An extremal property of the inf- and sup-convolutions regarding the Strong Maximum Principle*

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Abstract

In this paper we continue investigations started in [**?**] concerning the extension of the variational Strong Maximum Principle for lagrangeans depending on the gradient through a Minkowski gauge. We essentially enlarge the class of comparison functions, which substitute the identical zero when the lagrangean is not longer strictly convex at the origin.

1 Introduction

The Strong Maximum Principle, a well known property of the elliptic partial differential equations (see, e.g., [**?**, **?**] and the bibliography therein), can be formulated in the variational setting as was done by A. Cellina in 2002. Extending the main result in [**?**] we consider the integral functional

$$\int_{\Omega} f\left(\rho_F\left(\nabla u\left(x\right)\right)\right)\,dx, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ is an open bounded connected domain; $f:\mathbb{R}^+\to\mathbb{R}^+\cup\{+\infty\}$, $f(0)=0$, is a lower semicontinuous convex function; $F\subset\mathbb{R}^n$ is a convex closed bounded set with $0\in\mathrm{int}F$ (*interior* of F), and $\rho_F(\cdot)$ is the *Minkowski functional* (*gauge function*) associated to F,

$$\rho_F(\xi):=\inf\left\{\lambda>0:\xi\in\lambda F\right\}. \tag{1.2}$$

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In the traditional sense, the Strong Maximum Principle (SMP) for (**??**) means that there is no a nonconstant continuous minimizer of this functional on $u_0(\cdot) + W_0^{1,1}(\Omega)$ (with some Sobolev function $u_0(\cdot)$), admitting its *minimal* (*maximal*) value in Ω. In the case of a *rotationally invariant* lagrangean ($\rho_F(\xi) = \|\xi\|$) and $n > 1$ it was proven in [**?**] that this property is valid if and only if the function $f(\cdot)$ is strictly convex and smooth at the origin, or, in other words, if the equalities

$$\partial f^*(0) = \{0\} \tag{1.3}$$

and

$$\partial f(0) = \{0\} \tag{1.4}$$

hold. Here, as usual, ∂f stands for the *subdifferential* of the function $f(\cdot)$ in the sense of Convex Analysis, and $f^*(\cdot)$ is the *Legendre-Fenchel tranform* (*conjugate*) of $f(\cdot)$. Observe that in the case $n = 1$ the smoothness of $f(\cdot)$ at zero (assumption (**??**)) is not necessary, and the SMP is valid unless the function $f(\cdot)$ is not affine near the origin (the latter condition already contains (**??**)).

In [**?**] we proved that under the same hypotheses on $f(\cdot)$ the Strong Maximum Principle remains valid for a general functional (**??**), where the gauge F is not assumed to be either rotund, smooth or symmetric. Furthermore, we tried to extend the SMP to the case when the condition (**??**) fails.

Since the SMP can be equivalently reformulated as a comparison property:

$$\boxed{\begin{array}{l}\textbf{if a continuous nonnegative (nonpositive)}\\ \textbf{minimizer } u(\cdot) \textbf{ of the functional } (\mathbf{??}) \text{ on}\\ u_0(\cdot) + W_0^{1,1}(\Omega) \ \ \textbf{touches zero at some point}\\ x^* \in \Omega \textbf{ then necessarily } u(\cdot) \equiv 0 \textbf{ on } \Omega,\end{array}} \tag{1.5}$$

it is obviously violated whenever the lagrangean is no longer strictly convex at the origin (see also [**?**]). Nevertheless, we emphasized a class $\mathfrak{C}$ of continuous functions, which being themselves solutions of the variational problem can substitute in some sense the *identical zero* in the property (**??**). These functions (further called *test*, or *comparison,* functions) depend certainly on the subdifferential $\partial f^*(0)$ and reduce to constants when $\partial f^*(0)$ reduces to the singleton $\{0\}$. In the place of the null-function in (**??**), clearly, any constant can stand.

In the case $\partial f^*(0) \neq \{0\}$ so extended Strong Maximum Principle for a test function $\hat{u}(\cdot) \in \mathfrak{C}$ can be given as follows:

$$\boxed{\begin{array}{l}\textbf{each continuous minimizer of } (\mathbf{??}) \textbf{ on}\\ u_0(\cdot) + W_0^{1,1}(\Omega) \ \textbf{such that } u(x) \geq \hat{u}(x) \ \textbf{(respectively,}\\ u(x) \leq \hat{u}(x)\,),\ x \in \Omega\textbf{, having the same points}\\ \textbf{of } \textit{local minimum} \textbf{ (respectively, } \textit{local maximum}\text{)}\\ \textbf{as } \hat{u}(\cdot) \ \textbf{should coincide with } \hat{u}(\cdot) \ \textbf{everywhere on } \Omega.\end{array}} \tag{1.6}$$

Observe that all the functions $\hat{u}(\cdot) \in \mathfrak{C}$ are written in terms of the *polar set* F^0. Namely, in the simplest case of unique local minimum (local maximum) point $x_0 \in \Omega$ the functions $\hat{u}^+_{x_0,\mu}(x) := \mu + a\rho_{F^0}(x - x_0)$ and $\hat{u}^-_{x_0,\mu}(x) := \mu - a\rho_{F^0}(x_0 - x)$ belong to $\mathfrak{C}$ for each real μ. Here $a := \sup \partial f^*(0)$. If instead $\partial f^*(0) = \{0\}$ (equivalently, $a = 0$) then we can take $x_0 = x^*$ where x^* is an arbitrary "floating" point from Ω (see (**??**)), and we arrive at the traditional SMP although the property (**??**) is not formally applicable.

In [**?**] also a "multipoint" version of the Strong Maximum Principle was established when the comparison function $\hat{u}(\cdot)$ is the lower (upper) envelope of a finite number of functions $\hat{u}^+_{x_0,\mu}(\cdot)$ (respectively, $\hat{u}^-_{x_0,\mu}(\cdot)$) for various $x_0 \in \Omega$ and $\mu \in \mathbb{R}$. Notice that (**??**) takes place only for convex domains $\Omega \subset \mathbb{R}^n$ (or, at least, under a kind of *star-shapeness* hypothesis that can not be removed, see [**?**]).

Another restriction, under which validity of the property (**??**) was proven, is *smoothness* of the gauge function $\rho_F(\cdot)$, or, equivalently, *rotundity* of the polar set F^0. In fact, one of the tools we use in the proofs is the so named *modulus of rotundity*

$$\begin{aligned}\mathfrak{M}_{F^0}(r;\alpha,\beta) := \inf\{&1 - \rho_{F^0}(\xi + \lambda(\eta - \xi)) : \\ &\xi,\eta \in \partial F^0,\ \ \rho_{F^0}(\xi - \eta) \geq r,\ \ \alpha \leq \lambda \leq \beta\},\end{aligned} \tag{1.7}$$

which is strictly positive for all $r > 0$ and all $0 < \alpha \leq \beta < 1$ whenever F^0 is *rotund.*

In this paper we essentially enlarge the class $\mathfrak{C}$ envolving the infinite (continuous) envelopes of the functions $\hat{u}^{\pm}_{x_0,\mu}(\cdot)$ by such a way that the generalized SMP gets an *unique extremal extension principle* and unifies both properties (**??**) and (**??**). Namely, given an arbitrary function $\theta(\cdot)$ defined on a closed subset $\Gamma \subset \Omega$ and satisfying a natural *slope condition* w.r.t. F we prove in Section 3 that the *inf-convolution*

$$u^+_{\Gamma,\theta}(x) := \inf_{y\in\Gamma}\{\theta(y) + a\rho_{F^0}(x - y)\} \tag{1.8}$$

(respectively, the *sup-convolution*

$$u^-_{\Gamma,\theta}(x) := \sup_{y\in\Gamma}\{\theta(y) - a\rho_{F^0}(y - x)\}\ ) \tag{1.9}$$

is the only continuous minimizer $u(\cdot)$ of the functional (**??**) on $u_0(\cdot) + W^{1,1}_0(\Omega)$ such that $u(x) = \theta(x)$ on Γ and $u(x) \geq u^+_{\Gamma,\theta}(x)$ (respectively, $u(x) \leq u^-_{\Gamma,\theta}(x)$), $x \in \Omega$. The domain Ω is always assumed to be convex.

2 Preliminaries. Auxiliary statements

In what follows we assume that

$$a := \sup \partial f^*(0) > 0,$$

and so the second Cellina's hypothesis (**??**) is automatically fulfilled. Furthermore, we introduce the nondecreasing upper semicontinuous function

$$\varphi(t) := \sup \partial f^*(t).$$

So $\varphi(0) = a$ and $\varphi(t) < +\infty$ on the interior of the domain

$$\mathrm{dom} f^* := \left\{ t \in \mathbb{R}^+ : f^*(t) < +\infty \right\}.$$

The version of SMP we wish to prove is essentially based on the following local estimates of continuous minimizers of (**??**) obtained in [**?**] by using the dual properties of convex sets (see, e.g., [**?**] or [**?**]), being themselves an interesting result of Convex Analysis.

Theorem 1 *Given an open bounded region $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and a continuous admissible minimizer $\bar{u}(\cdot)$ of the functional (**??**) on $u_0(\cdot) + W_0^{1,1}(\Omega)$, assume a point $\bar{x} \in \Omega$ and real numbers $\beta > 0$ and μ to be such that*

$$\bar{u}(x) \geq \mu \quad \forall x \in \bar{x} - \beta F^0 \subset \Omega$$

and

$$\bar{u}(\bar{x}) > \mu + a\beta.$$

Then for some $\eta > 0$ the inequality

$$\bar{u}(x) \geq \mu + \varphi(\eta)\left(\beta - \rho_{F^0}(\bar{x} - x)\right) \tag{2.1}$$

holds for all $x \in \bar{x} - \beta F^0$.

Simmetrically, if a point $\bar{x} \in \Omega$ and numbers $\beta > 0$ and μ are such that

$$\bar{u}(x) \leq \mu \quad \forall x \in \bar{x} + \beta F^0 \subset \Omega$$

and

$$\bar{u}(\bar{x}) < \mu - a\beta,$$

then there exists $\eta > 0$ such that

$$\bar{u}(x) \leq \mu - \varphi(\eta)\left(\beta - \rho_{F^0}(x - \bar{x})\right) \tag{2.2}$$

for all $x \in \bar{x} + \beta F^0$.

Roughly speaking, the statement above means that for each continuous admissible minimizer $\bar{u}(\cdot)$ of (**??**) and for each point $\bar{x} \in \Omega$, which is not local extremum for $\bar{u}(\cdot)$, the deviation of $\bar{u}(\cdot)$ from the extremal level can be controlled near $\bar{x}$ by an affine transformation of the dual Minkowski gauge (see (**??**) and (**??**)). Recall that *admissible minimizers* are those giving finite values to functional (**??**).

In the case $a > 0$ (it is our standing assumption along with the paper) we have the following simple consequence of this theorem.

Corollary 1 *Given $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and $\bar{u}(\cdot)$ as in Theorem 1 let us assume that for some $x_0 \in \Omega$ and $\delta > 0$*

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in x_0 + \delta F^0 \subset \Omega. \tag{2.3}$$

Then

$$\bar{u}(x) = \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \qquad \forall x \in x_0 + \frac{\delta}{\|F\| \|F^0\| + 1} F^0.$$

*Similarly, if in the place of (**??**)*

$$\bar{u}(x) \leq \bar{u}(x_0) - a\rho_{F^0}(x_0 - x) \qquad \forall x \in x_0 - \delta F^0 \subset \Omega \tag{2.4}$$

then

$$\bar{u}(x) = \bar{u}(x_0) - a\rho_{F^0}(x_0 - x) \qquad \forall x \in x_0 - \frac{\delta}{\|F\| \|F^0\| + 1} F^0.$$

Here $\|F\| := \sup\{\|\xi\| : \xi \in F\}$.

As standing hypotheses in what follows we assume that $F \subset \mathbb{R}^n$ is a convex closed bounded set, $0 \in \mathrm{int}F$, with *smooth boundary* (the latter means that the Minkowski functional $\rho_F(\xi)$ is *Fréchet differentiable* at each $\xi \neq 0$), and that $\Omega \subset \mathbb{R}^n$ is an open convex bounded region.

Let us consider an arbitrary nonempty closed subset $\Gamma \subset \Omega$ and a function $\theta : \Gamma \to \mathbb{R}$ satisfying the *slope condition*:

$$\theta(x) - \theta(y) \leq a\sigma_F(x - y) \qquad \forall x, y \in \Gamma, \tag{2.5}$$

where

$$\sigma_F(\xi) := \sup_{v \in F} \langle v, \xi \rangle$$

is the *support function* of F ($\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{R}^n$). It is well known that

- $\left(F^0\right)^0 = F$;
- $\sigma_F(\xi) = \rho_{F^0}(\xi)$ whenever $\xi \in F^0$;
- the polar set F^0 is *rotund* (see Section 1).

We will use also the following property of the gauge function:

$$\frac{1}{\|F\|} \|\xi\| \leq \rho_F(\xi) \leq \left\|F^0\right\| \|\xi\|. \tag{2.6}$$

Let us define now *inf*- and *sup-convolutions* of $\theta(\cdot)$ with the gauge function $a\rho_{F^0}(\cdot)$ by the formulas (**??**) and (**??**). Observe first that the function $u^{\pm}_{\Gamma,\theta}(\cdot)$

is the minimizer of (**??**) on $u^{\pm}_{\Gamma,\theta}(\cdot) + W^{1,1}_0(\Omega)$. Indeed, it is obviously Lipschitz continuous on Ω, and for its (classic) gradient $\nabla u^{\pm}_{\Gamma,\theta}$ existing by Rademacher's theorem we have that

$$\nabla u^{\pm}_{\Gamma,\theta}(x) \in \partial^c u^{\pm}_{\Gamma,\theta}(x) \subset aF$$

for a.e. $x \in \Omega$ (see [**?**, Theorem 2.8.6]). Here ∂^c stands for the *Clarke's subdifferential* of a (locally) Lipschitzean function. Consequently,

$$f\left(\rho_F\left(\nabla u^{\pm}_{\Gamma,\theta}(x)\right)\right) = 0$$

a.e. on Ω, and the function $u^{\pm}_{\Gamma,\theta}(\cdot)$ gives to (**??**) the minimal possible value zero.

Due to the slope condition (**??**) it follows that $u^{\pm}_{\Gamma,\theta}(x) = \theta(x)$ for all $x \in \Gamma$. Moreover, $u^{\pm}_{\Gamma,\theta}(\cdot)$ is the (unique) *viscosity solution* of the *Hamilton-Jacobi equation*

$$\pm\left(\rho_F\left(\nabla u(x)\right) - a\right) = 0, \qquad u\,|_{\Gamma} = \theta,$$

(see, e.g., [**?**]).

Notice that Γ can be a finite set, say $\{x_1, x_2, ..., x_m\}$. In this case $\theta(\cdot)$ associates to each x_i a real number θ_i, $i = 1, ..., m$, and the condition (**??**) slightly strengthened (by assuming that the inequality in (**??**) is strict for $x_i \neq x_j$) means that all the simplest test functions $\theta_i + a\rho_{F^0}(x - x_i)$ (respectively, $\theta_i - a\rho_{F^0}(x_i - x)$) are essential (not superfluous) in constructing the respective lower or upper envelope. Then the extremal property established below is reduced to the extended SMP (**??**) (see [**?**, Theorem 6]).

On the other hand, if $\theta(\cdot)$ is a Lipschitz continuous function defined on a closed convex set $\Gamma \subset \Omega$ with nonempty interior then (**??**) holds iff

$$\nabla\theta(x) \in aF$$

for almost each (a.e.) $x \in \Gamma$. This immediately follows from Lebourg's mean value theorem (see [**?**, p. 41]) recalling the properties of the Clarke's subdifferential and from the separability theorem.

Certainly, the mixed (discrete and continuous) case can be considered as well, and all the situations are unified by the hypothesis (**??**).

In the particular case $\theta \equiv 0$ ((**??**) is trivially fulfilled) the function $u^{+}_{\Gamma,\theta}(x)$ is nothing else than the *minimal time* necessary to achieve the closed set Γ from the point $x \in \Omega$ by trajectories of the differential inclusion with the constant convex right-hand side

$$-a\dot{x}(t) \in F^0, \tag{2.7}$$

while $-u^{-}_{\Gamma,\theta}(x)$ is, contrarily, the *minimal time*, for which trajectories of (**??**) arrive at x starting from a point of Γ. Furthermore, if $F = \overline{B}$ is the closed unit ball centered at the origin then the gauge function $\rho_{F^0}(\cdot)$ is the euclidean norm in $\mathbb{R}^n$, and we have

$$u^{\pm}_{\Gamma,\theta}(x) = \pm a d_{\Gamma}(x)$$

where $d_\Gamma(\cdot)$ means the *distance* from a point to the set Γ.

Proving the main theorem below we essentially use the following simple property of extremums in (**??**) and (**??**), which generalizes a well-known property of metric projections (see [**?**, Lemma 1]).

Proposition 1 *Given an arbitrary nonempty closed set $\Gamma \subset \mathbb{R}^n$ and a real-valued function $\theta(\cdot)$ defined on Γ let us assume that for $x \in \mathbb{R}^n \setminus \Gamma$ the minimum of $y \mapsto \theta(y) + a\rho_{F^0}(x-y)$ (respectively, the maximum of $y \mapsto \theta(y) - a\rho_{F^0}(y-x)$) on Γ is attained at some point $\bar{y} \in \Gamma$. Then $\bar{y}$ is also a minimizer of $y \mapsto \theta(y) + a\rho_{F^0}(x_\lambda - y)$ (respectively, the maximizer of $y \mapsto \theta(y) - a\rho_{F^0}(y - x_\lambda)$) for all $\lambda \in [0,1]$, where $x_\lambda := \lambda x + (1-\lambda)\,\bar{y}$.*

3 Generalized Strong Maximum Principle

Now we are ready to deduce the extremal property of the functions $u^{\pm}_{\Gamma,\theta}(\cdot)$ announced above.

Theorem 2 *Under all the standing hypotheses formulated in the previous section let us assume that a continuous admissible minimizer $\bar{u}(\cdot)$ of functional (**??**) on $u_0(\cdot) + W^{1,1}_0(\Omega)$ is such that*

(i) $\bar{u}(x) = u^{+}_{\Gamma,\theta}(x) = \theta(x) \qquad \forall x \in \Gamma;$

(ii) $\bar{u}(x) \geq u^{+}_{\Gamma,\theta}(x) \qquad \forall x \in \Omega.$

Then $\bar{u}(x) \equiv u^{+}_{\Gamma,\theta}(x)$ on Ω.

Simmetrically, if a continuous admissible minimizer $\bar{u}(\cdot)$ satisfies the conditions

(i)$'$ $\bar{u}(x) = u^{-}_{\Gamma,\theta}(x) = \theta(x) \qquad \forall x \in \Gamma;$

(ii)$'$ $\bar{u}(x) \leq u^{-}_{\Gamma,\theta}(x) \qquad \forall x \in \Omega,$

then $\bar{u}(x) \equiv u^{-}_{\Gamma,\theta}(x)$ on Ω.

Proof. Let us prove the first part of Theorem only since the respective changements in the symmetric case are obvious.

Given a continuous admissible minimizer $\bar{u}(\cdot)$ satisfying conditions (**i**) and (**ii**) we suppose, on the contrary, that there exists $\bar{x} \in \Omega \setminus \Gamma$ with $\bar{u}(\bar{x}) > u^{+}_{\Gamma,\theta}(\bar{x})$.

Let us denote by

$$\Gamma^{+} := \left\{ x \in \Omega : \bar{u}(x) = u^{+}_{\Gamma,\theta}(x) \right\}$$

and claim that

$$u^{+}_{\Gamma,\theta}(x)=\inf_{y\in\Gamma^{+}}\left\{\bar{u}(y)+a\rho_{F^0}(x-y)\right\} \tag{3.1}$$

for each $x\in\Omega$. Indeed, the inequality "$\geq$" in (??) is obvious because $\Gamma^{+}\supset\Gamma$ and $\bar{u}(y)=\theta(y)$, $y\in\Gamma$. On the other hand, given $x\in\Omega$ take an arbitrary $y\in\Gamma^{+}$ and due to the compactness of Γ we find $y^{*}\in\Gamma$ such that

$$\bar{u}(y)=\theta(y^{*})+a\rho_{F^0}(y-y^{*}). \tag{3.2}$$

Then, by triangle inequality,

$$\begin{aligned}\bar{u}(y)+a\rho_{F^0}(x-y) &\geq \theta(y^{*})+a\rho_{F^0}(x-y^{*})\geq\\ &\geq u^{+}_{\Gamma,\theta}(x).\end{aligned} \tag{3.3}$$

Passing to infimum in (??) we prove the inequality "$\leq$" in (??) as well.

Since for arbitrary $x,y\in\Gamma^{+}$ and for $y^{*}\in\Gamma$ satisfying (??) we have

$$\begin{aligned}\bar{u}(x)-\bar{u}(y) &= u^{+}_{\Gamma,\theta}(x)-u^{+}_{\Gamma,\theta}(y)\leq a\rho_{F^0}(x-y^{*})-a\rho_{F^0}(y-y^{*})\leq\\ &\leq a\sigma_F(x-y),\end{aligned} \tag{3.4}$$

we can extend the function $\theta:\Gamma\to\mathbb{R}$ to the (closed) set $\Gamma^{+}\subset\Omega$ by setting

$$\theta(x)=\bar{u}(x),\ x\in\Gamma^{+},$$

and all the conditions on $\theta(\cdot)$ remain valid (see (??) and (??)). So, without loss of generality we can assume that the strict inequality

$$\bar{u}(x)>u^{+}_{\Gamma,\theta}(x) \tag{3.5}$$

holds for all $x\in\Omega\setminus\Gamma\neq\emptyset$.

Notice that the *convex hull* $K:=\operatorname{co}\Gamma$ is the compact set contained in Ω (due to the convexity of Ω). Let us choose now $\varepsilon>0$ such that $K\pm\varepsilon F^{0}\subset\Omega$ and denote by

$$\delta:=2\varepsilon\mathfrak{M}_{F^0}\left(\frac{2\varepsilon}{\Delta};\frac{\varepsilon}{\varepsilon+\Delta},\frac{\Delta}{\varepsilon+\Delta}\right)>0,$$

where $\mathfrak{M}_{F^0}$ is the modulus of rotundity associated to F^{0} (see (??)) and

$$\Delta:=\sup_{\xi,\eta\in\Omega}\rho_{F^0}(\xi-\eta)$$

is the ρ_{F^0}-diameter of the region Ω. Similarly as in [?] (see Step 1 in the proof of Theorem 5) we show that

$$\rho_{F^0}(y_1-x)+\rho_{F^0}(x-y_2)-\rho_{F^0}(y_1-y_2)\geq\delta \tag{3.6}$$

whenever $y_1,y_2\in\Gamma$ and $x\in\Omega\diagdown\left[\left(K+\varepsilon F^{0}\right)\cup\left(K-\varepsilon F^{0}\right)\right]$. Indeed, we obviously have

$$\varepsilon\leq\rho_1:=\rho_{F^0}(y_1-x)\leq\Delta\quad\text{and}\quad\varepsilon\leq\rho_2:=\rho_{F^0}(x-y_2)\leq\Delta,$$

and, consequently,

$$\lambda := \frac{\rho_2}{\rho_1 + \rho_2} \in \left[\frac{\varepsilon}{\varepsilon + \Delta}, \frac{\Delta}{\varepsilon + \Delta}\right]. \tag{3.7}$$

Setting $\xi_1 := \frac{y_1 - x}{\rho_1}$ and $\xi_2 := \frac{x - y_2}{\rho_2}$ we can write

$$\xi_1 - \xi_2 = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left(\frac{\rho_2}{\rho_1 + \rho_2} y_1 + \frac{\rho_1}{\rho_1 + \rho_2} y_2 - x\right),$$

and hence

$$\rho_{F^0}(\xi_1 - \xi_2) \geq \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\varepsilon \geq \frac{2\varepsilon}{\Delta}. \tag{3.8}$$

On the other hand,

$$\begin{aligned} & \rho_{F^0}(y_1 - x) + \rho_{F^0}(x - y_2) - \rho_{F^0}(y_1 - y_2) \\ = \;& (\rho_1 + \rho_2)\left[1 - \rho_{F^0}\left(\frac{\rho_1}{\rho_1 + \rho_2}\xi_1 + \frac{\rho_2}{\rho_1 + \rho_2}\xi_2\right)\right] \geq \\ \geq \;& 2\varepsilon\left[1 - \rho_{F^0}(\xi_1 + \lambda(\xi_2 - \xi_1))\right]. \end{aligned} \tag{3.9}$$

Combining (**??**), (**??**), (**??**) and the definition of the rotundity modulus (**??**) we arrive at (**??**).

Let us fix $\bar{x} \in \Omega \diagdown \Gamma$ and $\bar{y} \in \Gamma$ such that

$$u^{+}_{\Gamma,\theta}(\bar{x}) = \theta(\bar{y}) + a\rho_{F^0}(\bar{x} - \bar{y}).$$

Then by Proposition 1 the point $\bar{y}$ is also a minimizer on Γ of the function $y \mapsto \theta(y) + a\rho_{F^0}(x_\lambda - y)$, where $x_\lambda := \lambda\bar{x} + (1-\lambda)\bar{y}$, $\lambda \in [0,1]$, i.e.,

$$u^{+}_{\Gamma,\theta}(x_\lambda) = \theta(\bar{y}) + a\rho_{F^0}(x_\lambda - \bar{y}). \tag{3.10}$$

Define now the Lipschitz continuous function

$$\begin{aligned} \bar{v}(x) := \max\{\bar{u}(x), \min\{&\theta(\bar{y}) + a\rho_{F^0}(x - \bar{y}), \\ &\theta(\bar{y}) + a(\delta - \rho_{F^0}(\bar{y} - x))\}\} \end{aligned} \tag{3.11}$$

and claim that $\bar{v}(\cdot)$ minimizes the functional (**??**) on the set $\bar{u}(\cdot) + W^{1,1}_0(\Omega)$. In order to prove this we observe first that for each $x \in \Omega$, $x \notin K \pm \varepsilon F^0$, and for each $y \in \Gamma$, by the slope condition (**??**) and by (**??**), the inequality

$$\begin{aligned} & \theta(y) + a\rho_{F^0}(x - y) - \theta(\bar{y}) + a\rho_{F^0}(\bar{y} - x) \\ \geq \;& a(\rho_{F^0}(\bar{y} - x) + \rho_{F^0}(x - y) - \rho_{F^0}(\bar{y} - y)) \geq a\delta \end{aligned} \tag{3.12}$$

holds. Passing to infimum in (**??**) for $y \in \Gamma$ and taking into account the basic assumption (**ii**), we have

$$\begin{aligned} \bar{u}(x) &\geq \inf_{y \in \Gamma}\{\theta(y) + a\rho_{F^0}(x - y)\} \\ &\geq \theta(\bar{y}) + a(\delta - \rho_{F^0}(\bar{y} - x)), \end{aligned}$$

and, consequently, $\bar{v}(x) = \bar{u}(x)$ for all $x \in \Omega \setminus \left[\left(K + \varepsilon F^0\right) \cup \left(K - \varepsilon F^0\right)\right]$. In particular, $\bar{v}(\cdot) \in \bar{u}(\cdot) + W_0^{1,1}(\Omega)$. Furthermore, setting

$$\Omega' := \{x \in \Omega : \bar{v}(x) \neq \bar{u}(x)\},$$

by the well known property of the support function, we have $\nabla \bar{v}(x) \in aF$ for a.e. $x \in \Omega'$, while $\nabla \bar{v}(x) = \nabla \bar{u}(x)$ for a.e. $x \in \Omega \diagdown \Omega'$. Then

$$\begin{aligned} \int_\Omega f(\rho_F(\nabla \bar{v}(x)))\, dx &= \int_{\Omega \diagdown \Omega'} f(\rho_F(\nabla \bar{u}(x)))\, dx \\ &\leq \int_\Omega f(\rho_F(\nabla \bar{u}(x)))\, dx \leq \int_\Omega f(\rho_F(\nabla u(x)))\, dx \end{aligned}$$

for each $u(\cdot) \in \bar{u}(\cdot) + W_0^{1,1}(\Omega)$.

Finally, setting

$$\mu := \min\left\{\varepsilon, \frac{\delta}{(\|F\| \|F^0\| + 1)^2}\right\},$$

we see that the minimizer $\bar{v}(\cdot)$ satisfies the inequality

$$\bar{v}(x) \geq \theta(\bar{y}) + a\rho_{F^0}(x - \bar{y}) \tag{3.13}$$

on $\bar{y} + \mu\left(\|F\| \left\|F^0\right\| + 1\right) F^0$. Indeed, it follows from (**??**) that

$$\rho_{F^0}(x - \bar{y}) + \rho_{F^0}(\bar{y} - x) \leq \mu\left(\|F\| \left\|F^0\right\| + 1\right)^2 \leq \delta$$

whenever $\rho_{F^0}(x - \bar{y}) \leq \mu\left(\|F\| \left\|F^0\right\| + 1\right)$, implying that the minimum in (**??**) is equal to $\theta(\bar{y}) + a\rho_{F^0}(x - \bar{y})$. Since, obviously, $\bar{v}(\bar{y}) = \theta(\bar{y})$, applying Corollary 1 we deduce from (**??**) that

$$\bar{v}(x) = \theta(\bar{y}) + a\rho_{F^0}(x - \bar{y})$$

for all $x \in \bar{y} + \mu F^0 \subset K + \varepsilon F^0 \subset \Omega$. Taking into account (**??**), we have

$$\bar{u}(x) \leq \theta(\bar{y}) + a\rho_{F^0}(x - \bar{y}), \quad x \in \bar{y} + \mu F^0. \tag{3.14}$$

On the other hand, for some $\lambda_0 \in (0, 1]$ the points x_λ, $0 \leq \lambda \leq \lambda_0$, belong to $\bar{y} + \mu F^0$. Combining now (**??**) for $x = x_\lambda$ with (**??**) we obtain

$$\bar{u}(x_\lambda) \leq u^+_{\Gamma,\theta}(x_\lambda)$$

and hence (see the hypothesis (**ii**))

$$\bar{u}(x_\lambda) = u^+_{\Gamma,\theta}(x_\lambda),$$

$0 \leq \lambda \leq \lambda_0$, contradicting (**??**). ∎

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