ORIGINAL RESEARCH

Fourth Order Impulsive Periodic Boundary Value Problems

J. Fialho · F. Minhós

1 **Abstract** In this work it is presented an existence result for the impulsive problem composed
2 by the fourth order fully nonlinear equation

3
$$u^{(iv)}(x) = f\left(x, u(x), u'(x), u''(x), u'''(x)\right)$$

4 for a.e. $x \in [0, 1] \setminus \{x_1, \ldots, x_m\}$ where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function,
5 along with the periodic boundary conditions

6
$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3,$$

7 and the impulses

8
$$u\left(x_j^+\right) = g_j\left(u\left(x_j\right)\right),$$
$$u'\left(x_j^+\right) = h_j\left(u'\left(x_j\right)\right),$$
$$u''\left(x_j^+\right) = k_j\left(u''\left(x_j\right)\right),$$
$$u'''\left(x_j^+\right) = l_j\left(u'''\left(x_j\right)\right),$$

9 where $x_j \in (0, 1)$, for $j = 1, \ldots, m$, such that $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$,
10 and g_j, h_j, k_j, l_j are given real valued functions satisfying some adequate conditions. The
11 arguments used apply lower and upper solutions technique combined with an iterative and
12 non monotone technique.

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Introduction

This work will study the nonlinear periodic fourth order impulsive problem composed by the fully differential equation

$$u^{(iv)}(x) = f\left(x, u(x), u'(x), u''(x), u'''(x)\right) \tag{1}$$

for a.e. $x \in I \setminus \{x_1, \ldots, x_m\}$ with $I := [0, 1]$, where $f : I \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function. This equation is considered along with the boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3, \tag{2}$$

and the impulse conditions

$$
\begin{aligned}
u\left(x_j^+\right) &= g_j\left(u\left(x_j\right)\right), \\
u'\left(x_j^+\right) &= h_j\left(u'\left(x_j\right)\right), \\
u''\left(x_j^+\right) &= k_j\left(u''\left(x_j\right)\right), \\
u'''\left(x_j^+\right) &= l_j\left(u'''\left(x_j\right)\right),
\end{aligned}
\tag{3}
$$

where $x_j \in (0, 1)$, for $j = 1, \ldots, m$, such that $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$ and g_j, h_j, k_j and l_j are given real valued functions verifying some adequate conditions to be precise.

The theory of impulsive problems is experiencing a rapid development in the last few years, mainly because impulsive differential equations have been used to describe real phenomena where there are instantaneous changes at some time instants. These changes may occur not only at a micro scale but also at a global level. Applications can be found in a wide range of areas, such as physics, medicine, biology, control theory, population and/or gene dynamics (see [1,11,18] and the references therein).

The monotone iterative technique coupled with lower and upper solutions is a powerful method used to approximate solutions in several nonlinear problems in general and periodic ones in particular (see, for instance, [7,10,12,14–17,19]).

As far as we know, it is the first time that the above methods are applied to fourth-order periodic impulsive problem with a fully differential equation, that is, where the nonlinearity can depend in every derivatives till the third order. Moreover, this paper introduces several other new features:

- The assumptions on the nonlinear part f are more general than the ones usually considered. As example, we mention that no global monotonicity conditions are assumed, neither a Nagumo-type condition to control a subquadratic growth of f on the third derivative.
- An iterative technique, not necessarily monotone, is used to obtain the existence of solutions.
- The impulsive effects, given by (3), are given by general functions, not necessarily linear.

The arguments make use of lower and upper solutions approach to higher order problems suggested in [3–5,9,8], and an iterative method as in [2].

The work is organized in the following way: "Definitions and Auxiliary Results" section contains the definitions used and a uniqueness result for some fourth order problems. The main result, an existence and location theorem, is presented in "Existence of Solutions" section. An example is shown in last section, to illustrate the main theorem.

51 Definitions and Auxiliary Results

52 In this section some notations, definitions and auxiliary results, needed for the main existence
53 result, are presented.

54 For $m \in \mathbb{N}$, let $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$, $D = \{x_1, \ldots, x_m\}$ and

$$u(x_k^{\pm}) := \lim_{x \to x_k^{\pm}} u(x).$$

56 **Definition 1** Denote by $PC(I)$ the set of functions $u : I \to \mathbb{R}$ continuous on $I \setminus D$ where
57 $u(x_k^+)$ and $u(x_k^-)$ exist with $u(x_k^-) = u(x_k)$ for $k = 1, 2, \ldots, m$.

58 For $u \in PC(I)$, we define the norm by

$$\|u\| = \sup_{x \in I} |u(x)|.$$

60 Consider $PC^{(l)}(I)$, $l = 1, 2, 3$, as the space of the real-valued functions u such that
61 $u^{(j)} \in PC(I)$, $u^{(j)}(x_k^+)$ and $u^{(j)}(x_k^-)$ exist with $u^{(j)}(x_k^-) = u^{(j)}(x_k)$ for $j = 0, 1, 2, 3$
62 and $k = 1, 2, \ldots, m$.

63 Therefore $u \in PC^3(I)$ can be written as

$$u(x) = \begin{cases} u_0(x) & \text{if } x \in [0, x_1], \\ u_1(x) & \text{if } x \in (x_1, x_2], \\ \vdots \\ u_m(x) & \text{if } x \in (x_m, 1], \end{cases}$$

65 where $u_m = v_{m|(x_i, x_{i+1}]}$ with $v_m \in C^3([x_i, x_{i+1}])$ for $i = 0, 1, \ldots, m$.
66 Denote

$$PC_D^3(I) = \left\{ u \in PC^3(I) : u''' \in AC(x_i, x_{i+1}), \ i = 0, 1, \ldots, m \right\}$$

68 and for each $u \in PC_D^3(I)$ we set the norm

$$\|u\|_D = \|u\| + \|u'\| + \|u''\| + \|u'''\|.$$

70 Moreover for $p \in L^1(I)$ we consider the usual norm

$$\|p\|_1 := \int_I |p(t)| \, dt.$$

72 Throughout this paper the following hypothesis will be assumed:

73 (H1) $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is a L^1-Carathéodory function, that is, $f(x, \cdot, \cdot, \cdot, \cdot)$ is a contin-
74 uous function for a. e. $x \in I$;
75 $f(\cdot, y_0, y_1, y_2, y_3)$ is measurable for $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$; and for every $M > 0$ there
76 is a real-valued function $\psi_M \in L^1([0, 1])$ such that

$$|f(x, y_0, y_1, y_2, y_3)| \le \psi_M(x) \text{ for a. e. } x \in [0, 1]$$

78 and for every $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ with $|y_i| \le M$ for $i = 0, 1, 2, 3$.
79 (H2) the real valued functions g_j, h_j, k_j and l_j are nondecreasing for $j = 1, \ldots, m$.

80 **Definition 2** A function $u \in PC_D^3(I)$ is a solution of (1)–(3) if it satisfies (1) almost
81 everywhere in $I \setminus D$, the periodic conditions (2) and the impulse conditions (3).

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82 Next lemma will have a key role in the results presented forward.

83 **Lemma 3** *Let $p : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a L^1-Carathéodory function such that*

$$p(x, v) \text{ is nonincreasing on } v, \text{ for every } x \in [0, 1]. \tag{4}$$

85 *Then for each $a_j,\ b_j,\ c_j,\ d_j \in \mathbb{R},\ j = 0, 1, 2, \ldots, m,$ the initial value problem composed*
86 *by the equation*

$$u^{(iv)}(x) = p(x, u'''(x)) \quad \text{for} \quad a.e.\, x \in (0, 1) \tag{5}$$

88 *and the boundary conditions*

$$u\left(x_j^+\right) = a_j, \quad u'\left(x_j^+\right) = b_j, \quad u''\left(x_j^+\right) = c_j, \quad u'''\left(x_j^+\right) = d_j \tag{6}$$

90 *has a unique solution $u \in PC_D^3(I)$.*

91 *Proof* The solution of problem (5)–(6) can be written as

$$u(x) = a_j + b_j\left(x - x_j^+\right) + c_j \frac{\left(x - x_j^+\right)^2}{2}$$

$$+ d_j \frac{\left(x - x_j^+\right)^3}{3!} + \int_{x_j^+}^x \frac{(x - r)^3}{3!} p\left(r, u'''(r)\right) dr. \tag{7}$$

94 As $p(x)$ is bounded in I, we can define $N := \left\| p\left(x, u'''(x)\right) \right\|_1$ and the following estimates
95 can be obtained:

$$|u(x)| \leq |a_j| + |b_j| + \frac{|c_j|}{2} + \frac{|d_j|}{3!} + N,$$

$$|u'(x)| \leq |b_j| + |c_j| + \frac{|d_j|}{2} + N,$$

$$|u''(x)| \leq |c_j| + |d_j| + N,$$

$$|u'''(x)| \leq |d_j| + N.$$

100 Hence, for $\delta := |a_j| + 2|b_j| + \frac{5}{2}|c_j| + \frac{8}{3}|d_j| + 4N,$

$$\|u\|_D \leq \delta. \tag{8}$$

102 Let $u \in PC_D^3(I)$ such that $\|u\|_D \leq \delta.$
103 Define the operator $\mathcal{T} : PC_D^3(I) \to PC_D^3(I)$ given by

$$\mathcal{T}u := a_j + b_j\left(x - x_j^+\right) + c_j \frac{\left(x - x_j^+\right)^2}{2}$$

$$+ d_j \frac{\left(x - x_j^+\right)^3}{3!} + \int_{x_j^+}^x \frac{(x - r)^3}{3!} p\left(r, u'''(r)\right) dr. \tag{9}$$

106 As $p\left(x, u'''(x)\right)$ is a L^1-Carathéodory function, then $\mathcal{T}$ is continuous and, by (8),

$$\|\mathcal{T}u_n\|_D = \|\mathcal{T}u_n\| + \left\|(\mathcal{T}u_n)'\right\| + \left\|(\mathcal{T}u_n)''\right\| + \left\|(\mathcal{T}u_n)'''\right\| \leq \delta.$$

108 Given that $\mathcal{T}$ is uniformly bounded and equicontinuous, by Ascoli–Arzela's theorem, $\mathcal{T}$
109 is a compact operator. Then the set of solutions of the equation $u = \mathcal{T}u$ is bounded. By
110 Schauder fixed point theorem, $\mathcal{T}$ has a fixed point $u \in PC_D^3(I)$ which satisfies (7) and

$$u\left(x_j^+\right) = a_j, u'\left(x_j^+\right) = b_j, u''\left(x_j^+\right) = c_j \quad \text{and} \quad u'''\left(x_j^+\right) = d_j.$$

112 This proves the existence of solution for (5), (6).
113 To show uniqueness, we will assume that the problem (5), (6) has two solutions, u_1 and
114 u_2, define $z(x) = u_1'''(x) - u_2'''(x)$ for $x \in]x_j, x_{j+1}]$.
115 By (4),

$$z(x)z'(x) = \left[u_1'''(x) - u_2'''(x)\right]\left[p(x, u_1'''(x)) - p(x, u_2'''(x))\right] \leq 0$$

117 for $x \in]x_j, x_{j+1}]$
118 On the other hand as $z\left(x_j^+\right) = 0$,

$$\int_{x_j^+}^{x} z(t)z'(t)dt = \frac{(z(x))^2}{2} - \frac{\left(z(x_j^+)\right)^2}{2} \geq 0.$$

120 So $z(x) = 0$ for every $x \in]x_j, x_{j+1}]$. By integration and (6), $u_1'''(x) = u_2'''(x)$ for $x \in$
121 $]x_j, x_{j+1}]$.

122 Another important tool used in the proof of the main result is the upper and lower solutions
123 technique. Definition is as follows:

124 **Definition 4** A function $\alpha \in PC_D^3(I)$ is said to be a lower solution of the problem (1)–(3)
125 if:

126 (i) $\alpha^{(iv)}(x) \leq f\left(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x)\right)$ for a.e. $x \in (0, 1)$.
127 (ii) $\alpha(0) \leq \alpha(1), \alpha'(0) \leq \alpha'(1), \alpha''(0) \leq \alpha''(1), \alpha'''(0) \leq \alpha'''(1)$,
128 (iii) $\alpha\left(x_j^+\right) \leq g_j\left(\alpha(x_j)\right), \alpha'\left(x_j^+\right) \leq h_j\left(\alpha'(x_j)\right), \alpha''\left(x_j^+\right) \leq k_j\left(\alpha''(x_j)\right), \alpha'''\left(x_j^+\right) \leq$
129 $l_j\left(\alpha'''(x_j)\right)$,

130 A function $\beta \in PC_D^3(I)$ is an upper solution of problem (1)–(3) if the reversed inequalities
131 hold.

Existence of Solutions

133 In this section the main existence and location result is the following:

134 **Theorem 5** *Let* $\alpha, \beta \in PC_D^3(I)$ *be, respectively, lower and upper solutions of* (1)–(3) *such*
135 *that*

$$\alpha'''(x) \leq \beta'''(x) \quad on \quad I \setminus D \tag{10}$$

137 *and*

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \quad i = 0, 1, 2. \tag{11}$$

139 *Assume that*

$$f\left(x, \alpha, \alpha', \alpha'', y_3\right) \leq f(x, y_0, y_1, y_2, y_3) \leq f\left(x, \beta, \beta', \beta'', y_3\right) \tag{12}$$

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for fixed $(x, y_3) \in I \times \mathbb{R}$, $\alpha^{(i)} \leq y_i \leq \beta^{(i)}$ for $i = 0, 1, 2$ and $f(x, y_0, y_1, y_2, y_3)$ is nonincreasing on y_3 for $x \in [0, 1]$.

If conditions (H1) and (H2) hold, then the problem (1)–(3) has a solution $u(x) \in PC_D^3(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x), \quad on \ I, \quad for \quad i = 0, 1, 2, 3.$$

Remark 6 As one can notice by (11), the inequalities $\alpha^{(i)}(x) \leq \beta^{(i)}(x)$ hold for $i = 0, 1, 2$ and every $x \in I$.

Proof Consider the following modified problem composed by the equation

$$u^{(iv)}(x) = f\left(x, \delta_0(x, u(x)), \delta_1(x, u'(x)), \delta_2(x, u''(x)), \delta_3(x, u'''(x))\right)$$
$$-u'''(x) + \delta_3(x, u'''(x)) \tag{13}$$

for $x \in (0, 1)$ and $x \neq x_j$ where the continuous functions $\delta_i : \mathbb{R}^2 \to \mathbb{R}$, for $i = 0, 1, 2, 3$, are given by

$$\delta_i(x, y_i) = \begin{cases} \beta^{(i)}(x), & y_i > \beta^{(i)}(x), \\ y_i, & \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \\ \alpha^{(i)}(x), & y_i < \alpha^{(i)}(x), \end{cases} \tag{14}$$

with the boundary conditions (2) and the impulse conditions (3).

To prove the existence of solution for the problem (13), (2), (3) we apply an iterative technique, not necessarily monotone. Let $(u_n)_{n \in \mathbb{N}}$ be the sequence in $PC_D^3(I)$ defined as follows:

$$u_0(x) = \alpha(x), \tag{15}$$

and for $n = 1, 2, \ldots$ the problem composed by Eq. (11)

$$u_n^{(iv)}(x) = f\left(x, \delta_0(x, u_{n-1}(x)), \delta_1(x, u'_{n-1}(x)), \delta_2(x, u''_{n-1}(x)), \delta_3(x, u'''_n(x))\right)$$
$$+u'''_n(x) - \delta_3(x, u'''_n(x)) \quad for \ a.e. \ x \in (0, 1) \tag{16}$$

with the boundary conditions

$$u_n(0) = u_{n-1}(1), \quad u'_n(0) = u'_{n-1}(1),$$
$$u''_n(0) = u''_{n-1}(1), \quad u'''_n(0) = u'''_{n-1}(1), \tag{17}$$

and the impulsive conditions, for $j = 1, \ldots, m$,

$$u_n\left(x_j^+\right) = g_j\left(u_{n-1}(x_j)\right), \quad u'_n\left(x_j^+\right) = h_j\left(u'_{n-1}(x_j)\right),$$
$$u''_n\left(x_j^+\right) = k_j\left(u''_{n-1}(x_j)\right), \quad u'''_n\left(x_j^+\right) = l_j\left(u'''_{n-1}(x_j)\right). \tag{18}$$

Problem (5)–(6) is an initial value problem, however considering in (13) fixed (y_0, y_1, y_2) the function $f(y_0, y_1, y_2, y_3)$ verifies (5) and, therefore, the sequence $(u_n)_{n \in \mathbb{N}}$ is well defined by the uniqueness given by Lemma 3.

Remark that the initial value problem (16)–(18) will become the periodic impulsive problem (1)–(3), if the two following claims hold:

- *Every solution $u_n(x)$ of the problem (16)–(18) verifies*

$$\alpha^{(i)}(x) \leq u_n^{(i)}(x) \leq \beta^{(i)}(x) \quad for \ i = 0, 1, 2, 3,$$

175 *for all $n \in \mathbb{N}$ and every $x \in I$, which implies that*

$$\delta_i \left(x, u_n^{(i)}(x) \right) = u_n^{(i)}(x) \quad \text{for } i = 0, 1, 2, 3, n \in \mathbb{N} \text{ and every } x \in I.$$

176

177 • *The sequence $(u_n)_{n \in \mathbb{N}}$ is convergent to a solution $u \in PC_D^3$ of problem (16)–(18).*

178 <u>*Step 1*</u> - *Every solution of problem (16)–(18) verifies*

$$\alpha^{(i)}(x) \leq u_n^{(i)}(x) \leq \beta^{(i)}(x) \quad \text{for } i = 0, 1, 2, 3, \tag{19}$$

179

180 *for all $n \in \mathbb{N}$ and every $x \in I$.*

181 Let u be a solution of the problem (16)–(18). To prove inequalities (19) we will rely on
182 mathematical induction.

183 For $n = 0$, by (15),

$$\alpha'''(x) = u_0'''(x) \leq \beta'''(x) \quad \text{for } x \in I \tag{}$$

184

185 and by, Remark 6,

$$\alpha^{(i)}(x) \leq u_0^{(i)}(x) \leq \beta^{(i)}(x) \quad \text{for } i = 0, 1, 2, \text{ and } x \in I. \tag{20}$$

186

187 Suppose that for $k = 1, \ldots, n - 1$ and every $x \in I$ we have

$$\alpha'''(x) \leq u_k'''(x) \leq \beta'''(x) \tag{21}$$

188

189 for $x = 0$, by (17), (21) and Definition 4, we have

$$u_n'''(0) = u_{n-1}'''(1) \geq \alpha'''(1) \geq \alpha'''(0).$$

190

191 If $x = x_j^+$, $j = 1, \ldots, m$, from (18), (H2), (21) and Definition 4, then

$$u_n''' \left(x_j^+ \right) = l_j \left(u_{n-1}''' \left(x_j \right) \right) \geq l_j \left(\alpha''' \left(x_j \right) \right) \geq \alpha''' \left(x_j^+ \right).$$

192

193 For $x \in]x_j, x_{j+1}]$, $j = 1, 2, \ldots, m$ suppose, by contradiction, that there exists $x^* \in$
194 $]x_j, x_{j+1}]$ such that $\alpha'''(x^*) > u_n'''(x^*)$ and define

$$\min_{x \in]x_j, x_{j+1}]} u_n'''(x) - \alpha'''(x) := u_n''' \left(x^* \right) - \alpha''' \left(x^* \right) < 0.$$

195

196 As by (18), $u_n''' \left(x_j^+ \right) \geq \alpha''' \left(x_j^+ \right)$, then there is an interval $(\underline{x}, \overline{x}) \subset (x_j, x^*)$ such that

$$u_n'''(x) < \alpha'''(x) \quad \text{and } u_n^{(iv)}(x) \leq \alpha^{(iv)}(x) \, \forall x \in (\underline{x}, \overline{x}).$$

197

198 From (16) and (12) the following contradiction is obtained for $x \in (\underline{x}, \overline{x})$:

$$0 \geq u_n^{(iv)}(x) - \alpha^{(iv)}(x)$$

199

$$= f \left(x, \delta_0 \left(x, u_{n-1}(x) \right), \delta_1 \left(x, u_{n-1}'(x) \right), \delta_2 \left(x, u_{n-1}''(x) \right), \alpha'''(x) \right)$$

200

$$- u'''(x) + \alpha'''(x) - \alpha^{(iv)}(x)$$

201

$$\geq f \left(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x) \right) - u'''(x) + \alpha'''(x)$$

202

$$- f \left(x, \alpha(x), \alpha'(x), \alpha''(x), \alpha'''(x) \right)$$

203

$$\geq \alpha'''(x) - u'''(x) > 0.$$

204

205 As x^* is taken arbitrarily in $]x_j, x_{j+1}]$, then $u_n'''(x) \geq \alpha'''(x)$ for all $n \in \mathbb{N}$ and every
206 $x \in I$. In the same way it can be shown that $u_n'''(x) \leq \beta'''(x) \, \forall x \in I$, $\forall n \in \mathbb{N}$, and so (19)
207 is proved for $i = 3$.

208 Assuming that for $k = 1, \ldots, n-1$ and every $x \in I$,

209
$$\alpha''(x) \leq u_k''(x) \leq \beta''(x), \qquad (22)$$

210 then for $x \in [0, x_1]$, by integration of the inequality $u_n'''(x) \geq \alpha'''(x)$ in $[0, x]$ we have

211
$$u_n''(x) - u_n''(0) \geq \alpha''(x) - \alpha''(0).$$

212 By (17) and (22),

213
$$u_n''(x) \geq \alpha''(x) - \alpha''(0) + u_{n-1}''(1)$$

214
$$\geq \alpha''(x) - \alpha''(0) + \alpha''(1) \geq \alpha''(x)$$

215 hence $u_n''(x) \geq \alpha''(x)$, for all $x \in [0, x_1]$.

216 For $x \in]x_j, x_{j+1}]$, $j = 1, 2, \ldots, m$, by integration of the inequality $u_n'''(x) \geq \alpha'''(x)$ in
217 $x \in]x_j, x_{j+1}]$,

218
$$u_n''(x) \geq \alpha''(x) - \alpha''\left(x_j^+\right) + u_n''\left(x_j^+\right),$$

219 and by (18) and Definition 4

220
$$u_n''(x) \geq \alpha''(x) - \alpha''\left(x_j^+\right) + k_j\left(u_{n-1}''\left(x_j\right)\right) \geq \alpha''(x),$$

221 obtaining that $u_n''(x) \geq \alpha''(x)$ for all $n \in \mathbb{N}$ and every $x \in I$. Using similar arguments it can
222 be proved that $u_n''(x) \leq \beta''(x)$ and therefore

223
$$\alpha''(x) \leq u_n''(x) \leq \beta''(x) \quad \forall x \in I \quad \forall n \in \mathbb{N}. \qquad (23)$$

224 The remaining inequalities in (19) can be proved as above, by integration of (22) in $[0, x_1]$
225 applying the correspondent hypothesis of induction, conditions (17), (18), and Definition 4.
226 *Step 2 - The sequence $(u_n)_{n \in \mathbb{N}}$ is convergent to u solution of problem* (16)–(18).
227 For $i = 0, 1, 2, 3$, let $C_i = \max\left\{\left\|\alpha^{(i)}\right\|, \left\|\beta^{(i)}\right\|\right\}$. So there exists $M > 0$, with $M :=$
228 $\sum_{i=0}^{3} C_i$ and for all $n \in \mathbb{N}$

229
$$\|u_n\|_D \leq M. \qquad (24)$$

230 Let Ω be a compact subset of $\mathbb{R}^4$ given by

231
$$\Omega = \left\{(w_0, w_1, w_2, w_3) \in \mathbb{R}^4 : \|w_i\| \leq C_i, i = 0, 1, 2, 3\right\}.$$

232 As f is a L^1-Carathéodory function in Ω, then there exists a real-valued function $h_M(x) \in$
233 $L^1(I)$ such that

234
$$|f(x, w_0, w_1, w_2, w_3)| \leq h_M(x) \quad \text{for every } (w_0, w_1, w_2, w_3) \in \Omega. \qquad (25)$$

235 By Step 1 and (24), $\left(u_n, u_n', u_n'', u_n'''\right) \in \Omega$ for all $n \in \mathbb{N}$. From (16) and (25) we obtain

236
$$\left|u_n^{(iv)}(x)\right| \leq h_M(x) + 2C_3 \quad \text{for a.e. } x \in I,$$

237 hence $u_n^{(iv)}(x) \in L^1(I)$.
238 By integration in I we obtain that

239
$$u_n'''(x) = u_n'''(0) + \int_0^x u_n^{(iv)}(s)\,\mathrm{d}s + \sum_{0 < x_j \leq x} l_j\left(u_{n-1}'''\left(x_j\right)\right),$$

therefore $u_n''' \in AC\left(x_j, x_{j+1}\right)$ and $u_n \in PC_D^3(I)$. By Ascoli–Arzela's theorem there exists a subsequence, denoted by $(u_n)_{n \in \mathbb{N}}$, which converges to $u \in PC_D^3(I)$. Then $\left(u, u', u'', u'''\right) \in \Omega$.

Using the Lebesgue dominated convergence theorem, for $x \in \left(x_j, x_{j+1}\right)$,

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0\left(s, u_{n-1}(s)\right), \delta_1\left(s, u_{n-1}'(s)\right), \delta_2\left(s, u_{n-1}''(s)\right), \delta_3\left(s, u_n'''(s)\right)\right) \\ -u_n'''(s) + \delta_3\left(s, u_n'''(s)\right) \end{array} \right] ds$$

is convergent to

$$\int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0\left(s, u(s)\right), \delta_1\left(s, u'(s)\right), \delta_2\left(s, u''(s)\right), \delta_3\left(s, u'''(s)\right)\right) \\ -u'''(s) + \delta_3\left(s, u'''(s)\right) \end{array} \right] ds$$

as $n \to \infty$.

Therefore as $n \to \infty$

$$u_n'''(x)$$

$$= u_n'''\left(x_j\right) + \int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0\left(s, u_{n-1}(s)\right), \delta_1\left(s, u_{n-1}'(s)\right), \delta_2\left(s, u_{n-1}''(s)\right), \delta_3\left(s, u_n'''(s)\right)\right) \\ -u_n'''(s) + \delta_3\left(s, u_n'''(s)\right) \end{array} \right] ds.$$

is convergent to

$$u'''(x) = u'''\left(x_j\right) + \int_{x_j}^x \left[\begin{array}{c} f\left(s, \delta_0\left(s, u(s)\right), \delta_1\left(s, u'(s)\right), \delta_2\left(s, u''(s)\right), \delta_3\left(s, u'''(s)\right)\right) \\ -u'''(s) + \delta_3\left(s, u'''(s)\right) \end{array} \right] ds.$$

As the function f is L^1-Carathéodory function in $\left(x_j, x_{j+1}\right)$, then $u'''(x) \in AC\left(x_j, x_{j+1}\right)$. Therefore $u \in PC_D^3(I)$ and u is a solution of problem (16)–(18).

To prove that u is a solution of the boundary value problem (1)–(3) we note that taking the limit in (17) and (18), as $n \to \infty$, by the convergence of u_n then u verifies (2) and, by the continuity of of the impulsive functions, u verifies (3). By (14), Step 1 and the convergence of u_n, u verifies (1).

Then problem (1)–(3) has a solution $u(x) \in PC_D^3(I)$, such that

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x) \quad \text{for } i = 0, 1, 2, 3$$

for $x \in I$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example

Let us consider the following nonlinear periodic impulsive boundary value problem, composed by the equation:

$$u^{(iv)}(x) = (u(x))^3 + \arctan\left(u'(x) + 1\right) + 0.01\left(u''(x)\right)^5 + \eta\left|u'''(x)\right|^\theta, \qquad (26)$$

where $0 \le \theta \le 2$ and $\eta \le -677$ for all $x \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$ along with the boundary conditions (2) and for $x_j = \frac{1}{2}$ the impulse conditions

$$u \left(\frac{1}{2}^+ \right) = \mu_1 \left(u \left(\tfrac{1}{2} \right) \right)^3,$$

$$u' \left(\frac{1}{2}^+ \right) = \mu_2 \left(u' \left(\tfrac{1}{2} \right) \right),$$

$$u'' \left(\frac{1}{2}^+ \right) = \mu_3 \sqrt[3]{\left(u'' \left(\tfrac{1}{2} \right) \right)},$$

$$u''' \left(\frac{1}{2}^+ \right) = \mu_4 \left(u''' \left(\tfrac{1}{2} \right) \right)^5, \tag{27}$$

with $\mu_i \in \mathbb{R}^+$, $i = 1, 2, 3, 4$.

Obviously this problem is a particular case of (1)–(3) with

$$f (x, y_0, y_1, y_2, y_3) = (y_0)^3 + \arctan (y_1 + 1) + 0.01 (y_2)^5 + \eta |y_3|^\theta,$$

for all $x \in [0, 1] \setminus \left\{ \frac{1}{2} \right\}$, $m = 1$, $x_1 = \frac{1}{2}$ and the nondecreasing functions g_1, h_1, k_1 and l_1 are given by

$$g_1(x) = \mu_1 x^3, \quad h_1(x) = \mu_2 x, \quad k_1(x) = \mu_3 \sqrt[3]{x}, \quad l_1(x) = \mu_4 x^5.$$

The functions $\alpha (x) = 0$ and

$$\beta (x) = \begin{cases} x^3 + 3x^2 + 4x + \frac{3}{2}, & x \in \left[0, \frac{1}{2}\right], \\ x^3, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

are $PC_D^3 (I)$ for $D = \left\{ \frac{1}{2} \right\}$ with

$$\beta' (x) = \begin{cases} 3x^2 + 6x + 4, & x \in \left[0, \frac{1}{2}\right], \\ 3x^2, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

$$\beta'' (x) = \begin{cases} 6x + 6, & x \in \left[0, \frac{1}{2}\right], \\ 6x, & x \in \left(\frac{1}{2}, 1\right], \end{cases}$$

$$\beta''' (x) = 6, \quad x \in [0, 1]$$

are lower and upper solutions, respectively, for problem (26), (2), (27), with

$$\mu_1 \le \frac{64}{42875}, \quad \mu_2 \le \frac{3}{31}, \quad \mu_3 \le \sqrt[3]{3}, \quad \mu_4 \le \frac{1}{6^4}.$$

As f verifies (12), by Theorem 5 there is a non-negative solution $u (x) \in PC_D^3 (I)$, such that

$$0 \le u^{(i)} (x) \le \beta^{(i)} (x) \quad \text{for } i = 0, 1, 2, 3. \tag{28}$$

Remark that this solution cannot be a trivial periodic one, as the only constant verifying (26) is $-\sqrt[3]{\pi/4}$ and it does not satisfy (28).

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