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EXISTENCE AND MULTIPLICITY OF SOLUTIONS IN FOURTH ORDER BVPS WITH UNBOUNDED NONLINEARITIES

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Abstract. In this work the authors present some existence, non-existence and location results of the problem composed of the fourth order fully nonlinear equation

$$u^{(4)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = s\ p(x)$$

for $x \in [a,b]$, where $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$, $p : [a,b] \to \mathbb{R}^+$ are continuous functions and s a real parameter, with the boundary conditions

$$u(a) = A,\ u'(a) = B,\ u'''(a) = C,\ u'''(b) = D,$$

for $A, B, C, D \in \mathbb{R}$. In this work they use an Ambrosetti-Prodi type approach, with some new features: the existence part is obtained in presence of nonlinearities not necessarily bounded, and in the multiplicity result it is not assumed a speed growth condition or an asymptotic condition, as it is usual in the literature for these type of higher order problems.

The arguments used apply lower and upper solutions technique and topological degree theory.

An application is made to a continuous model of the human spine, used in aircraft ejections, vehicle crash situations, and some forms of scoliosis.

1. Introduction.

Let us consider the problem given by the equation

$$u^{(iv)}(x) + f(x, u(x), u'(x), u''(x), u'''(x)) = sp(x) \tag{1}$$

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with $x \in [a,b]$, where $f : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ and $p : [a,b] \to \mathbb{R}^+$ are continuous functions and s a real parameter, with the boundary conditions

$$u(a) = A,\ u'(a) = B,\ u'''(a) = C,\ u'''(b) = D, \tag{2}$$

with $A, B, C, D \in \mathbb{R}$. For problem (1)-(2) we use an Ambrosetti-Prodi type approach, that is, there are $s_0, s_1 \in \mathbb{R}$ such that (1)-(2) has no solution if $s < s_0$, it has at least one solution if $s = s_0$ and (1)-(2) has at least two solutions for $s \in]s_0, s_1]$.

The technique applied is suggested by several papers namely: [2], applied to second order periodic problems; [12, 15], to third order separated boundary value problems; [9, 11] to fourth order equations with two-point boundary conditions. The method is based on upper and lower solutions technique for higher order boundary value problems, as in [1, 4], and degree theory, [8].

This work improves the existing literature in this field due to the following features:

- An unilateral Nagumo condition is considered, as in [7], allowing that the nonlinear part of equation (1) can be unbounded.
- The multiplicity part of the result is obtained without extra monotonicity conditions on f , as is usual in higher order Ambrosetti-Prodi boundary value problems. For example, in [11] or [12] it is assumed that the nonlinearity growth in some variables is stronger than in other ones. In [5, Theorem 2.3], this type of "speed growth condition" is replaced by an asymptotic condition to guarantee the existence of a second solution. In the current result, neither of these additional hypothesis is assumed (see Theorem 3.3).
- The definition of strict lower and upper solutions is more general than usual, as it considers the strict relation only in the differential inequality (see Definition 3.1).

This type of problems has several applications such as in beam theory to study the bending of different types of support at the end points. Here, we consider a not so common "beam": a continuous model of the human spine, used in aircraft ejections, vehicle crash situations, and some forms of scoliosis under some loading forces (for details see [13, 14] and the references therein).

2. *A priori* bound and general results.

In this section we define an one-sided Nagumo-type growth condition assumed on the nonlinear part of the differential equation which will be an important tool to obtain an *a priori* bound for the third derivative of the corresponding solutions, even with unbounded functions.

In the following, $C^k([a,b])$ denotes the space of real valued functions with continuous i-derivative in $[a,b]$, for $i = 1, ..., k$, equipped with the norm

$$\|y\|_{C^k} = \max_{0 \le i \le k} \left\{ \left| y^{(i)}(x) \right| : x \in [a,b] \right\}.$$

By $C([a,b])$ we denote the space of continuous functions with the norm

$$\|y\| = \max_{x \in [a,b]} |y(x)|.$$

Definition 2.1. Given a subset $E \subset [a,b] \times \mathbb{R}^4$, a continuous function $f : E \to \mathbb{R}$ is said to satisfy the one-sided Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [k, +\infty[$, for some $k > 0$, such that

$$f(x, y_0, y_1, y_2, y_3) \ge -h_E(|y_3|),\ \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{3}$$

or

$$f(x, y_0, y_1, y_2, y_3) \leq h_E(|y_3|), \ \forall (x, y_0, y_1, y_2, y_3) \in E, \tag{4}$$

with

$$\int_0^{+\infty} \frac{t}{h_E(t)} dt = +\infty. \tag{5}$$

The *a priori* bound is given by the following lemma whose proof follows arguments suggested by [3].

Lemma 2.2. *Let $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, satisfying Nagumo-type conditions (3) and (5) in*

$$E = \left\{(x, y_0, y_1, y_2, y_3) \in [a, b] \times \mathbb{R}^4 : \gamma_i(x) \leq y_i \leq \Gamma_i(x), \ i = 0, 1, 2\right\},$$

where $\gamma_i(x)$ and $\Gamma_i(x)$ are continuous functions such that, for $i = 0, 1, 2$, $\gamma_i(x) \leq \Gamma_i(x)$, for all $x \in [a, b]$.

Then for every $\rho > 0$ there is $R > 0$ such that every solution $u(x)$ of equation (1) satisfying

$$u'''(a) \leq \rho \ , \ u'''(b) \geq -\rho \tag{6}$$

and $\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x)$, for all $x \in [a, b]$, for $i = 0, 1, 2$, satisfies $\|u'''\| < R$.

Remark 1. Observe that R depends only on the functions h_E, γ_2 and Γ_2 and not on the boundary conditions. Moreover, if s belongs to a bounded set, then R can be considered the same, independently of s.

Remark 2. The previous Lemma still holds if the one-sided Nagumo condition (3) is replaced by (4) and (6) by $u'''(a) \geq -\rho$, $u'''(b) \leq \rho$.

Lower and upper solutions will have an important role on the arguments.

Definition 2.3. Let $A, B, C, D \in \mathbb{R}$. The function $\alpha \in C^4([a, b])$ is a lower solution of the problem (1)-(2) if

$$\alpha^{(iv)}(x) \geq sp(x) - f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)),$$

and

$$\alpha(a) \leq A, \ \alpha'(a) \leq B, \quad \alpha'''(a) \geq C, \quad \alpha'''(b) \leq D.$$

The function $\beta \in C^4([a, b])$ is an upper solution of problem (1)-(2) if the reversed inequalities hold.

The following theorem provides a general existence and location result and follows the standard technique in lower and upper solutions method (see, for example [3]):

Theorem 2.4. *Suppose that there are upper and lower solutions of the problem (1)-(2), respectively, $\alpha(x)$ and $\beta(x)$, such that, $\alpha''(x) \leq \beta''(x)$, for every $x \in [a, b]$. Let $f : [a, b] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the one-sided Nagumo conditions (3) (or (4)), and (5) in*

$$E_* = \left\{(x, y_0, y_1, y_2, y_3) \in [a, b] \times \mathbb{R}^4 : \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), i = 0, 1, 2\right\}.$$

If f satisfies

$$f(x, \alpha, \alpha', y_2, y_3) \leq f(x, y_0, y_1, y_2, y_3) \leq f(x, \beta, \beta', y_2, y_3), \tag{7}$$

for

$$\alpha(x) \leq y_0 \leq \beta(x) \ \text{ and } \alpha'(x) \leq y_1 \leq \beta'(x) \text{ in } [a, b],$$

and for fixed $(x, y_2, y_3) \in [a,b] \times \mathbb{R}^2$, *then problem (1)-(2) has at least one solution* $u(x) \in C^4([a,b])$ *satisfying* $\alpha^{\{i\}}(x) \le u^{\{i\}}(x) \le \beta^{\{i\}}(x)$ *for* $i = 0,1,2$ *and every* $x \in [a,b]$.

The dependence of the solution on the parameter s will be discussed in $[0,1]$ and for the particular case $A = B = C = D = 0$, that is,

$$u(0) = u'(0) = u'''(0) = u'''(1) = 0. \tag{8}$$

Therefore, the corresponding definitions of lower and upper solutions will satisfy these restrictions.

The next theorem follows the method suggested by [9].

Theorem 2.5. *Let* $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ *be a continuous function that satisfy the one-sided Nagumo conditions (3) (or (4)), and (5). Let*

$$f(x, y_0, y_1, y_2, y_3) \text{ be non-decreasing in } y_0 \text{ and } y_1 \text{ and non-increasing in } y_2 \tag{9}$$

and there are $s_1 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\frac{f(x,0,0,0,0)}{p(x)} < s_1 < \frac{f(x, y_0, y_1, -r, 0)}{p(x)}, \tag{10}$$

for every $x \in [0,1]$ *and* $y_0, y_1 \le -r$. *Then there is* $s_0 < s_1$ *(with the possibility that* $s_0 = -\infty$*) such that:*

1) for $s < s_0$, *problem (1), (8) has no solution.*

2) for $s_0 < s \le s_1$, *problem (1), (8) has at least one solution.*

Proof. The proof is a particular case of [9, Theorem 2.6], assuming in the Sturm-Liouville part of boundary conditions $k_1 = k_3 = 0$ and $k_2 = k_4 = 1$. □

3. **Multiple solutions.** To prove the existence of at least a second solution it is necessary to introduce stronger lower and upper solutions:

Definition 3.1. The function $\alpha(x) \in C^4([0,1])$ is a strict lower solution of the problem (1), (8) if the following conditions are satisfied:

$$\alpha^{(iv)}(x) + f(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)) > sp(x) \tag{11}$$

and

$$\alpha(0) \le 0, \ \alpha'(0) \le 0, \ \alpha'''(0) \ge 0, \ \alpha'''(1) \le 0. \tag{12}$$

The function $\beta(x) \in C^4([0,1])$ is called a strict upper solution of problem (1), (8) if the reversed inequalities hold.

We remark that (12) is not a particular case of the strict lower solutions considered in [9, Theorem 2.6].

Let us consider the set

$$Y = \left\{ y \in C^3([0,1]) : y(0) = y'(0) = y'''(0) = y'''(1) = 0 \right\}$$

and the operator $L : dom\ L \to C([0,1])$ in which $dom\ L = C^4([0,1]) \cap Y$ given by

$$Lu = u^{(iv)} - u''.$$

For $s \in \mathbb{R}$ consider $N_s : C^3([0,1]) \cap Y \to C([0,1])$ defined by

$$N_s u = f(x, u, u', u'', u''') + \delta_2(x, u'') - sp(x),$$

with δ_2 the truncation given by

$$\delta_2(x, u'') = \max\{\alpha''(x), \min\{u''(x), \beta''(x)\}\}. \tag{13}$$

Thus, for $\Omega \subset Y$ open and bounded, the operator $L + N_s$ is L-compact in $\overline{\Omega}$. Observe that in the domain of L, the problem (1), (8) is equivalent to the equation

$$Lu + N_s u = 0.$$

The next lemma provides an important tool for multiplicity results.

Lemma 3.2. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function verifying Nagumo conditions, (5) and (3) (or (4)) and condition (9). Let us suppose that there are strict upper and lower solutions of the problem (1), (8), $\alpha(x)$ and $\beta(x)$ respectively, such that $\alpha''(x) < \beta''(x)$, for all $x \in [0,1]$. Thus, there is $\rho_2 > 0$ such that for*

$$\Omega = \left\{ u \in dom\ L : \alpha^{(i)}(x) < u^{(i)}(x) < \beta^{(i)}(x),\ i = 0,1,2,\ \left\| u'''(x) \right\|_\infty < \rho_3 \right\},$$

the degree $L + N_s$, relative to L, is well defined and given by

$$d_L(L + N_s, \Omega, 0) = \pm 1.$$

Proof. The proof is similar to the proof of [9, Lemma 3.2] and apply the topological degree properties. □

Remark 3. As long as s belongs to a bounded set and α and β are strict lower and upper solutions of (1), (8), respectively, the set Ω can be taken the same.

To obtain a multiplicity result, there is no need to consider extra assumptions on the monotone behavior of the nonlinearity.

Theorem 3.3. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function satisfying the conditions in Theorem 2.5. Suppose that there is $M > -r$ such that for every u solution of the problem (1), (8), with $s \leq s_1$, satisfies*

$$u''(x) < M,\ \forall x \in [0,1], \tag{14}$$

and there is $m \in \mathbb{R}$ such that

$$f(x, y_0, y_1, y_2, y_3) \geq mp(x), \tag{15}$$

for $(x, y_0, y_1, y_2, y_3) \in [0,1] \times [-r - |M|, r + |M|]^2 \times [-r, |M|] \times \mathbb{R}$, with given r by (10).

Then the s_0 given by Theorem 2.5 is finite and:

1) for $s < s_0$, the problem (1), (8) has no solution;
2) for $s = s_0$, (1), (8) has, at least, one solution.
3) for $s \in \left]s_0, s_1\right]$, (1), (8) has, at least, two solutions.

Proof. <u>**Step 1.**</u> **-** Every solution $u(x)$of problem (1), (8), for $s \in \left]s_0, s_1\right]$, satisfies

$$-r < u''(x) < M,\ -r < u^{(i)}(x) < |M|,\ \forall x \in [0,1],\ i = 0,1.$$

By (14), we only need to prove

$$-r < u''(x), \quad \text{for every } x \in [0,1].$$

Suppose by contradiction that there is $s \in \left]s_0, s_1\right]$, u solution of (1), (8) and $x_0 \in [0,1]$ such that

$$\min_{x \in [0,1]} u''(x) := u''(x_0) \leq -r.$$

By (9) and boundary conditions (8), $x_0 \in \left]0,1\right[$ and

$$\begin{aligned} 0 &\leq u^{(4)}(x_0) = sp(x_0) - f(x_0, u(x_0), u'(x_0), u''(x_0), 0) \\ &\leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0). \end{aligned} \tag{16}$$

If $u(x_0) \leq -r$ and $u'(x_0) \leq -r$, then by (10), we obtain the contradiction

$$0 \leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0) < 0.$$

Suppose that $u(x_0) > -r$ and $u'(x_0) \leq -r$. Thus by (9) and (10) we obtain the contradiction

$$\begin{aligned} 0 &\leq s_1 p(x_0) - f(x_0, u(x_0), u'(x_0), -r, 0) \\ &\leq s_1 p(x_0) - f(x_0, -r, u'(x_0), -r, 0) < 0. \end{aligned}$$

The other possible cases can be proven in the same way.
So,

$$-r < u''(x) < M, \ \forall x \in [0,1].$$

Integrating on $[0,x]$ we have

$$-r \leq -rx < \int_0^x u''(s)\, ds = u'(x) < |M|\, x \leq |M|$$

and

$$-r \leq \int_0^x u'(s)\, ds = u(x) \leq |M|.$$

Step 2. - s_0 is finite

Assume that $s_0 = -\infty$, i.e., by Theorem 2.5, problem (1), (8) has a solution for every $s \leq s_1$, denoted by $u(x)$. Then by (15),

$$u^{(4)}(x) = sp(x) - f(x, u(x), u'(x), u''(x), u'''(x)) \leq (s-m)\, p(x).$$

Consider s small enough such that

$$m - s > 0. \tag{17}$$

As $u'''(0) = 0 = u'''(1)$, there is $c \in \left]0,1\right[$, such that $u^{(4)}(c) = 0$.
By (17) we have the contradiction

$$0 = u^{(4)}(c) \leq (s-m)\, p(c) < 0.$$

Therefore, s_0 is finite.

Step 3. - For $s \in \left]s_0, s_1\right]$, there is at least a second solution of problem (1), (8).

By Step 2 and Theorem 2.5, there is $s_{-1} < s_0$ such that the problem (1), (8) has no solution for $s = s_{-1}$. By Lemma 3.2, consider $\rho_1 > 0$, large enough, such that the estimate

$$\|u'''(x)\| < \rho_1$$

holds for every u solution of (1), (8), with $s \in [s_{-1}, s_1]$.

For $M_1 := \max\{r, |M|\}$ and the set

$$\Omega_2 = \{y \in domL : \|y''\| < M_1, \ \|y'''\| < \rho_1\},$$

we have

$$d_L\left(L + N_{s_{-1}}, \Omega_2, 0\right) = 0. \tag{18}$$

By Step 1, if u is a solution of (1), (8), for $s \in [s_{-1}, s_1]$, then $u \notin \partial\Omega_2$. Therefore, defining the homotopy in the parameter s

$$H(\lambda) = (1-\lambda)\, s_{-1} + \lambda s_1,$$

the coincidence degree $d_L\left(L + N_{H(\lambda)}, \Omega_2, 0\right)$ is well defined for every $\lambda \in [0,1]$ and $s \in [s_{-1}, s_1]$. By the invariance under homotopy,

$$0 = d_L\left(L + N_{s_{-1}}, \Omega_2, 0\right) = d_L\left(L + N_s, \Omega_2, 0\right), \tag{19}$$

for $s \in [s_{-1}, s_1]$.

By Theorem 2.5, there are $\sigma \in]s_0, s_1] \subset [s_{-1}, s_1]$ and $u_\sigma(x)$ a solution of (1), (8), with $s = \sigma$. Moreover $u_\sigma(x)$ is a strict upper solution of problem (1), (8) for $\sigma < s \leq s_1$ since

$$\begin{aligned} u_\sigma^{(4)}(x) &= \sigma p(x) - f\left(x, u_\sigma(x), u_\sigma'(x), u_\sigma''(x), u_\sigma'''(x)\right) \\ &< sp(x) - f\left(x, u_\sigma(x), u_\sigma(x), u_\sigma''(x), u_\sigma'''(x)\right). \end{aligned}$$

By (10), it can be proved that the function

$$\alpha(x) = -r\frac{x^2}{2}$$

is a strict lower solution of the problem (1), (8) for $s \leq s_1$.

By Step 1 $-r < u_\sigma''(x)$ for every $x \in [0,1]$, and integrating in $[0,x]$ and (11),

$$\alpha^{(i)}(x) < u_\sigma^{(i)}(x), \; i = 0, 1, \text{ for all} x \in [0,1].$$

By Lemma 3.2, there is $\overline{\rho}_1 > 0$, independent of s, such that for

$$\Omega_r = \left\{x \in dom\ L : -r < y''(x) < u_\sigma''(x), \; \|y'''\| < \overline{\rho}_1\right\},$$

the coincidence degree of $L + N_s$ in Ω_r satisfies

$$d_L\left(L + N_s, \Omega_r, 0\right) = \pm 1, \quad \text{for } s \in]\sigma, s_1]. \tag{20}$$

Consider ρ_1 in Ω_2 large enough such that for $\Omega_r \subset \Omega_2$, by (19), (20) and the additivity of the degree, we have

$$d_L\left(L + N_s, \Omega_2 - \bar{\Omega}_r, 0\right) = \pm 1, \quad \text{for } s \in]\sigma, s_1]. \tag{21}$$

Then problem (1), (8) has at least two solutions u_1 and u_2 such that $u_1 \in \Omega_r$ and $u_2 \in \Omega_2 - \bar{\Omega}_r$, for $s \in]\sigma, s_1] \subset]s_0, s_1]$.

<u>Step 4</u> - *For $s = s_0$, the problem (1), (8) has one solution.*

Take a sequence (s_m), where $s_m \in]s_0, s_1]$ and $\lim s_m = s_0$. By Theorem 2.5, we conclude that for every s_m the problem (1), (8), with $s = s_m$, has a solution u_m. By Step 1, we have that

$$\left\|u_m^{(i)}\right\| < M_1,$$

for $i = 0, 1, 2$, independent of m, and so there is $\tilde{\rho}_1 > 0$ large enough that

$$\|u_m'''\| < \tilde{\rho}_1,$$

independently of m. Thus, the sequence $\left(u_m^{(4)}\right)_{m \in \mathbb{N}}$ is bounded in $C([0,1])$. Using the Arzéla-Ascoli Theorem, we can consider a subsequence (u_m) which converges in $C^3([0,1])$ for a solution $u_0(x)$ of (1), (8), with $s = s_0$. □

4. **Continuous human spine model.**

The influence of some forces on initially curved beam-column can be simulated by a continuum spine model provided an appropriate adjusted flexural rigidity factor (EI) is evaluate (for details see [13, 14]).

To be precise, the total lateral displacement of the beam-column, $y(x)$, is expressed as the sum of the initial lateral displacement, $y_0(x)$, and the lateral displacement due to axial and transverse loads, $y_1(x)$, i.e.,

$$y(x) = y_0(x) + y_1(x). \tag{22}$$

The function, $y_1(x)$, can be modeled by the differential equation

$$EI\ y_1^{(4)}(x) - P\ y_1''(x) = s + Q\left(y_1'''(x)\right) + P\ y_0''(x) \tag{23}$$

where P is the axial load, EI is the flexural rigidity, s is a parameter, and $Q\left(y_1'''(x)\right)$ is a continuous function representing the transverse load.

The boundary conditions for $y_1(x)$ take into account the shear force, noted by $y_1'''(x)$, and the column length L. They are given by

$$y_1\left(-\frac{L}{2}\right)=y_1'\left(-\frac{L}{2}\right)=y_1'''\left(-\frac{L}{2}\right)=y_1'''\left(\frac{L}{2}\right)=0. \tag{24}$$

This problem (23)-(24) is a particular case of (1)-(2) with $[a,b]=\left[-\frac{L}{2},\frac{L}{2}\right]$, $p(x)\equiv 1$, $A=B=C=D=0$ and

$$f(x,u,v,w,z)=-\frac{P}{EI}\left(w+y_0''(x)\right)-\frac{1}{EI}Q(z). \tag{25}$$

The functions

$$\alpha(x)=-\frac{x^4}{24}-L^2x^2-2L^3x-L^4$$

and

$$\beta(x)=\frac{x^4}{24}+L^2x^2+2L^3x+L^4$$

are lower and upper solutions of problem (23)-(24) for EI, P, Q, L , y_0'' and s such that

$$\begin{aligned} 1-\frac{P}{EI}\left(\frac{x^2}{2}+2L^2+\|y_0''\|\right)-\frac{1}{EI}Q(x)\le s \\ \le -1+\frac{P}{EI}\left(\frac{x^2}{2}+2L^2-\|y_0''\|\right)-\frac{1}{EI}Q(-x)\end{aligned} \tag{26}$$

holds for every $x\in\left[-\frac{L}{2},\frac{L}{2}\right]$.

Assuming that the function Q has a subquadratic growth, f given by (25) verifies Nagumo conditions in the set

$$E=\left\{(x,u,v,w,z)\in\left[-\frac{L}{2},\frac{L}{2}\right]\times\mathbb{R}^4:-\frac{x^2}{2}-2L^2\le w\le\frac{x^2}{2}+2L^2\right\} \tag{27}$$

and trivially satisfies (7). Therefore, by Theorem 2.4, problem (23)-(24) has a solution $y_1(x)$, for s satisfying (26), such that

$$\begin{aligned} -\frac{x^4}{24}-L^2x^2-2L^3x-L^4 &\le y_1(x)\le\frac{x^4}{24}+L^2x^2+2L^3x+L^4 \\ -\frac{x^3}{6}-2L^2x-2L^3 &\le y_1'(x)\le\frac{x^3}{6}+2L^2x+2L^3 \\ -\frac{x^2}{2}-2L^2 &\le y_1''(x)\le\frac{x^2}{2}+2L^2, \end{aligned}$$

for $x\in\left[-\frac{L}{2},\frac{L}{2}\right]$. Moreover, considering in (26), $L=0.4$,

$$EI=0.1,\ P=1,\ Q(x)=-\sqrt[3]{x+1}\quad\text{and}\quad\|y_0''\|=0.2, \tag{28}$$

the problem (23)-(24) has a solution for $s\in[7.6,8.4]$. Observe that this solution is nontrivial as for assumptions in (28) the null solution only exists for $s=0.85$.

Consider now that the differential equation (23) is defined in the normalized interval $[0,1]$ with the boundary conditions

$$y_1(0)=y_1'(0)=y_1'''(0)=y_1'''(1)=0, \tag{29}$$

and (28) holds. Therefore the number s_1, given by (10), is estimated by

$$11.5 < s_1 < 5r + 8.5, \text{ for } r > \frac{3}{5}. \tag{30}$$

The functions

$$\alpha(x) = -\frac{2}{3}x^2 \text{ and } \beta(x) = \frac{3}{4}x^2$$

are, respectively, strict lower and upper solutions of problem (23), (29) for

$$5.67 < s < 12.$$

Restricting the set of solutions of (23), (29) to the set

$$E^* = [0,1] \times [-1,1]^3 \times \mathbb{R}^+,$$

it can be said that every solution u of problem (23), (29) in E^* satisfies

$$-1 \leq u''(x) \leq 1$$

and

$$f(x,u,v,w,z) \geq -6.5, \text{ for } (x,u,v,w,z) \in E^*.$$

So, by Theorem 3.3, there is a finite $s_{0,}$ with $s_0 < 8.5$, such that problem (23), (29) has at least two solutions for $s \in]s_0, s_1]$, with s_1 in the interval defined in (30).

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