasing function such that $\alpha(t) \le t$, for all $t \in [0,+\infty)$.

The function ϕ belongs to $C^1([-r,0],\mathbb{R}^n)$.

(h_3) $I_k : \mathbb{R}^n \to \mathbb{R}^n$ and $J_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,l$ are continuous functions such that there are $D_I, D_J > 0$ for which

$$\left|I_k(v)\right| \le D_I, \quad \left|J_k(v)\right| \le D_J, \quad \forall v \in \mathbb{R}^n. \tag{14}$$

We will work in the Banach space

$$\begin{aligned} BPC_\Theta[-r,+\infty) := \{ y : [-r,+\infty) \longrightarrow \mathbb{R}^n \mid\ & y \text{ is bounded} \\ & \text{and continuous in } t \notin \Theta, \\ & \text{there exist } y(t_k^-) = y(t_k) \\ & \text{and } y(t_k^+) < \infty, t_k \in \Theta\}, \end{aligned} \tag{15}$$

endowed with the supremum norm $\|y\|_\infty = \sup\{|y(t)| : t \in [-r,+\infty)\}$.

We define for $x \in BPC_\Theta[-r,+\infty)$:

$$\begin{aligned} w_x(t) &:= \begin{cases} \int_0^t S(t-s) f(s,x(s),x(\alpha(s)))\,ds, & t \ge 0, \\ 0, & -r \le t \le 0 \end{cases} \\ (Ix)(t) &:= \begin{cases} \sum_{0<t_k<t} C(t-t_k) I_k(x(t_k)), & t > t_1, \\ 0, & -r \le t \le t_1 \end{cases} \\ (Jx)(t) &:= \begin{cases} \sum_{0<t_k<t} S(t-t_k) J_k(x(t_k)), & t > t_1, \\ 0, & -r \le t \le t_1. \end{cases} \end{aligned} \tag{16}$$

For any $x \in BPC_\Theta[-r,+\infty)$ let Tx be the function defined on $[-r,+\infty)$ by

$$(Tx)(t) := \begin{cases} C(t)\phi(0) + S(t)\eta + w_x(t) \\ +(Ix)(t) + (Jx)(t), & t \ge 0 \\ \phi(t), & -r \le t \le 0. \end{cases} \tag{17}$$

In the next propositions we show the properties of the operator T useful for us.

Proposition 7. *The operator T maps $BPC_\Theta[-r,+\infty)$ into $BPC_\Theta[-r,+\infty)$ and $(Tx)(t_k^+)-(Tx)(t_k^-) = I_k(x(t_k))$. Moreover Tx has derivative in $t \in [-r,+\infty) \setminus \Theta$, $(Tx)'(t_k^-) = (Tx)'(t_k)$, $(Tx)'(t_k^+) < \infty$ and $(Tx)'(t_k^+) - (Tx)'(t_k^-) = J_k(x(t_k))$.*

Proof. The continuity of f, S, and C guarantees that Tx is continuous in $t \in [-r,+\infty) \setminus \Theta$, and there exists $(Tx)'(t)$ for $t \in [-r,+\infty) \setminus \Theta$. Moreover, since p is integrable, for $t \in [-r,+\infty) \setminus \Theta$, one has

$$\begin{aligned} |(Tx)(t)| \le{}& M_C\left|\phi(0)\right| + M_S|\eta| + M_S\Psi(2\|x\|_\infty) \\ & \times \int_0^\infty p(s)\,ds + lM_C D_I + lM_S D_J, \end{aligned} \tag{18}$$

so $\|Tx\|_\infty < \infty$. We need to show that, for $k = 1,\dots,l$, there exist $(Tx)(t_k^+)$ and $(Tx)(t_k^-) = (Tx)(t_k)$. Note that

(i) $\lim_{t\to t_k} w_x(t) = w_x(t_k)$. Indeed, using (u5), for $t \to t_k$

$$\begin{aligned} &\left|w_x(t) - w_x(t_k)\right| \\ &\le \int_0^t \left|S(t-s) - S(t_k-s)\right| \\ &\quad \cdot \left|f(s,x(s),x(\alpha(s)))\right| ds \\ &\quad + \int_t^{t_k} \left|S(t_k-s)\right| \cdot \left|f(s,x(s),x(\alpha(s)))\right| ds \\ &(\text{by } (h_1)) \\ &\le \int_0^t \left|S(t-s) - S(t_k-s)\right| \Psi(2\|x\|_\infty)\, p(s)\,ds \\ &\quad + M_S\Psi(2\|x\|_\infty) \int_t^{t_k} p(s)\,ds \\ &\le \left(\left|S(t) - S(t_k)\right| + \left|C(t) - C(t_k)\right|\right) \\ &\quad \times \max\{M_S, M_C\}\Psi(2\|x\|_\infty) \int_0^t p(s)\,ds \\ &\quad + M_S\Psi(2\|x\|_\infty) \int_t^{t_k} p(s)\,ds \longrightarrow 0. \end{aligned} \tag{19}$$

(ii) It results that $(Ix)(t_k^-) = (Ix)(t_k) = \sum_{j=1}^{k-1} C(t_k - t_j) I_j(x(t_j))$. Indeed let $t_{k-1} < t < t_k$. Then

$$\begin{aligned} &\left|(Ix)(t) - (Ix)(t_k)\right| \\ &= \left|\sum_{0<t_j<t} C(t - t_j) I_j(x(t_j)) - \sum_{0<t_j<t_k} C(t_k - t_j) I_j(x(t_j))\right|, \\ &\text{(by } (h_3)) \\ &\leq D_I \sum_{j=1}^{k-1} \left|C(t - t_j) - C(t_k - t_j)\right| \longrightarrow 0. \end{aligned} \tag{20}$$

Analogously one can see that

$$(Ix)(t_k^+) = \sum_{j=1}^{k} C(t_k - t_j) I_j(x(t_j)), \tag{21}$$

in such a way that $(Ix)(t_k^+) - (Ix)(t_k^-) = I_k(x(t_k))$.

Since, by (u1), $S(t_k - t_k) = 0$, a similar proof permits to verify that

$$\begin{aligned} (Jx)(t_k^+) - (Jx)(t_k^-) = &\sum_{j=1}^{k} S(t_k - t_j) J_j(x(t_j)) \\ &+ \sum_{j=1}^{k-1} S(t_k - t_j) J_j(x(t_j)) = 0. \end{aligned} \tag{22}$$

By the above steps, it follows that $(Tx)(t_k^+) - (Tx)(t_k^-) = I_k(x(t_k))$.

Since there exists $(Tx)'(t)$ if $t \in [0, +\infty) \setminus \Theta$, to conclude our proof we show that there exists $(Tx)'(t_k^-) = (Tx)'(t_k)$, $(Tx)'(t_k^+)$ is finite, and $(Tx)'(t_k^+) - (Tx)'(t_k^-) = J_k(x(t_k))$. We also observe that, for $t \geq 0$ and $t \notin \Theta$, by (u6),

$$\begin{aligned} (Tx)'(t) = &AS(t)\phi(0) + C(t)\eta \\ &+ \int_0^t C(t-s) f(s, x(s), x(\alpha(s)))\,ds \\ &+ \sum_{0<t_k<t} AS(t - t_k) I_k(x(t_k)) \\ &+ \sum_{0<t_k<t} C(t - t_k) J_k(x(t_k)). \end{aligned} \tag{23}$$

Thus, following the same idea, when $t \to t_k$, we have

$$\begin{aligned} &\left|\int_0^t C(t-s) f(s, x(s), x(\alpha(s)))\,ds - \int_0^{t_k} C(t_k - s) f(s, x(s), x(\alpha(s)))\,ds\right| \\ &\leq \int_0^t \left|C(t-s) - C(t_k - s)\right| p(s) \Psi(2\|x\|_\infty)\,ds \\ &+ M_C \Psi(2\|x\|_\infty) \int_t^{t_k} p(s)\,ds \longrightarrow 0. \end{aligned} \tag{24}$$

Moreover, for $t \to t_k^-$

$$\begin{aligned} &\left|\sum_{0<t_j<t} AS(t - t_j) I_j(x(t_j)) - \sum_{j=1}^{k-1} AS(t_k - t_j) I_j(x(t_j))\right| \\ &\leq D_I |A| \sum_{j=1}^{k-1} \left|S(t - t_j) - S(t_k - t_j)\right| \longrightarrow 0, \end{aligned} \tag{25}$$

$$\lim_{t \to t_k^+} \left(\sum_{0<t_k<t} AS(t - t_k) I_k(x(t_k))\right) = \sum_{j=1}^{k-1} AS(t - t_k) I_j(x(t_j)).$$

Finally, for $t \to t_k^-$

$$\begin{aligned} &\left|\sum_{0<t_k<t} C(t - t_k) J_k(x(t_k)) - \sum_{j=1}^{k-1} C(t_k - t_j) J_j(x(t_j))\right| \\ &\leq D_J \sum_{j=1}^{k-1} \left|S(t - t_j) - S(t_k - t_j)\right| \longrightarrow 0, \end{aligned} \tag{26}$$

while

$$\lim_{t \to t_k^+} \left(\sum_{0<t_k<t} C(t - t_k) J_k(x(t_k))\right) = \sum_{j=1}^{k} C(t - t_k) J_j(x(t_j)), \tag{27}$$

and these permit to obtain that there exist $(Tx)'(t_k^+)$ and $(Tx)'(t_k^-)$ and that one has $(Tx)'(t_k^+) - (Tx)'(t_k^-) = J_k(x(t_k))$. □

Proposition 8. *The fixed points of T are solutions of the problem (P).*

Proof. In fact, $Tx|_{[-r,0]} = \phi$ and $(Tx)'(0) = \eta$. Moreover, by the hypotheses on f and on the sine and cosine families, one obtains that $(Tx)'$ is derivable in $t \in [-r, +\infty) \setminus \Theta$, and it results in the following:

$$\begin{aligned}(Tx)''(t) = {} & AC(t)\phi(0) + AS(t)\eta \\ & + \int_0^t AS(t-s) f(s, x(s), x(\alpha(s)))\, ds \\ & + f(t, x(t), x(\alpha(t))) \\ & + \sum_{0<t_k<t} AC(t-t_k) I_k(x(t_k)) \\ & + \sum_{0<t_k<t} AS(t-t_k) J_k(x(t_k)).\end{aligned} \tag{28}$$

So one can conclude that

$$(Tx)''(t) = ATx(t) + f(t, x(t), x(\alpha(t))), \tag{29}$$

that is, Tx satisfies the differential equation in problem (P). This combined with the results in Proposition 7 yields the thesis. □

In order to prove that T is a completely continuous operator, we make use of the following compactness criterion in $BPC_\Theta[-r, +\infty)$.

Proposition 9 (see [16]). *Let $T : BPC_\Theta[-r, +\infty) \to BPC_\Theta[-r, +\infty)$ be a continuous operator. Suppose that, for any bounded set $F \subset BPC_\Theta[-r, +\infty)$, $T(F)$ is a bounded set, and there exist ν bounded functions $\varphi_j : [-r, +\infty) \to \mathbb{R}^n$, $j = 1, 2, \dots, \nu$, such that, for all $t, s \in [-r, +\infty)$ and, for all $y \in F$*

$$\left|(Ty)(t) - (Ty)(s)\right| \le \sum_{j=1}^{\nu} \left|\varphi_j(t) - \varphi_j(s)\right|. \tag{30}$$

Then T is a compact operator.

Proof. The Banach space $BPC_\Theta[-r, +\infty)$ is isometric to the Banach space:

$$\begin{aligned}\widetilde{\Omega} = \{ & y \in BC(Q, \mathbb{R}^n) : \text{there exist } y(t_k^+) \\ & \text{and } y(t_k^-), \text{for } k = 1, \dots, l\},\end{aligned} \tag{31}$$

where $Q = [-r, +\infty) \setminus \Theta$. Of course, $\widetilde{\Omega}$ is closed in $BC(Q, \mathbb{R}^n)$. So the thesis follows from Theorem 6. □

Proposition 10. *T is a completely continuous operator.*

Proof. First we prove that T is a continuous operator. Let $(y_m)_{m\in\mathbb{N}}$ a sequence in $BPC_\Theta[-r, +\infty)$ such that $y_m \to y$ in $BPC_\Theta[-r, +\infty)$. We prove that $Ty_m \to Ty$. By the continuity of f it results that, for fixed $s \in [0, +\infty)$,

$$\begin{aligned}\sigma_m(s) := {} & f(s, y_m(s), y_m(\alpha(s))) \\ & - f(s, y(s), y(\alpha(s))) \longrightarrow 0.\end{aligned} \tag{32}$$

Moreover, by the monotonicity of Ψ, we have

$$\begin{aligned}\left|\sigma_m(s)\right| \le {} & \left|f(s, y_m(s), y_m(\alpha(s)))\right| \\ & + \left|f(s, y(s), y(\alpha(s)))\right| \\ \le {} & p(s)\left[\Psi(2\|y_m\|_\infty) + \Psi(2\|y\|_\infty)\right] \\ \le {} & 2p(s)\Psi(2\Lambda),\end{aligned} \tag{33}$$

where $\Lambda = \max\{\sup_{m\in\mathbb{N}}\|y_m\|_\infty, \|y\|_\infty$. Therefore $|\sigma_m(s)|$ is dominated by an integrable function that does not depend on $m \in \mathbb{N}$. Then, from the uniform boundedness of the sine family and by the dominated convergence theorem, we have

$$\begin{aligned}\left\|w_{y_m} - w_y\right\|_\infty &= \sup_{0\le t<+\infty} \left|w_{y_m}(t) - w_y(t)\right| \\ &= \sup_{0\le t<+\infty} \left|\int_0^t S(t-s)\sigma_m(s)\, ds\right| \\ &\le M_S \int_0^\infty \left|\sigma_m(s)\right| ds \longrightarrow 0.\end{aligned} \tag{34}$$

The uniform boundedness of the cosine family and the continuity of I_k and J_k imply that, as $m \to \infty$,

$$\begin{aligned}& \|Iy_m - Iy\|_\infty \\ & \quad \le \sup_{t\in[t_1,+\infty)} \left(\sum_{0<t_k<t} \left|C(t-t_k)\right| \left|I_k(y_m(t_k)) - I_k(y(t_k))\right|\right) \\ & \quad \le M_C \sum_{k=1}^{l} \left|I_k(y_m(t_k)) - I_k(y(t_k))\right| \longrightarrow 0, \\ & \qquad \|Jy_m - Jy\|_\infty \longrightarrow 0.\end{aligned} \tag{35}$$

Thus

$$\begin{aligned}\|Ty_m - Ty\|_\infty \le {} & \left\|w_{y_m} - w_y\right\|_\infty \\ & + \|Jy_m - Jy\|_\infty + \|Iy_m - Iy\|_\infty \longrightarrow 0.\end{aligned} \tag{36}$$

Now, let

$$B_\rho := \{u \in BPC_\Theta[-r, +\infty) : \|u\|_\infty \le \rho\}. \tag{37}$$

Thanks to Proposition 9, it is enough to show that $T(B_\rho)$ is a bounded set and that it is possible to control the oscillations of each function in $T(B_\rho)$ by means of a finite number of bounded functions. The boundedness of $T(B_\rho)$ follows by the inequalities:

$$\begin{aligned}\|w_x\|_\infty &\le M_S\Psi(2\rho)\int_0^\infty p(s)\, ds, \\ \|Ix\|_\infty &\le M_C \sum_{k=1}^{l} \left|I_k(x(t_k))\right| \le lM_C D_I, \\ \|Jx\|_\infty &\le M_S \sum_{k=1}^{l} \left|J_k(x(t_k))\right| \le lM_S D_J.\end{aligned} \tag{38}$$

To control the oscillations of $T(B_\rho)$ we should need to distinguish three cases: $0 \le \tau_1 < \tau_2$, $\tau_1 < 0 < \tau_2$, and $\tau_1 < \tau_2 \le 0$.

Here we study the case $0 \le \tau_1 < \tau_2$ only because the proofs of the other cases are similar.

Let $0 \le \tau_1 < \tau_2$ and $x \in B_\rho$. Then

$$\begin{aligned} &\left|(Tx)(\tau_1) - (Tx)(\tau_2)\right| \\ &\quad \le \left|C(\tau_1) - C(\tau_2)\right| \left|\phi(0)\right| + \left|S(\tau_1) - S(\tau_2)\right| \left|\eta\right| \\ &\qquad + \left|\int_0^{\tau_1} \left[S(\tau_1 - s) - S(\tau_2 - s)\right] f(s, x(s), x(\alpha(s)))\right| ds \\ &\qquad + \left|\int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, x(s), x(\alpha(s))) ds\right| \\ &\qquad + \left|(Ix)(\tau_1) - (Ix)(\tau_2)\right| + \left|(Jx)(\tau_1) - (Jx)(\tau_2)\right|. \end{aligned} \tag{39}$$

Now, by property (u5), one obtains that

$$\begin{aligned} &S(\tau_2 - s) - S(\tau_1 - s) \\ &\quad = \left[S(\tau_2) - S(\tau_1)\right] C(s) - \left[C(\tau_2) - C(\tau_1)\right] S(s), \end{aligned} \tag{40}$$

so we have

$$\begin{aligned} &\left|\int_0^{\tau_1} \left[S(\tau_1 - s) - S(\tau_2 - s)\right] f(s, x(s), x(\alpha(s))) ds\right| \\ &\quad \le \left|S(\tau_2) - S(\tau_1)\right| \\ &\qquad \times \int_0^{\tau_1} \left|C(s)\right| \left|f(s, x(s), x(\alpha(s)))\right| ds \\ &\qquad + \left|C(\tau_2) - C(\tau_1)\right| \\ &\qquad \times \int_0^{\tau_1} \left|S(s)\right| \left|f(s, x(s), x(\alpha(s)))\right| ds \\ &\quad \le \left|S(\tau_2) - S(\tau_1)\right| M_C \Psi(2\rho) \int_0^\infty p(s) ds \\ &\qquad + \left|C(\tau_2) - C(\tau_1)\right| M_S \Psi(2\rho) \int_0^\infty p(s) ds. \end{aligned} \tag{41}$$

Using (h_3) we obtain also that

$$\begin{aligned} &\left|\int_{\tau_1}^{\tau_2} S(\tau_2 - s) f(s, x(s), x(\alpha(s))) ds\right| \\ &\quad \le M_S \Psi(2\rho) \left[\int_0^{\tau_2} p(s) ds - \int_0^{\tau_1} p(s) ds\right]. \end{aligned} \tag{42}$$

To control the oscillations of the operators I and J, first we note that if $\tau_1 < t_1$, and $\tau_2 < t_1$ we have nothing to prove because $(Ix)(\tau_1) = (Ix)(\tau_2) = (Jx)(\tau_1) = (Jx)(\tau_2) = 0$. The boundedness of I and J solves also the case $\tau_1 < t_1 < \tau_2$ since we have

$$\begin{aligned} &\left|(Ix)(\tau_2) - (Ix)(\tau_1)\right| = \left|(Ix)(\tau_2)\right| \le D_I \sum_{0<t_k<\tau_2} M_C, \\ &\left|(Jx)(\tau_2) - (Jx)(\tau_1)\right| = \left|(Jx)(\tau_2)\right| \le D_J \sum_{0<t_k<\tau_2} M_S. \end{aligned} \tag{43}$$

Thus it remains to prove only the case $t_1 < \tau_1 < \tau_2$. We have that

$$\begin{aligned} &\left|(Ix)(\tau_1) - (Ix)(\tau_2)\right| \\ &\quad = \left|\sum_{0<t_k<\tau_1} C(\tau_1 - t_k) I_k(x(t_k)) \right. \\ &\qquad \left. - \sum_{0<t_k<\tau_2} C(\tau_2 - t_k) I_k(x(t_k))\right| \\ &\quad \le \left|\sum_{0<t_k<\tau_1} \left[C(\tau_1 - t_k) - C(\tau_2 - t_k)\right] I_k(x(t_k))\right| \\ &\qquad + \sum_{\tau_1 \le t_k < \tau_2} \left|C(\tau_2 - t_k)\right| \left|I_k(x(t_k))\right| \\ &\quad \le D_I \sum_{k=1}^{l} \left|C(\tau_1 - t_k) - C(\tau_2 - t_k)\right| \\ &\qquad + D_I \left[\sum_{0<t_k<\tau_2} M_C - \sum_{0<t_k<\tau_1} M_C\right]. \end{aligned} \tag{44}$$

In a similar way we can show that

$$\begin{aligned} &\left|(Jx)(\tau_1) - (Jx)(\tau_2)\right| \\ &\quad \le D_J \sum_{k=1}^{l} \left|S(\tau_1 - t_k) - S(\tau_2 - t_k)\right| \\ &\qquad + D_J \left[\sum_{0<t_k<\tau_2} M_S - \sum_{0<t_k<\tau_1} M_S\right]. \end{aligned} \tag{45}$$

So, defining

$$\begin{aligned} &\psi_1(t) = C(t)\phi(0), \qquad \psi_2(t) = S(t)\eta, \\ &\psi_3(t) = M_C K S(t), \qquad \psi_4(t) = M_S K C(t), \\ &\psi_5(t) = M_S \Psi(2\rho) \int_0^t p(s) ds, \\ &\psi_6(t) = D_I C(t - t_1), \\ &\psi_7(t) = D_I C(t - t_2), \ldots, \psi_{l+5}(t) = D_I C(t - t_l), \\ &\psi_{l+6}(t) = D_I M_C \sum_{0<t_k<t} 1, \\ &\psi_{l+7}(t) = D_J S(t - t_1), \\ &\psi_{l+8}(t) = D_J S(t - t_2), \ldots, \psi_{2l+6}(t) = D_J S(t - t_l), \\ &\psi_{2l+7}(t) = D_J M_S \sum_{0<t_k<t} 1, \end{aligned} \tag{46}$$

where $K = \Psi(2\rho) \int_0^\infty p(s) ds$, and we obtain that for all $x \in B_\rho$

$$\left|(Tx)(\tau_1) - (Tx)(\tau_2)\right| \le \sum_{k=1}^{2l+7} \left|\psi_k(\tau_1) - \psi_k(\tau_2)\right|. \tag{47}$$

□

4. Main Result

Theorem 11. *Assume that the hypotheses (h_0)–(h_3) hold. Then the problem (P) has at least one solution.*

Proof. Our problem (P) can be reduced, by Propositions 7 and 8, to find a fixed point for the operator T. Proposition 10 assures that T is a completely continuous operator.

To apply the Schaefer's fixed point theorem, it remains to prove that the set

$$\zeta(T) := \{x \in \mathbb{R}^n : x = \lambda T x, \text{for some } 0 < \lambda < 1\} \tag{48}$$

is bounded. The proof is based on an idea in [19].

Let $x = \lambda(Tx)$ with $\lambda \in (0,1)$. First of all, since $\{Tx|_{[-r,0]} : x \in BPC_\Theta[-r,+\infty)\} = \{\phi\}$, for $t \in [-r,0]$ we have

$$|x(t)| = \lambda |(Tx)(t)| < |(Tx)(t)| = |\phi(t)| \leq \|\phi\|_\infty. \tag{49}$$

We consider now that $t \geq 0$. Thus we have

$$\begin{aligned}|x(t)| = \lambda |(Tx)(t)| &< M_C\|\phi\|_\infty + M_S|\eta| \\ &+ M_S \int_0^t p(s)\Psi(|x(s)| + |x(\alpha(s))|)\,ds \\ &+ lM_C D_I + lM_S D_J.\end{aligned} \tag{50}$$

Consider the function $\mu_x : [0,+\infty) \to [0,+\infty)$ defined by

$$\mu_x(t) := \sup\{|x(\xi)| : 0 \leq \xi \leq t\}. \tag{51}$$

Observe that μ_x is not necessarily continuous in Θ but $\mu_x(t_k^+)$ and $\mu_x(t_k^-) = \mu_x(t_k)$ exist, for all $t_k \in \Theta$.

For $\xi \in [0,t]$, we have

$$\begin{aligned}|x(\xi)| \leq M_C\|\phi\|_\infty &+ M_S|\eta| \\ &+ M_S \int_0^t p(s)\Psi(|x(s)| + |x(\alpha(s))|)\,ds \\ &+ lM_S D_I + lM_C D_J.\end{aligned} \tag{52}$$

Moreover, being α increasing and $\alpha(t) \leq t$ for $t \geq 0$, one has

$$\begin{aligned}\sup_{0\leq s\leq t}|x(\alpha(s))| &= \sup_{\alpha(0)\leq s\leq \alpha(t)}|x(s)| \\ &\leq \sup_{0\leq s\leq t}|x(s)| = \mu_x(t)\end{aligned} \tag{53}$$

so, taking the supremum over $[0,t]$ in the inequality (52), we obtain that

$$\begin{aligned}\mu_x(t) \leq M_C\|\phi\|_\infty &+ M_S|\eta| \\ &+ M_S \int_0^t p(s)\Psi(2\mu_x(t))\,ds \\ &+ lM_S D_I + lM_C D_J.\end{aligned} \tag{54}$$

Denoting by $v_x(t)$ the right-hand side of the last inequality, we have that the function v_x is continuous:

$$c := v_x(0) = (1 + M_C)\|\phi\|_\infty + M_S|\eta| + lM_S D_I + lM_C D_J \tag{55}$$

and $\mu_x(t) \leq v_x(t)$ for $t \geq 0$.

Moreover, since Ψ is nondecreasing, for $t \notin \Theta$,

$$v_x'(t) \leq M_S p(t)\Psi(2v_x(t)). \tag{56}$$

This implies that

$$\frac{v_x'(t)}{\Psi(2v_x(t))} \leq M_S p(t), \quad t \notin \Theta \tag{57}$$

and so, for any $b > 0$,

$$\int_0^b \frac{v_x'(t)}{\Psi(2v_x(t))}dt \leq M_S \int_0^b p(t) := \Gamma_b < \infty. \tag{58}$$

Since v_x' is a continuous function for all $t \notin \Theta$, we have

$$\int_{2c}^{2v_x(b)} \frac{ds}{2\Psi(s)} \leq \Gamma_b. \tag{59}$$

This, together with condition (13), permits us to conclude that v_x is bounded by a constant Δ depending on the functions Ψ and p only.

Summarizing, $x \in \zeta(T)$ implies that $\|x\|_\infty \leq \max\{\|\phi\|_\infty, \Delta\}$. □

5. Functional Initial Conditions

Let us consider the problem

$$\begin{aligned}&x''(t) = Ax(t) + f(t, x(t), x(\alpha(t))), \\ &\qquad t \in [0,+\infty) \setminus \{t_1, \ldots, t_l\} \\ &x(t_k^+) - x(t_k^-) = I_k(x(t_k)), \quad k = 1,\ldots,l \\ &x'(t_k^+) - x'(t_k^-) = J_k(x(t_k)), \quad k = 1,\ldots,l \\ &Lx = H(x), \qquad x'(0) = \eta.\end{aligned} \tag{$\mathrm{P_F}$}$$

The modularity of the operator T used in the proof of our Theorem 11 permits to prove a result on the existence of solutions for Problem $(\mathrm{P_F})$ modifying few parts of the above proof.

For $u \in BPC_\Theta[-r,+\infty)$, we define the function $\widetilde{C}u : [-r,+\infty) \to \mathbb{R}^n$ as

$$(\widetilde{C}u)(t) := \begin{cases} u(t), & -r \leq t \leq 0 \\ C(t)u(0), & t \geq 0. \end{cases} \tag{60}$$

We suppose that

(h_4) $L : BPC_\Theta[-r,+\infty) \to C([-r,0],\mathbb{R}^n)$ is a bounded linear operator for which if $u, v \in BPC_\Theta[-r,+\infty)$ are such that $u|_{[-r,0]} = v|_{[-r,0]}$, then $Lu = Lv$.

$H : BPC_\Theta[-r, +\infty) \to C([-r, 0]; \mathbb{R}^n)$ is a completely continuous operator such that $\|H(u)\|_\infty \leq M_1$ for a certain constant M_1.

(h_5) For $u \in BPC_\Theta[-r, +\infty)$, $H(u) = (L\widetilde{C}H)(u)$.

Theorem 12. *Assume that the hypotheses (h_0)–(h_5) hold. Then the problem (P_F) has at least one solution.*

Proof. Let us consider the operator W defined on $BPC_\Theta[-r, +\infty)$ as

$$(Wx)(t) := \left(\widetilde{C}H(x)\right)(t) + S(t)\eta + w_x(t) + (Ix)(t) + (Jx)(t). \tag{61}$$

Following the proof of Theorem 11, it is necessary to study only the part of the operator arising from the functional initial condition. First, for $t \geq 0$, one has

$$\left|\left(\widetilde{C}H(x)\right)(t)\right| \leq M_C |(H(u))(t)| \leq M_C M_1 \Longrightarrow \left\|\widetilde{C}H(x)\right\|_\infty \leq M_C M_1 \tag{62}$$

and, by the continuity of $C(t)$, as $t \to t_k$

$$\left|\left(\widetilde{C}H(x)\right)(t) - \left(\widetilde{C}H(x)\right)(t_k)\right| \leq M_1 \left|C(t) - C(t_k)\right| \longrightarrow 0. \tag{63}$$

These, following Proposition 7, are enough to prove that $Wx \in BPC_\Theta[-r, +\infty)$ and $(Wx)(t_k^+) - (Wx)(t_k^-) = I_k(x(t_k))$.

Moreover, it is not difficult to verify that $(\widetilde{C}H(x))'(t) = AS(t)(H(u)(t))$ for $t \geq 0$, and this is enought to prove that $(\widetilde{C}H(x))'(t_k^+) = (\widetilde{C}H(x))'(t_k^-)$ so that one has

$$(Wx)'(t_k^+) - (Wx)'(t_k^-) = J_k(x(t_k)). \tag{64}$$

From the hypotheses on L one has

$$LS(t)\eta = Lw_x(t) = L(Ix)(t) = L(Jx)(t) = 0, \tag{65}$$

so, it follows from (h_5) that

$$\begin{aligned} L(Wx)(t) &= L\left(\widetilde{C}H(x)\right)(t) + LS(t)\eta \\ &\quad + Lw_x(t) + L(Ix)(t) + L(Jx)(t) \\ &= L\left(\widetilde{C}H(x)\right)(t) = (H(x))(t). \end{aligned} \tag{66}$$

Moreover, $(\widetilde{C}H(x))''(t) = A(\widetilde{C}H(x))(t)$ and so, using Proposition 8, the fixed points of W are solutions of $(\mathrm{P_F})$. It remains to prove that we can control the oscillations of Wx by a finite number of bounded functions.

We need to control only the oscillations of $\widetilde{C}H(x)$. If $0 < \tau_1 < \tau_2$ (the other case are similar), we have

$$\left|\left(\widetilde{C}H(x)\right)(\tau_1) - \left(\widetilde{C}H(x)\right)(\tau_2)\right| \leq \left|C(\tau_1) - C(\tau_2)\right| M_1. \tag{67}$$

So, by defining $\gamma(t) := M_1 C(t)$ we have

$$\left|\left(\widetilde{C}H(x)\right)(\tau_1) - \left(\widetilde{C}H(x)\right)(\tau_2)\right| \leq \gamma(\tau_1) - \gamma(\tau_2). \tag{68}$$

Hence W is a completely continuous operator. Following the same proof of Theorem 11 one can see that the set

$$\zeta(W) := \{x \in \mathbb{R}^n : x = \lambda Wx, \text{for some } 0 < \lambda < 1\} \tag{69}$$

is bounded, so W has fixed points that are solutions of $(\mathrm{P_F})$. This complete the proof. □

Remark 13. The reader is referred to [5, 9, 13, 19] for some examples and applications.

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Research Article

On Linear and Nonlinear Fourth-Order Eigenvalue Problems with Nonlocal Boundary Condition

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We determine the principal eigenvalue of the linear problem $u^{(4)}(t) + \beta u''(t) = \mu[u(t) - u''(t)]$, $t \in (0,1)$, $u(0) = u(1) = \int_0^1 p(s)u(s)\mathrm{d}s$, $u''(0) = u''(1) = \int_0^1 q(s)u''(s)\mathrm{d}s$, where $0 < \beta < \pi^2$ and $p, q \in L[0,1]$. Moreover, we investigate the existence of positive solutions for the corresponding nonlinear problem. The proofs of our main results are based upon the Krein-Rutman theorem and fixed point index theory.

1. Introduction

The deformations of an elastic beam can be described by the boundary value problems of the fourth-order ordinary differential equations. For example, an elastic beam in an equilibrium state whose both ends are simply supported can be described by the fourth-order boundary value problem of the form

$$\begin{aligned} &u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{1}$$

see Gupta [1, 2]. Owing to its significance in physics, it has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point index theory, and the method of lower and upper solutions; see, for example, [3–16].

Recently, Bai [3] investigated the existence of positive solutions for more general fourth-order nonlocal boundary value problem

$$\begin{aligned} &u^{(4)}(t) + \beta u''(t) = \lambda f(t, u(t), u''(t)), \quad t \in (0,1), \\ &u(0) = u(1) = \int_0^1 p(s)u(s)\,\mathrm{d}s, \\ &u''(0) = u''(1) = \int_0^1 q(s)u''(s)\,\mathrm{d}s. \end{aligned} \tag{2}$$

By using the Krasnoselskii fixed point theorem, the sufficient conditions for the existence of positive solutions of (2) are obtained. We can find that, however, those conditions do not involve the eigenvalues with respect to the relevant linear operator, and those sufficient conditions are not optimal.

The likely reason is that the spectrum structure of the linear eigenvalue problem

$$\begin{aligned} &u^{(4)}(t) + \beta u''(t) = \mu[u(t) - u''(t)], \quad t \in (0,1), \\ &u(0) = u(1) = \int_0^1 p(s)u(s)\,\mathrm{d}s, \\ &u''(0) = u''(1) = \int_0^1 q(s)u''(s)\,\mathrm{d}s \end{aligned} \tag{3}$$

is not clear.

It is the purpose of this paper to investigate the first eigenvalue of (3) by using the Krein-Rutman theorem [17, 18], and then we use this spectrum result to establish the existence of positive solutions of nonlinear nonlocal problem (2). The existence of positive solution is obtained by means of fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator. The obtained sufficient conditions in this paper are optimal. For the concepts and properties of fixed point index theory, we refer the reader to [19].

The rest of the paper is arranged as follows: in Section 2, we study the spectrum of (3). Finally, in Section 3, we give an application of our main result.

2. Spectrum of (3)

Let us assume that

(A1) $0 < \beta < \pi^2$, $p, q \in L[0,1]$, $p(t) \geq 0$, $q(t) \geq 0$, $t \in [0,1]$, $\int_0^1 p(s)ds < 1$, $\int_0^1 q(s)\sin\sqrt{\beta}s\,ds + \int_0^1 q(s)\sin\sqrt{\beta}(1-s)ds < \sin\sqrt{\beta}$.

(A2) $f \in C([0,1] \times [0,\infty) \times (-\infty,0], [0,\infty))$.

Definition 1. We say μ is an *eigenvalue* of linear problem (see (3)), if (3) has nontrivial solutions.

To study the spectrum of (3), we need several preliminary results.

Let

$$G_1(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1; \\ s(1-t), & 0 \leq s \leq t \leq 1; \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{\beta}t \sin\sqrt{\beta}(1-s)}{\sqrt{\beta}\sin\sqrt{\beta}}, & 0 \leq t \leq s \leq 1; \\ \dfrac{\sin\sqrt{\beta}s \sin\sqrt{\beta}(1-t)}{\sqrt{\beta}\sin\sqrt{\beta}}, & 0 \leq s \leq t \leq 1. \end{cases} \quad (4)$$

Lemma 2 (see [7]). *Assume (A1) holds. Then one has*

(i) $G_i(t,s) \geq 0$, *for* $t, s \in [0,1]$; $G_i(t,s) > 0$, *for* $t, s \in (0,1)$;

(ii) $b_i G_i(t,t) G_i(s,s) \leq G_i(t,s) \leq C_i G_i(t,t)$, *for* $t, s \in [0,1]$,

where $C_1 = 1$, $b_1 = 1$; $C_2 = 1/\sin\sqrt{\beta}$, $b_2 = \sqrt{\beta}\sin\sqrt{\beta}$.

Lemma 3 (see [3]). *Assume (A1) holds. Then for any* $g \in C[0,1]$, *u solves the problem*

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) = g(t), \quad t \in (0,1), \\ u(0) = u(1) = \int_0^1 p(s)u(s)\,ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)\,ds, \end{gathered} \quad (5)$$

if and only if

$$u(t) = \int_0^1 \int_0^1 K_1(t,s) K_2(s,\tau) g(\tau)\,d\tau\,ds, \quad (6)$$

where

$$K_1(t,s) = G_1(t,s) + \frac{1}{1 - \int_0^1 p(x)\,dx} \int_0^1 G_1(s,x) p(x)\,dx,$$

$$K_2(s,\tau) = G_2(s,\tau) + \rho(s) \int_0^1 G_2(\tau,x) q(x)\,dx,$$

$$\rho(s) = \frac{\sin\sqrt{\beta}s + \sin\sqrt{\beta}(1-s)}{\sin\sqrt{\beta} - \int_0^1 q(x)\sin\sqrt{\beta}x\,dx - \int_0^1 q(x)\sin\sqrt{\beta}(1-x)\,dx}. \quad (7)$$

Let

$$e(t) := -t^2 + t + \frac{\int_0^1 p(s)(s-s^2)\,ds}{1 - \int_0^1 p(s)\,ds}, \quad t \in [0,1]. \quad (8)$$

Then, there exist $c, \tilde{c} > 0$ such that

$$\int_0^1 K_1(t,s) e(s)\,ds \leq c e(t), \quad t \in [0,1], \quad (9)$$

$$\int_0^1 K_2(t,s) e(s)\,ds \leq \tilde{c} e(t), \quad t \in [0,1], \quad (10)$$

respectively.

In fact, from (ii) of Lemma 2, we can obtain

$$\begin{aligned} &\int_0^1 K_1(t,s) e(s)\,ds \\ &= \int_0^1 \left[G_1(t,s) + \frac{1}{1 - \int_0^1 p(x)\,dx} \times \int_0^1 G_1(s,x) p(x)\,dx \right] e(s)\,ds \end{aligned}$$

$$\leq \int_0^1 \left[G_1(t,t) + \frac{1}{1-\int_0^1 p(x)\,dx} \times \int_0^1 G_1(s,s)\,p(x)\,dx \right] e(s)\,ds$$
$$= G_1(t,t)\int_0^1 e(s)\,ds + \frac{\int_0^1 p(x)\,dx}{1-\int_0^1 p(x)\,dx} \times \int_0^1 G_1(s,s)\,e(s)\,ds$$
$$= \left(-t^2+t\right)\int_0^1 e(s)\,ds + \frac{\int_0^1 p(x)\,dx}{1-\int_0^1 p(x)\,dx} \times \int_0^1 \left(s-s^2\right) e(s)\,ds$$
$$\leq ce(t), \quad t\in[0,1], \tag{11}$$

where we can take

$$c = \max\left\{\int_0^1 e(s)\,ds, \frac{\int_0^1 p(s)\,ds}{\int_0^1 (s-s^2)\,p(s)\,ds} \times \int_0^1 \left(s-s^2\right)e(s)\,ds\right\}, \tag{12}$$

and also we obtain

$$\int_0^1 K_2(t,s)\,e(s)\,ds$$
$$= \int_0^1 \left[G_2(t,s) + \rho(t)\int_0^1 G_2(s,x)\,q(x)\,dx\right] e(s)\,ds$$
$$\leq \frac{1}{\sin\sqrt{\beta}}\int_0^1 \left[G_2(t,t) + \rho(t)\int_0^1 G_2(x,x)\,q(x)\,dx\right] \times e(s)\,ds$$
$$\leq c_1 \sin\sqrt{\beta}t\sin\sqrt{\beta}(1-t) + c_2$$
$$\leq \tilde{c}e(t), \tag{13}$$

where

$$c_1 = \frac{1}{\sqrt{\beta}\sin^2\sqrt{\beta}}\int_0^1 e(s)\,ds,$$
$$c_2 = \frac{1}{\sin\sqrt{\beta}}\int_0^1 e(s)\,ds\int_0^1 G_2(s,s)\,q(s)\,ds\max_{0\leq t\leq 1}\rho(t), \tag{14}$$
$$\tilde{c} = \frac{1-\int_0^1 p(s)\,ds}{\int_0^1 (s-s^2)\,p(s)\,ds}\left(c_1+c_2\right).$$

Let

$$X = \left\{\begin{array}{c} u\in C^2[0,1] : u(0) = u(1) = \int_0^1 p(s)\,u(s)\,ds \\ u''(0) = u''(1) = \int_0^1 q(s)\,u''(s)\,ds \\ -\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t) \end{array}\right\}, \tag{15}$$

for a certain $\varepsilon > 0$, and let Y be the Banach space $C[0,1]$ with the norm $\|u\|_\infty = \max_{0\leq t\leq 1}|u(t)|$. For $u\in X$, we have

$$u(t) = \int_0^1 K_1(t,s)\left(-u''(s)\right)ds. \tag{16}$$

Combining (16) and (9) with the fact that $-\varepsilon e(s) \leq -u''(s) \leq \varepsilon e(s)$, we conclude that

$$-c\varepsilon e(t) \leq u(t) \leq c\varepsilon e(t), \quad t\in[0,1]. \tag{17}$$

So, we may define the norm of $u\in X$ by

$$\|u\|_X := \inf\left\{\varepsilon \mid \varepsilon > 0, -\varepsilon e(t) \leq -u''(t) \leq \varepsilon e(t), t\in[0,1]\right\}. \tag{18}$$

This norm is so called u_0-norm.

It is easy to get the following lemmas.

Lemma 4. *$(X, \|\cdot\|_X)$ is a Banach space.*

Proof. Let $\{u_n\}$ be Cauchy sequence of $(X,\|\cdot\|_X)$. Then, we have

$$\|u_n - u_m\|_X \longrightarrow 0, \quad (n,m\longrightarrow\infty). \tag{19}$$

From (18) and (19), for all $\varepsilon > 0$, there exists $N > 0$ such that

$$-\varepsilon e(t) \leq -\left(u_n''(t) - u_m''(t)\right) \leq \varepsilon e(t), \quad t\in[0,1], \ \forall n,m > N. \tag{20}$$

Hence,

$$\left\|u_n'' - u_m''\right\|_\infty \leq \varepsilon\|e\|_\infty \longrightarrow 0, \quad (n,m\longrightarrow\infty), \tag{21}$$

which implies that $\{u_n''\}$ is a Cauchy sequence of $(C[0,1],\|\cdot\|_\infty)$. According to the completeness of $(C[0,1], \|\cdot\|_\infty)$, there exists $u_0\in C^2[0,1]$ such that

$$\left\|u_n'' - u_0''\right\|_\infty \longrightarrow 0, \quad (n\longrightarrow\infty). \tag{22}$$

Let $m\to\infty$ in (20). Then, we have

$$-\varepsilon e(t) \leq -\left(u_n''(t) - u_0''(t)\right) \leq \varepsilon e(t), \quad t\in[0,1], \ (n > N), \tag{23}$$

which implies that

$$\|u_n - u_0\|_X \leq \varepsilon, \quad (n > N). \tag{24}$$

Hence, for Cauchy sequence $\{u_n\}\subset(X,\|\cdot\|_X)$, there exists $u_0\in X$ satisfying

$$\|u_n - u_0\|_X \longrightarrow 0, \quad (n\longrightarrow\infty). \tag{25}$$

Therefore, $(X,\|\cdot\|_X)$ is a Banach space. □

Lemma 5. *$X \cap C^4[0,1]$ is compactly embedded in X.*

Proof. Let $\{u_n\} \subset X \cap C^4[0,1]$ be bounded. Then, according to the fact that $C^4[0,1]$ is compactly embedded in $C^2[0,1]$, there exist $\{u_{n_j}\} \subset \{u_n\}$ and $u_0 \in C^2[0,1]$ such that

$$\left\| u''_{n_j} - u''_0 \right\|_\infty \longrightarrow 0, \quad j \longrightarrow \infty. \tag{26}$$

From (26), for all $\varepsilon > 0$, there exists $N > 0$ such that

$$-\varepsilon \le -\left(u''_{n_j}(t) - u''_0(t) \right) \le \varepsilon, \quad t \in [0,1],\ n_j > N. \tag{27}$$

Since

$$e(t) \ge \frac{\int_0^1 (s - s^2)\, p(s)\, \mathrm{d}s}{1 - \int_0^1 p(s)\, \mathrm{d}s}, \quad t \in [0,1], \tag{28}$$

combining this fact with (27) we have

$$\begin{aligned} &-\frac{1 - \int_0^1 p(s)\, \mathrm{d}s}{\int_0^1 (s - s^2)\, p(s)\, \mathrm{d}s} \varepsilon e(t) \\ &\quad \le -\left(u''_{n_j}(t) - u''_0(t) \right) \\ &\quad \le \frac{1 - \int_0^1 p(s)\, \mathrm{d}s}{\int_0^1 (s - s^2)\, p(s)\, \mathrm{d}s} \varepsilon e(t), \quad t \in [0,1],\ n_j > N, \end{aligned} \tag{29}$$

which implies

$$\left\| u_{n_j} - u_0 \right\|_X \le \frac{1 - \int_0^1 p(s)\, \mathrm{d}s}{\int_0^1 (s - s^2)\, p(s)\, \mathrm{d}s} \varepsilon \longrightarrow 0, \quad n_j \longrightarrow \infty. \tag{30}$$

Therefore, $X \cap C^4[0,1]$ is compactly embedded in X. □

Let

$$P := \left\{ u \in X \mid u(t) \ge 0,\ u''(t) \le 0,\ t \in [0,1] \right\}. \tag{31}$$

Then, the cone P is normal and has nonempty interior $\operatorname{int} P$.

In the rest of this section, we will prove the existence of the first eigenvalue of (3). To wit, we get the following.

Theorem 6. *Assume (A1) holds. Then* (3) *has an algebraically simple eigenvalue μ_1, with an eigenfunction $\varphi_1 \in \operatorname{int} P$, and there is no other eigenvalue with a positive eigenfunction.*

Remark 7. If $p = q \equiv 0$, then μ_1 can be explicitly given by

$$\mu_1 = \frac{\pi^4 - \beta\pi^2}{1 + \pi^2} \tag{32}$$

and the corresponding eigenfunction $\varphi_1(t) = \sin \pi t$, $t \in [0,1]$.

Proof of Theorem 6. For $u \in X$, define a linear operator $T : X \to Y$ by

$$(Tu)(t) := \iint_0^1 K_1(t,s)\, K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau\, \mathrm{d}s. \tag{33}$$

Then, by the definition of K_1, we have

$$\begin{aligned} &(Tu)(t) \\ &= \int_0^1 \left\{ \left[G_1(t,s) + \frac{1}{1 - \int_0^1 p(x)\, \mathrm{d}x} \int_0^1 G_1(s,x)\, p(x)\, \mathrm{d}x \right] \right. \\ &\qquad \left. \times \int_0^1 K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}s \\ &= \int_0^t \left\{ \left[s(1-t) + \frac{1}{1 - \int_0^1 p(x)\, \mathrm{d}x} \int_0^1 G_1(s,x)\, p(x)\, \mathrm{d}x \right] \right. \\ &\qquad \left. \times \int_0^1 K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}s \\ &\quad + \int_t^1 \left\{ \left[t(1-s) + \frac{1}{1 - \int_0^1 p(x)\, \mathrm{d}x} \int_0^1 G_1(s,x)\, p(x)\, \mathrm{d}x \right] \right. \\ &\qquad \left. \times \int_0^1 K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}s. \end{aligned} \tag{34}$$

Therefore, we can obtain

$$\begin{aligned} (Tu)'(t) &= \int_0^t \left\{ -s \int_0^1 K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}s \\ &\quad + \int_t^1 \left\{ (1-s) \int_0^1 K_2(s,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \right\} \mathrm{d}s, \end{aligned} \tag{35}$$

$$\begin{aligned} (Tu)''(t) &= -t \int_0^1 K_2(t,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \\ &\quad + (t-1) \int_0^1 K_2(t,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau \\ &= -\int_0^1 K_2(t,\tau) \left[u(\tau) - u''(\tau) \right] \mathrm{d}\tau. \end{aligned} \tag{36}$$

We claim that $T : P \to P$. In fact, for $u \in X$, let $\|u\|_X = \alpha$. Then, from (18) and (17) we have, for $t \in [0,1]$,

$$\begin{aligned} -\alpha e(t) &\le -u''(t) \le \alpha e(t), \\ -c\alpha e(t) &\le u(t) \le c\alpha e(t). \end{aligned} \tag{37}$$

On the one hand, from (36)-(37) and (10) we have

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 K_2(t,s)\left[u(s)-u''(s)\right]\mathrm{d}s \\ &\le \int_0^1 K_2(t,s)\,[c\alpha+\alpha]\,e(s)\,\mathrm{d}s \\ &\le \tilde{c}\,(c\alpha+\alpha)\,e(t). \end{aligned} \tag{38}$$

On the other hand, from (36)-(37) and (10) we have

$$\begin{aligned} -(Tu)''(t) &= \int_0^1 K_2(t,s)\left[u(s)-u''(s)\right]\mathrm{d}s \\ &\ge -\int_0^1 K_2(t,s)\,[c\alpha+\alpha]\,e(s)\,\mathrm{d}s \\ &\ge -\tilde{c}\,(c\alpha+\alpha)\,e(t). \end{aligned} \tag{39}$$

Therefore, $T(X)\subseteq X$. If $u\in P$, then $u(s)\ge 0$, $u''(s)\le 0$ on $[0,1]$. According to (36) and the fact that

$$K_2(t,s)\ge 0,\quad 0\le t,\ s\le 1, \tag{40}$$

we have

$$(Tu)''(t) = -\int_0^1 K_2(t,s)\left[u(s)-u''(s)\right]\mathrm{d}s \le 0. \tag{41}$$

Furthermore, according to (41), the definition of T, and the fact that

$$K_1(t,s)\ge 0,\quad 0\le t,\ s\le 1, \tag{42}$$

we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 \left\{K_1(t,s)\int_0^1 K_2(s,\tau)\left[u(\tau)-u''(\tau)\right]\mathrm{d}\tau\right\}\mathrm{d}s \\ &= \int_0^1 K_1(t,s)\left[-(Tu)''(s)\right]\mathrm{d}s \ge 0. \end{aligned} \tag{43}$$

Then $u\in P$, and accordingly, $T(P)\subseteq P$.

Now, since $T(X)\subset X\cap C^4[0,1]$ and $X\cap C^4[0,1]$ is compactly embedded in X, we have that $T: X\to X$ is compact.

Next, we show that $T: P\to P$ is strongly positive.

For $u\in P\backslash\{\theta\}$, it is easy to check that there exist $\rho_{1u},\rho_{2u}>0$, such that for $t\in[0,1]$

$$\begin{aligned} (Tu)(t) &\ge \rho_{1u}e(t), \\ -(Tu)''(t) &\ge \rho_{2u}e(t). \end{aligned} \tag{44}$$

In fact, for every $u\in P\setminus\{\theta\}$, it follows from Lemma 2 that

$$\begin{aligned} &(Tu)(t) \\ &= \iint_0^1 \left[G_1(t,s)+\frac{1}{1-\int_0^1 p(x)\,\mathrm{d}x}\int_0^1 G_1(s,x)\,p(x)\,\mathrm{d}x\right] \\ &\quad\times\left[G_2(s,\tau)+\rho(s)\int_0^1 G_2(\tau,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[u(\tau)-u''(\tau)\right]\mathrm{d}\tau\,\mathrm{d}s \\ &\ge \iint_0^1 \left[G_1(t,t)\,G_1(s,s)\right. \\ &\qquad\left.+\frac{G_1(s,s)}{1-\int_0^1 p(x)\,\mathrm{d}x}\int_0^1 G_1(x,x)\,p(x)\,\mathrm{d}x\right] \\ &\quad\times\left[G_2(s,\tau)+\rho(s)\int_0^1 G_2(\tau,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[u(\tau)-u''(\tau)\right]\mathrm{d}\tau\,\mathrm{d}s \\ &= \rho_{1u}e(t), \end{aligned} \tag{45}$$

where

$$\begin{aligned} \rho_{1u} &= \iint_0^1 G_1(s,s) \\ &\quad\times\left[G_2(s,\tau)+\rho(s)\int_0^1 G_2(\tau,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[u(\tau)-u''(\tau)\right]\mathrm{d}\tau\,\mathrm{d}s>0, \end{aligned} \tag{46}$$

$$\begin{aligned} &-(Tu)''(t) \\ &= \int_0^1 \left[G_2(t,s)+\rho(t)\int_0^1 G_2(s,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[u(s)-u''(s)\right]\mathrm{d}s \\ &\ge \sqrt{\beta}\sin\sqrt{\beta}\int_0^1 \left[G_2(t,t)\,G_2(s,s)\right. \\ &\qquad\left.+\rho(t)\,G_2(s,s)\int_0^1 G_2(x,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[u(s)-u''(s)\right]\mathrm{d}s \\ &= \sqrt{\beta}\sin\sqrt{\beta}\left[G_2(t,t)+\rho(t)\int_0^1 G_2(x,x)\,q(x)\,\mathrm{d}x\right] \\ &\quad\times\left[\int_0^1 G_2(s,s)\left[u(s)-u''(s)\right]\mathrm{d}s\right] \end{aligned}$$

$$\begin{aligned}&\geq \rho(t)\left[\int_0^1 \sin\sqrt{\beta}x\sin\sqrt{\beta}(1-x)q(x)\,dx\right]\\&\quad\times\left[\int_0^1 G_2(s,s)\left[u(s)-u''(s)\right]ds\right]\\&\geq \rho_{2u}e(t),\end{aligned}\tag{47}$$

where

$$\begin{aligned}\rho_{2u}=\min_{t\in[0,1]}\rho(t)&\left[\int_0^1 \sin\sqrt{\beta}x\sin\sqrt{\beta}(1-x)q(x)\,dx\right.\\&\left.\times\int_0^1 G_2(s,s)\left[u(s)-u''(s)\right]ds\right]\\&\times\left(\frac{1}{4}+\frac{\int_0^1 p(s)\left(s-s^2\right)ds}{1-\int_0^1 p(s)\,ds}\right)^{-1}\\&>0.\end{aligned}\tag{48}$$

Therefore, it follows from (44) that $Tu\in\operatorname{int}P$.

Now, on the one hand, by the Krein-Rutman theorem ([17, Theorem 7.C] and [18, Theorem 19.3]), T has an algebraically simple eigenvalue $r(T)>0$ with an eigenfunction $\varphi_1\in\operatorname{int}P$. Moreover, there is no other eigenvalue with a positive eigenfunction. On the other hand, we have from the definition of T and Lemma 3 that (3) is equivalent to the integral equation

$$u(t)=\mu(Tu)(t).\tag{49}$$

Therefore, $\mu_1=(r(T))^{-1}$ with a positive eigenfunction φ_1 is a simple eigenvalue of (3). Moreover, for (3), there is no other eigenvalue with a positive eigenfunction. □

3. An Application of Theorem 6

For convenience, we introduce the following notations:

$$\begin{gathered}\overline{f}^{\infty}=\limsup_{|u|+|v|\to+\infty}\max_{t\in[0,1]}\frac{f(t,u,v)}{|u|+|v|},\\ \underline{f}_0=\liminf_{|u|+|v|\to0^+}\min_{t\in[0,1]}\frac{f(t,u,v)}{|u|+|v|},\\ \overline{f}^{0}=\limsup_{|u|+|v|\to0^+}\max_{t\in[0,1]}\frac{f(t,u,v)}{|u|+|v|},\\ \underline{f}_{\infty}=\liminf_{|u|+|v|\to+\infty}\min_{t\in[0,1]}\frac{f(t,u,v)}{|u|+|v|},\\ B_r=\{u\in X\mid \|u\|_X<r\},\\ \partial B_r=\{u\in X\mid \|u\|_X=r\}\quad(r>0).\end{gathered}\tag{50}$$

Theorem 8. *Assume that (A1) and (A2) hold, and $0\leq\overline{f}^{\infty}<\underline{f}_0\leq+\infty$. Then, for each λ satisfying*

$$\frac{\mu_1}{\underline{f}_0}<\lambda<\frac{\mu_1}{\overline{f}^{\infty}},\tag{51}$$

there exists at least one positive solution of (2).

Theorem 9. *Assume that (A1) and (A2) hold, and $0\leq\overline{f}^{0}<\underline{f}_{\infty}\leq+\infty$. Then, for each λ satisfying*

$$\frac{\mu_1}{\underline{f}_{\infty}}<\lambda<\frac{\mu_1}{\overline{f}^{0}},\tag{52}$$

there exists at least one positive solution of (2).

Remark 10. Bai [3] proved existence of positive solutions via Guo-Krasnoselskii fixed point theorem under some conditions which do not involve the eigenvalue of (3). While our Theorems 8 and 9 are established under (51) or (52) which are related to the eigenvalue of (3). Our Theorems 8 and 9 cover an undefined case in [3]. Consider the following boundary value problem:

$$\begin{aligned}&u^{(4)}(t)+\frac{\pi^2}{4}u''(t)\\&\quad=\frac{3\pi^4+1}{4(1+\pi^2)}\left[u(t)-u''(t)\right]\\&\qquad-\frac{2}{4(1+\pi^2)}\sin\left[u(t)-u''(t)\right],\quad t\in(0,1),\\&u(0)=u(1)=u''(0)=u''(1)=0.\end{aligned}\tag{53}$$

In this case, $\mu_1=3\pi^4/4(1+\pi^2)$, $f(t,u,v)=((3\pi^4+1)/4(1+\pi^2))(u-v)-(2/4(1+\pi^2))\sin(u-v)$, $\lambda=1$, and

$$\frac{\mu_1}{\underline{f}_{\infty}}=\frac{3\pi^4}{3\pi^4+1}<1=\lambda<\frac{3\pi^4}{3\pi^4-1}=\frac{\mu_1}{\overline{f}^{0}}.\tag{54}$$

According to Theorem 9, the above boundary value problem has at least one positive solution. For the above boundary value problem, however, we cannot obtain the above conclusion by [3, Theorem 3.1] since

$$\underline{f}_{\infty}=\frac{3\pi^4+1}{4(1+\pi^2)}<20<36<\frac{\pi^3\left(\sqrt{4+2\sqrt{2}}+2\right)}{2\sqrt{2}}=\frac{\eta_1}{\lambda}.\tag{55}$$

Moreover, (51) and (52) are optimal. In order to illustrate this point, consider the problem

$$\begin{aligned}&u^{(4)}(t)+\beta u''(t)=\frac{\pi^4-\beta\pi^2}{1+\pi^2}\left[u(t)-u''(t)\right]\\&\qquad\qquad+\rho\left(u(t),u''(t)\right),\quad t\in(0,1),\\&u(0)=u(1)=u''(0)=u''(1)=0,\end{aligned}\tag{56}$$

where

$$\rho(x,y)=\begin{cases}(x-y)^2, & 0\le x-y\le 1,\\ \sqrt{x-y}, & 1\le x-y<\infty.\end{cases}\quad (57)$$

In this case, $\mu_1=(\pi^4-\beta\pi^2)/(1+\pi^2)$, $f(t,u,v)=((\pi^4-\beta\pi^2)/(1+\pi^2))[u-v]+\rho(u,v)$, $\lambda=1$, and

$$\frac{\mu_1}{\underline{f}_0}=\lambda=\frac{\mu_1}{\overline{f}^{\infty}},\qquad \frac{\mu_1}{\underline{f}_{\infty}}=\lambda=\frac{\mu_1}{\overline{f}^{0}}.\quad (58)$$

However, (56) has no positive solution. In fact, suppose on the contrary that (56) has a positive solution u. Multiplying the first equation of (56) with $\sin\pi t$ and integrating from 0 to 1, we get

$$0=\int_0^1\rho\left(u(t),u''(t)\right)\sin\pi t\,dt>0,\quad (59)$$

which is a contradiction.

To prove Theorems 8 and 9, we need the following preliminary results.

Lemma 11. *For every $u\in X$, there exist $L_0,L_1>0$ such that*

$$L_0\|u\|_\infty\le\left\|u'\right\|_\infty\le\left\|u''\right\|_\infty\le L_1\|u\|_X.\quad (60)$$

Proof. (1) By $u(0)=\int_0^1 p(s)u(s)ds$, there is $u(t)=\int_0^t u'(s)ds+\int_0^1 p(s)u(s)ds$, so

$$\begin{aligned}|u(t)|&\le\int_0^1\left|u'(s)\right|ds+\int_0^1 p(s)|u(s)|ds\\&\le\left\|u'\right\|_\infty+\|u\|_\infty\int_0^1 p(s)ds.\end{aligned}\quad (61)$$

Hence

$$L_0\|u\|_\infty\le\left\|u'\right\|_\infty,\quad (62)$$

where $L_0=1-\int_0^1 p(s)ds$.

(2) By $u(0)=u(1)$, there is a $\xi\in(0,1)$ such that $u'(\xi)=0$, and so, for $t\in[0,1]$

$$-u'(t)=\int_t^\xi u''(s)ds.\quad (63)$$

Hence $|u'(t)|\le|\int_t^\xi|u''(s)|ds|\le\int_0^1|u''(s)|ds\le\|u''\|_\infty$, and accordingly,

$$\left\|u'\right\|_\infty\le\left\|u''\right\|_\infty.\quad (64)$$

(3) We have from (18) that $|u''(t)|\le\|u\|_X e(t)$, which implies that

$$\left|u''(t)\right|\le\left(\frac14+\frac{\int_0^1 p(s)\left(s-s^2\right)ds}{1-\int_0^1 p(s)ds}\right)\|u\|_X,\quad (65)$$

and consequently

$$\left\|u''\right\|_\infty\le L_1\|u\|_X,\quad (66)$$

where $L_1=(1/4)+(\int_0^1 p(s)(s-s^2)ds/(1-\int_0^1 p(s)ds))$. □

Let

$$\begin{aligned}(Au)(t)&:=\lambda\iint_0^1 K_1(t,s)K_2(s,\tau)f\left(\tau,u(\tau),u''(\tau)\right)d\tau ds,\\&\qquad t\in[0,1],\ \lambda>0.\end{aligned}\quad (67)$$

It is easy to show that $A:P\to P$ is a completely continuous operator. In addition, we can verify that the nonzero fixed points of the operator A are positive solutions of the problem (2).

Lemma 12 (see [19]). *Let E be Banach space, P a cone in E, and $\Omega(P)$ a bounded open set in P. Suppose that $A:\overline{\Omega(P)}\to P$ is a completely continuous operator. If there exists $u_0\in P\backslash\{\theta\}$ such that*

$$u-Au\ne\mu u_0,\quad\forall u\in\partial\Omega(P),\ \mu\ge0,\quad (68)$$

then the fixed point index $i(A,\Omega(P),P)=0$.

Lemma 13 (see [19]). *Let E be Banach space, P a cone in E, and $\Omega(P)$ a bounded open set in P with $\theta\in\Omega(P)$. Suppose that $A:\overline{\Omega(P)}\to P$ is a completely continuous operator. If*

$$Au\ne\mu u,\quad\forall u\in\partial\Omega(P),\ \mu\ge1,\quad (69)$$

then the fixed point index $i(A,\Omega(P),P)=1$.

Proof of Theorem 8. It follows from the first inequality of (51) that there exists $r_0>0$, such that

$$\lambda f(t,u,v)\ge\mu_1(|u|+|v|),\quad\forall t\in[0,1],\ 0\le|u|+|v|\le r_0.\quad (70)$$

Let φ_1 be the positive eigenfunction of T (see (33)) corresponding to μ_1. Thus $\varphi_1=\mu_1T\varphi_1$.

Let $r_1=L_0r_0/(1+L_0)L_1$. Then for every $u\in\partial B_{r_1}\cap P$, we have from Lemma 11 that

$$\begin{aligned}0\le u(t)-u''(t)&\le\|u\|_\infty+\left\|u''\right\|_\infty\\&\le\frac{(1+L_0)}{L_0}\|u\|_X\le r_0.\end{aligned}\quad (71)$$

It follows from (70) that

$$\begin{aligned}(Au)(t)&\ge\mu_1\iint_0^1K_1(t,s)K_2(s,\tau)\left[u(\tau)-u''(\tau)\right]d\tau ds\\&=\mu_1(Tu)(t),\quad t\in[0,1].\end{aligned}\quad (72)$$

We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). Now we show that

$$u - Au \neq \mu\varphi_1, \quad \forall u \in \partial B_{r_1} \cap P, \ \mu \geq 0. \tag{73}$$

Suppose the contrary, that there exists $u_1 \in \partial B_{r_1} \cap P$ and $\tau_1 \geq 0$ such that $u_1 - Au_1 = \tau_1\varphi_1$. Hence $\tau_1 > 0$ and

$$u_1 = Au_1 + \tau_1\varphi_1 \geq \tau_1\varphi_1. \tag{74}$$

Put

$$\tau^* := \sup\{\tau \mid u_1 \geq \tau\varphi_1\}. \tag{75}$$

It is easy to see that $\tau^* \geq \tau_1 > 0$ and $u_1 \geq \tau^*\varphi_1$. We find from $T(P) \subset P$ that

$$\mu_1 T u_1 \geq \tau^* \mu_1 T \varphi_1 = \tau^* \varphi_1. \tag{76}$$

Therefore, by (72), we have

$$u_1 = Au_1 + \tau_1\varphi_1 \geq \mu_1 T u_1 + \tau_1\varphi_1 \geq \tau^*\varphi_1 + \tau_1\varphi_1, \tag{77}$$

which contradicts the definition of τ^*. Hence (73) is true, and we have from Lemma 12 that

$$i\left(A, B_{r_1} \cap P, P\right) = 0. \tag{78}$$

It follows from the second inequality of (51) that there exists $0 < \sigma < 1$ and $r_2 > r_1$ such that

$$\lambda f(t,u,v) \leq \sigma\mu_1(|u| + |v|), \quad \forall t \in [0,1], \ |u| + |v| \geq r_2. \tag{79}$$

Let $T_1 u = \sigma\mu_1 T u$, $u \in X$. Then $T_1 : X \to X$ is a bounded linear operator and $T_1(P) \subset P$.

Let

$$\begin{aligned} M = {} & \left[\max_{0 \leq t,s,\tau \leq 1} K_1(t,s) K_2(s,\tau)\right] \\ & \times \left[\sup_{u \in \overline{B}_{r_3} \cap P} \lambda \int_0^1 f\left(\tau, u(\tau), u''(\tau)\right) d\tau\right], \end{aligned} \tag{80}$$

where $r_3 = L_0 r_2/(1 + L_0)L_1$. It is clear that $M < +\infty$.

Let

$$W = \{u \in P \mid u = \mu A u, \ 0 \leq \mu \leq 1\}. \tag{81}$$

In the following, we prove that W is bounded.

For any $u \in W$, set $E(\tau) = \{\tau \in [0,1] \mid u(\tau) - u''(\tau) \geq r_2\}$. Then,

$$\begin{aligned} u(t) &= \mu(Au)(t) \\ &\leq \lambda \iint_0^1 K_1(t,s) K_2(s,\tau) f\left(\tau, u(\tau), u''(\tau)\right) d\tau\, ds \\ &= \lambda \int_0^1 \int_{E(\tau)} K_1(t,s) K_2(s,\tau) f\left(\tau, u(\tau), u''(\tau)\right) d\tau\, ds \\ &\quad + \lambda \int_0^1 \int_{[0,1]\setminus E(\tau)} K_1(t,s) K_2(s,\tau) f \\ &\qquad \times \left(\tau, u(\tau), u''(\tau)\right) d\tau\, ds \\ &\leq \sigma\mu_1 \iint_0^1 K_1(t,s) K_2(s,\tau) \left[u(\tau) - u''(\tau)\right] d\tau\, ds \\ &\quad + \lambda \int_0^1 \int_{[0,1]\setminus E(\tau)} K_1(t,s) K_2(s,\tau) f \\ &\qquad \times \left(\tau, u(\tau), u''(\tau)\right) d\tau\, ds \\ &\leq (T_1 u)(t) + M, \quad t \in [0,1]. \end{aligned} \tag{82}$$

Thus, $((I - T_1)u)(t) \leq M$, for every $t \in [0,1]$. Since μ_1 is the first eigenvalue of T and $0 < \sigma < 1$, the first eigenvalue of T_1, $(r(T_1))^{-1} > 1$. Therefore, the inverse operator $(I - T_1)^{-1}$ exists and

$$(I - T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots. \tag{83}$$

It follows from $T_1(P) \subset P$ that $(I - T_1)^{-1}(P) \subset P$. So, we have $u(t) \leq (I - T_1)^{-1} M$, $t \in [0,1]$, and we conclude that W is bounded.

Select $r_4 > \max\{r_1, \sup W\}$. Then, from the invariance property of the fixed point index, we have

$$i\left(A, B_{r_4} \cap P, P\right) = i\left(\theta, B_{r_4} \cap P, P\right) = 1. \tag{84}$$

By (78) and (84), we have that

$$\begin{aligned} & i\left(A, \left(B_{r_4} \cap P\right) \setminus \left(\overline{B}_{r_1} \cap P\right), P\right) \\ & \quad = i\left(A, B_{r_4} \cap P, P\right) - i\left(A, B_{r_1} \cap P, P\right) = 1. \end{aligned} \tag{85}$$

Then A has at least one fixed point on $(B_{r_4} \cap P) \setminus (\overline{B}_{r_1} \cap P)$. This means that the boundary value problem (2) has at least one positive solution. □

Proof of Theorem 9. It follows from the second inequality of (52) that there exists $r_0 > 0$, such that

$$\lambda f(t,u,v) \leq \mu_1(|u| + |v|), \quad \forall t \in [0,1], \ 0 \leq |u| + |v| \leq r_0. \tag{86}$$

Let $r_1 = L_0 r_0/(1 + L_0)L_1$. Then, for every $u \in \partial B_{r_1} \cap P$, we have from Lemma 11 that

$$\begin{aligned} 0 &\leq u(t) - u''(t) \leq \|u\|_\infty + \left\|u''\right\|_\infty \\ &\leq \frac{(1 + L_0) L_1}{L_0} \|u\|_X \leq r_0. \end{aligned} \tag{87}$$

It follows from (86) that

$$(Au)(t) \leq \mu_1 \iint_0^1 K_1(t,s) K_2(s,\tau) \left[u(\tau) - u''(\tau)\right] \mathrm{d}\tau \,\mathrm{d}s$$
$$= \mu_1 (Tu)(t), \quad t \in [0,1]. \tag{88}$$

Suppose there exists $u_2 \in \partial B_{r_1} \cap P$ and $\tau_2 \geq 1$ satisfying $Au_2 = \tau_2 u_2$. We may suppose that A has no fixed points on $\partial B_{r_1} \cap P$ (otherwise, the proof is finished). So $\tau_2 > 1$. By (88), we have $\tau_2 u_2 = Au_2 \leq \mu_1 T u_2$. By induction, we have $\tau_2^n u_2 \leq \mu_1^n T^n u_2$, for all $n \in N$. Thus

$$\|T^n\| \geq \frac{\|T^n u_2\|_X}{\|u_2\|_X} \geq \frac{\tau_2^n \|u_2\|_X}{\mu_1^n \|u_2\|_X} = \frac{\tau_2^n}{\mu_1^n}. \tag{89}$$

By Gelfand's formula, we have

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} \geq \frac{\tau_2}{\mu_1} > \frac{1}{\mu_1}, \tag{90}$$

which is a contradiction with $r(T) = 1/\mu_1$. Hence

$$Au \neq \tau u, \quad \forall u \in \partial B_{r_1} \cap P, \ \tau \geq 1, \tag{91}$$

and we have from Lemma 13 that

$$i\left(A, B_{r_1} \cap P, P\right) = 1. \tag{92}$$

It follows from the first inequality of (52) that there exist $\varepsilon > 0$ and $R_0 > 0$ such that

$$\lambda f(t,u,v) \geq (\mu_1 + \varepsilon)(|u| + |v|), \quad |u| + |v| \geq R_0. \tag{93}$$

From (A2), we know that there exists $d \geq 0$ such that

$$\lambda f(t,u,v) \geq (\mu_1 + \varepsilon)(|u| + |v|) - d,$$
$$t \in [0,1], \quad 0 \leq |u| + |v| < +\infty. \tag{94}$$

We can take $R > r_1$ large enough, such that

$$\rho_{1u} \geq \frac{d \iint_0^1 K_1(s,s) K_2(s,\tau) \mathrm{d}\tau \,\mathrm{d}s}{\varepsilon\left(\int_0^1 (s - s^2) p(s) \mathrm{d}s / \left(1 - \int_0^1 p(s) \mathrm{d}s\right)\right)}, \tag{95}$$

where ρ_{1u} is defined as in (46). Then by (94) and (33), for all $u \in P$, $\|u\|_X \geq R$, $t \in [0,1]$, we have

$$\begin{aligned}(Au)(t) &= \lambda \iint_0^1 K_1(t,s) K_2(s,\tau) f\left(\tau, u(\tau), u''(\tau)\right) \mathrm{d}\tau \,\mathrm{d}s \\ &\geq (\mu_1 + \varepsilon) \iint_0^1 K_1(t,s) K_2(s,\tau) \\ &\qquad \times \left[u(\tau) - u''(\tau)\right] \mathrm{d}\tau \,\mathrm{d}s \\ &\quad - d \iint_0^1 K_1(t,s) K_2(s,\tau) \mathrm{d}\tau \,\mathrm{d}s \\ &\geq \mu_1 (Tu)(t) + \varepsilon \rho_{1u} e(t) \\ &\quad - d \iint_0^1 K_1(s,s) K_2(s,\tau) \mathrm{d}\tau \,\mathrm{d}s \\ &\geq \mu_1 (Tu)(t) + \varepsilon \rho_{1u} \frac{\int_0^1 \left(s - s^2\right) p(s) \mathrm{d}s}{1 - \int_0^1 p(s) \mathrm{d}s} \\ &\quad - d \iint_0^1 K_1(s,s) K_2(s,\tau) \mathrm{d}\tau \,\mathrm{d}s \\ &\geq \mu_1 (Tu)(t).\end{aligned} \tag{96}$$

Hence, for every $u \in P$, $\|u\|_X \geq R$, we have

$$(Au)(t) \geq \mu_1 (Tu)(t), \quad t \in [0,1]. \tag{97}$$

Let φ_1 be the positive eigenfunction of T corresponding to μ_1. Thus $\varphi_1 = \mu_1 T \varphi_1$.

We may suppose that A has no fixed points on $\partial B_R \cap P$ (otherwise, the proof is finished). Now, we show that

$$u - Au \neq \tau \varphi_1, \quad \forall u \in \partial B_R \cap P, \ \tau \geq 0. \tag{98}$$

Suppose the contrary, that there exists $u_3 \in \partial B_R \cap P$ and $\tau_3 \geq 0$ such that

$$u_3 - Au_3 = \tau_3 \varphi_1. \tag{99}$$

Hence $\tau_3 > 0$ and

$$u_3 = Au_3 + \tau_3 \varphi_1 \geq \tau_3 \varphi_1. \tag{100}$$

Put

$$\tau^* := \sup\left\{\tau \mid u_3 \geq \tau \varphi_1\right\}. \tag{101}$$

It is easy to see that $\tau^* \geq \tau_3 > 0$ and $u_3 \geq \tau^* \varphi_1$. We find from $T(P) \subset P$ that

$$\mu_1 T u_3 \geq \tau^* \mu_1 T \varphi_1 = \tau^* \varphi_1. \tag{102}$$

Therefore by (97), we have

$$u_3 = Au_3 + \tau_3 \varphi_1 \geq \mu_1 T u_3 + \tau_3 \varphi_1 \geq \tau^* \varphi_1 + \tau_3 \varphi_1, \tag{103}$$

which contradicts the definition of τ^*. Hence (98) is true, and we have from Lemma 12 that

$$i\left(A, B_R \cap P, P\right) = 0. \tag{104}$$

By (92) and (104), we have that

$$\begin{aligned} i&\left(A, \left(B_R \cap P\right) \setminus \left(\overline{B}_{r_1} \cap P\right), P\right) \\ &= i\left(A, B_R \cap P, P\right) - i\left(A, B_{r_1} \cap P, P\right) = -1. \end{aligned} \tag{105}$$

Then, A has at least one fixed point on $(B_R \cap P) \setminus (\overline{B}_{r_1} \cap P)$. This means that the boundary value problem (2) has at least one positive solution. □

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Research Article

Initial Boundary Value Problem of the General Three-Component Camassa-Holm Shallow Water System on an Interval

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We study the initial boundary value problem of the general three-component Camassa-Holm shallow water system on an interval subject to inhomogeneous boundary conditions. First we prove a local in time existence theorem and present a weak-strong uniqueness result. Then, we establish a asymptotic stabilization of this system by a boundary feedback. Finally, we obtain a result of blow-up solution with certain initial data and boundary profiles.

1. Introduction

It is well known that the Camassa-Holm equation has attracted much attention in the past decade. It is a nonlinear dispersive wave equation that models the propagation of unidirectional irrotational shallow water waves over a flat bed, as well as water waves moving over an underlying shear flow. It was first introduced by Fokas and Fuchssteiner as a bi-Hamiltonian model. Cauchy problem and initial boundary value problem for Camassa-Holm equation have been studied extensively in a number of papers (see [1–15] and the references within).

Fu and Qu in [16] proposed a coupled Camassa-Holm equation,

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv)_x + nv_x, \\ n_t &= 2nv_x + n_x v + (nu)_x + mu_x; \end{aligned} \tag{1}$$

with $m = u - u_{xx}$, $n = v - v_{xx}$, which has peakon solitons in the form of a superposition of multipeakons and may as well be integrable. They investigated the local well-posedness and blow-up solutions of system (1) by means of Kato's semigroup approach to nonlinear hyperbolic evolution equation and obtained a criterion and condition on the initial data guaranteeing the development of singularities in finite time for strong solutions of system (1) by energy estimates. Recently the initial boundary value problem for the system (1) has been established in [17]; moreover, the local well-posedness and blow-up phenomena for the coupled Camassa-Holm equation were also established in [16, 18–32]. In [33], Tian and Xu obtained the compact and bounded absorbing set and the existence of the global attractor for viscous system (1) with the periodic boundary condition in by uniform prior estimate.

Recently, Fu and Qu in [34] introduced a general three-component Camassa-Holm equation as follows:

$$\begin{aligned} m_t &= 2mu_x + m_x u + (mv + mw)_x + nv_x + lw_x, \\ n_t &= 2nv_x + n_x v + (nu + nw)_x + mu_x + lw_x, \\ l_t &= 2lw_x + l_x w + (lu + lv)_x + mu_x + nv_x, \end{aligned} \tag{2}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, and $l = w - w_{xx}$. Equation (2) also has peakon solitons in the form of a superposition of multipeakons. Such system also conserves the H^1-norm conservation law. Moreover, the well-posedness and blow-up phenomena for system (2) with peakons have been established in [35]. To our knowledge, the initial boundary value problem of (2) has not been studied yet. The first aim of this paper is to consider an initial boundary value problem of the following

$$\begin{aligned} &m_t = 2mu_x + m_x u + (mv + mw)_x + nv_x + lw_x, \\ &n_t = 2nv_x + n_x v + (nu + nw)_x + mu_x + lw_x, \\ &l_t = 2lw_x + l_x w + (lu + lv)_x + mu_x + nv_x, \\ &m(0,\cdot) = m_0, \quad n(0,\cdot) = n_0, \quad l(0,\cdot) = l_0, \\ &m(\cdot,0)_{|\Gamma_l} = m_l, \quad n(\cdot,0)_{|\Gamma_l} = n_l, \quad l(\cdot,0)_{|\Gamma_l} = l_l, \\ &m(\cdot,1)_{|\Gamma_r} = m_r, \quad n(\cdot,1)_{|\Gamma_r} = n_r, \quad l(\cdot,1)_{|\Gamma_r} = l_r, \end{aligned} \tag{3}$$

where $\Gamma_l = \{t \in [0,T] \mid (u+v+w)(t,x) < 0\}$, $\Gamma_r = \{t \in [0,T] \mid (u+v+w)(t,x) > 0\}$.

Then, we will consider the asymptotic stabilization of (3) by means of a stationary feedback law acting on the inhomogeneous boundary condition. Following the step in [11], we convert the initial boundary value problem of (3) on the interval into an ODE system and two PDE systems. Then, we can consider the system (3) easily. Consequently, we obtain a local in time existence theorem, a weak-strong uniqueness result, asymptotic stabilization result on the interval, and a result of blow-up solution, respectively.

Our paper is organized as follows. In Section 2, we will consider an initial boundary value problem and the uniqueness of the solution to (3). By using the feedback law enjoyed by (3), the asymptotic stabilization on an interval is considered in Section 3. Finally, in Section 4, a result of blow-up solution with certain initial data and boundary profiles will be established.

First, we begin with a general remark that will be used many times later.

Remark 1. Let T be a positive number and $\Omega_T = [0,T] \times [0,1]$. Changing $u(t,x)$ in $-u(t,1-x)$, $v(t,x)$ in $-v(t,1-x)$, $w(t,x)$ in $-w(t,1-x)$, and t in $T-t$, and it will be more convenient for us to analysis the system, if we define the following sets

$$\begin{aligned} P_l &= \{t \in [0,T] \mid (u+v+w)(t,0) = 0\}, \\ P_r &= \{t \in [0,T] \mid (u+v+w)(t,1) = 0\}. \end{aligned} \tag{4}$$

Let $\Lambda = (1-\partial_x^2)^{1/2}$, then the operator Λ^{-2} can be expressed as

$$\Lambda^{-2} f(x) = G * f(x) = \frac{1}{2}\int_0^1 e^{-|x-y|} f(y)\,dy, \tag{5}$$

where $G = (1/2)e^{-|x|}$. Now, let $A_i = \Lambda^{-2}B_i = G*B_i$, $i = 1,2,3$, where B_i is an auxiliary function which lifts the boundary values m_l, m_r, n_l, n_r, and l_l defined by

$$\begin{aligned} &B_i(t,x) = 0, \quad (t,x) \in [0,T]\times[0,1], \\ &G * B_i(t,0) = v_{l_i}(t), \quad G * B_i(t,1) = v_{r_i}(t), \quad \forall t \in [0,T], \end{aligned} \tag{6}$$

where $i = 1,2,3$.

Setting $u = p + G*B_1$, $v = q + G*B_2$, and $w = r + G*B_3$, we can further rewrite the system (3) as

$$\begin{aligned} &p = G*m, \qquad q = G*n, \qquad r = G*l, \\ &p(t,0) = p(t,1) = 0, \qquad q(t,0) = q(t,1) = 0, \\ &r(t,0) = r(t,1) = 0, \end{aligned} \tag{7}$$

$$\begin{aligned} m_t &= (p + G*B_1 + v + w)\,m_x \\ &\quad + \left[2(p + G*B_1)_x + v_x + w_x\right] m + nv_x + lw_x, \\ n_t &= (q + G*B_2 + u + w)\,n_x \\ &\quad + \left[2(q + G*B_2)_x + u_x + w_x\right] n + mu_x + lw_x, \\ l_t &= (r + G*B_3 + u + v)\,l_x \\ &\quad + \left[2(r + G*B_3)_x + u_x + v_x\right] l + mu_x + nv_x, \\ &m(0,\cdot) = m_0, \quad m(\cdot,0)_{|\Gamma_l} = m_l, \quad m(\cdot,1)_{|\Gamma_r} = m_r, \\ &n(0,\cdot) = n_0, \quad n(\cdot,0)_{|\Gamma_l} = n_l, \quad n(\cdot,1)_{|\Gamma_r} = n_r, \\ &l(0,\cdot) = l_0, \quad l(\cdot,0)_{|\Gamma_l} = l_l, \quad l(\cdot,1)_{|\Gamma_r} = l_r, \end{aligned} \tag{8}$$

where functions m_l, m_r, n_l, n_r, l_l, and l_r in $C^0([0,1],R)$ are the boundary values and m_0, n_0, and l_0 in $L^\infty(0,1)$ are the initial datum.

Lemma 2. *We have $A_i = G*B_i \in C^0([0,T];C^\infty[0,1]) \in C^0([0,T];C^\infty[0,1])$ and $\tilde{p},\tilde{q},\tilde{r} \in L^\infty((0,T),C^{1,1}([0,1])) \cap \mathrm{Lip}((0,T),H_0^1(0,1))$, $i = 1,2,3$, $m,n,l \in L^\infty(\Omega_T) \cap \mathrm{Lip}((0,T),H^{-1}(0,1))$. Moreover, we also have the bounds*

$$\begin{aligned} &\|A_i\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \\ &\quad \le \frac{\cosh(1)}{\sinh(1)}\left(\|v_{r_i}\|_{L^\infty(0,T)} + \|v_{l_i}\|_{L^\infty(0,T)}\right), \\ &\qquad (i = 1,2,3), \end{aligned} \tag{9}$$

$$\begin{aligned} &\|\tilde{p}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|m\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{p}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t m\|_{L^\infty((0,T);\,H^{-1}[0,1])}, \\ &\|\tilde{q}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|n\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{q}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t n\|_{L^\infty((0,T);\,H^{-1}[0,1])}, \\ &\|\tilde{r}\|_{L^\infty((0,T);\,C^{1,1}[0,1])} \le 2(1+\cosh(1))\,\|l\|_{L^\infty(\Omega_T)}, \\ &\|\partial_t\tilde{r}\|_{L^\infty((0,T);\,H_0^1[0,1])} \le \|\partial_t l\|_{L^\infty((0,T);\,H^{-1}[0,1])}. \end{aligned} \tag{10}$$

Proof. $A_i, \widetilde{p}, \widetilde{q}, \widetilde{r}, (i = 1, 2, 3)$ can be expressed, respectively, as

$$A_i(t,x) = G * B_i(t,x) = \frac{1}{\sinh(1)}\left(\sinh(x) v_{r_i}(t) + \sinh(1-x) v_{l_i}(t)\right), \quad (i = 1,2,3),$$

$$\widetilde{p}(t,x) = -\int_0^x \sinh(x-\widetilde{x}) m(t,\widetilde{x}) d\widetilde{x} + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\widetilde{x}) m(t,\widetilde{x}) d\widetilde{x},$$

$$\widetilde{q}(t,x) = -\int_0^x \sinh(x-\widetilde{x}) n(t,\widetilde{x}) d\widetilde{x} + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\widetilde{x}) n(t,\widetilde{x}) d\widetilde{x},$$

$$\widetilde{r}(t,x) = -\int_0^x \sinh(x-\widetilde{x}) l(t,\widetilde{x}) d\widetilde{x} + \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(1-\widetilde{x}) l(t,\widetilde{x}) d\widetilde{x}. \tag{11}$$

Estimates (9) and (10) can be easily obtained from the above expressions. □

2. Initial Boundary Value Problem

First, we define what we mean by a weak solution to (8). Our test functions will be in the space:

$$\mathrm{Adm}(\Omega_T) = \left\{\varphi \in C^1(\Omega_T) \mid \varphi(t,x) = 0 \text{ on } [0,T] \setminus \Gamma_l \times \{0\} \cup [0,T] \setminus \Gamma_r \times \{0\} \cup \{T\} \times [0,1]\right\}. \tag{12}$$

Definition 3. When $(p,q,r) \in L^\infty((0,T); \mathrm{Lip}[0,1]) \times L^\infty((0,T); \mathrm{Lip}[0,1]) \times L^\infty((0,T); \mathrm{Lip}[0,1])$, the function $(m,n,l) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ is the weak solution to (8) if for all $\varphi \in \mathrm{Adm}(\Omega_T)$:

$$\iint_{\Omega_T} m\left(\varphi_t - (u+v+w)\varphi_x + \left(p_x + \partial_x G * B_1\right)\varphi\right) dt\,dx$$
$$= -\iint_{\Omega_T} (nv_x + lw_x)\varphi(t,x)\,dt\,dx - \int_0^1 m_0(x)\varphi(0,x)\,dt$$
$$+ \int_0^T \big((u+v+w)(t,0)\varphi(t,0)m(t,0) - (u+v+w)(t,1)\varphi(t,1)m(t,1)\big)\,dt,$$

$$\iint_{\Omega_T} n\left(\varphi_t - (u+v+w)\varphi_x + \left(q_x + \partial_x G * B_2\right)\varphi\right) dt\,dx$$
$$= -\iint_{\Omega_T} (mu_x + lw_x)\varphi(t,x)\,dt\,dx - \int_0^1 n_0(x)\varphi(0,x)\,dx$$
$$+ \int_0^T \big((u+v+w)(t,0)\varphi(t,0)n(t,0) - (u+v+w)(t,1)\varphi(t,1)n(t,1)\big)\,dt,$$

$$\iint_{\Omega_T} l\left(\varphi_t - (u+v+w)\varphi_x + \left(r_x + \partial_x G * B_3\right)\varphi\right) dt\,dx$$
$$= -\iint_{\Omega_T} (mu_x + nv_x)\varphi(t,x)\,dt\,dx - \int_0^1 l_0(x)\varphi(0,x)\,dx$$
$$+ \int_0^T \big((u+v+w)(t,0)\varphi(t,0)l(t,0) - (u+v+w)(t,1)\varphi(t,1)l(t,1)\big)\,dt. \tag{13}$$

It is obvious that $C_0^1(\Omega_T) \subset \mathrm{Adm}(\Omega_T)$; therefore, a weak solution to system (8) is also a solution to (8) in the distribution sense. And it is clear that a regular weak solution is a classical solution.

Definition 4. For $(t,x) \in \Omega_T$, we consider $\omega(\cdot,t,x)$ the maximal solution satisfying

$$\omega_t = -(u+v+w)(t,\omega(t,x)), \quad \omega(0,x) = x. \tag{14}$$

We consider that ω is the flow of $(u(t,x), v(t,x), w(t,x))$. For $(t,x) \in \Omega_T$, $\omega(\cdot,t,x)$ is defined on a set $[e(t,x), h(t,x)]$. Here $e(t,x)$ is basically the entrance time in Ω_T of the characteristic curve going through (t,x).

Remark 5. Obviously $e(t,x) > 0$ implies that $\omega(e(t,x),t,x) \in \{0,1\}$.

In the following, we consider a partition of Ω_T, which allows us to distinguish the different influence zones in Ω_T.

Definition 6. Let $P = \{(t,x) \in \Omega_T \mid \exists s \in [e(t,x), h(t,x)]$ such that $\omega \in \{0,1\}$ and $(u+v+w)(s,\omega(s,t,x)) = 0\} \cup \{(s,\omega(s,0,0)) \mid$ for all $s \in [0,T]\} \cup \{(s,\omega(s,0,1)) \mid$ for all $s \in [0,T]\}$,

$$I = \left\{(t,x) \in \Omega_T \setminus p \mid e(t,x) = 0\right\},$$
$$L = \left\{(t,x) \in \Omega_T \setminus p \mid \omega(e(t,x),t,x) = 0\right\}, \tag{15}$$
$$R = \left\{(t,x) \in \Omega_T \setminus p \mid \omega(e(t,x),t,x) = 1\right\}.$$

Those points of the set P are tangent to the boundary, which are precisely the singular points of e and h. It's obviously that the sets P, I, L, and R constitute a partition of Ω_T. Furthermore, if $(t,x) \in L$, then $e(t,x) \in \Gamma_l$, and if $(t,x) \in R$, then $e(t,x) \in \Gamma_r$.

Definition 7. Here, we consider the case of data $(u,v,w) \in L^\infty([0,T]; C^1([0,1])) \times L^\infty([0,T]; C^1([0,1])) \times$

$L^\infty([0,T];C^1([0,1]))$, $(m_l,n_l,l_l) \in C_c^1(\Gamma_l)\times C_c^1(\Gamma_l)\times C_c^1(\Gamma_l)$; $(m_r,n_r,l_r) \in C_c^1(\Gamma_r)\times C_c^1(\Gamma_r)\times C_c^1(\Gamma_r)$, $(m_0,n_0,l_0) \in C_c^1(0,1)\times C_c^1(0,1)\times C_c^1(0,1)$. We define the functions m,n, and l in the following way.

When $(t,x)\in P$, $m(t,x)=0$, $n(t,x)=0$, and $l(t,x)=0$, when $(t,x)\in I$,

$$
\begin{aligned}
m(t,x) &= m_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_0^t (nv_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
n(t,x) &= n_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_0^t (mu_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left([2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
l(t,x) &= l_0(\omega(0,t,x)) \\
&\quad\times \exp\left(\int_0^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_0^t (mu_x+nv_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s',\omega(s',t,x))\,ds'\right) ds,
\end{aligned}
\tag{16}
$$

when $(t,x)\in L$,

$$
\begin{aligned}
m(t,x) &= m_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_{e(t,x)}^t (nv_x+lw_x)(r,\omega(r,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) dr, \\
n(t,x) &= n_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_{e(t,x)}^t (mu_x+lw_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(q_x+\partial_x G * B_2)+u_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) ds, \\
l(t,x) &= l_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_{e(t,x)}^t (mu_x+nv_x)(s,\omega(s,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(r_x+\partial_x G * B_3)+u_x+v_x] \times (s',\omega(s',t,x))\,ds'\right) ds,
\end{aligned}
\tag{17}
$$

when $(t,x)\in R$,

$$
\begin{aligned}
m(t,x) &= m_l(e(t,x)) \\
&\quad\times \exp\left(\int_{e(t,x)}^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s,\omega(s,t,x))\,ds\right) \\
&\quad+\int_{e(t,x)}^t (nv_x+lw_x)(r,\omega(r,t,x)) \\
&\quad\times \exp\left(\int_s^t [2(p_x+\partial_x G * B_1)+v_x+w_x] \times (s',\omega(s',t,x))\,ds'\right) dr,
\end{aligned}
$$

$$\begin{aligned}
n(t,x) &= n_l(e(t,x)) \\
&\quad\times \exp\Bigg(\int_{e(t,x)}^{t}\left[2\left(q_x+\partial_x G * B_2\right)+u_x+w_x\right] \\
&\qquad\times (s,\omega(s,t,x))\,ds\Bigg) \\
&\quad+\int_{e(t,x)}^{t}\left(mu_x+lw_x\right)(s,\omega(s,t,x)) \\
&\qquad\times \exp\left(\int_{s}^{t}\left[2\left(q_x+\partial_x G * B_2\right)+u_x+w_x\right]\right. \\
&\qquad\left.\times\left(s',\omega\left(s',t,x\right)\right)ds'\right)ds, \\
l(t,x) &= l_l(e(t,x)) \\
&\quad\times \exp\Bigg(\int_{e(t,x)}^{t}\left[2\left(r_x+\partial_x G * B_3\right)+u_x+v_x\right] \\
&\qquad\times (s,\omega(s,t,x))\,ds\Bigg) \\
&\quad+\int_{e(t,x)}^{t}\left(mu_x+nv_x\right)(s,\omega(s,t,x)) \\
&\qquad\times \exp\left(\int_{s}^{t}\left[2\left(r_x+\partial_x G * B_3\right)+u_x+v_x\right]\right. \\
&\qquad\left.\times\left(s',\omega\left(s',t,x\right)\right)ds'\right)ds.
\end{aligned} \tag{18}$$

Lemma 8. *Since $(m,n,l) \in L^{\infty}(\Omega_T)\times L^{\infty}(\Omega_T)\times L^{\infty}(\Omega_T)$ and satisfies (8), we immediately get that (m,n,l) is the weak solution of (8) and $(m,n,l) \in W^{1,\infty}(0,T,H^{-1}(0,1)) \times W^{1,\infty}(0,T,H^{-1}(0,1)) \times W^{1,\infty}(0,T,H^{-1}(0,1))$. However, the functions m,n, and l satisfy the following estimates:*

$$\begin{aligned}
&\|m\|_{C^0(\Omega_T)} \\
&\quad\le \left[\max\left(\|m_0\|_{L^\infty},\|m_l\|_{L^\infty},\|m_r\|_{L^\infty}\right)+\|nv_x+lw_x\|_{L^\infty(0,1)}T\right] \\
&\qquad\times e^{T\|2(p+G*B_1)_x+v_x+w_x\|_{C^0(\Omega_T)}}, \\
&\|n\|_{C^0(\Omega_T)} \\
&\quad\le \left[\max\left(\|n_0\|_{L^\infty},\|n_l\|_{L^\infty},\|n_r\|_{L^\infty}\right)+\|mu_x+lw_x\|_{L^\infty(0,1)}T\right] \\
&\qquad\times e^{T\|2(q+G*B_2)_x+u_x+w_x\|_{C^0(\Omega_T)}}, \\
&\|l\|_{C^0(\Omega_T)} \\
&\quad\le \left[\max\left(\|m_0\|_{L^\infty},\|m_l\|_{L^\infty},\|m_r\|_{L^\infty}\right)+\|mu_x+nv_x\|_{L^\infty(0,1)}T\right] \\
&\qquad\times e^{T\|2(r+G*B_3)_x+u_x+v_x\|_{C^0(\Omega_T)}},
\end{aligned}$$

$$\begin{aligned}
&\|\partial_t m\|_{C^0(\Omega_T)} \\
&\quad\le \Big\{2\max\left(\|m_0\|_{L^\infty},\|m_l\|_{L^\infty},\|m_r\|_{L^\infty}\right) \\
&\qquad\times\left(\|p+G*B_1\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right. \\
&\qquad\quad\left.+\|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad+\Big[\left(\|u\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad+\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad+\left(\|w\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\Big] \\
&\qquad\times\Big(1+2\Big(\|p+G*B_1\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} \\
&\qquad\quad+\|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\Big)T\Big\} \\
&\qquad\times\exp\Big(2T\Big(\|(p+G*B_1)_x\|_{L^\infty(\Omega_T)} \\
&\qquad\quad+\|v_x\|_{L^\infty(\Omega_T)}+\|w_x\|_{L^\infty(\Omega_T)}\Big)\Big), \\
&\|\partial_t n\|_{C^0(\Omega_T)} \\
&\quad\le \Big\{2\max\left(\|n_0\|_{L^\infty},\|n_l\|_{L^\infty},\|n_r\|_{L^\infty}\right) \\
&\qquad\times\left(\|q+G*B_2\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right. \\
&\qquad\quad\left.+\|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad+\Big[\left(\|u\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad+\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\qquad\quad+\left(\|w\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\Big] \\
&\qquad\times\Big(1+2\Big(\|q+G*B_2\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} \\
&\qquad\quad+\|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])} \\
&\qquad\quad+\|w\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\Big)T\Big\} \\
&\qquad\times\exp\Big(2T\Big(\|(q+G*B_2)_x\|_{L^\infty(\Omega_T)} \\
&\qquad\quad+\|p_x\|_{L^\infty(\Omega_T)}+\|w_x\|_{L^\infty(\Omega_T)}\Big)\Big), \\
&\|\partial_t l\|_{C^0(\Omega_T)} \\
&\quad\le \Big\{2\max\left(\|l_0\|_{L^\infty},\|l_l\|_{L^\infty},\|l_r\|_{L^\infty}\right) \\
&\qquad\times\left(\|r+G*B_3\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right. \\
&\qquad\quad\left.+\|u\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|v\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right) \\
&\qquad+\Big[\left(\|u\|_{L^\infty(\Omega_T)}+\left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2
\end{aligned}$$

$$+\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial^2_{xx}v\right\|_{L^\infty(\Omega_T)}\right)^2$$
$$+\left(\|w\|_{L^\infty(\Omega_T)}+\left\|\partial^2_{xx}w\right\|_{L^\infty(\Omega_T)}\right)^2\Big]$$
$$\times\left(1+2\left(\|r+G*B_3\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|p\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}+\|q\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}\right)\right)T\Big\}$$
$$\times\exp\left(2T\left(\left\|(r+G*B_3)_x\right\|_{L^\infty(\Omega_T)}+\|u_x\|_{L^\infty(\Omega_T)}+\|v_x\|_{L^\infty(\Omega_T)}\right)\right). \tag{19}$$

Definition 9. We can define operator E and a domain for the system (8) by: for all $p,q,r\in L^\infty((0,T);C^{1,1}([0,1]))\cap\mathrm{Lip}([0,T];H^1_0(0,1))$,

$$\begin{aligned}
E(p)&=\widetilde{p}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\cap\mathrm{Lip}\left([0,T];H^1_0(0,1)\right),\\
E(q)&=\widetilde{q}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\cap\mathrm{Lip}\left([0,T];H^1_0(0,1)\right),\\
E(r)&=\widetilde{r}\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\cap\mathrm{Lip}\left([0,T];H^1_0(0,1)\right),\\
&C_{M_0,M_1,T}\\
&=\Big\{p,q,r\in L^\infty\left((0,T);C^{1,1}([0,1])\right)\cap\mathrm{Lip}\left([0,T];H^1_0(0,1)\right)\mid\|d\|_{L^\infty((0,T);C^{1,1}[0,1])}\\
&\qquad\le M_0,\|d\|_{\mathrm{Lip}((0,T);H^1_0(0,1))}\le M_1\Big\},
\end{aligned}\tag{20}$$

where

$$d(t,x)=\max\left(p(t,x),q(t,x),r(t,x)\right),\quad (t,x)\in[0,T]\times[0,1]. \tag{21}$$

Obviously $C_{M_0,M_1,T}$ is convex and $C_{M_0,M_1,T}$ is compact with respect to the norm $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. We will endow $C_{M_0,M_1,T}$ with the norm $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}([0,1]))}$. There exist positive numbers M_0,M_1, and T such that E maps $C_{M_0,M_1,T}$ into itself.

Theorem 10. *There exists $T>0$, and (m,n,l) is a weak solution of (8) with $p,q,r\in L^\infty((0,T);C^{1,1}([0,1]))\cap\mathrm{Lip}([0,T];H^1_0(0,1))$ and $m,n,l\in L^\infty(\Omega_T)$. Moreover, any such solution (p,q,r) is in fact in $C^0([0,T];W^{2,P}(0,1))\cap C^1([0,T];W^{1,p}_0(0,1))$, for all $p<+\infty$. Furthermore, the existence time of a maximal solution $T\ge\min(T^*,\widetilde{T})$, with*

$$\begin{aligned}
T^*&=\max_{\alpha>0,\ \beta>0}\left(\frac{1}{6\alpha}\ln\left(\frac{|\alpha-C_1|}{4(1+\cosh(1))\alpha}\right)\right),\\
\widetilde{T}&=\max_{\alpha>0,\ \beta>0}\left(\frac{1}{6\alpha}\ln\left(\frac{|\beta|}{12\alpha^2}\right)\right),\\
C_1&=\max\left[\frac{\cosh(1)}{\sinh(1)}\left(\left\|v_{r_i}\right\|_{L^\infty(0,T)}+\left\|v_{l_i}\right\|_{L^\infty(0,T)}\right)\right],\quad i=1,2,3.
\end{aligned}\tag{22}$$

Proof. For $\widetilde{T}>0$, we consider m_l,m_r,n_l,n_r,l_l, and l_r in $C^0([0,\widetilde{T}])$ such that the sets P_l and P_r have only a finite number of connected components.

Let $C_0=\max(\|m_i\|_{L^\infty(0,1)},\|n_i\|_{L^\infty(\Gamma_l)},\|l_i\|_{L^\infty(\Gamma_r)})$, where $i=0,r,l$ and

$$C_1=\max\left[\frac{\cosh(1)}{\sinh(1)}\left(\left\|v_{r_i}\right\|_{L^\infty(0,T)}+\left\|v_{l_i}\right\|_{L^\infty(0,T)}\right)\right],\quad i=1,2,3. \tag{23}$$

Now, if $u,v,w\in C_{M_0,M_1,T}$ (see (21)), we have

$$\begin{aligned}
&\left\|2(p+G*B_1)_x+v_x+w_x\right\|_{C^\infty(\Omega_T)}\\
&\quad\le2\left(\|\partial_xu\|_{C^\infty(\Omega_T)}+\|\partial_xv\|_{C^\infty(\Omega_T)}+\|\partial_xw\|_{C^\infty(\Omega_T)}\right),\\
&\left\|2(q+G*B_2)_x+u_x+w_x\right\|_{C^\infty(\Omega_T)}\\
&\quad\le2\left(\|\partial_xu\|_{C^\infty(\Omega_T)}+\|\partial_xv\|_{C^\infty(\Omega_T)}+\|\partial_xw\|_{C^\infty(\Omega_T)}\right),\\
&\left\|2(r+G*B_3)_x+u_x+v_x\right\|_{C^\infty(\Omega_T)}\\
&\quad\le2\left(\|\partial_xu\|_{C^\infty(\Omega_T)}+\|\partial_xv\|_{C^\infty(\Omega_T)}+\|\partial_xw\|_{C^\infty(\Omega_T)}\right).
\end{aligned}\tag{24}$$

For all $u,v,w\in L^\infty((0,T);W^{2,\infty}(0,1))$, we have

$$\begin{aligned}
\|\partial_xu\|_{L^\infty(\Omega_T)}&\le2\sqrt{\|u\|_{L^\infty(\Omega_T)}\left\|\partial^2_{xx}u\right\|_{L^\infty(\Omega_T)}}\\
&\le\left(\|u\|_{L^\infty(\Omega_T)}+\left\|\partial^2_{xx}u\right\|_{L^\infty(\Omega_T)}\right),\\
\|\partial_xv\|_{L^\infty(\Omega_T)}&\le2\sqrt{\|v\|_{L^\infty(\Omega_T)}\left\|\partial^2_{xx}v\right\|_{L^\infty(\Omega_T)}}\\
&\le\left(\|v\|_{L^\infty(\Omega_T)}+\left\|\partial^2_{xx}v\right\|_{L^\infty(\Omega_T)}\right),
\end{aligned}$$

$$\begin{aligned}
\|\partial_x w\|_{L^\infty(\Omega_T)} &\le 2\sqrt{\|w\|_{L^\infty(\Omega_T)}\|\partial_{xx}^2 w\|_{L^\infty(\Omega_T)}} \\
&\le \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right), \\
&\|nv_x + lw_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \right.\\
&\quad + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\quad \left. + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\right], \\
&\|mu_x + lw_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \right.\\
&\quad + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\quad \left. + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\right], \\
&\|mu_x + nv_x\|_{L^\infty(0,1)} \\
&\le \left[\left(\|u\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 u\right\|_{L^\infty(\Omega_T)}\right)^2 \right.\\
&\quad + \left(\|v\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 v\right\|_{L^\infty(\Omega_T)}\right)^2 \\
&\quad \left. + \left(\|w\|_{L^\infty(\Omega_T)} + \left\|\partial_{xx}^2 w\right\|_{L^\infty(\Omega_T)}\right)^2\right].
\end{aligned} \tag{25}$$

We also define that $\tilde{d}(t,x) = \max(\tilde{p}(t,x), \tilde{q}(t,x), \tilde{r}(t,x))$, $(t,x) \in \Omega_T$. If $p, q, r \in C_{M_0,M_1,T}$, then from Lemmas 2 and 8, we derive that

$$\begin{aligned}
&\|\tilde{d}\|_{L^\infty((0,T);C^{1,1}[0,1])} \\
&\quad \le 2(1+\cosh(1))\left(C_0 + 12T^*(M_0+C_1)^2\right) \\
&\qquad \times \exp\left(6T^*(M_0+C_1)\right), \\
&\|\partial_t \tilde{d}\|_{C^0(\Omega_T)} \\
&\quad \le \left[6C_0(M_0+C_1) + 12(M_0+2C_1)^2 \right.\\
&\qquad \left. \times \left(1 + 6(M_0+C_1)\tilde{T}\right)\right] \exp\left(6\tilde{T}(M_0+C_1)\right).
\end{aligned} \tag{26}$$

Finally, to obtain $\tilde{p}, \tilde{q}, \tilde{r} \in C_{M_0,M_1,T}$, it is sufficient to show that

$$\begin{aligned}
&2(1+\cosh(1))\left(C_0 + 12T^*(M_0+C_1)^2\right) \\
&\quad \times \exp\left(6T^*(M_0+C_1)\right) \le M_0, \\
&M_0 + \left[6C_0(M_0+C_1) + 12(M_0+C_1)^2 \right.\\
&\qquad \left. \times \left(1 + 6\tilde{T}(M_0+C_1)\right)\right] \times \exp\left(6\tilde{T}(M_0+C_1)\right) \\
&\quad \le M_1,
\end{aligned} \tag{27}$$

if we have chosen T and M_0; it is easy to choose M_1 to satisfy the second inequality. For the above two inequalities, we just choose M_0 and M_1 sufficiently large and then T close to 0. More precisely:

$$\begin{gathered}
M_0 > 2(1+\cosh(1))C_0, \\
T^* \le \frac{1}{6(M_0+C_1)} \ln\left(\frac{M_0}{4(1+\cosh(1))(M_0+C_1)}\right), \\
\tilde{T} \le \frac{1}{6(M_0+C_1)} \ln\left(\frac{M_1 - M_0}{12(M_0+C_1)^2}\right).
\end{gathered} \tag{28}$$

Maximizing the bound of T, we can get minimum existence. Then, we get the result announced, where $\alpha = M_0 + C_1$, $\beta = M_1 - M_0$. □

Lemma 11. *The operator $E : C_{M_0,M_1,T} \to C_{M_0,M_1,T}$ is continuous with respect to $\|\cdot\|_{L^\infty((0,T);\mathrm{Lip}[0,1])}$.*

Proof. The proof is omitted here; one can see a similar proof in [8, Proposition 2.4].

Now, we can apply Shauder's fixed point theorem to the operator E, and we get the result that there exist fixed points p, q, r such that $E(p) = p$, $E(q) = q$, and $E(r) = r$, so we know that there exists a wake solution of (9).

$$p, q, r \in L^\infty\left((0,T); C^{1,1}([0,1])\right) \cap \mathrm{Lip}\left([0,T]; H_0^1(0,1)\right). \tag{29}$$

□

2.1. Uniqueness. We will prove the weak-strong uniqueness of weak solution of (8) in the following.

Theorem 12. *Let $(p,m), (q,n), (r,l) \in L^\infty((0,T); C^{1,1}([0,1])) \cap \mathrm{Lip}([0,T]; H_0^1(0,1) \times L^\infty((0,T); \mathrm{Lip}([0,1]))$ be the weak solution of (7)-(8), then it is unique in $L^\infty((0,T); C^{1,1}([0,1])) \times L^\infty(\Omega_T)$.*

Proof. Define $\Phi = m - \tilde{m}$, $\Psi = n - \tilde{n}$, $\Upsilon = l - \tilde{l}$ and $P = p - \tilde{p}$, $Q = q - \tilde{q}$, $H = r - \tilde{r}$, then we have

$$\begin{gathered}
P(t,\cdot) = G * \Phi(t,\cdot), \\
Q(t,\cdot) = G * \Psi(t,\cdot), \\
\mathrm{H}(t,\cdot) = G * \Upsilon(t,\cdot),
\end{gathered} \tag{30}$$

where $P, Q, H \in \mathrm{Lip}([0,T]; H_0^1(0,1))$, and $(\Phi, \Psi, \Upsilon) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ is the unique weak solution of

$$\begin{aligned}
\Phi_t &= (p + G * B + v + w)\,\Phi_x \\
&\quad + [2(p + G * B_1)_x + v_x + w_x]\,\Phi \\
&\quad + (P + Q + H)\,\widetilde{m}_x + (2P_x + Q_x + H_x)\,\widetilde{m} \\
&\quad + \Psi v_x + \widetilde{n} Q_x + \Upsilon w_x + \widetilde{l} H_x, \\
\Psi_t &= (q + G * B + u + w)\,\Psi_x \\
&\quad + [2(q + G * B_2)_x + u_x + w_x]\,\Psi \\
&\quad + (P + Q + H)\,\widetilde{n}_x + (P_x + 2Q_x + H_x)\,\widetilde{n} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Upsilon w_x + \widetilde{l} H_x, \\
\Upsilon_t &= (r + G * B + u + v)\,\Upsilon_x \\
&\quad + [2(r + G * B_3)_x + u_x + v_x]\,\Upsilon \\
&\quad + (P + Q + H)\,\widetilde{l}_x + (P_x + Q_x + 2H_x)\,\widetilde{l} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Psi v_x + \widetilde{n} Q_x.
\end{aligned} \tag{31}$$

Let

$$\begin{aligned}
b_1 &= 2(p + G * B_1)_x + v_x + w_x, \\
b_2 &= 2(q + G * B_2)_x + u_x + w_x, \\
b_3 &= 2(r + G * B_3)_x + u_x + v_x, \\
f_1 &= (P + Q + H)\,\widetilde{m}_x + (2P_x + Q_x + H_x)\,\widetilde{m} \\
&\quad + \Psi v_x + \widetilde{n} Q_x + \Upsilon w_x + \widetilde{l} H_x, \\
f_2 &= (P + Q + H)\,\widetilde{n}_x + (P_x + 2Q_x + H_x)\,\widetilde{n} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Upsilon w_x + \widetilde{l} H_x, \\
f_3 &= (P + Q + H)\,\widetilde{l}_x + (P_x + Q_x + 2H_x)\,\widetilde{l} \\
&\quad + \Phi u_x + \widetilde{m}\Phi_x + \Psi v_x + \widetilde{n} Q_x,
\end{aligned} \tag{32}$$

with $i_0 = 0$, $i_l = 0$, and $i_r = 0$, where $i = \Phi, \Psi, \Upsilon$.

For $(t,x) \in P$, we have $\Phi(t,x) = 0, \Psi(t,x) = 0$, and $\Upsilon(t,x) = 0$.

Then, we get the uniqueness result.

For $(t,x) \in I$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_0^t f_1(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_0^t f_2(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_0^t f_3(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{33}$$

For $(t,x) \in L$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_{e(t,x)}^t f_1(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_{e(t,x)}^t f_2(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_{e(t,x)}^t f_3(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{34}$$

For $(t,x) \in R$, we have

$$\begin{aligned}
\Phi(t,x) &= \int_{e(t,x)}^t f_1(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_1(s', \omega(s',t,x))\, ds'\right) ds, \\
\Psi(t,x) &= \int_{e(t,x)}^t f_2(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_2(s', \omega(s',t,x))\, ds'\right) ds, \\
\Upsilon(t,x) &= \int_{e(t,x)}^t f_3(s, \omega(s,t,x)) \\
&\quad \times \exp\left(\int_r^t b_3(s', \omega(s',t,x))\, ds'\right) ds.
\end{aligned} \tag{35}$$

Now since $\|P(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Phi(t,\cdot)\|_{L^\infty(0,1)}$, $\|Q(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Psi(t,\cdot)\|_{L^\infty(0,1)}$, $\|\mathrm{H}(t,\cdot)\|_{L^\infty(0,1)} \le 5\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}$ and $\widetilde{m}, \partial_x\widetilde{m}, \widetilde{n}, \partial_x\widetilde{n}, \widetilde{l}, \partial_x\widetilde{l}$ bounded, we see that for some $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$,

$$\begin{aligned}
&\|f_1(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_1 \left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}\right), \\
&\|f_2(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_2 \left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}\right), \\
&\|f_3(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \le \lambda_3 \left(\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)+\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)}}\right),
\end{aligned} \tag{36}$$

since p, q, r, and $B_i, (i = 1, 2, 3)$ are bounded, we get that for some $\lambda_1' > 0, \lambda_2' > 0, \lambda_3' > 0$,

$$\begin{aligned}
&\|\Phi(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \leq \lambda_1' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds, \\
&\|\Psi(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \leq \lambda_2' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds, \\
&\|\Upsilon(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \leq \lambda_3' \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds.
\end{aligned} \tag{37}$$

We can obtain that

$$\begin{aligned}
&\|\Phi(t,\cdot)\|_{L^\infty(0,1)} + \|\Psi(t,\cdot)\|_{L^\infty(0,1)} + \|\Upsilon(t,\cdot)\|_{L^\infty(0,1)} \\
&\quad \leq \left(\lambda_1' + \lambda_2' + \lambda_3'\right) \\
&\quad \times \int_0^t \left(\|\Phi(s,\cdot)\|_{L^\infty(0,1)} + \|\Psi(s,\cdot)\|_{L^\infty(0,1)} \right. \\
&\qquad \left. + \|\Upsilon(s,\cdot)\|_{L^\infty(0,1)}\right) ds.
\end{aligned} \tag{38}$$

As a result, we get the result of the uniqueness by Gronwall's inequality when $(t, x) \in I$, $(t, x) \in L$, $(t, x) \in R$. Then, we complete the proof of the uniqueness results. □

3. Asymptotic Stabilization

3.1. Preliminary Results. The equilibrium state that we want to stabilize is $m = n = l = 0$, $p = q = r = G * B_i = 0$, and $i = 1, 2, 3$. A natural idea is using Lyapunov indirection method to investigate whether the linearized system around the equilibrium state is stabilizable or not. Its stabilization would provide a local stabilization result on the nonlinear system. However, there is a difficulty in the stabilization problem. We have to prescribe y_l, and we just need to make a continuous transition at $(t, x) = (0, 0)$, and that y_l asymptotically converge in time. For convenience, the system (6)–(8) can rewrite in the following

$$\begin{gathered}
\partial_t y - \breve{a}(t,x)\,\partial_x y = \breve{b}(t,x)\,y + \breve{f}(t,x), \\
y(0,\cdot) = y_0, \qquad y(\cdot,0) = y_l, \qquad y(1,\cdot) = y_r, \\
\breve{g}(t,x) = G * y(t,x), \qquad \breve{g}(t,0) = \breve{g}(t,1) = 0, \\
\breve{B}(t,x) = 0, \qquad G * \breve{B}(t,0) = v_l(t), \\
G * \breve{B}(t,1) = v_r(t),
\end{gathered} \tag{39}$$

where

$$\begin{gathered}
\breve{b}(t,x) = \begin{pmatrix} 2\left(p + G * \breve{B}\right)_x + v_x + w_x & 0 & 0 \\ 0 & 2\left(q + G * \breve{B}\right)_x + u_x + w_x & 0 \\ 0 & 0 & 2\left(r + G * \breve{B}\right)_x + u_x + v_x \end{pmatrix}, \\
\breve{a}(t,x) = \begin{pmatrix} \left(p + G * \breve{B} + v + w\right) & 0 & 0 \\ 0 & \left(u + q + G * \breve{B} + w\right) & 0 \\ 0 & 0 & \left(u + v + r + G * \breve{B}\right) \end{pmatrix}, \\
y = \begin{pmatrix} m \\ n \\ l \end{pmatrix}, \qquad y_0 = \begin{pmatrix} m_0 \\ n_0 \\ l_0 \end{pmatrix}, \\
\breve{f}(t,x) = \begin{pmatrix} nv_x + lw_x \\ mu_x + lw_x \\ mu_x + nv_x \end{pmatrix} = \begin{pmatrix} 0 & n & l \\ m & 0 & l \\ m & n & 0 \end{pmatrix} \begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix}, \\
\breve{g}(t,x) = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \qquad y_l = \begin{pmatrix} m_l \\ n_l \\ l_l \end{pmatrix}, \qquad y_r = \begin{pmatrix} m_r \\ n_r \\ l_r \end{pmatrix}.
\end{gathered} \tag{40}$$

Our feedback law for (3) reads

$$y \in C^0([0,1]) \longmapsto \begin{cases} v_l(y) = A_l \|y\|_{C^0([0,1])}, \\ v_r(y) = A_r \|y\|_{C^0([0,1])}, \\ \partial_t y_l = M y_l, \end{cases} \tag{41}$$

where $A_l > 2\sinh(1)$, $A_r > A_l \cosh(1) + \sinh(2)$, $M > 0$, $T > 0$, and $M = \left(\begin{smallmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{smallmatrix}\right)$, $\mu < 0$, a symmetric matrix, is the unique matrix solution to the matrix function:

$$PM + M^T P = -Z, \tag{42}$$

for some symmetric positive-definite matrices P and Z. Indeed, let $V(t, y_l) = y_l^T P y_l$ be the Lyapunov candidate, and that y_l asymptotically converges in time is equivalent to that the time derivative of the V,$\dot{V}(t, y_l) = -y_l^T Z y_l$ is strictly negative. A fixed-point strategy will be used again to prove the existence of a solution to the closed-loop system, we begin by defining the domain of the operator.

Definition 13. Let X be the space of $(g, N) \in C^0([0,T] \times [0,1]) \times C^0([0,1])$ satisfying

(1) for all $(t, x) \in [0,T] \times [0,1]$, $g(0, x) = y_0(x)$, $g(t, 0) = y_0(0)e^{Mt}$,

(2) for all $t \in [0,T]$, $\|g(t,\cdot)\|_{C^0([0,1])} \leq N(t)$,

(3) N is nonincreasing, and $N(0) \leq \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}$.

Lemma 14. *The domain X is nonempty, convex, bounded, and closed with respect to the uniform topology.*

The proof is elementary and one notices that $(y_0(x)e^{Mt}, \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}e^{Mt}) \in X$, so X is nonempty.

Now for $(y,N)\in X$, we define $\overset{\vee}{g}$ and $G* \overset{\vee}{B}$ as the solutions of

$$\begin{gathered}\forall (t,x)\in\Omega_T,\quad \overset{\vee}{g}(t,x) = G*y(t,x),\\ \overset{\vee}{g}(t,0) = \overset{\vee}{g}(t,1) = 0,\quad \overset{\vee}{B}(t,x) = 0,\\ G*\overset{\vee}{B}(t,0) = A_l N(t),\quad G*\overset{\vee}{B}(t,1) = A_r N(t).\end{gathered} \tag{43}$$

One has the following exact formulas:

$$\begin{aligned}&\forall (t,x)\in\Omega_T,\\ &\overset{\vee}{g}(t,x) = -\int_0^x \sinh(x-\tilde{x})\,y(t,\tilde{x})\,d\tilde{x} - \frac{\sinh(x)}{\sinh(1)}\int_0^1 \sinh(\tilde{x}-1)\,y(t,\tilde{x})\,d\tilde{x},\\ &G*\overset{\vee}{B}(t,x) = \frac{N(t)}{\sinh(1)}\left(A_r\sinh(x) + A_l\sinh(1-x)\right).\end{aligned} \tag{44}$$

Therefore, we have the following inequalities:

$$\begin{aligned}&\forall (t,x)\in[0,T]\times[0,1],\\ &\left|\overset{\vee}{g}(t,x)\right| \leq 2(1+\cosh(1))\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial_x \overset{\vee}{g}(t,x)\right| \leq 2\cosh(1)\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial^2_{xx} \overset{\vee}{g}(t,x)\right| \leq [2(\cosh(1)+1)+1] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ &\left|\partial_x\left(G*\overset{\vee}{B}\right)(t,x)\right| \geq \frac{A_r - 2\cosh(1)A_l}{\sinh(1)}N(t),\\ &\left|G*\overset{\vee}{B}(t,x)\right| \geq A_l N(t).\end{aligned} \tag{45}$$

Let $\overset{\vee}{c}(t,x) = G*\overset{\vee}{B}(t,x) + \overset{\vee}{g}(t,x)$, where $\overset{\vee}{c}(t,x) = \begin{pmatrix}u\\v\\w\end{pmatrix}(t,x)$, and in turn those provide

$$\begin{aligned}\overset{\vee}{c}(t,x) &\leq \left[2(1+\cosh(1)) + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l)\right] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ \partial_x \overset{\vee}{c}(t,x) &\leq \frac{\sinh(2) + 2A_l\cosh(1) - A_r}{\sinh(1)} \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])},\\ \partial^2_{xx} \overset{\vee}{c}(t,x) &\leq \left[2(1+\cosh(1)) + 1 + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l)\right] \times \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}.\end{aligned} \tag{46}$$

Now, if ω is the flow of $\overset{\vee}{c}$, ω is C^1, and since $\overset{\vee}{c} \geq 0$, $\omega(\cdot,t,x)$ is nondecreasing. This allows us to define the entrance time and then the operator S as follows. Let $e(t,x) = \min\{s\in[0,t] \mid \omega(s,t,x) = 0\}$.

Now, for for all $(t,x)\in[0,T]\times[0,1]$, $S(y,N) = (\tilde{y},\widetilde{N})$ with the following:

(1) *if $x \geq \omega(t,0,0)$,*

$$\begin{aligned}y(t,x) &= y_0(\omega(0,t,x))\exp\left(\int_0^t \overset{\vee}{b}(r,\omega(r,t,x))\,dr\right) + \int_0^t \overset{\vee}{f}(r,\omega(r,t,x)) \times \exp\left(\int_r^t \overset{\vee}{b}(r',\omega(r',t,x))\,dr'\right)dr,\end{aligned} \tag{47}$$

(2) *if $x \leq \omega(t,0,0)$,*

$$\begin{aligned}y(t,x) &= y_0(0)e^{Me(t,x)}\exp\left(\int_0^t \overset{\vee}{b}(r,\omega(r,t,x))\,dr\right) + \int_{e(t,x)}^t \overset{\vee}{f}(r,\omega(r,t,x)) \times \exp\left(\int_r^t \overset{\vee}{b}(r',\omega(r',t,x))\,dr'\right)dr,\end{aligned} \tag{48}$$

(3) $N(t) = \|\tilde{y}(t,\cdot)\|_{C^0([0,1])}$.

Lemma 15. *(1) The operator S maps X to X.*

(2) The family $S(X)$ is uniformly bounded and equicontinuous.

(3) S is continuous w.r.t. the uniform topology.

The proof is very similar to [10], except for the state y here is a three-component vector and the proof is omitted.

Now, we can apply Schauder's fixed point theorem to S and get (y, N) fixed point of S.

3.2. Stabilization and Global Existence

Theorem 16. *For any $y_0 \in C^0([0,1]) \times C^0([0,1]) \times C^0([0,1])$, there exists $y \in C^0(\Omega_T) \times C^0([0,T], C^2([0,1]))$ a weak solution of (39) satisfying*

$$\forall x \in [0,1] \quad y(0,x) = y_0(x). \tag{49}$$

Furthermore, any maximal solution of (39) and (41) is global, and if we let

$$\begin{aligned} k &= \max\left(2(1+\cosh(1)) + 1 + \frac{\cosh(1)}{\sinh(1)}(A_r + A_l), \right. \\ &\qquad \left. \frac{\sinh(2) + 2A_l\cosh(1) - A_r}{\sinh(1)}\right), \\ \tau &= \frac{1}{\|M\|_3} \ln\left(\frac{\|M\|_3^2}{8k^3\|y_0\|^2_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}}\right), \end{aligned} \tag{50}$$

then we have

$$\begin{aligned} \forall t \geq \tau \quad & \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\ & \leq \frac{\|M\|_3}{2k}\left(1 + 2k\|M\|_3\right) e^{\|M\|_3(t-x)}. \end{aligned} \tag{51}$$

To finish the proof of Theorem (39), we have to prove the global existence of a maximal solution and the estimate (51).

Proof. First, we rewrite (46) as the following:

$$\begin{aligned} \forall (t,x) \in \Omega_T \quad & \overset{\vee}{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])}, \\ & \partial_x \overset{\vee}{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])}, \\ & \partial_{xx} \overset{\vee}{c}(t,x) \leq k\|y(t,\cdot)\|_{C^0([0,1])}, \end{aligned} \tag{52}$$

where $k = \max(2(1+\cosh(1)) + 1 + (\cosh(1)/\sinh(1))(A_r + A_l), (\sinh(2) + 2A_l\cosh(1) - A_r)/\sinh(1))$.

For y is the solution of the transport (39) and it satisfies

$$\begin{aligned} y(t,x) &= y(s,\omega(s,t,x)) \exp\left(\int_0^t \overset{\vee}{b}(r,\omega(r,t,x))\,dr\right) \\ &\quad + \int_0^t \overset{\vee}{f}(r,\omega(r,t,x)) \\ &\qquad \times \exp\left(\int_r^t \overset{\vee}{b}\left(r',\omega\left(r',t,x\right)\right)dr'\right)dr. \end{aligned} \tag{53}$$

Combining those facts, we get for $t \geq s$ the following:

$$\begin{aligned} & |y(t,x)| \\ & \quad \leq |y(s,\omega(s,t,x))| \left(1 + 4k^2|y(s,\omega(s,t,x))|t\right) \\ & \qquad \times \exp\left(2\int_s^t k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}dr\right). \end{aligned} \tag{54}$$

We have also imposed $y(t,0) = y(s,0)e^{M(t-s)}$ and thanks to the existence theorem that a maximal solution of the closed loop system is global. To get a more precise statement, we consider all the between time t and s, and we obtain.

For $0 \leq s \leq t$,

$$\begin{aligned} & \|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\ & \leq \|y(s,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\ & \times \max\left[e^{\|M\|_3(r-x)}\left(1 + 4k^2\|y(s,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r\right)\right] \\ & \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}d\alpha\right). \end{aligned} \tag{55}$$

We define

$$\begin{aligned} g(r) &= \left[e^{\|M\|_3(r-s)}\left(1 + 4k^2\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r\right)\right] \\ &\quad \times \exp\left(2k\int_s^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}d\alpha\right), \end{aligned} \tag{56}$$

and we set $g(r) = g_1(r) + g_2(r)$, where

$$\begin{aligned} g_1(r) &= e^{\|M\|_3(r-s)} \\ &\quad \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right), \\ g_2(r) &= e^{\|M\|_3(r-s)}4k^2\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}r \\ &\quad \times \exp\left(2k\int_r^t \|y(\alpha,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right). \end{aligned} \tag{57}$$

Then, we have

$$\begin{aligned} & g'(r) \\ & \quad = \left(\|M\|_3 - 2k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right) g_1(r) \\ & \qquad + \left(\frac{1}{r} + \|M\|_3 - 2k\|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])}\right) \\ & \qquad \times g_2(r), \end{aligned} \tag{58}$$

as long as the quantity $\|y(r,\cdot)\|_{C^0([0,1])}$ is not equal to zero, it strictly decreases, so if $\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} >$

$\|M\|_3/2k$, for t small enough $\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \geq \|M\|_3/2k$, and we have the following.

$$\begin{aligned}
&\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\
&\quad \leq \|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\
&\qquad \times \left(1 + 4k^2\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} t\right) e^{\|M\|_3 t}.
\end{aligned} \tag{59}$$

If we define $\tau = (1/\|M\|_3)\ \ln(\|M\|_3^2/8k^3\|y_0\|^2_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])})$, we get that

$$\|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \frac{\|M\|_3}{2k}. \tag{60}$$

This provides $\tau \leq s \leq t$, the inequality (which was clear when $\|y_0\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \leq \|M\|_3/2k$)

$$\begin{aligned}
&\|y(t,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\
&\quad \leq \|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} \\
&\qquad \times \left(1 + 4k^2\|y(\tau,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} t\right) \\
&\qquad \times \exp\left(2k\int_x^t \|y(r,\cdot)\|_{C^0([0,1])\times C^0([0,1])\times C^0([0,1])} dr\right) \\
&\quad \leq \frac{\|M\|_3}{2k}\left(1 + 2k\|M\|_3\right) e^{\|M\|_3(t-x)}.
\end{aligned} \tag{61}$$

□

4. Blow-Up Phenomena

In this section, we present a result with the initial data and boundary profiles under a special condition that ensure strong solutions to following system blow-up in finite time as follows:

$$\begin{aligned}
&\partial_t m - (u+v+w)(t,x)\,\partial_x m \\
&\quad = \left(2\left(p + G * B\right) + v_x + w_x\right)(t,x)\, m \\
&\qquad + \left(n v_x + l w_x\right)(t,x), \\
&\partial_t n - (u+v+w)(t,x)\,\partial_x n \\
&\quad = \left(2\left(q + G * B\right) + u_x + w_x\right)(t,x)\, n \\
&\qquad + \left(m u_x + l w_x\right)(t,x), \\
&\partial_t l - (u+v+w)(t,x)\,\partial_x l \\
&\quad = \left(2\left(r + G * B\right) + u_x + v_x\right)(t,x)\, l \\
&\qquad + \left(m u_x + n v_x\right)(t,x), \\
&m(0,\cdot) = m_0, \quad n(0,\cdot) = n_0, \quad l(0,\cdot) = l_0, \quad x \in [0,1], \\
&m(t,0) = m_l = m(t,1) = m_r, \quad t \in [0,T], \\
&n(t,0) = n_l = n(t,1) = n_r, \quad t \in [0,T], \\
&l(t,0) = l_l = l(t,1) = l_r, \quad t \in [0,T],
\end{aligned} \tag{62}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$, $l = w - w_{xx}$. This imply that

$$\begin{aligned}
u(t,0) &= u(t,1), \\
v(t,0) &= v(t,1), \\
w(t,0) &= w(t,1).
\end{aligned} \tag{63}$$

From Definition 3, we can also define

$$\begin{aligned}
&\omega_t = -(u+v+w)\left(t, \omega(t,y)\right), \\
&\qquad (t,x) \in [0,T]\times[0,1], \\
&\omega(0,x)\,1 = x, \quad x \in [0,1];
\end{aligned} \tag{64}$$

where u, v, and w denote the solution to (62). Applying classical results in the theory of ordinary differential equations, one can obtain a result on which is crucial in studying blow-up phenomena.

From (62), we obtain that

$$\begin{aligned}
m_t - \partial_x(mu + mv + mw) &= m u_x + n v_x + l w_x, \\
n_t - \partial_x(mu + nv + mw) &= m u_x + n v_x + l w_x, \\
l_t - \partial_x(lu + lv + lw) &= m u_x + n v_x + l w_x.
\end{aligned} \tag{65}$$

Lemma 17. *Let $u, v, w \in C([0,T);H^s)\cap C^1([0,T);H^{s-1})$, $(s \geq 2)$, then (64) has a unique solution $\omega \in C([0,T)\times[0,1])$. Moreover, the map $\omega(t,\cdot)$ is an increasing diffeomorphism with*

$$\begin{aligned}
\omega_x(t,x) = \exp&\left\{-\int_0^t \left(u_x(s,\omega(s,x))\right.\right. \\
&\left.\left. + v_x(s,\omega(s,x)) + w_x(s,\omega(s,x))\,ds\right)\right\} \\
&> 0, \\
\omega_x(0,x) &= 1, \quad \forall x \in [0,1].
\end{aligned} \tag{66}$$

Proof. The proof is omitted here, one can see a similar proof in [12]. □

Now, we have the following lemma that the potential $m-n$, $n-l$, $m-l$ with compactly supported initial datum $m_0 - n_0$, $n_0 - l_0$, $m_0 - l_0$ also has compact x support as long as it exists.

Lemma 18. *Assume that $u_0(t,x), v_0(t,x), w_0(t,x) \in H^s \times H^s$ with $s > (3/2)$, (u,v,w) is the corresponding solution, if $m_0 - n_0, n_0 - l_0, m_0 - l_0$ has compact support, then $m-n, n-l, m-l$ also has compact support, moreover, we can obtain that*

$$\begin{aligned}
\|m(t,\cdot) - n(t,\cdot)\| &\leq e^{3KT}\left\|m_0(\cdot) - n_0(\cdot)\right\|, \\
\|m(t,\cdot) - l(t,\cdot)\| &\leq e^{3KT}\left\|m_0(\cdot) - l_0(\cdot)\right\|, \\
\|n(t,\cdot) - l(t,\cdot)\| &\leq e^{3KT}\left\|n_0(\cdot) - l_0(\cdot)\right\|.
\end{aligned} \tag{67}$$

Proof. Since

$$\begin{aligned}
&\frac{d}{dt}\left(m\left(t,\omega\left(t,x\right)\right)\omega_x\right)\\
&\quad=\left(m_t+m_x\omega_t\right)\omega_x+m\omega_{xt}\\
&\quad=\left[m_t-\partial_x\left(mu+mv+mw\right)\right]\omega_x\\
&\quad=\left(mu_x+nv_x+lw_x\right)\omega_x.
\end{aligned}\tag{68}$$

Similarly,

$$\begin{aligned}
&\frac{d}{dt}\left(n\left(t,\omega\left(t,x\right)\right)\omega_x\right)=\left(mu_x+nv_x+lw_x\right)\omega_x,\\
&\frac{d}{dt}\left(l\left(t,\omega\left(t,x\right)\right)\omega_x\right)=\left(mu_x+nv_x+lw_x\right)\omega_x.
\end{aligned}\tag{69}$$

So it follows that

$$\begin{aligned}
&\frac{d}{dt}\left(\left(m\left(t,\omega\left(t,x\right)\right)-n\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right)=0,\\
&\frac{d}{dt}\left(\left(m\left(t,\omega\left(t,x\right)\right)-l\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right)=0,\\
&\frac{d}{dt}\left(\left(n\left(t,\omega\left(t,x\right)\right)-l\left(t,\omega\left(t,x\right)\right)\right)\omega_x\right)=0.
\end{aligned}\tag{70}$$

We obtain

$$\begin{aligned}
\left[m\left(t,\omega\left(t,x\right)\right)-n\left(t,\omega\left(t,x\right)\right)\right]\omega_x&=m_0\left(x\right)-n_0\left(x\right),\\
\left[m\left(t,\omega\left(t,x\right)\right)-l\left(t,\omega\left(t,x\right)\right)\right]\omega_x&=m_0\left(x\right)-l_0\left(x\right),\\
\left[n\left(t,\omega\left(t,x\right)\right)-l\left(t,\omega\left(t,x\right)\right)\right]\omega_x&=n_0\left(x\right)-l_0\left(x\right).
\end{aligned}\tag{71}$$

From Lemma 17, we have

$$\begin{aligned}
\omega_x=\exp\Big(-\int_0^t &u_x\left(s,\omega\left(s,x\right)\right)\\
&+v_x\left(s,\omega\left(s,x\right)\right)+w_x\left(s,\omega\left(s,x\right)\right)ds\Big).
\end{aligned}\tag{72}$$

If there exist four constants K, K_1, K_2, and K_3 such that $u_x\le K_1$, $v_x\le K_2$, $w_x\le K_3$, and $K=\max(K_1,K_2,K_3)$, we can get that

$$\begin{aligned}
&\left\|m\left(t,\cdot\right)-n\left(t,\cdot\right)\right\|_{L^\infty}\\
&\quad=\left\|m\left(t,\omega\left(t,\cdot\right)\right)-n\left(t,\omega\left(t,\cdot\right)\right)\right\|_{L^\infty}\\
&\quad=\Big\|\exp\Big(\int_0^t u_x\left(s,\omega\left(s,x\right)\right)+v_x\left(s,\omega\left(s,x\right)\right)\\
&\qquad\qquad+w_x\left(s,\omega\left(s,x\right)\right)ds\Big)\times\left(m_0\left(x\right)-n_0\left(x\right)\right)ds\Big\|_{L^\infty}\\
&\quad\le e^{3KT}\left\|m_0\left(\cdot\right)-n_0\left(\cdot\right)\right\|.
\end{aligned}\tag{73}$$

Similarly,

$$\begin{aligned}
\left\|m\left(t,\cdot\right)-l\left(t,\cdot\right)\right\|&\le e^{3KT}\left\|m_0\left(\cdot\right)-l_0\left(\cdot\right)\right\|,\\
\left\|n\left(t,\cdot\right)-l\left(t,\cdot\right)\right\|&\le e^{3KT}\left\|n_0\left(\cdot\right)-l_0\left(\cdot\right)\right\|.
\end{aligned}\tag{74}$$

□

Theorem 19. *Let $u_0(x), v_0(x)$, and $w_0(x)$, $s>3/2$ and $u(t,x), v(t,x), w(t,x)$ be the solution in (62) with time T. Then, T is finite if and only if*

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[u_x\left(x,t\right)\right]\right\}=-\infty\tag{75}$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[v_x\left(x,t\right)\right]\right\}=-\infty\tag{76}$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[w_x\left(x,t\right)\right]\right\}=-\infty.\tag{77}$$

Proof. Let $u_0(x), v_0(x)$, and $w_0(x)$, $s>3/2$ and $u(t,x), v(t,x), w(t,x)$ be the solution (62) with time T. We know that $u(t,0)=u(t,1), v(t,0)=v(t,1)$, and $w(t,0)=w(t,1)$. By the definition of m, n, and l, we have

$$\begin{aligned}
\|m\|_{L^2}^2&=\int_0^1\left(u-u_{xx}\right)^2dx=\int_0^1\left(u^2+2u_x^2+u_{xx}^2\right)dx,\\
\|n\|_{L^2}^2&=\int_0^1\left(v-v_{xx}\right)^2dx=\int_0^1\left(v^2+2v_x^2+v_{xx}^2\right)dx,\\
\|l\|_{L^2}^2&=\int_0^1\left(w-w_{xx}\right)^2dx=\int_0^1\left(w^2+2w_x^2+w_{xx}^2\right)dx.
\end{aligned}\tag{78}$$

Hence, $\|u\|_{H^2}^2\le\|m\|_{L^2}^2\le2\|u\|_{H^2}^2, \|v\|_{H^2}^2\le\|n\|_{L^2}^2\le2\|v\|_{H^2}^2, \|w\|_{H^2}^2\le\|l\|_{L^2}^2\le2\|w\|_{H^2}^2$.

Multiplying the first equation by m, the second one by n, and the third one by l, after integration by parts and adding up the results, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left(m^2+n^2+l^2\right)dx\\
&=\int_0^1\Big[\left(2u_x+v_x+w_x\right)m^2+\left(2v_x+u_x+w_x\right)n^2\\
&\qquad+\left(2w_x+u_x+v_x\right)l^2+\left(mm_x+nn_x+ll_x\right)\\
&\qquad\times\left(u+v+w\right)+mn\left(u_x+v_x\right)\\
&\qquad+lm\left(u_x+w_x\right)+\ln\left(v_x+w_x\right)\Big]dx\\
&=\int_0^1\Big[\Big(2u_x+v_x+w_x-\frac{1}{2}u_x-\frac{1}{2}v_x-\frac{1}{2}w_x\Big)m^2
\end{aligned}$$

$$
\begin{aligned}
&+\left(2v_x+u_x+w_x-\frac{1}{2}u_x-\frac{1}{2}v_x-\frac{1}{2}w_x\right)n^2\\
&+\left(2w_x+u_x+v_x-\frac{1}{2}u_x-\frac{1}{2}v_x-\frac{1}{2}w_x\right)l^2\\
&+mn\left(u_x+v_x\right)+lm\left(u_x+w_x\right)\ln\left(v_x+w_x\right)\Big]\,dx\\
=&\int_0^1\left[\left(\frac{3}{2}u_x+\frac{1}{2}v_x+\frac{1}{2}w_x\right)m^2+\left(\frac{3}{2}v_x+\frac{1}{2}u_x+\frac{1}{2}w_x\right)n^2\right.\\
&+\left(\frac{3}{2}w_x+\frac{1}{2}u_x+\frac{1}{2}v_x\right)l^2+mn\left(u_x+v_x\right)\\
&\left.+lm\left(u_x+w_x\right)+\ln\left(v_x+w_x\right)\right]dx\\
=&\int_0^1\left[\frac{1}{2}\left(u_x+v_x+w_x\right)\left(m^2+n^2+l^2\right)\right.\\
&+m^2u_x+n^2v_x+l^2w_x+mn\left(u_x+v_x\right)\\
&\left.+lm\left(u_x+w_x\right)+\ln\left(v_x+w_x\right)\right]dx\\
=&\int_0^1\left[\frac{1}{2}\left(u_x+v_x\right)(m+\mathrm{n})^2+\frac{1}{2}\left(u_x+w_x\right)(m+l)^2\right.\\
&+\frac{1}{2}\left(v_x+w_x\right)(l+n)^2+\frac{1}{2}u_xm^2\\
&\left.+\frac{1}{2}v_xn^2+\frac{1}{2}w_xl^2\right]dx
\end{aligned}
\tag{79}
$$

So, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1\left(m^2+n^2+l^2\right)dx\\
&\le\left(K_1+K_2+K_3\right)\\
&\quad\times\int_0^1\left[(m+n)^2+(m+l)^2\right.\\
&\quad\left.+(l+n)^2+m^2+n^2+l^2\right]dx\\
&\le 5\left(K_1+K_2+K_3\right)\int_0^1\left(m^2+n^2+l^2\right)dx.
\end{aligned}
\tag{80}
$$

By Gronwall's inequality, we get

$$
\begin{aligned}
&\|u\|_{H^2}^2+\|v\|_{H^2}^2+\|w\|_{H^2}^2\\
&\le\int_0^1\left(m^2+n^2+l^2\right)dx\\
&\le\exp\left[5T\left(K_1+K_2+K_3\right)\right]\\
&\quad\times\int_0^1\left(m_0^2+n_0^2+l_0^2\right)dx\\
&\le 2\exp\left[5T\left(K_1+K_2+K_3\right)\right]\\
&\quad\times\left(\|u_0\|_{H^2}^2+\|v_0\|_{H^2}^2+\|w_0\|_{H^2}^2\right),
\end{aligned}
\tag{81}
$$

The above inequality, Soblev's embedding theorem, ensure that the solution $(u(t,\mathrm{x}), v(t,x), w(t,x))$ cannot blow up in finite time.

On the other hand, if

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[u_x(x,t)\right]\right\}=-\infty \tag{82}$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[v_x(x,t)\right]\right\}=-\infty \tag{83}$$

or

$$\liminf_{t\to T}\left\{\inf_{x\in[0,1]}\left[w_x(x,t)\right]\right\}=-\infty, \tag{84}$$

then the solution will blow up in finite time. □

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Research Article

Positive Solutions of a Singular Third-Order m-Point Boundary Value Problem

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This paper is concerned with the existence and nonexistence of positive solutions to the singular third-order m-point boundary value problem $u'''(t) + a(t)f(u(t)) = 0$, $0 < t < 1$, $u(0) = u'(0) = 0$, $u'(1) - \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) = \lambda$, where $\xi_i \in [0, 1)$, $\alpha_i \in [0, \infty)$ $(i = 1, 2, \ldots, m-2)$ are constants, $\lambda \in (0, 1)$ is a parameter, $f : [0, \infty) \to [0, \infty)$ is continuous and $a(\cdot)$ is allowed to be singular at $t = 0$ and $t = 1$. The results here essentially extend and improve some known results.

1. Introduction and the Main Results

Singular boundary value problems for nonlinear ordinary differential equations arise in a variety of areas of applied mathematics, physics, chemistry, and so on. For earlier works, see [1–3]. Nonsingular third-order multipoint boundary value problems have been studied by many authors by using different type of techniques, see, for example, [4–9] and the references therein. In recent years, singular third-order multipoint boundary value problems have also received much attention, see [10–12].

Very recently, motivated by Ma [13], Sun [14] considered the third-order three-point boundary value problem

$$\begin{aligned} &u'''(t) + a(t) f(u(t)) 0 = 0, \quad t \in (0, 1), \\ &u(0) = u'(0) = 0, \qquad u'(1) - \alpha u'(\eta) = \lambda, \end{aligned} \tag{P_λ}$$

where $\eta \in (0, 1)$, $\alpha \in [0, 1/\eta)$ are constants and $\lambda \in (0, \infty)$ is a parameter. Under the following assumptions:

(H_1) $a \in C((0, 1), [0, \infty))$ and $0 < \int_0^1 (1 - s)sa(s)ds < \infty$;

(H_2) $f \in C([0, \infty), [0, \infty))$;

(H_4^*) f is superlinear, that is, $f_0 = 0$, $f_\infty = \infty$;

(H_5^*) f is sublinear, that is, $f_0 = \infty$, $f_\infty = 0$,

where $f_0 = \lim_{r \to 0^+}(f(r)/r)$, $f_\infty = \lim_{r \to +\infty}(f(r)/r)$.

By using Guo-Krasnosel'skii fixed point theorem, the author established the following results.

Theorem A (see [14, Theorem 3.1]). *Suppose that (H_1), (H_2), and (H_4^*) hold. Then the problem (P_λ) has at least one positive solution for λ small enough and has no positive solution for λ large enough.*

Theorem B (see [14, Theorem 3.2]). *Suppose that (H_1), (H_2), and (H_4^*) hold. If f nondecreasing, then there exists a positive constant λ^* such that the problem (P_λ) has at least one positive solution for $\lambda \in (0, \lambda^*)$ and has no positive solution for $\lambda \in (\lambda^*, \infty)$.*

Theorem C (see [14, Theorem 3.3]). *Suppose that (H_1), (H_2), and (H_5^*) hold. Then the problem (P_λ) has at least one positive solution for any $\lambda \in (0, \infty)$.*

Being directly inspired by the previously mentioned works, we will consider the existence and nonexistence of positive solutions to the following third-order m-point BVP:

$$u'''(t) + a(t) f(u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) = \lambda, \tag{2_λ}$$

where $\xi_i \in [0, 1)$, $\alpha_i \in [0, \infty)$ $(i = 1, 2, \ldots, m-2)$ are constants and $\lambda \in (0, 1)$ is a parameter, $a(\cdot)$ is allowed to be singular at $t = 0$ and $t = 1$. Here, the solution u of BVP of (1),

(2_λ) is called positive solution if $u(t)$ is positive on $(0,1)$ and satisfies (1) and the boundary conditions (2_λ).

We assume that (H_1), (H_2) hold and make the following additional assumptions:

(H_3) $0 < \sum_{i=1}^{m-2} \alpha_i \xi_i < 1$;

(H_4) $\lim_{u\to 0^+} \sup (f(u)/u) < \Lambda_1$ and $\lim_{u\to +\infty} \inf (f(u)/u) > \Lambda_2$;

(H_5) $\lim_{u\to 0^+} \inf (f(u)/u) > 0$ and $\lim_{u\to +\infty} \sup (f(u)/u) < L$,

where

$$\Lambda_1 = 2(1+\|h\|)^{-1} L, \quad \|h\| = \frac{1}{2}\gamma,$$
$$L = \left[\left(1+\gamma\sum_{i=1}^{m-2}\alpha_i\right)\int_0^1 (1-s)\, s a(s)\, ds\right]^{-1},$$
$$\gamma = \left(1-\sum_{i=1}^{m-2}\alpha_i\xi_i\right)^{-1}, \tag{2}$$
$$\Lambda_2 = \frac{1}{4}\left[\theta^2 \int_\theta^1 (1-s)\, s a(s)\, ds\right]^{-1}.$$

From (H_1), we know that there exists $t_0 \in (0,1)$ such that $a(t_0) > 0$. Let θ satisfy $0 < \theta < t_0 < 1$.

Our main results are the following.

Theorem 1. *Let (H_1)–(H_4) hold. Then there exists a positive number λ^* such that BVP of (1), (2_λ) has at least one positive solution for $\lambda \in (0,\lambda^*)$ and none for $\lambda \in (\lambda^*,\infty)$.*

Theorem 2. *Let (H_1)–(H_3) and (H_5) hold. Then BVP of (1), (2_λ) has at least one positive solution for any $\lambda \in (0,\infty)$.*

The proof of previous theorems is based on the Schauder fixed-point theorem.

Remark 3. BVP (P_λ) is a special case of (1), (2_λ) with $\alpha_1 = \alpha$, $\xi_1 = \eta$, and $\alpha_2 = \cdots = \alpha_{m-2} = 0$, $\xi_2 = \cdots = \xi_{m-2} = 0$.

Remark 4. (H_4) allows but do not require the nonlinearity $f(u)$ to be sublinear at zero and infinity; (H_5) allows but do not require the nonlinearity $f(u)$ to be sublinear at zero and infinity.

Remark 5. We do not assume any monotonicity condition on the nonlinearity as in [14]. We find that the nondecreasing condition of f can be removed from Theorem 3.2 in [14], and the same result is obtained in Theorem 1.

Remark 6. It is obvious that Theorem 1 is an extension and complement of Theorems 3.1 and 3.2; furthermore, Theorem 2 is also an extension of Theorem 3.3 in [14].

2. Preliminary Lemmas

In this section, we present some notation and preliminary lemmas.

Let $C^+[0,1] = \{u \in C[0,1] \mid u(t) \geq 0,\ t \in [0,1]\}$ equipped with the norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$.

Lemma 7 (see [15, Lemma 2.1]). *Suppose that $\phi \in \{\varphi \in L^1_{\text{loc}}[0,1] \mid \int_0^1 t(1-t)|\varphi(t)|dt < \infty\}$.*

(i) *Then $\int_0^t s\phi(s)ds$, $\int_t^1 (1-s)\phi(s)ds \in L^1(0,1)$ and*

$$\int_0^1\int_0^t s\phi(s)\, ds\, dt = \int_0^1\int_t^1 (1-s)\phi(s)\, ds\, dt = \int_0^1 s(1-s)\phi(s)\, ds. \tag{3}$$

(ii) *Let $r \in (0,1)$. Then*

$$\lim_{t\to 0^+} v(t)\int_t^1 (1-s)\phi(s)\, ds = 0, \tag{4}$$

for every $v \in C^1[0,r]$ with $v(0)=0$, and

$$\lim_{t\to 1^-} w(t)\int_0^t s\phi(s)\, ds = 0, \tag{5}$$

for every $w \in C^1[r,1]$ with $w(1)=0$.

Lemma 8. *Suppose that (H_1)–(H_3) hold, then BVP*

$$u'''(t) + a(t) f(u(t)) = 0, \quad 0<t<1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{m-2}\alpha_i u'(\xi_i) = 0, \tag{6}$$

has a unique nonnegative solution $u \in C^1[0,1] \cap C^3(0,1)$ which can be represented as

$$u(t) = \int_0^1 G(t,s)\, a(s) f(u(s))\, ds + \frac{1}{2}\gamma\delta_f t^2, \tag{7}$$

where

$$G(t,s) := \frac{1}{2}\begin{cases}\left(2t-t^2-s\right)s, & 0\leq s\leq t\leq 1,\\ (1-s)t^2, & 0\leq t\leq s\leq 1,\end{cases}$$
$$G_1(t,s) := \begin{cases}(1-t)s, & 0\leq s\leq t\leq 1,\\ (1-s)t, & 0\leq t\leq s\leq 1,\end{cases} \tag{8}$$

and $\delta_f = \sum_{i=1}^{m-2}\alpha_i \int_0^1 G_1(\xi_i,s)a(s)f(u(s))ds$.

Proof. The proof of the uniqueness is standard and hence is omitted here. Now we prove the existence of the solution.

From (H_1)–(H_3) and Lemma 7, we conclude that the integration in (7) is well defined. Let $v = u'$; then BVP (6) may be reduced to boundary value problems

$$v''(t) + a(t) f(u(t)) = 0, \quad 0<t<1, \tag{9}$$

$$v(0) = 0, \qquad v(1) - \sum_{i=1}^{m-2}\alpha_i v(\xi_i) = 0, \tag{10}$$

$$u'(t) = v(t), \quad 0<t<1, \tag{11}$$

$$u(0) = 0. \tag{12}$$

We claim that (9), (10) have a nonnegative solution v which can be represented as

$$v(t) = \int_0^1 G_1(t,s)\,a(s)\,f(u(s))\,ds + \gamma\delta_f t. \tag{13}$$

In fact, from (H_1)–(H_3) and Lemma 7, for each $r \in (0,1)$, $sa(s)f(u(s)) \in L^1[0,r]$ and $(1-s)a(s)f(u(s)) \in L^1[r,1]$. Combining the continuity of $sa(s)f(u(s))$ and $(1-s)a(s)f(u(s))$, we have

$$\begin{gathered}\int_0^t sa(s)\,f(u(s))\,ds \in C^1(0,r], \\ \int_t^1 (1-s)\,a(s)\,f(u(s))\,ds \in C^1[r,1).\end{gathered} \tag{14}$$

Thus $v(t) \in C^1(0,1)$. Moreover

$$\begin{aligned}v'(t) = &-\int_0^t sa(s)\,f(u(s))\,ds \\ &+ \int_t^1 (1-s)\,a(s)\,f(u(s))\,ds + \gamma\delta_f.\end{aligned} \tag{15}$$

Similarity, $v'(t) \in C^1(0,1)$. From (15), we get $v''(t) = -a(t)f(u(t))$, $t \in (0,1)$.

By Lemma 7, we have from (13) that

$$\begin{aligned}v(0) &= \lim_{t\to 0^+} v(t) \\ &= \lim_{t\to 0^+}(1-t)\int_0^t sa(s)\,f(u(s))\,ds \\ &\quad + \lim_{t\to 0^+} t\int_t^1 (1-s)\,a(s)\,f(u(s))\,ds \\ &\quad + \lim_{t\to 0^+} t\gamma\delta_f = 0.\end{aligned} \tag{16}$$

Again applying (13), we have

$$v(1) = \lim_{t\to 1^-} v(t) = \gamma\delta_f. \tag{17}$$

This together with (13) implies that $v(1) = \sum_{i=1}^{m-2}\alpha_i v(\xi_i)$. The claim is proved.

By Lemma 7, we obtain from (11), (12), and (13) that

$$\begin{aligned}u(t) &= \int_0^t v(\tau)\,d\tau \\ &= \int_0^t\int_0^\tau (1-\tau)\,sa(s)\,f(u(s))\,ds\,d\tau \\ &\quad + \int_0^t\int_\tau^1 (1-s)\,\tau a(s)\,f(u(s))\,ds\,d\tau + \frac{1}{2}\gamma\delta_f t^2 \\ &= \frac{1}{2}\int_0^t \left(2t-t^2-s\right) sa(s)\,f(u(s))\,ds \\ &\quad + \frac{1}{2}\int_t^1 (1-s)\,t^2 a(s)\,f(u(s))\,ds + \frac{1}{2}\gamma\delta_f t^2.\end{aligned} \tag{18}$$

It is easy to see that $u \in C^1[0,1] \cap C^3(0,1)$, and moreover, u is a nonnegative solution of the BVP (6).

The proof is complete. □

Lemma 9 (see [14, Lemmas 2.2 and 2.3]). *For any $(t,s) \in [0,1]\times[0,1]$, one has*

(i) *$q(t)\,G(1,s) \le G(t,s) \le G(1,s) = (1/2)(1-s)s$, where $q(t) = t^2$,*

(ii) *$(\partial/\partial t)\,G(t,s) = G_1(t,s)$, and $0 \le G_1(t,s) \le G_1(s,s) = (1-s)s$.*

Lemma 10. *Suppose that (H_1)–(H_3) hold; then the unique nonnegative solution u of (6) satisfies*

$$\min_{t\in[\theta,1]} u(t) \ge \theta^2\,\|u\|. \tag{19}$$

The proof is similar to Lemma 2.4 in [14].

Lemma 11. *Suppose that (H_1)–(H_3) hold. Let $j \in \{1,2,\ldots,m-2\}$ and $r \in (-\infty,0)$. Then BVP*

$$w''(t) = 0, \quad 0 < \tau < t < 1, \tag{20}$$

$$w(\tau) = 0, \qquad w(1) - \sum_{i=j}^{m-2}\alpha_i w(\xi_i) = r \tag{21}$$

has a unique solution w satisfying $w(t) \le 0$ on $[\tau,1]$.

Proof. From (20) and (21), we obtain $w(t) = (t-\tau)w'(\tau)$. Again applying (21), we have

$$\begin{gathered}(1-\tau)\,w'(\tau) - \sum_{i=j}^{m-2}\alpha_i(\xi_i-\tau)\,w'(\tau) = r, \\ w'(\tau)\left(1-\tau-\sum_{i=j}^{m-2}\alpha_i(\xi_i-\tau)\right) = r.\end{gathered} \tag{22}$$

Now set $d := 1-\tau-\sum_{i=j}^{m-2}\alpha_i(\xi_i-\tau)$; then we have that

$$\begin{aligned}d &> 1-\tau-\sum_{i=j}^{m-2}\alpha_i(\xi_i-\tau\xi_i) \\ &= (1-\tau)\left(1-\sum_{i=j}^{m-2}\alpha_i\xi_i\right) > 0.\end{aligned} \tag{23}$$

This together with the fact that $r \in (-\infty,0)$ implies that $w'(\tau) < 0$. Thus $w(t) \le 0$ on $[\tau,1]$. The proof is complete. □

3. Proof of the Main Results

In this section, we will prove our main results.

Proof of Theorem 1. We divide the proof into three steps.

Step 1. We first prove the existence of positive solutions to (1), (2_λ) for sufficiently small $\lambda : \lambda > 0$.

Let h be the unique solution of

$$u'''(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) = 1. \tag{24}$$

Then $h(t) = (1/2)\gamma t^2$. Let $v = u - \lambda h$; then u is a positive solution of BVP (1), (2_λ) if and only if $v = u - \lambda h$ is a nonnegative solution of BVP

$$v'''(t) + a(t) f(v(t) + \lambda h(t)) = 0, \quad 0 < t < 1,$$
$$v(0) = v'(0) = 0, \qquad v'(1) - \sum_{i=1}^{m-2} \alpha_i v'(\xi_i) = 0. \tag{25}$$

Let $\widetilde{f}(x) = \sup_{0 \le s \le x} f(s)$. Since $\lim_{u \to 0^+} \sup(\widetilde{f}(u)/u) < \Lambda_1$; then there exists a positive number λ_1 such that

$$\widetilde{f}(\lambda_1 + \lambda_1 \|h\|) \le \Lambda_1 (\lambda_1 + \lambda_1 \|h\|) = 2\lambda_1 L. \tag{26}$$

Define a closed convex subset in $C^+[0,1]$ by

$$D = \{v \in C^+[0,1] \mid v(t) \le \lambda_1,\ t \in [0,1]\} \tag{27}$$

and an operator $T : D \to C^+[0,1]$ by

$$Tv(t) := \int_0^1 G(t,s) a(s) f(v(s) + \lambda h(s)) ds + \frac{1}{2}\gamma \delta_f t^2, \tag{28}$$

where $\delta_f = \sum_{i=1}^{m-2} \alpha_i \int_0^1 G_1(\xi_i, s) a(s) f(v(s) + \lambda h(s)) ds$. Modeling the proof of Lemma 2.3 in [10], we can show that T is a completely continuous operator. From Lemma 8, we know that v is a nonnegative solution of (25) if and only if v is a fixed point of T.

Suppose that $\lambda < \lambda_1$; we claim that $T : D \to D$.

In fact, from Lemma 9 and (28), we have

$$\begin{aligned} 0 &\le Tv(t) \\ &\le \widetilde{f}(\lambda_1 + \lambda_1 \|h\|) \left[\frac{1}{2} \int_0^1 (1-s) s a(s) ds + \frac{1}{2}\gamma \sum_{i=1}^{m-2} \alpha_i \int_0^1 (1-s) s a(s) ds \right] \\ &= \frac{1}{2} \widetilde{f}(\lambda_1 + \lambda_1 \|h\|) L^{-1} \le \lambda_1. \end{aligned} \tag{29}$$

The claim is proved. Using the Schauder fixed point theorem, we conclude that T has a fixed point v in D, and then $u = v + \lambda h$ is a positive solution of (1), (2_λ).

Step 2. We verify that BVP of (1), (2_λ) has no positive solutions for λ large enough.

Suppose to the contrary that BVP of (1), (2_λ) has at least one positive solution for any $\lambda > 0$. Then there exist $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, with $\lim_{n\to\infty} \lambda_n = \infty$, such that for any positive integer n, BVP of (1), (2_λ) has a positive solution u_n. Thus $v_n = u_n - \lambda_n h$ is a nonnegative solution to (25). On the one hand, we have

$$\|v_n + \lambda_n h\| \ge \lambda_n \|h\| = \frac{1}{2}\gamma \lambda_n \longrightarrow \infty, \quad (n \longrightarrow \infty). \tag{30}$$

On the other hand, since $\lim_{u \to +\infty} \inf(f(u)/u) > \Lambda_2$, there exists $r_0 > 0$ such that $f(u) \ge \Lambda_2 u$, for any $u \in [r_0, \infty)$. Let n be large enough that $\theta^2 \|v_n + \lambda_n h\| \ge r_0$. By Lemma 10, we have

$$\inf_{t\in[\theta,1]} v_n(t) \ge \theta^2 \|v_n\|,$$
$$\inf_{t\in[\theta,1]} h(t) = \frac{1}{2}\gamma\theta^2 = \theta^2 \|h\|. \tag{31}$$

This implies that

$$\inf_{t\in[\theta,1]} (v_n(t) + \lambda_n h(t)) \ge \theta^2 (\|v_n\| + \lambda_n \|h\|) \ge \theta^2 \|v_n + \lambda_n h\|. \tag{32}$$

Thus

$$\begin{aligned} &\|v_n + \lambda_n h\| \\ &\ge \|v_n\| = \int_0^1 G(1,s) a(s) f(v_n(s) + \lambda_n h(s)) ds \\ &\quad + \frac{1}{2}\gamma \sum_{i=1}^{m-2} \alpha_i \int_0^1 G_1(\xi_i, s) a(s) f(v_n(s) + \lambda_n h(s)) ds \\ &\ge \int_\theta^1 G(1,s) a(s) f(v_n(s) + \lambda_n h(s)) ds \\ &\ge \frac{1}{2} \Lambda_2 (v_n(s) + \lambda_n h(s)) \int_\theta^1 (1-s) s a(s) ds \\ &= 2 \|v_n + \lambda_n h\|, \end{aligned} \tag{33}$$

which is a contradiction.

Step 3. Let $B = \{\lambda \mid$ BVP of (1), (2_λ) has at least one positive solution$\}$ and $\lambda^* = \sup B$; then $0 < \lambda^* < \infty$. We show that (1), (2_λ) have positive solution for any $\lambda \in (0, \lambda^*)$. From the definition of λ^*, we know that, for any $\lambda \in (0, \lambda^*)$, there exists $\widetilde{\lambda} > \lambda$ such that (1), (2_λ) have positive solution $u_{\widetilde{\lambda}}$.

Now we consider the following third-order m-point boundary value problem:

$$u'''(t) + a(t) (\widetilde{F}u)(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{m-2} \alpha_i u'(\xi_i) = \lambda, \tag{34}$$

where

$$(\widetilde{F}u)(t) = \begin{cases} f(u_{\widetilde{\lambda}}(t)), & \text{if } u(t) > u_{\widetilde{\lambda}}(t), \\ f(u(t)), & \text{if } 0 \le u(t) \le u_{\widetilde{\lambda}}(t), \\ f(0), & \text{if } u(t) < 0. \end{cases} \tag{35}$$

Since $\widetilde{F}$ is bounded, by Schauder fixed point theorem, the problem (34) has a solution u_λ.

By Lemma 8, u_λ satisfies

$$u_\lambda(t) = \int_0^1 G(t,s)\,a(s)\left(\widetilde{F}u\right)(s)\,ds + \frac{1}{2}\gamma t^2 \sum_{i=1}^{m-2} \alpha_i \int_0^1 G_1(\xi_i,s)\,a(s) \times \left(\widetilde{F}u\right)(s)\,ds + \frac{1}{2}\gamma\lambda t^2, \quad (36)$$

thus $u_\lambda \geq 0$.

Let $I = \{t \in (0,1] \mid u_\lambda(t) > u_{\widetilde{\lambda}}(t)\}$ and $\Omega = \{t \in (0,1] \mid w(t) > 0\}$, where $w(t) = u'_\lambda(t) - u'_{\widetilde{\lambda}}(t)$, $t \in [0,1]$. We will show that $I = \emptyset$. Noticing that, if $u'_\lambda(t) > u'_{\widetilde{\lambda}}(t)$ holds for any $t \in (0,1)$, combining with $u'_\lambda(0) = u'_{\widetilde{\lambda}}(0)$, we get $u_\lambda(t) > u_{\widetilde{\lambda}}(t)$. Thus we prove that $\Omega = \emptyset$; then we have $I = \emptyset$.

Suppose to the contrary that $\Omega \neq \emptyset$.

If $u'_\lambda(1) < u'_{\widetilde{\lambda}}(1)$, then, from $\Omega \neq \emptyset$, and the continuity of $w(t)$, there exists $(a,b) \subset \Omega$ such that $w'(a) = w(b) = 0$. Moreover, $w''(t) = 0$ in (a,b). Thus $w(t) \equiv 0$ in (a,b). This contradicts with the fact that $w(t) > 0$ in (a,b).

If $u'_\lambda(1) > u'_{\widetilde{\lambda}}(1)$, we claim that there exists $j \in \{1,2,\ldots,m-2\}$ such that $w(\xi_j) > 0$.

In fact, from the fact that $w(1) - \sum_{i=1}^{m-2}\alpha_i w(\xi_i) = \lambda - \widetilde{\lambda} < 0$ and $w(1) > 0$, we have that $\sum_{i=1}^{m-2}\alpha_i w(\xi_i) > 0$. Thus, there exists $j \in \{1,2,\ldots,m-2\}$ such that $w(\xi_j) > 0$.

Let $j_0 = \min\{j \mid j \in \{1,2,\ldots,m-2\}$ such that $w(\xi_j) > 0\}$; then, we only need to deal with the following four cases.

Case 1. $w(t) > 0$ in $(0,1)$. In this case, we have

$$\begin{gathered} w''(t) = 0, \quad 0 < t < 1, \\ w(0) = 0, \qquad w(1) - \sum_{i=1}^{m-2}\alpha_i w(\xi_i) = \lambda - \widetilde{\lambda} < 0. \end{gathered} \quad (37)$$

We easily verify that

$$w(t) = \left(\lambda - \widetilde{\lambda}\right)\gamma t < 0, \quad \text{in } (0,1). \quad (38)$$

This contradicts with the fact that $\Omega \neq \emptyset$.

Case 2. There exists $\tau \in (0,1)$ such that $w(\tau) = 0$ and $w(t) > 0$ in $(0,\tau)$. In this case, we have

$$\begin{gathered} w''(t) = 0, \quad 0 < t < \tau, \\ w(0) = 0, \qquad w(\tau) = 0. \end{gathered} \quad (39)$$

We easily obtain $w(t) \equiv 0$ in $[0,\tau]$, a contradiction again.

Case 3. There exists $\tau \in (0,1)$ such that $w(\tau) = 0$ and $w(t) > 0$ in $(\tau, 1]$. In this case, if $j_0 > 1$, then, for any $i \in \{1,2,\ldots,j_0 - 1\}$, we have $w(\xi_i) < 0$. Thus

$$\begin{aligned} r :&= w(1) - \sum_{i=j_0}^{m-2}\alpha_i w(\xi_i) \\ &= w(1) - \sum_{i=1}^{m-2}\alpha_i w(\xi_i) + \sum_{i=1}^{j_0-1}\alpha_i w(\xi_i) \\ &= \lambda - \widetilde{\lambda} + \sum_{i=1}^{j_0-1}\alpha_i w(\xi_i) \\ &\leq \lambda - \widetilde{\lambda} \\ &< 0. \end{aligned} \quad (40)$$

If $j_0 = 1$, then $r := w(1) - \sum_{i=j_0}^{m-2}\alpha_i w(\xi_i) = \lambda - \widetilde{\lambda} < 0$, and $w(t)$ satisfies

$$\begin{gathered} w''(t) = 0, \quad \tau < t < 1, \\ w(\tau) = 0, \qquad w(1) - \sum_{i=j_0}^{m-2}\alpha_i w(\xi_i) = r < 0. \end{gathered} \quad (41)$$

By Lemma 11, we also have $w(t) \leq 0$ in $[\tau, 1]$, a contradiction again.

Case 4. There exists $[a,b] \subset (0,1)$ such that $w(a) = w(b) = 0$ and $w(t) > 0$ in (a,b). The same as Case 2, we can lead to a contradiction.

Summarizing the previous discussion, we assert that $\Omega = \emptyset$; thus $I = \emptyset$. Up to now, the problem (1), (2_λ) has a solution u_λ. □

Proof of Theorem 2. Since $\lim_{u\to 0^+}\inf(f(u)/u) > 0$, there exists $\mu > 0$, and $r_1 > 0$ such that $f(u) \geq \mu u$, $u \in [0, r_1]$. Next we consider two cases: f is bounded or f is unbounded.

Case 1. Suppose that f is bounded, that is, $f(u) \leq M$, for all $u \in [0,\infty)$. By Schauder fixed point theorem the problem of (1), (2_λ) has a positive solution.

Case 2. If f is unbounded. Since $\lim_{u\to+\infty}\sup(f(u)/u) < L$, there exists a positive number r_2 such that $f(u) \leq Lu$, for $u \in [r_2, \infty)$. Since f is unbounded, for any $\lambda \in (0,\infty)$, we are able to choose

$$R_\lambda \geq \max\{2r_1, r_2, \lambda\|h\|\} \quad (42)$$

such that

$$f(u) \leq f(2R_\lambda), \quad \text{for } u \in [0, 2R_\lambda]. \quad (43)$$

Defining a closed convex subset in $C^+[0,1]$ by

$$D_1 = \{v \in C^+[0,1] \mid v(t) \leq R_\lambda,\ t \in [0,1]\}. \quad (44)$$

For each $v \in D_1$, we have $0 \le v + \lambda h \le 2R_\lambda$. By (28) and Lemma 9, we obtain that

$$\begin{aligned} Tv(t) &\le \frac{1}{2} f(2R_\lambda) \left(\int_0^1 (1-s) sa(s)\, ds \right. \\ &\qquad \left. + \gamma \sum_{i=1}^{m-2} \alpha_i \int_0^1 (1-s) sa(s)\, ds \right) \\ &\le \frac{1}{2} L \cdot 2R_\lambda L^{-1} = R_\lambda. \end{aligned} \tag{45}$$

That is, $Tv(t) \in D_1$. By using the Schauder fixed point theorem, we assert that T has a fixed point $v \in D_1$, and then $u = v + \lambda h$ is a positive solution of BVP of (1), (2_λ). □

4. Example

Consider the boundary value problem

$$u'''(t) + \frac{1}{t(1-t)} \cdot \frac{u^2(t) + u(t)}{u(t) + 176} (7 + \cos u(t)) = 0, \quad 0 < t < 1, \tag{46}$$

$$u(0) = u'(0) = 0, \qquad u'(1) - \sum_{i=1}^{4} i u'\left(\frac{1}{8i}\right) = \lambda, \tag{47_λ}$$

where $\alpha_i = i$, $\xi_i = (1/8i)$ $(i = 1, 2, 3, 4)$, $a(t) = 1/t(1-t)$, $t \in (0, 1)$, and $f(u) = ((u^2 + u)/(u + 176))(7 + \cos u)$, $u \in [0, +\infty)$. Obviously (H_1), and (H_2), and (H_3) hold. By calculating, we have $\gamma = 2$, $\|h\| = 1$, $L = 1/21$, $\Lambda_1 = 1/21$. Let $\theta = 1/4$, then $\Lambda_2 = 16/3$. We easily verify that

$$\lim_{u \to 0^+} \sup \frac{f(u)}{u} = \frac{1}{22} < \Lambda_1, \qquad \lim_{u \to +\infty} \inf \frac{f(u)}{u} = 6 > \Lambda_2, \tag{47}$$

that is, (H_4) is satisfied. Therefore, Theorem 1 now guarantees that there exists a positive number λ^* such that BVP of (46), (47_λ) has at least one positive solution for $\lambda \in (0, \lambda^*)$ and none for $\lambda \in (\lambda^*, \infty)$.

But we cannot apply Theorem B [14, Theorem 3.2]. In fact, f does not satisfy monotonicity condition. Moreover, condition (H_4^*) of Theorem B does not hold.

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Research Article

Existence Results for a Riemann-Liouville-Type Fractional Multivalued Problem with Integral Boundary Conditions

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We discuss the existence of solutions for a boundary value problem of Riemann-Liouville fractional differential inclusions of order $\alpha \in (2,3]$ with integral boundary conditions. We establish our results by applying the standard tools of fixed point theory for multivalued maps when the right-hand side of the inclusion has convex as well as nonconvex values. An illustrative example is also presented.

1. Introduction

In the last few decades, fractional calculus is found to be an effective modeling tool in many branches of physics, economics, and technical sciences [1–3]. A fractional-order differential operator is nonlocal in its character in contrast to its counterpart in classical calculus. It means that the future state of a dynamical system or process based on fractional-order derivative depends on both its current and past states. Thus, the application of fractional calculus in various materials and processes enables an investigator to study the complete behavior (ranging from past to current states) of such stuff. This is indeed an important feature that makes fractional-order models more realistic and practical than the integer-order models and has accounted for the popularity of the subject. For some recent development on the topic, see [4–17] and the references therein.

Differential inclusions appear in the mathematical modeling of certain problems in economics, optimal control, and so forth and are widely studied by many authors. Examples and details can be found in a series of papers [18–23] and the references cited therein.

In this paper, we study the following boundary value problem:

$$D_{0+}^{\alpha} x(t) \in F(t, x(t)), \quad 0 < t < 1, \; 2 < \alpha \leq 3,$$
$$x(0) = x'(0) = 0,$$
$$x(1) = \int_0^1 g(s)\, x(s)\, ds, \tag{1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$, $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and g is a continuous function.

Here we remark that the present work is motivated by a recent paper [17], where problem (1) is considered with F as single valued and the results on existence and nonexistence of positive solutions are obtained.

The main tools of our study include nonlinear alternative of Leray-Schauder type, a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps, and Covitz and Nadler's fixed point theorem for contraction multivalued maps. The application of these results is new in the framework of the problem at hand. We recall some preliminaries in Section 2 while the main results are presented in Section 3.

2. Preliminaries

2.1. Fractional Calculus. Let us recall some basic definitions of fractional calculus [1, 2].

Definition 1. The Riemann-Liouville derivative of fractional order q is defined as

$$D_{0+}^{q}g(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-q-1}g(s)\,ds,\qquad n-1<q<n,\quad n=[q]+1, \tag{2}$$

provided the integral exists, where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$I_{0+}^{q}g(t)=\frac{1}{\Gamma(q)}\int_{0}^{t}\frac{g(s)}{(t-s)^{1-q}}ds,\quad q>0, \tag{3}$$

provided the integral exists.

Definition 3. A function $x\in AC^{2}([0,1],\mathbb{R})$ is called a solution of problem (1) if there exists a function $v\in L^{1}([0,1],\mathbb{R})$ with $v(t)\in F(t,x(t))$, a.e. $[0,1]$ such that $D^{\alpha}x(t)=v(t)$, a.e. $[0,1]$ and $x(0)=x'(0)=0$, $x(1)=\int_{0}^{1}g(t)x(t)dt$.

Lemma 4 (see [17]). *Given $y\in L([0,1],\mathbb{R})$, then the unique solution of the problem*

$$\begin{gathered}D^{\alpha}x(t)+y(t)=0,\quad 0<t<1,\ 2<\alpha\leq 3,\\ x(0)=x'(0)=0,\\ x(1)=\int_{0}^{1}g(t)x(t)\,dt\end{gathered} \tag{4}$$

is given by

$$x(t)=\int_{0}^{1}G(t,s)\,y(s)\,ds, \tag{5}$$

where

$$G(t,s)=G_{1}(t,s)+G_{2}(t,s), \tag{6}$$

$$G_{1}(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1}, & 0\leq t\leq s\leq 1,\\ [t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}, & 0\leq s\leq t\leq 1,\end{cases} \tag{7}$$

$$\begin{gathered}G_{2}(t,s)=\frac{t^{\alpha-1}}{1-\sigma}\int_{0}^{1}G_{1}(\tau,s)\,g(\tau)\,d\tau,\\ \sigma=\int_{0}^{1}s^{\alpha-1}g(s)\,ds.\end{gathered} \tag{8}$$

Lemma 5 (see [17]). *The functions $G_1(t,s)$, $G(t,s)$ have the following properties:*

(i) $G_{1}(t,s)\geq 0$, *for all* $t,s\in[0,1]$;

(ii) $G_{1}(t,s)\leq((\alpha-1)s(1-s)^{\alpha-1})/\Gamma(\alpha)$ *for all* $t,s\in[0,1]$;

(iii) $G(t,s)\leq(((\alpha-1)s(1-s)^{\alpha-1})/\Gamma(\alpha))(1+(1/(1-\sigma))\int_{0}^{1}g(\tau)d\tau)$ *for all* $t,s\in[0,1]$.

2.2. Basic Concepts of Multivalued Maps. Let $(\mathcal{X},\|\cdot\|)$ be a normed space and let $H:\mathcal{X}\to\mathcal{P}(\mathcal{X})$ be a multivalued map. H is said to be

(i) convex (closed) valued if $H(x)$ is convex (closed) for all $x\in\mathcal{X}$;

(ii) bounded on bounded sets if $H(\mathbb{B})=\cup_{x\in\mathbb{B}}H(x)$ is bounded in $\mathcal{X}$ for all $\mathbb{B}\in\mathcal{P}_{b}(\mathcal{X})$, where $\mathcal{P}_{b}(\mathcal{X})=\{\mathcal{A}\in\mathcal{P}(\mathcal{X}):\mathcal{A}\text{ is bounded}\}$, (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|w|:w\in H(x)\}\}<\infty$);

(iii) upper semicontinuous (u.s.c.) on $\mathcal{X}$ if the set $H(a_0)$ is a nonempty closed subset of $\mathcal{X}$ for each $a_0\in\mathcal{X}$ and if for each open set M of $\mathcal{X}$ containing $H(a_0)$ there exists an open neighborhood M_0 of a_0 such that $H(M_0)\subseteq M$;

(iv) completely continuous if $H(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathcal{P}_{b}(\mathcal{X})$.

Definition 6. If the multivalued map H is completely continuous with nonempty compact values, then H is u.s.c. if and only if H has a closed graph; that is, $w_n\in H(x_n)$ imply that $w_*\in H(x_*)$ when $x_n\to x_*$, $w_n\to w_*$.

Definition 7. H has a fixed point if there is $x\in\mathcal{X}$ such that $x\in H(x)$. The fixed point set of the multivalued operator H will be denoted by $\operatorname{Fix}H$.

Definition 8. Let $\mathcal{P}_{cl}(\mathbb{R})=\{W\in\mathcal{P}(\mathbb{R}):W\text{ is closed}\}$. A multivalued map $H:[0;1]\to\mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if the function

$$t\longmapsto d(y,H(t))=\inf\{|y-z|:z\in H(t)\} \tag{9}$$

is measurable for every $y\in\mathbb{R}$.

In the sequel, by $C([0,1])$ we mean a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\|=\sup_{t\in[0,1]}|x(t)|$ whereas $L^{1}([0,1],\mathbb{R})$ is the Banach space of measurable functions $x:[0,1]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1}=\int_{0}^{1}|x(t)|dt$.

Definition 9. A multivalued map $\mathcal{H}:[0,1]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is said to be Carathéodory if the map $t\mapsto\mathcal{H}(t,x)$ is measurable for each $x\in\mathbb{R}$ and the map $x\mapsto\mathcal{H}(t,x)$ is upper semicontinuous for almost all $t\in[0,1]$. Further, a Carathéodory function $\mathcal{H}$ is called L^1-Carathéodory if for each $\alpha>0$, there exists $\varphi_\alpha\in L^{1}([0,1],\mathbb{R}^{+})$ such that $\|\mathcal{H}(t,x)\|=\sup\{|v|:v\in\mathcal{H}(t,x)\}\leq\varphi_\alpha(t)$ for all $\|x\|\leq\alpha$ and for a. e. $t\in[0,1]$.

Definition 10. For each $w\in C([0,1],\mathbb{R})$, the set of selections of $\mathcal{H}$ is defined by

$$S_{\mathcal{H},w}:=\{v\in L^{1}([0,1],\mathbb{R}):v(t)\in\mathcal{H}(t,w(t))\ \text{for a.e. } t\in[0,1]\}. \tag{10}$$

Definition 11. For a nonempty closed subset V of a Banach space E, let $\mathcal{H}_c:V\to\mathcal{P}(E)$ be a nonempty multivalued operator with closed values. We call $\mathscr{H}_c$ to be lower semi-continuous (l.s.c.) if the set $\{v \in V : \mathscr{H}_c(v) \cap B \neq \emptyset\}$ is open for any open set U in E.

Definition 12. A subset $\mathscr{A}$ of $[0,1] \times \mathbb{R}$ is $\mathscr{L} \otimes \mathscr{B}$ measurable if $\mathscr{A}$ belongs to the σ-algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in $[0,1]$ and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$.

Definition 13. A subset $\mathscr{A}_d$ of $L^1([0,1],\mathbb{R})$ is said be decomposable if for all a, $b \in \mathscr{A}_d$ and measurable $\mathscr{K} \subset [0,1]$, the function $a\phi_{\mathscr{K}} + b\phi_{[0,1]\setminus\mathscr{K}} \in \mathscr{A}_d$, where $\phi_{\mathscr{K}}$ stands for the characteristic function of $\mathscr{K}$.

Definition 14. A multivalued operator $\mathscr{H}_{bc} : W \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ has a property (BC) if $\mathscr{H}_{bc}$ is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values, where W is a separable metric space.

Definition 15. Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. We say that F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathscr{F}_N$ is lower semi-continuous and has nonempty closed and decomposable values, where

$$\mathscr{F}_N(x) = \Big\{u \in L^1([0,1],\mathbb{R}) : \\ u(t) \in F(t,x(t)) \quad \text{for a.e. } t \in [0,1]\Big\}. \tag{11}$$

Definition 16. Let $(\mathscr{X}, d)$ be a metric space induced from the normed space $(\mathscr{X}; \|\cdot\|)$ and let $\mathscr{M}_d : \mathscr{P}(\mathscr{X}) \times \mathscr{P}(\mathscr{X}) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$\mathscr{M}_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{12}$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{b,cl}(\mathscr{X}), \mathscr{M}_d)$ is a metric space and $(\mathscr{P}_{cl}(\mathscr{X}), M_d)$ is a generalized metric space (see [24]).

Definition 17. A multivalued operator $\mathscr{H} : \mathscr{X} \to \mathscr{P}_{cl}(\mathscr{X})$ is called γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$\mathscr{M}_d(N(x), N(y)) \leq \gamma d(x,y) \quad \text{for each } x, y \in \mathscr{X}, \tag{13}$$

and a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For further details on multi-valued maps, we refer the reader to the books [25, 26].

3. Existence Results for the Multivalued Problem

In this section, we present some existence results for the problem (1). Our first result deals with the case when F is Carathéodory. We make use of the following known results to establish the proof.

Lemma 18 (see [27]). *Let X be a Banach space. Let $F : [0,T] \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map with compact and convex values and let θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\theta \circ S_F : C([0,1],X) \longrightarrow \mathscr{P}_{cp,c}(C([0,1],X)), \\ x \longmapsto (\theta \circ S_F)(x) = \theta(S_{F,x}) \tag{14}$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Lemma 19 (nonlinear alternative for Kakutani maps [28]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to P_{c,cv}(C)$ is an upper semicontinuous compact map; here $P_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Theorem 20. *Assume that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \\ \leq p(t)\psi(\|x\|) \quad \textit{for each} \quad (t,x) \in [0,1] \times \mathbb{R}; \tag{15}$$

(H_3) *there exists a constant $\widetilde{M} > 0$ such that*

$$\widetilde{M}\left(\psi(\widetilde{M})\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1 + \frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \right. \\ \left. \times \int_0^1 s(1-s)^{\alpha-1} p(s)\,ds\right)^{-1} > 1. \tag{16}$$

Then the boundary value problem (1) *has at least one solution on* $[0,1]$.

Proof. In view of Lemma 4, we define an operator $\mathscr{Q} : C([0,1],\mathbb{R}) \to \mathscr{P}(C([0,1],\mathbb{R}))$ by

$$\mathscr{Q}(x) = \left\{h \in C([0,1],\mathbb{R}) : h(t) = \int_0^1 G(t,s)v(s)\,ds,\ v \in S_{F,x}\right\} \tag{17}$$

and show that it satisfies the hypotheses of Lemma 19. Since $S_{F,x}$ is convex (F has convex values), therefore, it can be easily shown that $\mathscr{Q}$ is convex for each $x \in C([0,1],\mathbb{R})$.

As a next step, we prove that $\mathscr{Q}$ maps the bounded sets (balls) into bounded sets in $C([0,1],\mathbb{R})$. For a positive number ξ, let $B_\xi = \{x \in C([0,1],\mathbb{R}) : \|x\| \leq \xi\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \mathscr{Q}(x)$, $x \in B_\xi$, there exists $v \in S_{F,x}$ such that

$$h(t) = \int_0^1 G(t,s)v(s)\,ds. \tag{18}$$

Then, for $t \in [0,1]$, we have

$$|h(t)| \le \int_0^1 G(t,s)|v(s)|\,ds \le \psi(\|x\|)\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1+\frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \times \int_0^1 s(1-s)^{\alpha-1}p(s)\,ds \le \psi(\xi)\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1+\frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \times \int_0^1 s(1-s)^{\alpha-1}p(s)\,ds. \quad (19)$$

Thus,

$$\|h\| \le \psi(\xi)\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1+\frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \times \int_0^1 s(1-s)^{\alpha-1}p(s)\,ds. \quad (20)$$

Now we show that $\mathcal{Q}$ maps the bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$.

Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_\xi$. For each $h \in \mathcal{Q}$, we obtain

$$\left|h(t'') - h(t')\right| \le \int_0^1 \left|G(t',s) - G(t'',s)\,\|\,v(s)\right|ds \le \int_0^1 \left|G(t',s) - G(t'',s)\right| p(s)\,\psi\,\|x\|\,ds \le \psi(\xi)\int_0^1 \left|G(t',s) - G(t'',s)\right| p(s)\,ds. \quad (21)$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_\xi$ as $t'' - t' \to 0$. As $\mathcal{Q}$ satisfies the above three assumptions, therefore, by the Ascoli-Arzelá theorem, it follows that $\mathcal{Q} : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that $\mathcal{Q}$ has a closed graph. Let $x_n \to x_*$, $h_n \in \mathcal{Q}(x_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{Q}(x_*)$. Associated with $h_n \in \mathcal{Q}(x_n)$, there exists $v_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$h_n(t) = \int_0^1 G(t,s)\,v_n(s)\,ds. \quad (22)$$

Thus, it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$h_*(t) = \int_0^1 G(t,s)\,v_*(s)\,ds. \quad (23)$$

Define a linear operator $\varphi : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ as

$$f \longmapsto \varphi(v)(t) = \int_0^1 G(t,s)\,v(s)\,ds. \quad (24)$$

Notice that

$$\|h_n(t) - h_*(t)\| = \left\|\int_0^1 G(t,s)\left(v_n(s) - v_*(s)\right)ds\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (25)$$

Thus, it follows from Lemma 18 that $\varphi \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \varphi(S_{F,x_n})$, since $x_n \to x_*$. Thus, for some $v_* \in S_{F,x_*}$, we have

$$h_*(t) = \int_0^1 G(t,s)\,v_*(s)\,ds. \quad (26)$$

In the last step, we show that there exists an open set $\mathcal{V} \subseteq C([0,1],\mathbb{R})$ with $x \notin \mathcal{Q}(x)$ for any $\mu \in (0,1)$ and all $x \in \partial\mathcal{V}$. Let $\mu \in (0,1)$ and $x \in \mu\mathcal{Q}(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that for $t \in [0,1]$, we have

$$x(t) = \mu\int_0^1 G(t,s)\,v(s)\,ds, \quad (27)$$

and using the computations used in the second step, we obtain

$$|x(t)| \le \psi(\|x\|)\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1+\frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \times \int_0^1 s(1-s)^{\alpha-1}p(s)\,ds. \quad (28)$$

In consequence, we have

$$\|x\|\left(\psi(\|x\|)\frac{(\alpha-1)}{\Gamma(\alpha)}\left(1+\frac{1}{1-\sigma}\int_0^1 g(s)\,ds\right) \times \int_0^1 s(1-s)^{\alpha-1}p(s)\,ds\right)^{-1} \le 1. \quad (29)$$

By the assumption (H_3), there exists $\widetilde{M}$ such that $\|x\| \ne \widetilde{M}$. Let us set

$$\mathcal{V} = \left\{x \in C([0,1],\mathbb{R}) : \|x\| < \widetilde{M}\right\}. \quad (30)$$

Observe that the operator $\mathcal{Q} : \overline{\mathcal{V}} \to \mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of $\mathcal{V}$, there is no $x \in \partial\mathcal{V}$ such that $x \in \mu\mathcal{Q}(x)$ for some $\mu \in (0,1)$. Consequently, by Lemma 19, we have that $\mathcal{Q}$ has a fixed point $x \in \overline{\mathcal{V}}$ which is a solution of the problem (1). This completes the proof. □

Example 21. Consider the following boundary value problem:

$$\begin{gathered} {}^cD^{5/2}x(t) \in F(t,x(t)), \quad t \in [0,1], \\ x(0) = x'(0) = 0, \\ x(1) = \int_0^1 \sqrt{t}\,x(t)\,dt, \end{gathered} \quad (31)$$

where $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$x \longrightarrow F(t,x) = \left[\frac{t|x|^3}{\left(|x|^3+3\right)}, \frac{t|x|}{2(|x|+1)} \right]. \quad (32)$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{t|x|^3}{\left(|x|^3+3\right)}, \frac{t|x|}{2(|x|+1)} \right) \leq \sqrt{t}, \quad x \in \mathbb{R}. \quad (33)$$

Here $\sigma = \int_0^1 s^{\alpha-1} g(s) ds = \int_0^1 s^{3/2} s^{1/2} ds = (1/3) < 1$, $\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq t = p(t)\psi(\|x\|)$, $x \in \mathbb{R}$, with $p(t) = t$, $\psi(\|x\|) = 1$, and $\int_0^1 s(1-s)^{3/2} s\, ds = 16/315$. Using the given values in the condition (H_3):

$$M\left(\psi(M) \frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \frac{1}{1-\sigma} \int_0^1 g(s) ds\right) \times \int_0^1 s(1-s)^{\alpha-1} p(s) ds\right)^{-1} > 1, \quad (34)$$

we find that

$$M > \frac{96}{315\sqrt{\pi}}. \quad (35)$$

Clearly, all the conditions of Theorem 20 are satisfied. Hence, the conclusion of Theorem 20 applies to the problem (31).

In our next result, we assume that F is not necessarily convex valued. We complete the proof of this result by applying the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [29] for lower semi-continuous maps with decomposable values, which is stated below.

Lemma 22 (see [29, 30]). *Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection; that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 23. *Suppose that (H_2) and (H_3) hold. In addition, we assume the following condition:*

(H_4) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- (a) *$(t,x) \mapsto F(t,x)$ is $\mathscr{L} \otimes \mathscr{B}$ measurable,*
- (b) *$x \mapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the problem (1) *has at least one solution on* $[0,1]$.

Proof. Observe that the assumptions (H_2) and (H_4) imply that F is of l.s.c. type. Then, by Lemma 22, there exists a continuous function $v : AC^2([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $v(x) \in \mathscr{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Let us consider the problem

$$D^\alpha x(t) = v(x(t)), \quad t \in [0,1],$$
$$x(0) = x'(0) = 0, \quad (36)$$
$$x(1) = \int_0^1 g(t) x(t) dt.$$

One can note that if $x \in AC^2([0,1],\mathbb{R})$ is a solution of (36), then x is a solution to the problem (1). To convert the problem (36) to a fixed point problem, we define an operator $\overline{\mathscr{Q}}$ as

$$\overline{\mathscr{Q}}x(t) = \int_0^1 G(t,s) v(x(s)) ds. \quad (37)$$

It is easy to show that the operator $\overline{\mathscr{Q}}$ is continuous and completely continuous. The rest of the proof is similar to that of Theorem 20. So we omit it. This completes the proof. □

Finally we show the existence of solutions for the problem (1) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps according to Covitz and Nadler [31].

Lemma 24 ([31]). *Let (X,d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then* $\text{Fix}\, N \neq \emptyset$.

Theorem 25. *Assume that*

(H_5) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [0,1] \to \mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_6) *$\mathscr{M}_d(F(t,x),F(t,y)) \leq m(t)|x-y|$ for almost all $t \in [0,1]$ and $x, y \in \mathbb{R}$ with $m \in L^1([0,1],\mathbb{R}^+)$ and $d(0,F(t,0)) \leq m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (1) *has at least one solution on* $[0,1]$ *if*

$$\frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \frac{1}{1-\sigma} \int_0^1 g(s) ds\right) \int_0^1 s(1-s)^{\alpha-1} m(s) ds < 1. \quad (38)$$

Proof. By the assumption (H_5), it follows that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$. So F has a measurable selection (see [32, Theorem III.6]). Now it will be shown that the operator $\mathscr{Q}$ defined by (17) satisfies the hypotheses of Lemma 24. To show that $\mathscr{Q}(x) \in \mathscr{P}_{cl}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\geq 0} \in \mathscr{Q}(x)$ be such that $u_n \to u (n \to \infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$u_n(t) = \int_0^1 G(t,s) v_n(s) ds. \quad (39)$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$, we have

$$v_n(t) \longrightarrow v(t) = \int_0^1 G(t,s) v(s) ds. \quad (40)$$

Hence, $u \in \mathscr{Q}$.

Next, we show that there exists $\delta < 1$ such that

$$\mathcal{M}_d(\mathcal{Q}(x), \mathcal{Q}(y)) \leq \delta \|x - y\| \quad \text{for each } x, y \in AC^2([0,1], \mathbb{R}). \tag{41}$$

Let $x, y \in AC^2([0,1], \mathbb{R})$ and $h_1 \in \mathcal{Q}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \int_0^1 G(t,s) v_1(s)\, ds. \tag{42}$$

By (H_6), we have that

$$\mathcal{M}_d(F(t,x), F(t,y)) \leq m(t) |x(t) - y(t)|. \tag{43}$$

So, there exists $w \in F(t, y(t))$ such that

$$|v_1(t) - w| \leq m(t) |x(t) - y(t)|, \quad t \in [0,1]. \tag{44}$$

Define $\mathcal{W} : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{W}(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t) |x(t) - y(t)|\}. \tag{45}$$

Since the multivalued operator $\mathcal{W}(t) \cap F(t, y(t))$ is measurable ([32, Proposition III.4]), there exists a function $v_2(t)$ which is a measurable selection for $\mathcal{W}$. So $v_2(t) \in F(t, y(t))$ and for each $t \in [0,1]$, we have that $|v_1(t) - v_2(t)| \leq m(t)|x(t) - y(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^1 G(t,s) v_2(s)\, ds. \tag{46}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \int_0^1 G(t,s) |v_1(s) - v_2(s)|\, ds \\ &\leq \frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \frac{1}{1-\sigma} \int_0^1 g(s)\, ds\right) \\ &\quad \times \int_0^1 s(1-s)^{\alpha-1} m(s)\, ds\, |v_1(s) - v_2(s)|\, ds. \end{aligned} \tag{47}$$

Hence,

$$\begin{aligned} \|h_1 - h_2\| &\leq \frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \frac{1}{1-\sigma} \int_0^1 g(s)\, ds\right) \\ &\quad \times \int_0^1 s(1-s)^{\alpha-1} m(s)\, ds\, \|x - y\|. \end{aligned} \tag{48}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} \mathcal{M}_d(\mathcal{Q}(x), \mathcal{Q}(y)) &\leq \delta \|x - y\| \\ &\leq \left\{\frac{(\alpha-1)}{\Gamma(\alpha)} \left(1 + \frac{1}{1-\sigma} \int_0^1 g(s)\, ds\right)\right. \\ &\quad \left. \times \int_0^1 s(1-s)^{\alpha-1} m(s)\, ds\right\} \|x - y\|. \end{aligned} \tag{49}$$

Since $\mathcal{Q}$ is a contraction, it follows from Lemma 24 that $\mathcal{Q}$ has a fixed point x which is a solution of (1). This completes the proof. □

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Research Article

Oscillatory Criteria for Higher Order Functional Differential Equations with Damping

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We investigate a class of higher order functional differential equations with damping. By using a generalized Riccati transformation and integral averaging technique, some oscillation criteria for the differential equations are established.

1. Introduction

In this paper, we consider the following higher order functional differential equations with distributed deviating arguments of the form as follows:

$$x^{(n)}(t) + p(t)\,x^{(n-1)}(t) + \int_a^b q(t,\xi) \times f\left(x\left[g_1(t,\xi)\right],\dots,x\left[g_m(t,\xi)\right]\right)d\mu(\xi) = 0, \quad t \ge t_0, \tag{1}$$

where $n \ge 2$ is an even number, $p(t) \in C^1([t_0,\infty),R_+)$, $q(t,\xi) \in C([t_0,\infty)\times[a,b],R_+)$, $g_i(t,\xi) \in C([t_0,\infty)\times[a,b],R)$, $\liminf_{t\to\infty,\xi\in[a,b]} g_i(t,\xi) = \infty$ for $i \in I_m = \{1,2,\dots,m\}$, and $f(u_1,\dots,u_m) \in C(R^m,R)$ has the same sign as $u_1,\dots,u_m$; when they have the same sign, $\mu(\xi) \in C([a,b],R)$ is nondecreasing, and the integral of (1) is a Stieltjes one.

We restrict our attention to those solutions $x(t)$ of (1) which exist on same half liner $[t_\mu,\infty)$ with $\sup\{x(t) : t \ge T\} \ne 0$ for any $T \ge t_\mu$ and satisfy (1). As usual, a solution $x(t)$ of (1) is called oscillatory if the set of its zeros is unbounded from above, otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all solutions are oscillatory.

In recent years, there has been an increasing interest in studying the oscillation behavior of solutions for the differential equations with distributed deviating arguments, and a number of results have been obtained (refer to [1–3] and their references). However, to the best of our knowledge, very little is known for the case of higher order differential equations with damping. The purpose of this paper is to establish some new oscillation criteria for (1) by introducing a class of functions $\Phi(t,s,r)$ defined in [2] and a generalized Riccati technique.

Firstly, we define the following two class functions.

We say that a function $\Phi = \Phi(t,u,v)$ belongs to the function class X, denoted by $\Phi \in X$, if $\Phi \in C(E,R)$, where $E = \{(t,u,v) : t_0 \le v \le u \le t < \infty\}$, which satisfies $\Phi(t,t,v) = \Phi(t,v,v) = 0$, $\Phi(t,u,v) \ne 0$, $v < u < t$ and has the partial derivative $\partial\Phi/\partial u$ on E that is locally integrable with respect to u in E.

Let $D_0 = \{(t,u) : t_0 \le u < t\}$, $D = \{(t,u) : t_0 \le u \le t\}$. We say that a function $H = H(t,u)$ belongs to the function class Y, denoted by $H \in Y$, if $H(t,t) = 0$ for $t \ge t_0$, $H(t,u) \ne 0$ in D_0, H has continuous partial derivative in D_0 with respect to t and s.

In order to prove the main theorems, we need the following lemmas.

Lemma 1 (see [4]). *Let $x(t) \in C^n([t_0,\infty),R_+)$, if $x^{(n)}(t)$ is of constant sign and not identically zero on any ray $[t_\mu,+\infty)$ for $t_\mu \ge t_0$, then there exists a $t_\sigma \ge t_\mu$, an integer $l(0 \le l \le n)$, with $n+l$ even for $x(t)x^{(n)}(t) \ge 0$ or $n+l$ odd for $x(t)x^{(n)}(t) \le 0$; and for $t \ge t_\sigma \ge t_\mu$, $x(t)x^{(k)}(t) > 0$, $0 \le k \le l$, and $(-1)^{k+l}x(t)x^{(k)}(t) > 0$, $l \le k \le n$.*

Lemma 2 (see [5]). *If the function $x(t)$ is as in Lemma 1 and $x^{(n-1)}(t)x^{(n)}(t) \le 0$ for $t \ge t_\sigma \ge t_\mu$, then there exists a constant $\theta \in (0,1)$ such that for sufficiently large t, there exists a constant $M_\theta > 0$, satisfying*

$$x'(\theta t) \ge M_\theta t^{n-2} x^{(n-1)}(t). \tag{2}$$

Lemma 3 (see [3]). *Suppose that $x(t)$ is a nonoscillatory solution of (1). If*

$$\lim_{t\to\infty}\int_{T_0}^{t}\exp\left\{-\int_{T_0}^{s} p(\tau)\,d\tau\right\}ds = \infty, \quad T_0 \ge t_0, \tag{3}$$

then $x(t)x^{(n-1)}(t) > 0$ for any large t.

2. Main Results

Theorem 4. *Assume that (3) holds, and*

(A$_1$) *there exists a function $\sigma(t) \in C^n([t_0,\infty), R_+)$ such that $\sigma(t) = \min_{1\le i\le m}\{t, \inf_{\xi\in[a,b]} g_i(t,\xi)\}$, $\lim_{t\to\infty}\sigma(t) = \infty$. $\sigma'(t) \ge \sigma > 0$, $\sigma^{n-2}(t) \ge \rho^{n-2} > 0$, $(M_\theta/2)\sigma\rho^{n-2} =: k > 0$, where σ, ρ, and k are constants.*

(A$_2$) *$f(u_1,\dots,u_m)$ is nondecreasing with u_i, $i \in I_m$, and there exist constants $N > 0$ and $\lambda > 0$ such that*

$$\liminf_{u_{i_0}\to\infty}\frac{f(u_1,\dots,u_m)}{u_{i_0}} \ge \lambda, \quad u_i \ge N,\ i \ne i_0. \tag{4}$$

If there exists a function $\Phi(t,u,v) \in X$, such that for any $l(t) \in C^1([t_0,\infty), R_+)$, $r(t) \in C^1([t_0,\infty), R)$ and $T_0 \ge t_0$,

$$\limsup_{t\to\infty}\int_{T_0}^{t}\left\{\Phi^2(s,u,v)\psi(s) - \frac{\phi(s)}{k}\times\left[\Phi_s'(s,u,v) + \frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]^2\right\}ds > 0, \tag{5}$$

where

$$\phi(t) = l(t)\exp\left\{-2k\int_{T_0}^{t} r(s)\,ds\right\},$$
$$\psi(t) = \phi(t)\left\{\lambda\int_a^b q(t,\xi)\,d\mu(\xi) + kr^2(t) - r'(t) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}\right\}. \tag{6}$$

Then (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. From the conditions of $g_i(t,\xi)$ and $f(u_1,\dots,u_m)$, there exists a $T_0 \ge t_0$, such that

$$x(t) > 0, \quad x[g_i(t,\xi)] > 0,$$
$$f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)]) > 0, \quad t \ge T_0,\ i \in I_m. \tag{7}$$

By Lemma 3, there exists a $t_1 \ge T_0$ such that $x^{(n-1)}(t) > 0$, $t \ge t_1$. Thus, we have

$$x^{(n)}(t) = -p(t)x^{(n-1)}(t) - \int_a^b q(t,\xi) f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)]) \times d\mu(\xi) \le 0, \quad t \ge t_1. \tag{8}$$

By Lemma 1, there exists a $t_2 > t_1$ such that $x'(t) > 0$, $t \ge t_2$. Further, by Lemma 2, there exist constant $M_\theta > 0$ and a $t_3 \ge t_2$, such that

$$x'\left[\frac{\sigma(t)}{2}\right] \ge M_\theta\sigma^{n-2}(t)x^{(n-1)}[\sigma(t)] \ge M_\theta\sigma^{n-2}(t)x^{(n-1)}(t), \quad t \ge t_3. \tag{9}$$

Set

$$y(t) = \phi(t)\left\{\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} + r(t) + \frac{p(t)}{2k}\right\}, \tag{10}$$

then

$$\begin{aligned} y'(t) = {} & \frac{\phi'(t)}{\phi(t)}y(t) + \phi(t) \\ & \times\left\{-\int_a^b q(t,\xi)\frac{f(x[g_1(t,\xi)],\dots,x[g_m(t,\xi)])}{x[\sigma(t)/2]} \times d\mu(\xi) - p(t)\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} - \frac{\sigma'(t)x^{(n-1)}(t)x'[\sigma(t)/2]}{2x^2[\sigma(t)/2]} + r'(t) + \frac{p'(t)}{2k}\right\}. \end{aligned} \tag{11}$$

In view of (A$_1$), (A$_2$) and the definition of $y(t)$, $\phi(t)$, we have

$$\begin{aligned} y'(t) \le {} & \frac{\phi'(t)}{\phi(t)}y(t) + \phi(t) \\ & \times\left\{-p(t)\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]} - \lambda\int_a^b q(t,\xi)\,d\mu(\xi) - k\left[\frac{x^{(n-1)}(t)}{x[\sigma(t)/2]}\right]^2 + r'(t) + \frac{p'(t)}{2k}\right\} \\ = {} & -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)y(t)}{l(t)} - \psi(t), \end{aligned} \tag{12}$$

where $\psi(t) = \phi(t)\{\lambda\int_a^b q(t,\xi)d\mu(\xi) + kr^2(t) - r'(t) - p'(t)/2k - p^2(t)/4k\}$.

Multiplying (12) by $\Phi^2(t,u,v)$ and integrating from t_3 to t, we have

$$\int_{t_3}^{t}\Phi^2(s,u,v)\,\psi(s)\,ds \le \int_{t_3}^{t}\Phi^2(s,u,v)\left[-y'(s)+\frac{l'(s)}{l(s)}y(s)\right]ds - \int_{t_3}^{t}\Phi^2(s,u,v)\frac{ky^2(s)}{\phi(s)}ds. \tag{13}$$

Integrating by parts and using integral averaging technique, we have

$$\begin{aligned}\int_{t_3}^{t}\Phi^2(s,u,v)\,\psi(s)\,ds &\le \int_{t_3}^{t}\frac{\phi(s)}{k}\left[\Phi_s'(s,u,v)+\frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]^2ds \\ &\quad - \int_{t_3}^{t}\left\{\Phi(s,u,v)\,y(s)\sqrt{\frac{k}{\phi(s)}}-\sqrt{\frac{\phi(s)}{k}}\right. \\ &\qquad\left.\times\left[\Phi_s'(s,u,v)+\frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]\right\}^2ds \\ &\le \int_{t_3}^{t}\frac{\phi(s)}{k}\left[\Phi_s'(s,u,v)+\frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]^2ds,\end{aligned} \tag{14}$$

thus

$$\limsup_{t\to\infty}\int_{t_3}^{t}\left\{\Phi^2(s,u,v)\,\psi(s)-\frac{\phi(s)}{k}\times\left[\Phi_s'(s,u,v)+\frac{l'(s)\Phi(s,u,v)}{2l(s)}\right]^2\right\}ds\le 0, \tag{15}$$

which contradicts (5). This completes the proof of Theorem 4. □

If we choose $\Phi(t,u,v) = \sqrt{H_1(t,u)H_2(u,v)}$, where H_1, $H_2 \in Y$. By Theorem 4, we have the following results.

Corollary 5. *Assume that (3), (A_1), and (A_2) hold. If there exist $H_1, H_2 \in Y$ such that for each $T_0 \ge t_0$,*

$$\limsup_{t\to\infty}\int_{T_0}^{t}H_1(s,u)\,H_2(u,v)\times\left\{\psi(s)-\frac{[h_1(s,u)+h_2(u,v)]^2}{4k}\right\}ds>0, \tag{16}$$

where h_1 and h_2 are defined by $\partial H_1(t,u)/\partial u = h_1(t,u)H_1(t,u)$, $\partial H_2(u,v)/\partial u = h_2(u,v)H_2(u,v)$, and

$$\psi(t)=\lambda\int_a^b q(t,\xi)\,d\mu(\xi)-\frac{p'(t)}{2k}-\frac{p^2(t)}{4k}. \tag{17}$$

Then (1) is oscillatory.

If we choose $l(t) = 1$, $r(t) = 0$, and let $\Phi(t,u,v) = (t-u)(u-v)^{\alpha}$, $\alpha > 1/2$, by Theorem 4, we have the following corollary.

Corollary 6. *Assume that (3), (A_1), and (A_2) hold. If there exists a constant $\alpha > 1/2$ such that for each $T_0 \ge t_0$,*

$$\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_{T_0}^{t}(s-u)^2(u-v)^{2\alpha}\times k\psi(s)\,ds>\frac{\alpha}{(2\alpha-1)(2\alpha+1)}, \tag{18}$$

where $\psi(t)$ is defined as in Corollary 5. Then (1) is oscillatory.

Theorem 7. *Assume that (3) holds, and*

- (A_3) *there exist functions $\gamma_i(t) \in C^n([t_0,\infty),R_+)$, such that $\gamma_i(t) \le \min_{1\le i\le m}\{t, \inf_{\xi\in[a,b]}g_i(t,\xi)\}$, $\lim_{t\to\infty}\gamma_i(t) = \infty$, $\gamma_i'(t) \ge \gamma_i > 0$, where γ_i are constants, $i \in I_m$;*
- (A_4) *there exist constants $\lambda_i \in [0,1]$ and $\overline{\lambda} > 0$, such that*

$$\sum_{i=1}^{m}\lambda_i\gamma_i(t)>0,\qquad \frac{f(u_1,\ldots,u_m)}{\lambda_1u_1+\cdots+\lambda_mu_m}\ge\overline{\lambda}. \tag{19}$$

$\gamma_i^{n-2}(t) \ge \rho_i^{n-2} \ge 0$, $(M_\theta/2)\sum_{i=1}^{m}\lambda_i\gamma_i\rho_i^{n-2}(t) =: k > 0$, where ρ_i and k are constants, $i \in I_m$.

If there exists a function $\Phi \in X$, such that for any $l(t) \in C^1([t_0,\infty),R_+)$, $r(t) \in C^1([t_0,\infty),R)$, and $T_0 \ge t_0$, and (5) holds, where $\phi(t)$ is defined as in Theorem 4:

$$\psi(t)=\phi(t)\left\{\overline{\lambda}\int_a^b q(t,\xi)\,d\mu(\xi)+kr^2(t)-r'(t)-\frac{p'(t)}{2k}-\frac{p^2(t)}{4k}\right\}. \tag{20}$$

Then (1) is oscillatory.

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Similar to the proof of Theorem 4, there exists a $T_0 \ge t_0$, such that $x[g_i(t,\xi)] > 0$, $x[\gamma_i(t)] > 0$, $x'(t) > 0$, $f(x[g_1(t,\xi)],\ldots,x[g_m(t,\xi)]) > 0$, $x^{(n-1)}(t) > 0$, and $x^{(n)}(t) \le 0$, for $t \ge T_0$, $i \in I_m$. Set

$$y(t)=\phi(t)\left\{\frac{x^{(n-1)}(t)}{\sum_{i=1}^{m}\lambda_i x[\gamma_i(t)/2]}+r(t)+\frac{p(t)}{2k}\right\}, \tag{21}$$

then

$$y'(t) = \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{ -p(t) \frac{x^{(n-1)}(t)}{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]} - \int_a^b q(t,\xi) \frac{f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)])}{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]} \times d\mu(\xi) - \frac{x^{(n-1)}(t)}{2\{\sum_{i=1}^{m} \lambda_i x[\gamma_i(t)/2]\}^2} \times \sum_{i=1}^{m} \lambda_i x'[\gamma_i(t)] \gamma_i'(t) + r'(t) + \frac{p'(t)}{2k} \right\}. \quad (22)$$

In view of (A_3), (A_4) and the definition of $y(t)$, $\phi(t)$, we have

$$y'(t) \leq -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)\, y(t)}{l(t)} - \psi(t). \quad (23)$$

The following proof is similar to Theorem 4, and we omit the details. This completes the proof of Theorem 7. □

Similar to Corollaries 5 and 6, we have the following corollaries.

Corollary 8. *Assume that* (3), *(A_3), and (A_4) hold. If there exist $H_1, H_2 \in Y$ such that for each $T_0 \geq t_0$, and* (16) *holds, where h_1, h_2 are defined as in Corollary 5:*

$$\psi(t) = \overline{\lambda} \int_a^b q(t,\xi)\, d\mu(\xi) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}. \quad (24)$$

Then (1) *is oscillatory.*

Corollary 9. *Assume that* (3), *(A_3), and (A_4) hold. If there exists a constant $\alpha > 1/2$ such that for each $T_0 \geq t_0$, and* (18) *holds. where $\psi(t)$ is defined as in Corollary 8, then* (1) *is oscillatory.*

For the case of the function $f(u_1, \ldots, u_m)$ with monotonicity, we have the following theorem.

Theorem 10. *Assume that* (3), *(A_3) hold, and*

(A_5) *there exist $\partial f/\partial u_i$ and $\partial f/\partial u_i \geq \lambda_i \geq 0$, such that $\sum_{i=1}^{m} \lambda_i \gamma_i'(t) > 0$, where λ_i is constants. $\gamma_i^{n-2} \geq \rho_i^{n-2} \geq 0$, $(M_\theta/2) \sum_{i=1}^{m} \lambda_i \gamma_i \rho_i^{n-2}(t) =: k > 0$, in which ρ_i and k are constants,$i \in I_m$.*

If there exists a function $\Phi \in X$, such that for any $l(t) \in C^1([t_0,\infty), R_+)$, $r(t) \in C^1([t_0,\infty), R)$, and (5) *holds, where $\phi(t)$ is defined as in Theorem 4:*

$$\psi(t) = \phi(t) \left\{ \int_a^b q(t,\xi)\, d\mu(\xi) + kr^2(t) - r'(t) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k} \right\}. \quad (25)$$

Then (1) *is oscillatory.*

Proof. Suppose to the contrary that (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Similar to the proof of Theorem 4, there exists a $T_0 \geq t_0$, when $t \geq T_0$, and we have $x[g_i(t,\xi)] > 0$, $x[\gamma_i(t)] > 0$, $x'(t) > 0$, $f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)]) > 0$, $x^{(n-1)}(t) > 0$, and $x^{(n)}(t) \leq 0$, $i \in I_m$. Set

$$y(t) = \phi(t) \left\{ \frac{x^{(n-1)}(t)}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} + r(t) + \frac{p(t)}{2k} \right\}, \quad (26)$$

then

$$y'(t) = \frac{\phi'(t)}{\phi(t)} y(t) + \phi(t) \times \left\{ -p(t) \frac{x^{(n-1)}(t)}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} - \int_a^b q(t,\xi) \frac{f(x[g_1(t,\xi)], \ldots, x[g_m(t,\xi)])}{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])} \times d\mu(\xi) - \frac{x^{(n-1)}(t)}{2\{f(x[\gamma_1(t)/2] + \cdots + x[\gamma_m(t)/2])\}^2} \times \sum_{i=1}^{m} \lambda_i x'[\gamma_i(t)] \gamma_i'(t) + r'(t) + \frac{p'(t)}{2k} \right\}. \quad (27)$$

In view of (A_3), (A_5) and the definition of $y(t)$, $\phi(t)$, we have

$$y'(t) \leq -\frac{ky^2(t)}{\phi(t)} + \frac{l'(t)\, y(t)}{l(t)} - \psi(t). \quad (28)$$

The following proof is similar to Theorem 4, we omit the details. This completes the proof of Theorem 10. □

Similar to Corollaries 5 and 6, we have the following corollaries.

Corollary 11. *Assume that* (3), (A_3), *and* (A_5) *hold. If there exist* $H_1, H_2 \in Y$ *such that for each* $T_0 \geq t_0$, *and* (16) *holds, where* h_1, h_2 *are defined as in Corollary 5:*

$$\psi(t) = \int_a^b q(t,\xi)\,d\mu(\xi) - \frac{p'(t)}{2k} - \frac{p^2(t)}{4k}. \tag{29}$$

Then (1) *is oscillatory.*

Corollary 12. *Assume that* (3), (A_3), *and* (A_5) *hold. If there exists a constant* $\alpha > 1/2$ *such that for each* $T_0 \geq t_0$, *and* (18) *holds. where* $\psi(t)$ *is defined as in Corollary 11. Then* (1) *is oscillatory.*

3. Examples

Example 13. Consider the following equation

$$x^{(4)}(t) + \frac{1}{t}x^{(3)}(t)\int_0^{\pi/2}\frac{\xi}{t^2}\frac{2x(t+\sin\xi)}{2-\exp\left(-x^2(t+\cos\xi)\right)}d\xi = 0, \quad t \geq 1, \tag{30}$$

where $u_1 = x(t+\cos\xi)$, $u_2 = x(t+\sin\xi)$, obviously $f(u_1,u_2)/u_2 \geq 1 = \lambda$. Choosing $\sigma = 1$, $\rho = 1$, then $k = M_\theta/2$, and

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_0^t (s-u)^2(u-v)^{2\alpha}\left(\frac{M_\theta}{2s^2}+\frac{1}{4s^2}\right)ds \\
&\quad = \left(\frac{M_\theta}{2}+\frac{1}{4}\right)\limsup_{t\to\infty}\frac{1}{t^{2\alpha+1}}\int_0^t (s-u)^2(u-v)^{2\alpha}\frac{1}{s^2}ds \\
&\quad = \left(\frac{M_\theta}{2}+\frac{1}{4}\right)\frac{1}{\alpha(2\alpha+1)(2\alpha-1)}.
\end{aligned} \tag{31}$$

Thus, there exists a constant $\alpha > 1/2$, such that $(M_\theta/2+1/4) > \alpha^2$, that is,

$$\left(\frac{M_\theta}{2}+\frac{1}{4}\right)\frac{1}{\alpha(2\alpha+1)(2\alpha-1)} > \frac{\alpha}{(2\alpha+1)(2\alpha-1)}. \tag{32}$$

By Corollary 6, then (30) is oscillatory.

Example 14. Consider the following equation

$$\begin{aligned}
&x^{(4)}(t) + \frac{1}{t}x^{(3)}(t) + \int_0^1 \frac{\xi}{t^2} \\
&\quad \times\left[x(t-\xi)+x(t+\xi)+x^3(t-\xi)+x^5(t+\xi)\right]d\xi = 0,
\end{aligned} \tag{33}$$

where $f(u_1,u_2) = u_1 + u_2 + u_1^3 + u_2^5$, obviously $\partial f/\partial u_1 = 1 + 3u_1^2 \geq 1$, $\partial f/\partial u_2 = 1 + 5u_2^4 \geq 1$. Choosing $\lambda_i = 1$, $\gamma_i = 1$, $\rho_i = 1$, and $i = 1, 2$, then $k = M_\theta$. By Corollary 12, then (33) is oscillatory.

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Research Article

Uniqueness of Positive Solutions for a Perturbed Fractional Differential Equation

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We are concerned with the existence and uniqueness of positive solutions for the following nonlinear perturbed fractional two-point boundary value problem: $D_{0+}^{\alpha}u(t) + f(t,u,u',\dots,u^{(n-2)}) + g(t) = 0$, $0 < t < 1$, $n-1 < \alpha \leq n$, $n \geq 2$, $u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0$, where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Our analysis relies on a fixed-point theorem of generalized concave operators. An example is given to illustrate the main result.

1. Introduction

In this paper, we are interested in the existence and uniqueness of positive solutions for the following nonlinear perturbed fractional two-point boundary value problem:

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f\left(t,u,u',\dots,u^{(n-2)}\right) + g(t) = 0, \quad 0 < t < 1,\ n-1 < \alpha \leq n,\ n \geq 2, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0, \end{gathered} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and $g : [0,1] \to [0,+\infty)$ is continuous.

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering, and biological sciences; see [1–15], for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [16–26] and the references therein. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors; see [19–22, 25], for example.

When $g(t) \equiv 0$, Yang and Chen [22] investigated the existence and uniqueness of positive solutions for the problem (1.1) by means of a fixed-point theorem for u_0 concave operators. They present the following result.

Theorem 1.1 (see [22]). *Assume that*

(H_1) $f \in C([0,1] \times [0,+\infty) \times R^{n-2} \rightarrow [0,+\infty))$, $f(t, y_1, y_2, \ldots, y_{n-1})$ *is increasing for* $y_i \geq 0$, $i = 1, 2, \ldots, n-1$, *and* f *is not identically vanishing;*

(H_2) *for any* $t \in [0,1]$, $y_i \geq 0$, $i = 1, 2, \ldots, n-1$, *there exist constants* m_1, m_2, $m_1 \leq 0 \leq m_2 < 1$ *such that*

$$c^{m_2} f(t, y_1, y_2, \ldots, y_{n-1}) \leq f(t, cy_1, cy_2, \ldots, cy_{n-1}) \leq c^{m_1} f(t, y_1, y_2, \ldots, y_{n-1}), \quad \forall 0 < c \leq 1; \tag{1.2}$$

(H_3) $m_2 < (2m_1 + 1)/3$.

Then the problem (1.1) *with* $g(t) \equiv 0$ *has a unique positive solution.*

In this paper, we will remove the condition (H_3) and improve on (H_2). And we will use a fixed-point theorem of generalized concave operators to show the existence and uniqueness of positive solutions for the problem (1.1). Our main result is summarized in the following theorem.

Theorem 1.2. *Assume that* (H_1) *holds and*

(H_4) *for any* $c \in (0,1)$ *and* $y_i \geq 0$, $i = 1, 2, \ldots, n-1$, *there exists a number* $\varphi(c) \in (c, 1)$ *such that*

$$f(t, cy_1, cy_2, \ldots, cy_{n-1}) \geq \varphi(c) f(t, y_1, y_2, \ldots, y_{n-1}). \tag{1.3}$$

Then the problem (1.1) *has a unique positive solution* $u^* = I_{0+}^{n-2} v^* \in C([0,1], [0,+\infty))$ *which satisfies* $\lambda(v^*) t^{\alpha-n+1}(1-t) \leq v^*(t) \leq \mu(v^*) t^{\alpha-n+1}(1-t)$, $t \in [0,1]$, *where* $\mu(v^*) \geq \lambda(v^*) > 0$.

Remark 1.3. Some examples of $\varphi(t)$ which satisfy the condition (H_4) are

(1) $\varphi(t) = t^r$, for all $t \in (0,1)$, where $r \in (0,1)$,

(2) $\varphi(t) = t(1 + \eta(t))$ with $0 < \eta(t) \leq 1/t - 1$, for all $t \in (0,1)$.

Remark 1.4. It is easy to see that the condition (H_4) is weaker than the condition (H_2). Moreover, we do not need the condition (H_3) in our main result.

2. Preliminaries and Previous Results

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proof of our theorem.

Definition 2.1 (see [4, Definition 2.1]). The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0, \tag{2.1}$$

is called the Riemann-Liouville fractional integral of order α, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2.2 (see [4, page 36-37]). For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \tag{2.2}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

By using the same method in [21], the problem (1.1) can be transformed into the following boundary value problem:

$$\begin{gathered} D_{0+}^{\alpha-n+2} v(t) + f\left(t, I_{0+}^{n-2} v(t), I_{0+}^{n-3} v(t), \ldots, I_{0+}^{1} v(t), v(t)\right) + g(t) = 0, \quad 0 < t < 1, \\ v(0) = v(1) = 0. \end{gathered} \tag{2.3}$$

Moreover, from Lemma 2.5 and Lemma 2.7 in [21], we can easily obtain the following result.

Lemma 2.3. *If $v \in C([0,1], [0, +\infty))$ is a positive solution of the problem* (2.3)*, then $u(t) = I_{0+}^{n-2} v(t)$ is a positive solution of the problem* (1.1)*. On the other hand, if $v \in C([0,1], [0, +\infty))$ is a positive solution of the problem* (2.3)*, then the solution is*

$$v(t) = \int_0^1 G(t,s) \left[f\left(s, I_{0+}^{n-2} v(s), I_{0+}^{n-3} v(s), \ldots, I_{0+}^{1} v(s), v(s)\right) + g(s) \right] ds, \quad t \in [0,1], \tag{2.4}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha - n + 2)} \begin{cases} (t(1-s))^{\alpha-n+1} - (t-s)^{\alpha-n+1}, & 0 \le s \le t \le 1, \\ (t(1-s))^{\alpha-n+1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.5}$$

Here $G(t,s)$ is called the Green function of the problem (2.3)*. Evidently, $G(t,s) \ge 0$ for $t, s \in [0,1]$.*

The following property of the Green function plays important roles in this paper.

Lemma 2.4 (see [22]). *The Green function $G(t,s)$ in Lemma 2.3 has the following property:*

$$\frac{\alpha-n+1}{\Gamma(\alpha-n+2)}s(1-s)^{\alpha-n+1}t^{\alpha-n+1}(1-t) \\ \leq G(t,s) \leq \frac{1}{\Gamma(\alpha-n+2)}(1-s)^{\alpha-n}t^{\alpha-n+1}(1-t) \quad \text{for } t,s\in[0,1]. \tag{2.6}$$

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed-point theorem which we will be used later. For convenience of readers, we suggest that one refers to [27, 28] for details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P\subset E$, that is, $x\leq y$ if and only if $y-x\in P$. If $x\leq y$ and $x\neq y$, then we denote $x<y$ or $y>x$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P\subset E$ is a cone if it satisfies (i) $x\in P$, $\lambda\geq 0\Rightarrow \lambda x\in P$; (ii) $x\in P$, $-x\in P\Rightarrow x=\theta$.

P is called normal if there exists a constant $M>0$ such that, for all $x,y\in E$, $\theta\leq x\leq y$ implies $\|x\|\leq M\|y\|$; in this case M is called the normality constant of P. If $x_1,x_2\in E$, the set $[x_1,x_2]=\{x\in E\mid x_1\leq x\leq x_2\}$ is called the order interval between x_1 and x_2.

For all $x,y\in E$, the notation $x\sim y$ means that there exist $\lambda>0$ and $\mu>0$ such that $\lambda x\leq y\leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h>\theta$ (i.e., $h\geq\theta$ and $h\neq\theta$), we denote by P_h the set $P_h=\{x\in E\mid x\sim h\}$. It is easy to see that $P_h\subset P$ is convex and $\lambda P_h=P_h$ for all $\lambda>0$.

In a recent paper [28], Zhai et al. considered the following operator equation:

$$x = Ax + x_0. \tag{2.7}$$

They established the existence and uniqueness of positive solutions for the above equation, and they present the following interesting result.

Theorem 2.5 (see Theorem 2.1 in [28]). *Let $h>\theta$ and P be a normal cone. Assume that*

$(D_1)A: P\to P$ is increasing;

$(D_2)x_0\in P$ satisfies $Ah+x_0\in P_h$;

(D_3) for any $x\in P$ and $t\in(0,1)$, there exists $\varphi(t)\in(t,1)$ such that $A(tx)\geq\varphi(t)Ax$.

Then the operator equation (2.7) has a unique solution in P_h.

Remark 2.6. An operator A is said to be *generalized concave* if A satisfies condition (D_3).

3. Proof of Theorem 1.2

In this section, we apply Theorem 2.5 to study the problem (1.1), and we obtain a new result on the existence and uniqueness of positive solutions. The method used here is new to the literature and so is the existence and uniqueness result to the fractional differential equations.

In our considerations we will work in the Banach space $C[0,1] = \{x : [0,1] \to \mathbf{R}$ is continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \le y \Longleftrightarrow x(t) \le y(t) \quad \text{for } t \in [0,1]. \tag{3.1}$$

Set $P = \{x \in C[0,1] \mid x(t) \ge 0, t \in [0,1]\}$, the standard cone. It is clear that P is a normal cone in $C[0,1]$ and the normality constant is 1.

Proof of Theorem 1.2. Let $h(t) = t^{\alpha-n+1}(1-t)$, $t \in [0,1]$. Then

$$P_h = \{x \in P \mid \text{and there exist } \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \le x \le \mu(x)h\}. \tag{3.2}$$

For any $v \in P$, we define

$$Av(t) = \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) ds, \qquad x_0(t) = \int_0^1 G(t,s) g(s) ds, \tag{3.3}$$

where $G(t,s)$ is given as in Lemma 2.3. Noting that $I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s) \ge 0$ and $G(t,s) \ge 0$, it follows from (H_1) that $A : P \to P$. In the sequel we check that A and x_0 satisfy all assumptions of Theorem 2.5.

Firstly, we prove that $A : P \to P$ is an increasing operator. In fact, for $v_i \in P$, $i = 1,2$ with $v_1 \le v_2$, we know that $v_1(t) \le v_2(t)$, $t \in [0,1]$, by the monotonicity of Riemann-Liouville fractional integral I_{0+}^{δ}, $\delta > 0$ and (H_1),

$$\begin{aligned} Av_1(t) &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v_1(s), I_{0+}^{n-3}v_1(s), \dots, I_{0+}^{1}v_1(s), v_1(s)\right) ds \\ &\le \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}v_2(s), I_{0+}^{n-3}v_2(s), \dots, I_{0+}^{1}v_2(s), v_2(s)\right) ds = Av_2(t). \end{aligned} \tag{3.4}$$

That is $Av_1 \le Av_2$. Hence, the condition (D_1) in Theorem 2.5 is satisfied.

Next we show that the condition (D_3) holds. From (H_4), for any $\gamma \in (0,1)$ and $v \in P$, we obtain

$$\begin{aligned} A(\gamma v)(t) &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}\gamma v(s), I_{0+}^{n-3}\gamma v(s), \dots, I_{0+}^{1}\gamma v(s), \gamma v(s)\right) ds \\ &= \int_0^1 G(t,s) f\left(s, \gamma I_{0+}^{n-2}v(s), \gamma I_{0+}^{n-3}v(s), \dots, \gamma I_{0+}^{1}v(s), \gamma v(s)\right) ds \\ &\ge \int_0^1 G(t,s) \varphi(\gamma) f\left(s, I_{0+}^{n-2}v(s), I_{0+}^{n-3}v(s), \dots, I_{0+}^{1}v(s), v(s)\right) ds \\ &= \varphi(\gamma) Av(t), \quad t \in [0,1]. \end{aligned} \tag{3.5}$$

That is $A(\gamma v) \geq \varphi(\gamma) A v$, for all $v \in P$, $\gamma \in (0,1)$. So the condition (D_3) in Theorem 2.5 is satisfied. Now we show that the condition (D_2) is also satisfied. On one hand, it follows from (H_1) and Lemma 2.4 that

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds \\ &\geq h(t) \int_0^1 \frac{\alpha-n+1}{\Gamma(\alpha-n+2)} s(1-s)^{\alpha-n+1}\ f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds \\ &= \frac{\alpha-n+1}{\Gamma(\alpha-n+2)} h(t) \\ &\quad \times \int_0^1 s(1-s)^{\alpha-n+1} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds, \quad t \in [0,1]. \end{aligned} \tag{3.6}$$

On the other hand, also from (H_1) and Lemma 2.4, we obtain

$$Ah(t) \leq \frac{1}{\Gamma(\alpha-n+2)} h(t) \int_0^1 (1-s)^{\alpha-n} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds, \quad t \in [0,1]. \tag{3.7}$$

Let

$$\begin{aligned} \lambda(h) &= \frac{\alpha-n+1}{\Gamma(\alpha-n+2)} \int_0^1 s(1-s)^{\alpha-n+1} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds, \\ \mu(h) &= \frac{1}{\Gamma(\alpha-n+2)} \int_0^1 (1-s)^{\alpha-n} f\left(s, I_{0+}^{n-2}h(s), I_{0+}^{n-3}h(s), \dots, I_{0+}^{1}h(s), h(s)\right) ds. \end{aligned} \tag{3.8}$$

Since f is continuous and $f \not\equiv 0$, we can get $0 < \lambda(h) \leq \mu(h)$. Consequently,

$$\lambda(h) h(t) \leq Ah(t) \leq \mu(h) h(t). \tag{3.9}$$

Next we consider x_0. If $g(t) \equiv 0$, then $x_0(t) \equiv 0$; if $g(t) \not\equiv 0$, let $l = \max_{t \in [0,1]} g(t)$, then $l > 0$. It is easy to prove that

$$0 \leq x_0(t) \leq \frac{l}{\Gamma(\alpha-n+2)} h(t) \int_0^1 (1-s)^{\alpha-n} ds = \frac{l}{(\alpha-n+1)\Gamma(\alpha-n+2)} h(t). \tag{3.10}$$

Hence,

$$0 \leq x_0 \leq \frac{l}{(\alpha-n+1)\Gamma(\alpha-n+2)} h. \tag{3.11}$$

Further,

$$\lambda(h)h \leq x_0 + Ah \leq \left[\mu(h) + \frac{l}{(\alpha - n + 1)\Gamma(\alpha - n + 2)}\right]h. \tag{3.12}$$

Hence $x_0 + Ah \in P_h$. Finally, using Theorem 2.5, $v = Av + x_0$ has a unique solution v^* in P_h. That is, v^* is a unique positive solution of the problem (2.3) in P_h. So there are $\mu(v^*)$, $\lambda(v^*)$ with $\mu(v^*) \geq \lambda(v^*) > 0$ such that $\lambda(v^*)t^{\alpha-n+1}(1-t) \leq v^*(t) \leq \mu(v^*)t^{\alpha-n+1}(1-t)$, $t \in [0,1]$. From Lemma 2.3, $u^* = I_{0+}^{n-2}v^*$ is the solution of the problem (1.1). Evidently, $u^* \in C([0,1],[0,+\infty))$ is a unique positive solution of the problem (1.1). □

Remark 3.1. Let $f \equiv C > 0$. Then the conditions (H_1), (H_4) are satisfied and the problem (2.3) has a unique solution $v(t) = \int_0^1 G(t,s)[C + g(t)]ds$, $t \in [0,1]$. From Lemma 2.4, the unique solution v is a positive solution and satisfies $v \in P_h = P_{t^{\alpha-n+1}(1-t)}$. So $u = I_{0+}^{n-2}v$ is a unique positive solution of the problem (1.1).

To illustrate how our main result can be used in practice we present an example.

Example 3.2. Consider the following problem:

$$\begin{gathered} D_{0+}^{5/2}u(t) + a(t)\left\{u^{1/3}(t) + \left[u'(t)\right]^{1/4}\right\} + g(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \end{gathered} \tag{3.13}$$

where $a, g : [0,1] \to [0,+\infty)$ are continuous with $a \not\equiv 0$.

In this example, we have $\alpha = 5/2$. Let $f(t,x,y) = a(t)[x^{1/3} + y^{1/4}]$. Evidently, $f(t,x,y)$ is increasing in x for $t \in [0,1]$, $y \geq 0$, and increasing in y for $t \in [0,1]$, $x \geq 0$. Moreover, $f \not\equiv 0$. Set $\varphi(\gamma) = \gamma^{5/12}, \gamma \in (0,1)$. Then

$$f(t,\gamma x,\gamma y) = a(t)\left[\gamma^{1/3}x^{1/3} + \gamma^{1/4}y^{1/4}\right] \geq \varphi(\gamma)f(t,x,y), \quad t \in [0,1], x, y \geq 0. \tag{3.14}$$

Hence, all the conditions of Theorem 1.2 are satisfied. An application of Theorem 1.2 implies that the problem (3.13) has a unique positive solution.

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Research Article

Spectral Analysis of q-Sturm-Liouville Problem with the Spectral Parameter in the Boundary Condition

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This paper is concerned with q-Sturm-Liouville boundary value problem in the Hilbert space with a spectral parameter in the boundary condition. We construct a self-adjoint dilation of the maximal dissipative q-difference operator and its incoming and outcoming spectral representations, which make it possible to determine the scattering matrix of the dilation. We prove theorems on the completeness of the system of eigenvalues and eigenvectors of operator generated by boundary value problem.

1. Introduction

Spectral analysis of Sturm-Liouville and Schrödinger differential equations with a spectral parameter in the boundary conditions has been analyzed intensively (see [1–16]). Then spectral analysis of discrete equations became an interesting subject in this field. So there is a substantial literature on this subject (see [10, 17–19]).

There has recently been great interest in quantum calculus and many works have been devoted to some problems of q-difference equation. In particular, we refer the reader to consult the reference [20] for some definitions and theorems on q-derivative, q-integration, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and Gamma functions, Euler-Maclaurin formula, anf so forth. In [21], Adıvar and Bohner investigated the eigenvalues and the spectral singularities of non-selfa-djoint q-difference equations of second order with spectral singularities. In [12], Huseynov and Bairamov examined the properties of eigenvalues and eigenvectors of a quadratic pencil of q-difference equations. In [22], Agarwal examined spectral analysis of self-adjoint equations. In [23], Shi and Wu presented several classes of explicit self-adjoint Sturm-Liouville difference operators with either a non-Hermitian leading coefficient function, or a non-Hermitian potential function, or a nondefinite weight function, or a non-self-adjoint boundary condition. In [24], Annaby and Mansour studied a q-analogue of Sturm-Liouville eigenvalue problems and formulated a self-adjoint q-difference operator in a Hilbert space. They also discussed properties of the eigenvalues and the eigenfunctions.

In this paper, we consider q-Sturm-Liouville Problem and define an adequate Hilbert space. Our main target of the present paper is to study q-Sturm-Liouville boundary value problem in case of dissipation at the right endpoint of $(0, a)$ and with the spectral parameter at zero. The maximal dissipative q-Sturm-Liouville operator is constructed using [25, 26] and Lax-Phillips scattering theory in [27]. Then we constructed a functional model of dissipative operator by means of the incoming and outcoming spectral representations and defined its characteristic function in terms of the solutions of the corresponding q-Sturm-Liouville equation. By combining the results of Nagy-Foiaş and Lax-Phillips, characteristic function is expressed with scattering matrix and the dilation of dissipative operator is set up. Finally, we give theorems on completeness of the system of eigenvectors and associated vectors of the dissipative q-difference operator.

Let q be a positive number with $0 < q < 1$, $A \subset \mathbb{R}$, and $a \in \mathbb{C}$. A q-difference equation is an equation that contains q-derivatives of a function defined on A. Let $y(x)$ be a complex-valued function on $x \in A$. The q-difference operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{\mu(x)}, \quad \forall x \in A, \tag{1.1}$$

where $\mu(x) = (q-1)x$. The q-derivative at zero is defined by

$$D_q y(0) = \lim_{n \to \infty} \frac{y(q^n x) - y(0)}{q^n x}, \quad x \in A, \tag{1.2}$$

if the limit exists and does not depend on x. A right inverse to Dq, the Jackson q-integration, is given by

$$\int_0^x f(t) d_q t = x(1-q) \sum_{n=0}^{\infty} q^n f(q^n x), \quad x \in A, \tag{1.3}$$

provided that the series converges, and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in A. \tag{1.4}$$

Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)| d_q x \right)^{1/2} < \infty. \tag{1.5}$$

The space $L_q^2(0,a)$ is a separable Hilbert space with the inner product

$$(f,g) := \int_0^a f(x)\overline{g(x)}d_qx, \quad f,g \in L_q^2(0,a). \tag{1.6}$$

We will consider the basic Sturm-Liouville equation

$$l(y) := -\frac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x), \quad 0 \le x \le a < +\infty, \tag{1.7}$$

where $v(x)$ is defined on $[0,a]$ and continuous at zero. The q-Wronskian of $y_1(x),\ y_2(x)$ is defined to be

$$W_q(y_1,y_2)(x) := y_1(x)D_qy_2(x) - y_2(x)D_qy_1(x), \quad x \in [0,a]. \tag{1.8}$$

Let L_0 denote the closure of the minimal operator generated by (1.7) and by D_0 its domain. Besides, we denote by D the set of all functions $y(x)$ from $L_q^2(0,a)$ such that $y(x)$ and $D_qy(x)$ are continuous in $[0,a)$ and $l(y) \in L_q^2(0,a)$; D is the domain of the maximal operator L. Furthermore, $L = L_0^*$ [2, 4, 13]. Suppose that the operator L_0 has defect index $(2,2)$.

For every $y,z \in D$ we have q-Lagrange's identity [24]

$$(Ly,z) - (y,Lz) = [y,\overline{z}](a) - [y,\overline{z}](0), \tag{1.9}$$

where $[y,\overline{z}] := y(x)\overline{D_{q^{-1}}z(x)} - D_{q^{-1}}y(x)\overline{z(x)}$.

2. Construction of the Dissipative Operator

Consider boundary value problem governed by

$$(ly) = \lambda y, \quad y \in D, \tag{2.1}$$

subject to the boundary conditions

$$y(a) - hD_{q^{-1}}y(a) = 0, \quad \operatorname{Im} h > 0, \tag{2.2}$$

$$\alpha_1 y(0) - \alpha_2 D_{q^{-1}}y(0) = \lambda\left(\alpha_1' y(0) - \alpha_2' D_{q^{-1}}y(0)\right), \tag{2.3}$$

where λ is spectral parameter and $\alpha_1,\alpha_2,\alpha_1',\alpha_2' \in \mathbb{R}$ and α is defined by

$$\alpha := \begin{vmatrix} \alpha_1' & \alpha_1 \\ \alpha_2' & \alpha_2 \end{vmatrix} = \alpha_1'\alpha_2 - \alpha_1\alpha_2' > 0. \tag{2.4}$$

For convenience we assume

$$
\begin{aligned}
R_0(y) &:= \alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0), \\
R_0'(y) &:= \alpha_1' y(0) - \alpha_2' D_{q^{-1}} y(0), \\
N_1^a(y) &:= y(a), \\
N_2^a(y) &:= D_{q^{-1}} y(a), \\
N_1^0(y) &:= y(0), \\
N_2^0(y) &:= D_{q^{-1}} y(0)_\infty, \\
R_a(y) &:= N_2^a(y) - h N_1^a(y).
\end{aligned}
\tag{2.5}
$$

Lemma 2.1. *For arbitrary $y, z \in D$, let one suppose that $R_0(\overline{z}) = \overline{R_0(z)}, R_0'(\overline{z}) = \overline{R_0'(z)}$, then one has the following.*

Proof.

$$
[y,z]_0 = \frac{1}{\alpha}\left[R_0(y)\overline{R_0'(z)} - R_0'(y)\overline{R_0(z)}\right],
\tag{2.6}
$$

$$
\begin{aligned}
&\frac{1}{\alpha}\left[R_0(y)R_0'\overline{(z)} - R_0'(y)\overline{R_0(z)}\right] \\
&\quad= \frac{1}{\alpha}\left[\begin{array}{c} \left(\alpha_1 y(0) - \alpha_2 D_{q^{-1}} y(0)\right)\left(\overline{\alpha_1' z(0) - \alpha_2' D_{q^{-1}} z(0)}\right) \\ -\left(\alpha_1' y(0) - \alpha_2' D_{q^{-1}} y(0)\right)\left(\overline{\alpha_1 z(0) - \alpha_2 D_{q^{-1}} z(0)}\right) \end{array}\right] \\
&\quad= \frac{1}{\alpha}\left[\left(\alpha_1'\alpha_2 - \alpha_1\alpha_2'\right)\left(\overline{y(0)} D_{q^{-1}} z(0) - D_{q^{-1}} y(0)\overline{z(0)}\right)\right] \\
&\quad= [y,z]_0.
\end{aligned}
\tag{2.7}
$$

Let θ_1, θ_2 denote the solutions of (2.1) satisfying the conditions

$$
N_1^0(\theta_2) = \alpha_2 - \alpha_2'\lambda, \qquad N_2^0(\theta_2) = \alpha_1 - \alpha_1'\lambda, \qquad N_1^a(\theta_1) = h, \qquad N_2^a(\theta_1) = 1.
\tag{2.8}
$$

Then from (2.3) we have

$$
\begin{aligned}
\Delta(\lambda) &= [\theta_1,\theta_2]_x = -[\theta_2,\theta_1]_x = -[\theta_2,\theta_1]_0 \\
&= -\frac{1}{\alpha}\left[R_0(\theta_1)\overline{R_0'(\theta_2)} - R_0'(\theta_1)\overline{R_0(\theta_2)}\right] \\
&= R_0(\theta_2) - \lambda R_0'(\theta_2),
\end{aligned}
$$

$$\begin{aligned}\Delta(\lambda) &= [\theta_1, \theta_2]_x = -[\theta_2, \theta_1]_x = -[\theta_2, \theta_1]_a \\ &= -\left(y(a)\overline{D_{q^{-1}} z(a)} - z(a)\overline{D_{q^{-1}} y(a)}\right) \\ &= -\left(y(a) - h\overline{D_{q^{-1}} y(a)}\right) = -\left(N_2^a(\theta_1) - hN_1^a(\theta_1)\right).\end{aligned} \tag{2.9}$$

We let

$$G(x, \xi, \lambda) = \frac{-1}{\Delta(\lambda)} \begin{Bmatrix} \theta_2(\xi, \lambda)\theta_1(x, \lambda), & x < \xi \\ \theta_1(x, \lambda)\theta_2(\xi, \lambda), & \xi < x \end{Bmatrix}. \tag{2.10}$$

It can be shown that $G(x, \xi, \lambda)$ satisfies (2.1) and boundary conditions (2.2)–(2.3). $G(x, \xi, \lambda)$ is a Green function of the boundary value problem (2.1)–(2.3). Thus, we obtain that the $G(x, \xi, \lambda)$ is a Hilbert-Schmidt kernel and the solution of the boundary value problem can be expressed by

$$y(x, \lambda) = \int_0^a G(x, \xi, \lambda) y(\xi, \lambda) d\xi = R_\lambda y. \tag{2.11}$$

Thus R_λ is a Hilbert Schmidt operator on space $L_q^2(0, a)$. The spectrum of the boundary value problem coincides with the roots of the equation $\Delta(\lambda) = 0$. Since Δ is analytic and not identical to zero, it means that the function Δ has at most a countable number of isolated zeros with finite multiplicity and possible limit points at infinity. □

Suppose that $f^{(1)} \in L_2[0, a)$, $f^{(2)} \in \mathbb{C}$, then we denote linear space $H = L_q^2(0, a) \oplus \mathbb{C}$ with two component of elements of $\widehat{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}$. If $\alpha > 0$ and $\widehat{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}$, $\widehat{g} = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix} \in H$, then the formula

$$\left(\widehat{f}, \widehat{g}\right) = \int_0^a f^{(1)}(x)\overline{g}^{(1)} d_q x + \frac{1}{\alpha} f^{(2)}\overline{g}^{(2)} \tag{2.12}$$

defines an inner product in Hilbert space H. Let us define operator of $A_h : H \to H$ with equalities suitable for boundary value problem

$$\begin{gathered} D(A_h) = \left\{ \widehat{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in H : f^{(1)} \in D, R_a\left(f^{(1)}\right) = 0, f^{(2)} = R_0'\left(f^{(1)}\right) \right\}, \\ A_h \widehat{f} = \widetilde{l}\left(\widehat{f}\right) := \begin{pmatrix} l\left(f^{(1)}\right) \\ R_0\left(f^{(1)}\right) \end{pmatrix}. \end{gathered} \tag{2.13}$$

Remind that a linear operator A_h with domain $D(A_h)$ in Hilbert space H is called dissipative if $\operatorname{Im}(A_h f, f) \geq 0$ for all $f \in D(A_h)$ and maximal dissipative if it does not have a proper extension.

Definition 2.2. If the system of vectors of $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 is

$$
\begin{gathered}
l(y_0) = \lambda_0 y_0, \qquad R_0(y_0) - \lambda R_0'(y_0) = 0, \qquad R_a(y_0) = 0, \\
l(y_s) - \lambda_0 y_s - y_{s-1} = 0, \qquad R_0(y_s) - \lambda R_0'(y_s) - R_0'(y_{s-1}) = 0, \\
R_a(y_s) = 0, \qquad s = 1, 2, \ldots, n,
\end{gathered}
\tag{2.14}
$$

then the system of vectors of $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 is called a chain of eigenvectors and associated vectors of boundary value problem (2.2)–(2.12).

Since the operator A_h is dissipative in H and from Definition 2.2, we have the following.

Lemma 2.3. *The eigenvalue of boundary value problem* (2.1)–(2.3) *coincides with the eigenvalue of dissipative A_h operator. Additionally each chain of eigenvectors and associated vectors $y_0, y_1, y_2, \ldots, y_n$ corresponding to the eigenvalue λ_0 corresponds to the chain eigenvectors and associated vectors $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$ corresponding to the same eigenvalue λ_0 of dissipative A_h operator. In this case, the equality*

$$
\widehat{y}_k = \begin{pmatrix} y_k \\ R_0'(y_k) \end{pmatrix}, \quad k = 0, 1, 2, \ldots, n
\tag{2.15}
$$

holds.

Proof. $\widehat{y}_0 \in D(A_h)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, then the equality $l(y_0) = \lambda_0 y_0$, $R_0(y_0) - \lambda R_0'(y_0) = 0$, $R_1(y_0) = R_2(y_0) = 0$ takes place; that is, y_0 is an eigenfunction of the problem. Conversely, if conditions (2.14) are realized, then $\begin{pmatrix} y_0 \\ R_0'(y_0) \end{pmatrix} = \widehat{y}_0 \in D(A_h)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $\widehat{y}_0$ is an eigenvector of the operator A_h. If $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$ are a chain of the eigenvectors and associated vectors of the operator A_h corresponding to the eigenvalue λ_0, then by implementing the conditions $\widehat{y}_k \in D(A_h)(k = 0, 1, 2, \ldots, n)$ and equality $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $A_h \widehat{y}_s = \lambda_0 \widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, \ldots, n$, we get the equality (2.15), where $y_0, y_1, y_2, \ldots, y_n$ are the first components of the vectors $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n$. On the contrary, on the basis of the elements $y_0, y_1, y_2, \ldots, y_n$ corresponding to (2.1)–(2.3), one can construct the vectors $\widehat{y}_k = \begin{pmatrix} y_k \\ R_0'(y_k) \end{pmatrix}$ for which $\widehat{y}_k \in D(A_h)(k = 0, 1, 2, \ldots, n)$ and $A_h \widehat{y}_0 = \lambda_0 \widehat{y}_0$, $A_h \widehat{y}_s = \lambda_0 \widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, \ldots, n$. □

Theorem 2.4. *The operator A_h is maximal dissipative in the space H.*

Proof. Let $\widehat{y} \in D(A_h)$. From (2.6), we have

$$
\begin{aligned}
(A_h \widehat{y}, \widehat{y}) - (\widehat{y}, A_h \widehat{y}) &= [y_1, y_1]_a - [y_1, \overline{y_1}]_0 + \frac{1}{\alpha}\left[R_0(y_1)\overline{R_0'(y_1)} - R_0'(y_1)\overline{R_0(y_1)}\right] \\
&= [y_1, y_1]_a = 2\,\mathrm{Im}\, h\left(D_{q^{-1}} y_1(a)\right)^2.
\end{aligned}
\tag{2.16}
$$

It follows from that $\operatorname{Im}(A_h\widehat{y},\widehat{y}) = \operatorname{Im} h(D_{q^{-1}}y_1(a))^2 \geq 0$, A_h is a dissipative operator in H. Let us prove that A_h is maximal dissipative operator in the space H. It is sufficient to check that

$$(A_h - \lambda I)D(A_h) = H, \qquad \operatorname{Im}\lambda < 0. \tag{2.17}$$

To prove (2.17), let $F \in H$, $\operatorname{Im}\lambda < 0$ and put

$$\Gamma = \begin{pmatrix} \left(\widetilde{G}_x, \overline{F}\right) \\ R'_0\left[\left(\widetilde{G}_x, \overline{F}\right)\right] \end{pmatrix}, \tag{2.18}$$

where

$$\begin{gathered} \widetilde{G}_x = \begin{pmatrix} G(x,\xi,\lambda) \\ R'_0[G(x,\xi,\lambda)] \end{pmatrix} = \begin{pmatrix} G(x,\xi,\lambda) \\ \dfrac{-1}{\Delta(\lambda)}\theta_1(x,\lambda)\alpha \end{pmatrix}, \\ G(x,\xi,\lambda) = \frac{-1}{\Delta(\lambda)}\begin{Bmatrix} \theta_2(\xi,\lambda)\theta_1(x,\lambda), & x < \xi \\ \theta_1(x,\lambda)\theta_2(\xi,\lambda), & \xi < x \end{Bmatrix}. \end{gathered} \tag{2.19}$$

The function $x \to (G(x,\xi,\lambda), \overline{F_1})$ satisfies the equation $l(y) - \lambda y = F_1 (0 \leq x < \infty)$ and the boundary conditions (2.1)–(2.3). Moreover, for all $F \in H$ and for $\operatorname{Im}\lambda < 0$, we arrive at $\Gamma \in D(A_h)$. For each $F \in H$ and for $\operatorname{Im}\lambda < 0$, we have $(A_h - \lambda I)\Gamma = F$. Consequently, in the case of $\operatorname{Im}\lambda < 0$, the result is $(A_h - \lambda I)D(A_h) = H$. Hence, Theorem 2.4 is proved. □

3. Self-Adjoint Dilation of Dissipative Operator

We first construct the self-adjoint dilation of the operator A_h. Let us add the "incoming" and "outgoing" subspaces $D_- = L^2(-\infty, 0]$ and $D_+ = L^2[0,\infty)$ to $H = L_q^2(0,a) \oplus \mathbb{C}$. The orthogonal sum $\mathrm{H} = D_- \oplus H \oplus D_+$ is called *main Hilbert space of the dilation*. In the space $\mathscr{H}$ we consider the operator $\mathcal{L}_h$ on the set $D(\mathcal{L}_h)$, its elements consisting of vectors $w = \langle \varphi_-, y, \varphi_+ \rangle$, generated by the expression

$$\mathcal{L}\langle \varphi_-, \widehat{y}, \varphi_+ \rangle = \left\langle i\frac{d\varphi_-}{d\xi}, \widetilde{l}(\widehat{y}), i\frac{d\varphi_+}{d\xi} \right\rangle. \tag{3.1}$$

satisfying the conditions: $\varphi_- \in W_2^1(-\infty, 0]$, $\varphi_+ \in W_2^1[0,\infty)$, $\widehat{y} \in H$, $\widehat{y} = \left(\begin{smallmatrix} y_1(x) \\ y_2 \end{smallmatrix}\right)$, $y_1 \in D$, $y_2 = R_0(y_1)$, and

$$y(a) - hD_{q^{-1}}y(a) = \beta\varphi_-(0), \qquad y(a) - \overline{h}D_{q^{-1}}y(a) = \beta\varphi_+(0), \tag{3.2}$$

where $W_2^1(\cdot,\cdot)$ are Sobolev spaces and $\beta^2 := 2\operatorname{Im} h$, $\beta > 0$. Then we have the following.

Theorem 3.1. *The operator $\mathcal{L}_h$ is self-adjoint in $\mathscr{H}$ and it is a self-adjoint dilation of the operator A_h.*

Proof. We first prove that $\mathcal{L}_h$ is symmetric in $\mathcal{H}$. Namely $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Let $f, g \in D(\mathcal{L}_h)$, $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle$ and $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle$. Then we have

$$\begin{aligned}
(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} &= (\mathcal{L}\langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \langle \psi_-, \widehat{z}, \psi_+ \rangle) - (\langle \varphi_-, \widehat{y}, \varphi_+ \rangle, \mathcal{L}\langle \psi_-, \widehat{z}, \psi_+ \rangle), \\
&= [y_1, \overline{z_1}]_a - [y_1, \overline{z_1}]_0 + \frac{1}{\alpha}\Big[R_0(y_1)\overline{R_0'(z_1)} - R_0'(y_1)\overline{R_0(z_1)}\Big] \\
&\quad + i\varphi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0), \\
(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} &= [y_1, \overline{z_1}]_a + i\varphi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0).
\end{aligned} \tag{3.3}$$

On the other hand,

$$\begin{aligned}
i\varphi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0) &= \frac{i}{\beta^2}\left(y(a) - hD_{q^{-1}}y(a)\right)\overline{\left(z(a) - hD_{q^{-1}}z(a)\right)} \\
&\quad - \frac{i}{\beta^2}\Big(y(a) - \overline{h}D_{q^{-1}}y(a)\Big)\overline{\Big(z(a) - \overline{h}\ D_{q^{-1}}z(a)\Big)}, \\
&= \frac{i}{\beta^2}\Big[\Big(h - \overline{h}\Big)y(a)\overline{D_{q^{-1}}z(a)} - D_{q^{-1}}y(a)\overline{z(a)}\Big].
\end{aligned} \tag{3.4}$$

By (3.3), we have

$$i\varphi_-(0)\overline{\varphi}_-(0) - i\varphi_+(0)\overline{\varphi}_+(0) = -[y_1, \overline{z_1}]_a. \tag{3.5}$$

From equalities (3.3) and (3.5), we have $(\mathcal{L}_h f, g)_{\mathcal{H}} - (f, \mathcal{L}_h g)_{\mathcal{H}} = 0$. Thus, $\mathcal{L}_h$ is a symmetric operator. To prove that $\mathcal{L}_h$ is self-adjoint, we need to show that $\mathcal{L}_h \subseteq \mathcal{L}_h^*$. We consider the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ on elements $g = \langle \psi_-, \widehat{z}, \psi_+ \rangle \in D(\mathcal{L}_h^*)$, where $f = \langle \varphi_-, \widehat{y}, \varphi_+ \rangle \in D(\mathcal{L}_h)$, $\varphi_\mp \in W_2^1(\mathbb{R}_\mp), \varphi_\mp(0) = 0$. Integrating by parts, we get $\mathcal{L}_h^* g = \langle i(d\psi_-/d\xi), \widehat{z}^*, i(d\psi_+/d\xi) \rangle$, where $\psi_\mp \in W_2^1(\mathbb{R}_\mp)$, $\widehat{z}^* \in H$. Similarly, if $f = \langle 0, \widehat{y}, 0 \rangle \in D(\mathcal{L}_h)$, then integrating by parts in $(\mathcal{L}_h f, g)_{\mathcal{H}}$, we obtain

$$\mathcal{L}_h^* g = \mathcal{L}^*\langle \psi_-, \widehat{z}, \psi_+ \rangle = \left\langle i\frac{d\psi_-}{d\xi}, \widetilde{l}(\widehat{z}), i\frac{d\psi_+}{d\xi} \right\rangle, \quad z_1 \in D, z_2 = R_0'(z_1). \tag{3.6}$$

Consequently, we have $(\mathcal{L}_h f, g)_{\mathcal{H}} = (f, \mathcal{L}_h g)_{\mathcal{H}}$, for each $f \in D(\mathcal{L}_h)$ by (3.6), where the operator $\mathcal{L}$ is defined by (3.1). Therefore, the sum of the integrated terms in the bilinear form $(\mathcal{L}_h f, g)_{\mathcal{H}}$ must be equal to zero:

$$[y_1, \overline{z_1}]_a - [y_1, \overline{z_1}]_0 + \frac{1}{\alpha}\Big[R_0(y_1)\overline{R_0^{'}(z_1)} - R_0'(y_1)\overline{R_0(z_1)}\Big] + i\varphi_-(0)'\overline{\psi}_-(0) - i\varphi'_+(0)\overline{\psi}_+(0) = 0. \tag{3.7}$$

Then by (2.6), we get

$$[y_1, \overline{z_1}]_a + i\varphi_-(0)'\overline{\psi}_-(0) - i\varphi'_+(0)\overline{\psi}_+(0) = 0. \tag{3.8}$$

From the boundary conditions for $\mathcal{L}_h$, we have

$$\begin{gathered} y(a) = \beta\varphi_-(0) - \frac{h_1}{i\beta}\left(\varphi_-(0) - \varphi_+(0)\right), \\ D_{q^{-1}}y(a) = \frac{i}{\beta}\left(\varphi_-(0) - \varphi_+(0)\right). \end{gathered} \tag{3.9}$$

Afterwards, by (3.8) we get

$$\begin{aligned} &\beta\varphi_-(0) - \frac{h_1}{i\beta}\left(\varphi_-(0) - \varphi_+(0)\right)\overline{z(a)} - \frac{i}{\beta}\left(\varphi_-(0) - \varphi_+(0)\right)\overline{D_{q^{-1}}z(a)} \\ &\quad = i\varphi_+(0)\overline{\varphi}_+(0) - i\varphi_-(0)\overline{\varphi}_-(0). \end{aligned} \tag{3.10}$$

Comparing the coefficients of $\varphi_-(0)$ in (3.10), we obtain

$$\frac{i\beta^2 - h_1}{\beta}z(a) + \frac{1}{\beta}\overline{D_{q^{-1}}z(a)} = \varphi_-(0) \tag{3.11}$$

or

$$z(a) - hD_{q^{-1}}z(a) = \beta\varphi_-(0). \tag{3.12}$$

Similarly, comparing the coefficients of $\varphi_+(0)$ in (3.10) we get

$$z(a) - \overline{h}D_{q^{-1}}z(a) = \beta\varphi_+(0). \tag{3.13}$$

Therefore conditions (3.12) and (3.13) imply $D(\mathcal{L}_h^*) \subseteq D(\mathcal{L}_h)$, hence $\mathcal{L}_h = \mathcal{L}_h^*$.

The self-adjoint operator $\mathcal{L}_h$ generates on $\mathcal{H}$ a unitary group $U_t = \exp(i\mathcal{L}_h t)$ $(t \in \mathbb{R}_+ = (0,\infty))$. Let us denote by $P : \mathcal{H} \to H$ and $P_1 : H \to \mathcal{H}$ the mapping acting according to the formulae $P : \langle \varphi_-, \widehat{y}, \varphi_+ \rangle \to \widehat{y}$ and $P_1 : \widehat{y} \to \langle 0, \widehat{y}, 0\rangle$. Let $Z_t := PU_tP_1, t \geq 0$, by using U_t. The family $\{Z_t\}(t \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contraction on H. Let us denote by B_h the generator of this semigroup: $B_h\widehat{y} = \lim_{t\to+0}(it)^{-1}(Z_t\widehat{y} - \widehat{y})$. The domain of B_h consists of all the vectors for which the limit exists. The operator B_h is dissipative. The operator $\mathcal{L}_h$ is called the self-adjoint dilation of B_h (see [2, 9, 18]). We show that $B_h = A_h$, hence $\mathcal{L}_h$ is self-adjoint dilation of B_h. To show this, it is sufficient to verify the equality

$$P(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = (A_h - \lambda I)^{-1}\widehat{y}, \quad \widehat{y} \in H, \operatorname{Im} h < 0. \tag{3.14}$$

For this purpose, we set $(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = g = \langle \psi_-, \widehat{z}, \psi_+\rangle$ which implies that $(\mathcal{L}_h - \lambda I)g = P_1\widehat{y}$, and hence $\widetilde{l}(\widehat{z}) - \lambda\widehat{z} = \widehat{y}, \psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\xi) = \psi_+(0)e^{-i\lambda\xi}$. Since $g \in D(\mathcal{L}_h)$, then $\psi_- \in L^2(-\infty,0)$, and it follows that $\psi_-(0) = 0$, and consequently z satisfies the boundary condition $z(a) - hD_{q^{-1}}z(a) = 0$. Therefore, $\widehat{z} \in D(A_h)$, and since point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of dissipative operator, it follows that $\varphi_+(0)$ is obtained from the formula $\varphi_+(0) = \beta^{-1}(z(a) - \overline{h}D_{q^{-1}}z(a))$. Thus

$$(\mathcal{L}_h - \lambda I)^{-1}P_1\widehat{y} = \left\langle 0, (A_h - \lambda I)^{-1}\widehat{y}, \beta^{-1}\left(z(a) - \overline{h}D_{q^{-1}}z(a)\right)\right\rangle \tag{3.15}$$

for $\widehat{y}$ and $\operatorname{Im}\lambda < 0$. On applying the mapping P, we obtain (3.14), and

$$\begin{aligned}(A_h - \lambda I)^{-1} &= P(\mathcal{L}_h - \lambda I)^{-1}P_1 = -iP\int_0^{\infty} U_t e^{-i\lambda t}dtP_1 \\ &= -i\int^{\infty} Z_t e^{-i\lambda t}dt = (B_h - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned} \tag{3.16}$$

so this clearly shows that $A_h = B_h$. □

The unitary group $\{U_t\}$ has an important property which makes it possible to apply it to the Lax-Phillips [27], that is, it has orthogonal incoming and outcoming subspaces $D_- = \langle L^2(-\infty, 0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$ having the following properties:

(1) $U_tD_- \subset D_-$, $t \leq 0$ and $U_tD_+ \subset D_+$, $t \geq 0$;

(2) $\cap_{t\leq 0}U_tD_- = \cap_{t\geq 0}U_tD_+ = \{0\}$;

(3) $\cup_{t\geq 0}U_tD_- = \cup_{t\leq 0}U_tD_+ = \mathscr{H}$;

(4) $D_- \perp D_+$.

To be able to prove property (1) for D_+ (the proof for D_- is similar), we set $\mathcal{R}_\lambda = (\mathcal{L}_h - \lambda I)^{-1}$. For all λ, with $\operatorname{Im}\lambda < 0$ and for any $f = \langle 0, 0, \varphi_+\rangle \in D_+$, we have

$$\mathcal{R}_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^{\xi} e^{i\lambda s}\varphi_+(s)ds\right\rangle, \tag{3.17}$$

as $\mathcal{R}_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (\mathcal{R}_\lambda f, g)_{\mathscr{H}} = -i\int_0^{\infty} e^{-i\lambda t}(U_t f, g)_{\mathscr{H}}dt, \quad \operatorname{Im}\lambda < 0 \tag{3.18}$$

which implies that $(U_t f, g)_{\mathscr{H}} = 0$ for all $t \geq 0$. Hence, for $t \geq 0$, $U_tD_+ \subset D_+$, and property (1) has been proved.

In order to prove property (2), we define the mappings $P^+ : \mathscr{H} \to L^2(0,\infty)$ and $P_1^+ : L^2(0,\infty) \to D_+$ as follows: $P^+ : \langle \varphi_-, \widehat{y}, \varphi_+\rangle \to \varphi_+$ and $P_1^+ : \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We take into consideration that the semigroup of isometries $U_t^+ := P^+U_tP_1^+(t \geq 0)$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $i(d/d\xi)$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator S of the semigroup of isometries $U_t^+(t \geq 0)$ is the operator $S\varphi = P^+\mathcal{L}_hP_1^+\varphi =$

$P^+\mathcal{L}_h\langle 0,0,\varphi\rangle = P^+\langle 0,0,i(d/d\xi)\varphi\rangle = i(d/d\xi)\varphi$, where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. Since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and hence

$$\bigcap_{t\geq 0} U_t D_+ = \left\langle 0,0,\bigcap_{t\leq 0} V_t L^2(0,\infty) \right\rangle = \{0\}, \tag{3.19}$$

so, the proof of property (2) is completed.

Definition 3.2. The linear operator A with domain $D(A)$ acting in the Hilbert space H is called *completely non-self-adjoint (or simple)* if there is no invariant subspace $M \subseteq D(A)(M \neq \{0\})$ of the operator A on which the restriction A to M is self-adjoint.

To prove property (3) of the incoming and outcoming subspaces, let us prove following lemma.

Lemma 3.3. *The operator A_h is completely non-self-adjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which A_h induces a self-adjoin t operator A'_h with domain $D(A'_h) = H' \cap D(A_h)$. If $\widehat{f} \in D(A'_h)$, then $\widehat{f} \in D(A'^*_h)$ and

$$\begin{aligned} \frac{d}{dt}\left\| e^{iA'_h t}\widehat{f}\right\|_H^2 &= \frac{d}{dt}\left(e^{iA'_h t}\widehat{f}, e^{iA'_h t}\widehat{f}\right)_H \\ &= i\left(A'_h e^{iA'_h t}\widehat{f}, e^{iA'_h t}\widehat{f}\right) - i\left(e^{iA'_h t}\widehat{f}, A'_h e^{iA'_h t}\widehat{f}\right) \end{aligned} \tag{3.20}$$

and taking $\widehat{g} = e^{iA'_h t}\widehat{f}$, we have

$$\begin{aligned} 0 &= i\left(A'_h\widehat{g},\widehat{g}\right)_H - i\left(\widehat{g}, A'_h\widehat{g}\right)_H \\ &= i\left[g_1,\overline{g_1}\right]_a - i\left[g_1,\overline{g_1}\right]_0 + \frac{i}{\alpha}\left[R_0(g_1)\overline{R'_0(g_1)} - R'_0(y_1)\overline{R_0(g_1)}\right] \\ &= -2\operatorname{Im} h\left(D_{q^{-1}}y_1(a)\right)^2 \\ &= -\beta^2\left(D_{q^{-1}}y_1(a)\right)^2. \end{aligned} \tag{3.21}$$

Since $\widehat{f} \in D(A'_h)$, A'_h holds condition above. Moreover, eigenvectors of the operator A'_h should also hold this condition. Therefore, for the eigenvectors $\widehat{y}(\lambda)$ of the operator A_h acting in H' and the eigenvectors of the operator A'_h, we have $D_{q^{-1}}y_1(a) = 0$. From the boundary conditions, we get $y_1(a) = 0$ and $\widehat{y}(x,\lambda) = 0$. Consequently, by the theorem on expansion in the eigenvectors of the self-adjoint operator A'_h, we obtain $H' = \{0\}$. Hence the operator A_h is simple. The proof is completed. □

Let us define $H_- = \overline{\cup_{t\geq 0} U_t D_-}$, $H_+ = \overline{\cup_{t\leq 0} U_t D_+}$.

Lemma 3.4. *The equality $H_- + H_+ = \mathscr{H}$ holds.*

Proof. Considering property (1) of the subspace D_+, it is easy to show that the subspace $\mathcal{H}' = \mathcal{H} \ominus (H_- + H_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') was nontrivial, then the unitary group $\{U_t\}$ restricted to this subspace would be a unitary part of the group $\{U_t'\}$, and hence the restriction B_h' of B_h to H' would be a self-adjoint operator in H'. Since the operator B_h is simple, it follows that $H' = \{0\}$. The lemma is proved. □

Assume that $\varphi(\lambda)$ and $\psi(\lambda)$ are solutions of $l(y) = \lambda y$ satisfying the conditions

$$\varphi_1(0,\lambda) = 0, \qquad \varphi_2(0,\lambda) = 1, \qquad \psi_1(0,\lambda) = 1, \qquad \psi_2(0,\lambda) = 0.$$
$$\theta(x,\lambda) = \varphi(x,\lambda) + m_a(\lambda)\psi(x,\lambda) \in L_q^2(0,a), \quad \operatorname{Im}\lambda > 0. \tag{3.22}$$

The Titchmarsh-Weyl function $m_a(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $m_a(\lambda)$ possesses the following properties: $\operatorname{Im} m_a(\lambda) \geq 0$ for all $\operatorname{Im}\lambda > 0$, and $\overline{m_a(\lambda)} = m_a(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles $m_a(\lambda)$. We set

$$S_h(\lambda) := \frac{m_a(\lambda) - h}{m_a(\lambda) - \overline{h}}, \tag{3.23}$$

$$U_\lambda^-(x,\xi,\zeta) = \langle e^{-i\lambda\xi}, (m_a(\lambda) - h)^{-1}\alpha\theta(x,\lambda), \overline{S_h}(\lambda)e^{-i\lambda\zeta}\rangle. \tag{3.24}$$

We note that the vectors $U_\lambda^-(x,\xi,\zeta)$ for real λ do not belong to the space $\mathcal{H}$. However, $U_\lambda^-(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. By means of vector $U_\lambda^-(x,\xi,\zeta)$, we define the transformation $F_- : f \to \widetilde{f}_-(\lambda)$ by

$$(F - f)(\lambda) := \widetilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_{\overline{\lambda}}^-)_{\mathcal{H}}, \tag{3.25}$$

on the vectors $f = \langle\varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$, $y(x)$ are smooth, compactly supported functions.

Lemma 3.5. *The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_-$ the Parseval equality and the inversion formulae hold:*

$$(f,g)_{\mathcal{H}} = \left(\widetilde{f}_-, \widetilde{g}_-\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_-(\lambda)U_{\overline{\lambda}}^- d\lambda, \tag{3.26}$$

where $\widetilde{f}_-(\lambda) = (F_- f)(\lambda)$ and $\widetilde{g}_-(\lambda) = (F_- g)(\lambda)$.

Proof. For $f, g \in D_-$, $f = \langle\varphi_-, 0, 0\rangle$, $g = \langle\varphi_+, 0, 0\rangle$, with Paley-Wiener theorem, we have

$$\widetilde{f}_-(\lambda) = \frac{1}{\sqrt{2\pi}}(f, U_{\overline{\lambda}}^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\varphi_-(\xi)e^{-i\lambda\xi}d\xi \in H_-^2, \tag{3.27}$$

and by using usual Parseval equality for Fourier integrals

$$(f,g)_{\mathscr{H}} = \int_{-\infty}^{\infty} \varphi_-(\xi)\overline{\varphi_-(\xi)}d\xi = \int_{-\infty}^{\infty} \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)}d\lambda = (F_-f, F_-g)_{L^2}. \tag{3.28}$$

Here, $H_\pm^2$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendible to the upper and lower half-planes, respectively.

We now extend to the Parseval equality to the whole of H_-. We consider in H_- the dense set of H'_- of the vectors obtained as follows from the smooth, compactly supported functions in $D_- : f \in H'_-$ if $f = U_T f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $T = T_f$ is a nonnegative number depending on f. If $f, g \in H'_-$, then for $T > T_f$ and $T > T_g$ we have $U_{-T}f, U_{-T}g \in D_-$; moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators $U_t (t \in \mathbb{R})$ are unitary, by the equality

$$F_-U_tf = (U_tf, U_{\lambda}^-)_{\mathscr{H}} = e^{i\lambda t}(f, U_\lambda^-)_{\mathscr{H}} = e^{i\lambda t}F_-f, \tag{3.29}$$

we have

$$\begin{gathered}(f,g)_{\mathscr{H}} = (U_{-T}f, U_{-T}g)_{\mathscr{H}} = (F_-U_{-T}f, F_-U_{-T}g)_{L^2} \\ \left(e^{i\lambda T}F_-f, e^{i\lambda T}F_-g\right)_{L^2} = \left(\widetilde{f}, \widetilde{g}\right)_{L^2}.\end{gathered} \tag{3.30}$$

By taking the closure (3.30), we obtain the Parseval equality for the space H_-. The inversion formula is obtained from the Parseval equality if all integrals in it are considered as limits in the of integrals over finite intervals. Finally $F_-H_- = \overline{\cup_{t\geq 0}F_-U_tD_-} = \overline{\cup_{t\geq 0}e^{i\lambda t}H_-^2} = L^2(\mathbb{R})$, that is, F_- maps H_- onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

We set

$$U_\lambda^+(x,\xi,\zeta) = \left\langle S_h(\lambda)e^{-i\lambda\xi}, \left(m_a(\lambda) - \overline{h}\right)^{-1}\alpha\theta(x,\lambda), e^{-i\lambda\zeta}\right\rangle. \tag{3.31}$$

We note that the vectors $U_\lambda^+(x,\xi,\zeta)$ for real λ do not belong to the space $\mathscr{H}$. However, $U_\lambda^+(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}U = \lambda U$ and the corresponding boundary conditions for the operator $\mathcal{L}_H$. With the help of vector $U_\lambda^+(x,\xi,\zeta)$, we define the transformation $F_+ : f \to \widetilde{f}_+(\lambda)$ by $(F_+f)(\lambda) := \widetilde{f}_+(\lambda) := (1/\sqrt{2\pi})(f, U_\lambda^+)_{\mathscr{H}}$ on the vectors $f = \langle \varphi_-, \widehat{y}, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$ and $y(x)$ are smooth, compactly supported functions.

Lemma 3.6. *The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $f, g \in H_+$ the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}} = \left(\widetilde{f}_+, \widetilde{g}_+\right)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)\overline{\widetilde{g}_+(\lambda)}d\lambda, \quad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{f}_+(\lambda)U_\lambda^+ d\lambda, \tag{3.32}$$

where $\widetilde{f}_+(\lambda) = (F_+f)(\lambda)$ and $\widetilde{g}_+(\lambda) = (F_+g)(\lambda)$.

Proof. The proof is analogous to Lemma 3.5. □

It is obvious that the matrix-valued function $S_h(\lambda)$ is meromorphic in $\mathbb{C}$ and all poles are in the lower half-plane. From (3.23), $|S_h(\lambda)| \leq 1$ for $\operatorname{Im}\lambda > 0$; and $S_h(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Therefore, it explicitly follows from the formulae for the vectors U_λ^- and U_λ^+ that

$$U_\lambda^+ = S_h(\lambda)U_\lambda^-. \tag{3.33}$$

It follows from Lemmas 3.5 and 3.6 that $H_- = H_+$. Together with Lemma 3.4, this shows that $H_- = H_+ = \mathscr{H}$; therefore, property (3) above has been proved for the incoming and outcoming subspaces. Finally property (4) is clear.

Thus, the transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t are transformed into the operators of multiplication by $e^{i\lambda t}$. This means that F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for the group $\{U_t\}$. It follows from (3.33) that the passage from the F_- representation of an element $f \in \mathscr{H}$ to its F_+ representation is accomplished as $\widetilde{f}_+(\lambda) = S_h^{-1}(\lambda)\widetilde{f}_-(\lambda)$. Consequently, according to [27] we have proved the following.

Theorem 3.7. *The function $S_h^{-1}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the self-adjoint operator L_H).*

Let $S(\lambda)$ be an arbitrary nonconstant inner function on the upper half-plane (the analytic function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|S_h(\lambda)| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $|S_h(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). Define $K = H_+^2 \ominus SH_+^2$. Then $K \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of operators $Z_t (t \geq 0)$ acting in K according to the formula $Z_t\varphi = P[e^{i\lambda t}\varphi], \varphi = \varphi(\lambda) \in K$, where P is the orthogonal projection from H_+^2 onto K. The generator of the semigroup $\{Z_t\}$ is denoted by

$$T\varphi = \lim_{t \to +0} (it)^{-1}(Z_t\varphi - \varphi), \tag{3.34}$$

in which T is a maximal dissipative operator acting in K and with the domain $D(T)$ consisting of all functions $\varphi \in K$, such that the limit exists. The operator T is called *a model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax-Phillips [27], is a special case of a more general model dissipative operator constructed by Nagy and Foiaş [26]). The basic assertion is that $S(\lambda)$ is the *characteristic function* of the operator T.

Let $K = \langle 0, H, 0 \rangle$, so that $\mathscr{H} = D_- \oplus K \bigoplus D_+$. It follows from the explicit form of the unitary transformation F_- under the mapping F_-

$$\begin{gathered}\mathscr{H} \longrightarrow L^2(\mathbb{R}), \qquad f \longrightarrow \widetilde{f}_-(\lambda) = (F_-f)(\lambda), \qquad D_- \longrightarrow H_-^2, \qquad D_+ \longrightarrow S_hH_+^2, \\ K \longrightarrow H_+^2 \ominus S_GH_+^2, \qquad U_tf \longrightarrow \left(F_-U_tF_-^{-1}\widetilde{f}_-\right)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda).\end{gathered} \tag{3.35}$$

The formulae (3.35) show that operator A_h is a unitarily equivalent to the model dissipative operator with the characteristic function $S_h(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operator coincide (see [26]), we have thus proved the following theorem.

Theorem 3.8. *The characteristic function of the maximal dissipative operator A_h coincides with the function $S_h(\lambda)$ defined in* (3.23).

Using characteristic function, the spectral properties of the maximal dissipative operator A_h can be investigated. The characteristic function of the maximal dissipative operator A_h is known to lead to information of completeness about the spectral properties of this operator. For instance, the absence of a singular factor $s(\lambda)$ of the characteristic function $S_h(\lambda)$ in the factorization $\det S_h(\lambda) = s(\lambda)B(\lambda)$, where $B(\lambda)$ is a Blaschke product, ensures completeness of the system of eigenvectors and associated vectors of the operator A_h in the space $L_q^2(0,a)$ (see [25]).

Theorem 3.9. *For all the values of h with* $\operatorname{Im} h > 0$, *except possibly for a single value $h = h_0$, the characteristic function $S_h(\lambda)$ of the maximal dissipative operator A_h is a Blaschke product. The spectrum of A_h is purely discrete and belongs to the open upper half-plane. The operator A_h has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenvectors and associated vectors of the operator A_h is complete in the space H.*

Proof. From (3.23), it is clear that $S_h(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be factored in the form

$$S_h(\lambda) = e^{i\lambda c}B_h(\lambda), \quad c = c(h) \geq 0, \tag{3.36}$$

where $B_h(\lambda)$ is a Blaschke product. It follows from (3.36) that

$$|S_h(\lambda)| = \left|e^{i\lambda c}\right| |B_h(\lambda)| \leq e^{-b(h)\operatorname{Im}\lambda}, \quad \operatorname{Im}\lambda \geq 0. \tag{3.37}$$

Further, for $m_a(\lambda)$ in terms of $S_h(\lambda)$, we find from (3.23) that

$$m_a(\lambda) = \frac{h - \overline{h}S_h(\lambda)}{S_h(\lambda) - 1}. \tag{3.38}$$

If $c(h) > 0$ for a given value $h(\operatorname{Im} h > 0)$, then (3.37) implies that $\lim_{t\to+\infty} S_h(it) = 0$, and then (3.24) gives us that $\lim_{t\to+\infty} m_a(it) = -G$. Since $m_a(\lambda)$ does not depend on h, this implies that $c(h)$ can be nonzero at not more than a single point $h = h_0$ (and further $h_0 = -\lim_{t\to+\infty} m_a(it)$). The theorem is proved.

Due to Theorem 2.4, since the eigenvalues of the boundary value problem (2.1)–(2.3) and eigenvalues of the operator A_h coincide, including their multiplicity and, furthermore, for the eigenfunctions and associated functions the boundary problems (2.1)–(2.3), then theorem is interpreted as follows. □

Corollary 3.10. *The spectrum of the boundary value problem* (2.1)–(2.3) *is purely discrete and belongs to the open upper half-plane. For all the values of h with* $\operatorname{Im}\lambda > 0$, *except possible for a single value $h = h_0$, the boundary value problem* (2.1)–(2.3) $(h \neq h_0)$ *has a countable number of isolated* *eigenvalues with finite multiplicity and limit points and infinity. The system of the eigenfunctions and associated functions of this problem* $(h \neq h_0)$ *is complete in the space* $L_q^2(0, a)$.

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