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Immunization Using a Parametric Model of the Term Structure

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Abstract/Resumo:

In this paper, we develop a new immunization model based on a parametric specification of the term structure of interest rates. The model extends traditional duration analysis to account for both parallel and non-parallel term structure shifts that have an economic meaning. Contrary to most interest rate risk models, we analyse both first-order and second-order conditions for bond portfolio immunization and conclude that the key to successful protection will be to build up a bond portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. In addition, we provide explicit formulae for new parametric interest rate risk measures and present alternative approaches to implement the immunization strategy. Furthermore, we provide useful expressions for the sensitivity of interest rate risk measures to changes in term structure shape parameters.

1 Introduction

Immunization, which may be defined as the protection of the nominal value of a portfolio (or the net value of a firm) against changes in the term structure of interest rates, is a well-known area of portfolio management. Immunization allows the setting up and management of a bond portfolio in such a way that this portfolio reaches a predetermined goal. This goal can either be to guarantee a set of future payments, to obtain a certain rate of return for the investment or, in certain cases, to replicate the performance of a bond market index. Immunization models (also known as interest-rate risk or duration models) control risk through duration and convexity measures, which capture the sensitivity of bond returns to changes in one or more interest rate risk factors.

The traditional approach to immunization employs duration measures derived analytically from prior assumptions regarding specific changes in the term structure of interest rates.[1] Unfortunately, in this case the effectiveness of the strategy is compromised, since the investment is protected only against the particular type of interest rate change assumed. Fong and Vasicek (1983, 1984) developed the M-Squared model in order to minimize the immunization risk due to non-parallel (slope) shifts in the term structure of interest rates. The authors show in particular that by setting the duration of a bond portfolio equal to its planning horizon and by minimizing a quadratic cash flow dispersion measure, the immunization risk due to adverse term structure shifts can be reduced.[2] More recently, new immunization risk (dispersion) measures were proposed by Nawalkha and Chambers (1996), Balbás and Ibáñez (1998) and Balbás et al.(2002).

In recent years researchers have redirected their attention towards the development of alternative formulations which try to capture more effectively the interest rate risk faced by fixed-income portfolios, without relying on any particular assumptions as to the type of stochastic process which governs interest rate movements. A popular approach is to assume that interest rate changes can be accurately described by shifts in the level of a limited number of segments (vertices or yield

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[1]For example, the duration measure developed by Fisher and Weil (1971) assumes a parallel and instantaneous shift in the term structure of interest rates. Non-parallel shifts were proposed by Bierwag (1977), Khang (1979) and Babbel (1983) or, in an equilibrium setting, by Cox et al. (1979), Ingersoll et al. (1978), Brennan and Schwartz (1983), Nelson and Schaefer (1983) and Wu (2000), among others.

[2]Nawalkha and Chambers (1997) and Nawalkha, Soto and Zhang (2003) derive a multiple-factor extension to the M-Squared model termed M-Vector Model.

curve drivers) into which the term structure is subdivided, generalizing then the concepts of duration and convexity to a multivariate context by considering the portfolio's joint exposure to these key rates. Specifically, we refer to the directional duration and to the partial duration models of Reitano (1990, 1991a, 1991b, 1992), to the key-rate duration model of Ho (1992) and to the reshaping duration model suggested by Klaffky et al.. (1992). In these models, the direction of interest rate shifts can be set on an *a priori* basis, or can be based on real data. In the latter case, the historical movements in the term structure of interest rates are used to identify a limited number of state variables, observable or not, which govern the yield curve.[3]

An alternative line of attack to the problem of immunization involves the use of parametric duration models. In this kind of formulation, which has its roots in the work of Cooper (1977), all that is assumed is that at each moment in time, the term structure of interest rates adheres to a particular functional form, which expresses itself as a function of time and a limited number of shape parameters. In this line of thought, provided that the mathematical function fits most yield curves accurately all interest rate movements can be expressed in terms of changes in one or more shape parameters that characterize this function. In other words, it is apparent that in this kind of models, the interest rate risk uncertainty is reflected by the unknown nature of future parameter values. Differentiating the bond price with respect to each shape parameter we obtain a vector of parametric interest rate risk measures. Choosing a particular functional form obviously involves some pricing errors. The difference is that in this case the errors can be quantified and controlled systematically, as long as we are able to choose the appropriate specification for the yield curve, in which by 'appropriate' we mean the specification that minimizes immunization risk.

Since the work of Cooper, there has been little research in this area. Garbade (1985), Chambers et al. (1988) and Prisman and Shores (1988) assume that a polynomial may be used to fit the term structure of interest rates as a first step to derive a vector of interest rate risk measures - termed a duration vector -, in which each element basically corresponds to the moment of order k of a bond[4]. Although simple, the use of polynomial functions to estimate the yield curve has been subject to great criticism, since it can lead to forward curves that exhibit undesirable (and

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[3]See, for example, Gultekin and Rogalski (1984), Elton et al. (1990), Garbade (1986), Litterman and Scheinkman (1991), Knez et al. (1994), D'Ecclesia and Zenios (1994), Barber and Copper (1996) and Bravo and Silva (2005).

[4]The moment of order k of a bond is defined as the weighted average of the kth power of its times of payments, the weights being the shares of the bond's cash flows in present value in the bond's present value. Chambers et al.

unrealistic) properties for long maturities, namely high instability. In Willner (1996), the actual yield curve risk exposure of a bond portfolio is decomposed using the Nelson and Siegel (1987) parametrization of the yield curve, a mathematical function that expresses interest rates in terms of four parameters and is compatible with standard increasing, decreasing, flat and inverted yield curve shapes.

In this paper, we develop a new immunization model based on the Svensson (1994) specification of the yield curve. The model is parametric by nature, i.e., the interest rate risk factors correspond to the parameters of the mathematical function used to represent the yield curve. Furthermore it adopts a multivariate setting, being compatible with both parallel and non-parallel term structure shifts. Since we do not impose any previous assumptions as to the way the yield curve changes, the model is applicable in virtually all yield curve environments. In addition, the model is intuitive and relatively easy to apply.

This paper is related to Willner (1996), but there are some important differences. First, we adopt Svensson's parametrization instead of Nelson and Siegel's mathematical function. As shown by Svensson (1994), the extended form allows more flexibility in the yield curve estimation, in particular in the short-term end of the yield curve. In addition, the model assumes that every movement in the term structure of interest rates can be approximated by changes in a small number of factors and that these factors can be directly interpreted as representing parallel, slope and curvature shifts in the yield curve. Second, contrary to Willner and most interest rate risk models, we formally analyze both first-order and second-order conditions for bond portfolio immunization, emphasizing that the key to successful immunization will be to build up a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. Moreover, we provide explicit formulae for new parametric interest rate risk measures and present alternative approaches to implement the immunization strategy. Finally, we extend previous analysis on the sensitivity of a bond's duration to changes in the yield to maturity by developing useful expressions for the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters.

The paper is organized as follows. In Section 2, we briefly characterize Svensson's specification of the yield curve and theoretically justify its use in the context of the immunization problem.

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(1988) perform immunization tests for the U.S. market over single-and multi-period horizons and conclude that the improvement in the immunization performance is considerable when at least four interest rate risk measures are added.

In Section 3, we introduce the concepts of parametric duration and parametric convexity and formally derive first-order and second-order conditions for immunization. We show that it is impossible to achieve immunization simply by meeting first-order conditions and that second-order conditions must be addressed appropriately. In Section 4, we provide simple expressions for the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters. Section 5 summarizes the main conclusions of the paper.

2   Term Structure Specification

The Svensson (1994) model is a parametric model which assumes that the instantaneous forward rate, $f(t, \mathbf{a})$:

$$f(t, \mathbf{a}) = a_0 + a_1 \left[\exp\left(-\frac{t}{a_4}\right) \right] + a_2 \left[\frac{t}{a_4} \exp\left(-\frac{t}{a_4}\right) \right] + a_3 \left[\frac{t}{a_5} \exp\left(-\frac{t}{a_5}\right) \right], \tag{1}$$

is a function of both the time to maturity t and a (line) vector of parameters $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)$ to be estimated, with $(a_0, a_4, a_5) > 0$. To increase the flexibility of the curves and to improve the fit, Svensson extended Nelson and Siegel's functional form by adding a potential extra hump in the forward curve. It is well known that the Nelson-Siegel method admits the existence of only one extremum and one point of inflection in the concavity. This means that when there are disturbances in the money market that lead to curves with two local extrema, the fit in the short segment of the yield curve turns out to be very poor. Given its higher adjustment capacity, the Svensson model has proven to be more adequate in estimating the term structure of interest rates and it is widely used by practitioners and major central banks.[5]

The parameters in the forward rate function are estimated by solving a non-linear optimization procedure to data observed on a trade day, which consists of minimizing the sum of squared yield (or price) deviations between observed and theoretical yields (or prices) as estimated with the model. The optimization problem can be solved using either a grid search procedure or a partial estimation technique[6]. In most practical applications, fitting was found to be relatively insensitive

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[5]The Bank of International Settlements (1999) notes that ten Central Banks (of twelve surveyed) routinely use either the Nelson and Siegel (1987) and/or the Svensson (1994) model as their primary method for analyzing the yield curve. See Bravo (2001), Barrett et al. (1995), Diebold and Li (2003) for other uses of the NS model.

[6]For more details on the estimation process see, for example, Nelson and Siegel (1987), BIS (1999) and Bolder and Stréliski (1999).

to changes in parameters a_4 and a_5 (e.g. Barrett et al., 1995, Willner, 1996 and Diebold and Li, 2003). Without loss of generality, we will follow standard practise and assume at any stage that these parameters are fixed at prespecified values. Note also that by setting a_3 equal to zero in (1), we obtain the Nelson and Siegel forward rate function.

Regardless of their popularity, the Nelson-Siegel-Svensson family of curves has been criticized because of two theoretical shortcomings. The first, pointed out by Björk and Christensen (1999) and Filipovic (1999, 2000), is that models fitted sequentially to cross-sectional data are not inter-temporally consistent with the dynamics of a given interest rate model. Björk and Christensen (1999) prove, for instance, that the Nelson-Siegel family of curves is inconsistent with the Ho-Lee interest rate model and with the Hull-White extension of the Vasicek model. This feature weakens the validity of the model for applications that involve a time-series context. It can be shown, however, that a simple manifold expansion (i.e. the addition of appropriate functions of maturity) is sufficient to make the Nelson and Siegel model consistent with given interest rate models, particularly with the generalized Vasicek short rate model.[7] Nonetheless, these adjustments impose additional constraints on the estimation of the models to cross-sectional data, thus leading to a non-trivial deterioration of the fitting performance when compared with that provided by the Nelson-Siegel-Svensson family of curves. On the other hand, it is not obvious to us that the use of arbitrage-free models is necessary or desirable for accomplishing good immunization performance. Indeed, if the theoretical superiority of equilibrium term structure models is unquestionable, when compared to traditional immunizing duration models, the truth is that a number of papers, such as Ingersoll (1983), Nelson and Schaefer (1983) and Brennan and Schwartz (1983), have shown that their immunization performance is rather similar. In addition, Brandt and Yaron (2003) prove that typical no-arbitrage models are actually time-inconsistent because their parameters are assumed constant for pricing purposes, even though the parameters change each time the model is recalibrated to data observed on a given date. Moreover, recent studies (e.g. Duffie, 2002 and Dai and Singleton, 2002) have shown that affine no-arbitrage models can produce poor forecasts.

The second theoretical shortcoming is that these models apparently lack a fundamental economic foundation, which leads researchers to be cautious when interpreting the parameters in conjunction with economic variables, and may explain why their use has been limited to cross-sectional applications, namely, yield-curve fitting and interest rate risk management. An ex-

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[7]See Björk and Christensen (1999), Filipovic (2000) and Krippner (2005a).

ception is given by Diebold and Li (2003), who use variations on the Nelson-Siegel framework to model the entire yield curve on an inter-temporally basis as a three-dimensional parameter evolving dynamically.[8] The authors prove, first, that the model is consistent with standard stylized facts regarding the yield curve and, second, that the three-time varying parameters may be roughly interpreted as factors corresponding to level, slope and curvature, a result consistent with previous studies on this subject.[9]

From (1), the continuously compounded zero-coupon curve $r(t, \mathbf{a})$ can be derived, noting that $r(t, \mathbf{a}) = \frac{1}{t} \int_0^t f(t, \mathbf{a}) dt$:

$$r(t, \mathbf{a}) = a_0 + a_1 \frac{a_4}{t} \left(1 - e^{-\frac{t}{a_4}} \right) + a_2 \frac{a_4}{t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] + a_3 \frac{a_5}{t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right], \quad (2)$$

whereas the discount function $d(t, \mathbf{a})$ is defined as:

$$d(t, \mathbf{a}) = \exp \left[-r(t, \mathbf{a}) t \right]. \quad (3)$$

Each parameter in (1) has a particular impact on the shape of the forward rate curve. Parameter a_0, which represents the asymptotic value of $f(t, \mathbf{a})$ (i.e., $\lim_{t \to \infty} f(t, \mathbf{a}) = a_0$), can actually be regarded as a long-term (consol) interest rate. Parameter a_1 defines the speed with which the curve tends towards its long-term value. The yield curve will be upward sloping if $a_1 < 0$ and downward-sloping if $a_1 > 0$. The higher the absolute value of a_1 the steeper the yield curve. Notice also that the sum of a_0 and a_1 corresponds to the instantaneous forward rate with an infinitesimal maturity ($\lim_{t \to 0} f(t, \mathbf{a}) = a_0 + a_1$), i.e., it defines the intercept of the curve. Parameters a_2 and a_3 have similar significance and influence the shape of the yield curve. They determine the magnitude and the direction of the first and second humps, respectively. For example, if a_2 is positive, a hump will occur at a_4 whereas, if a_2 is negative, a U-shape value will emerge at a_4. Parameters a_4 and a_5, which are always positive, have similar roles and define the position of the first and second humps, respectively.

The Svensson model is very intuitive since parameters a_0, a_1, a_2 and a_3 (the interest rate factors) can directly be linked to parallel displacements, slope changes and curvature shifts in the

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[8]See also Krippner (2005b).

[9]See, for instance, Litterman and Scheinkman (1991), Barber and Copper (1996), Knez et al. (1994), D'Ecclesia and Zenios (1994) and Bravo and Silva (2005).

yield curve, given that scale coefficients are fixed. To perceive this behavior, **Figure 1** displays the sensitivity $S_k = \frac{\partial f(t,\mathbf{a})}{\partial a_k}$ of forward rates to each parameter a_k, for $k = 0, ..., 3$.

[Insert **Figure 1**, about here]

As can be seen, the sensitivity of forward rates with respect to the consol rate is constant across the whole maturity spectrum, which means that it can actually be regarded as a level factor. In other words, the level factor S_0 fundamentally represents a parallel displacement in the term structure of interest rates. The sensitivity of interest rates to changes in parameter a_1 shows a descending shape, first larger for shorter maturities, then declining exponentially toward zero as maturity increases. In this sense, factor S_1 is a slope factor and represents changes in the steepness of the yield curve. Finally, factors S_2 and S_3 have different impacts on intermediate rates as opposed to extreme maturities (short and long), reaching a maximum on those points (a_4 and a_5, respectively) where the yield curve has humps. Hence, these factors may be interpreted as curvature factors. In brief, the Svensson model assumes that: (i) every movement in the term structure of interest rates can be approximated by changes in only four factors; (ii) these factors take familiar shapes, that is parallel shifts, changes in steepness, and changes in the curvature of the yield curve.

3   Constructing Immunized Portfolios

Consider an investor who has a position in a number L of default-free bonds. Let c_{lt} denote the nominal cash flow (in monetary units) received from bond l ($l = 1, ..., L$) at time t ($t = 1, ..., N$). Let $t = 0$ be the current date, and H a known, finite investment horizon, measured in years. Assuming that the initial term structure is known and described by the parametric function (2), which assigns a spot rate to each payment date t, the present value of bond l, $B_0^l(\mathbf{a})$, is given by:

$$
\begin{aligned}
B_0^l(\mathbf{a}) &= \sum_{t=1}^{N} c_{lt} \exp\left[-r(t,\mathbf{a})t\right] = \\
&= \sum_{t=1}^{N} c_{lt} e^{-\left\{a_0 + a_1 \frac{a_4}{t}\left(1 - e^{-\frac{t}{a_4}}\right) + a_2 \frac{a_4}{t}\left[1 - e^{-\frac{t}{a_4}}\left(1 + \frac{t}{a_4}\right)\right] + a_3 \frac{a_5}{t}\left[1 - e^{-\frac{t}{a_5}}\left(1 + \frac{t}{a_5}\right)\right]\right\} t},
\end{aligned} \tag{4}
$$

8

where we have stressed the functional relationship between the bond price $B^l(\mathbf{a})$ and the initial vector $\mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5)^T$ of parameters of the forward rate function. Let n_l represent the number of type l bonds in the portfolio. In this case, the present value (at time 0) of this bond portfolio, $P_0(\mathbf{a})$, is given by:

$$
\begin{aligned}
P_0(\mathbf{a}) &= \sum_{l=1}^{L} n_l B_0^l(\mathbf{a}) = \sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{a})t\right] = \\
&= \sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} e^{-\left\{a_0 + a_1 \frac{a_4}{t}\left(1 - e^{-\frac{t}{a_4}}\right) + a_2 \frac{a_4}{t}\left[1 - e^{-\frac{t}{a_4}}\left(1 + \frac{t}{a_4}\right)\right] + a_3 \frac{a_5}{t}\left[1 - e^{-\frac{t}{a_5}}\left(1 + \frac{t}{a_5}\right)\right]\right\}t}
\end{aligned}
\tag{5}
$$

For simplicity of exposition, consider now that the investor is interested only in his wealth position at some future time H (where H might represent, for example, the due date on a single liability payment). The value of this portfolio at time H, under the expectations hypothesis of the term structure assuming no change in the yield curve, $P_H(\mathbf{a})$, will be:

$$
\begin{aligned}
P_H(\mathbf{a}) &= P_0(\mathbf{a})\exp\left[r(H,\mathbf{a})H\right] = \\
&= \left[\sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{a})t\right]\right]\exp\left[r(H,\mathbf{a})H\right]
\end{aligned}
\tag{6}
$$

Suppose now that at time τ, immediately after the investor purchased the portfolio, the spot rate function has undergone a variation, which may be viewed here as a vector $d\mathbf{A}$ of multiple random shifts and represent both parallel and nonparallel shifts, such that the new term structure, represented again by Svensson's model, is $r_\tau(t,\mathbf{A}) = r(t,\mathbf{a} + d\mathbf{A})$:

$$
\begin{aligned}
r_\tau(t,\mathbf{A}) &= A_0 + A_1 \frac{A_4}{t}\left(1 - e^{-\frac{t}{A_4}}\right) + A_2 \frac{A_4}{t}\left[1 - e^{-\frac{t}{A_4}}\left(1 + \frac{t}{A_4}\right)\right] \\
&\quad + A_3 \frac{A_5}{t}\left[1 - e^{-\frac{t}{A_5}}\left(1 + \frac{t}{A_5}\right)\right],
\end{aligned}
\tag{7}
$$

where $\mathbf{A} = (A_0, ..., A_5)^T$ denotes the new vector of coefficients of the spot rate function estimated at time τ. The new terminal value of the portfolio, $P_H(\mathbf{A})$, keeps the same form as above, except that vector $\mathbf{A}$ now replaces the initial vector of parameters $\mathbf{a}$:

$$
P_H(\mathbf{A}) = \left[\sum_{l=1}^{L}\sum_{t=1}^{N} n_l c_{lt} \exp\left[-r(t,\mathbf{A})t\right]\right]\exp\left[r(H,\mathbf{A})H\right]
\tag{8}
$$

9

The traditional definition of immunization (e.g. Fisher and Weil, 1971) for the case of a single liability establishes that a portfolio of default-free bonds is said to be immunized against any type of interest rate shifts if its accumulated value at the end of the planning horizon is at least as great as the target value, where the target value is defined as the portfolio value at the horizon date under the scenario of no change in the spot (and forward) rates. Stated more formally, by immunization we mean selection of a bond portfolio such that the actual future value of the income stream $P_H(\mathbf{A})$ at time H will exceed the initially expected value $P_H(\mathbf{a})$, i.e., $P_H(\mathbf{A}) \geq P_H(\mathbf{a})$ (or equivalently, $\Delta P_H = P_H(\mathbf{A}) - P_H(\mathbf{a}) \geq 0$), if the interest rates $r(t, \mathbf{A})$ shift to their new value $r_\tau(t, \mathbf{A})$.

Under the assumption that interest rates only change by a parallel shift, the main conclusion of Fisher and Weil was that immunization is achieved when the duration of the portfolio is set equal to the length of the investment horizon. The assumption that interest rates can change only by a parallel shift is very restrictive and can carry serious risks. In this paper we offer a more generalized approach to immunization by deriving the conditions under which the investment is protected against both parallel and non-parallel yield curve shifts.

3.1   First-Order Conditions

Let $P_H(\mathbf{A})$ be a multivariate price function, assumed to be twice continuously differentiable. The idea is to use a Taylor series expansion of $P_H(\mathbf{A})$ around the initial vector of parameters in order to evaluate the necessary and sufficient conditions for a local minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. For most practical applications, an expansion up to the second order is sufficient to obtain a reasonable approximation. The quadratic approximation for (8) is then given by:[10]

$$dP_H(\mathbf{A}) = P_H(\mathbf{A}) - P_H(\mathbf{a}) = \nabla P_H(\mathbf{a})^T \cdot d\mathbf{a} + \frac{1}{2} d\mathbf{a}^T \cdot \nabla^2 P_H(\mathbf{a}) \cdot d\mathbf{a} + R_2(\mathbf{a}, d\mathbf{A}), \qquad (9)$$

where $d\mathbf{A} = (da_i)_{i=0,\ldots,5}^T$ denotes the (column) vector of variations of parameters $\mathbf{a}$, $\cdot$ denotes the inner product of two vectors and $R_2(\mathbf{a}, d\mathbf{A})$ represents the remaining terms of the series. Terms $\nabla P_H(\mathbf{A})$ and $\nabla^2 P_H(\mathbf{A})$ represent, respectively, the gradient vector and the Hessian matrix of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. Alternatively, if we divide (9) by $P_H(\mathbf{A})$, we obtain the percentage change in the terminal value of the bond portfolio:

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[10]Note that the change in the portfolio value resulting from the passage of time is ignored here due to its deterministic nature.

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{P_H(\mathbf{A})}\nabla P_H(\mathbf{a})^T \cdot d\mathbf{a} + \frac{1}{2}d\mathbf{a}^T \cdot \frac{1}{P_H(\mathbf{A})}\nabla^2 P_H(\mathbf{a}) \cdot d\mathbf{a} + R_2^*(\mathbf{a},d\mathbf{A}), \qquad (10)$$

where $R_2^*(\mathbf{a},d\mathbf{A}) = R_2(\mathbf{a},d\mathbf{A})/P_H(\mathbf{A})$. Let now $c_t = \sum_{l=1}^{L} n_l c_{lt}$ denote the total nominal cash flows received by the holder of the portfolio at time t. To determine the nature of the horizon value near the origin, we compute the first-order partial derivative of (8) with respect to A_k ($k = 0, ..., 5$). This yields the generic element of the gradient vector $\frac{\partial P_H(\mathbf{A})}{\partial A_k}$:

$$\begin{aligned}\frac{\partial P_H(\mathbf{A})}{\partial A_k} &= \sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]}\left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - t\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right] \qquad (11)\\ &= P_H(\mathbf{A})\left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right] - \left[\sum_{t=1}^{N} tc_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\end{aligned}$$

which, after dividing by $P_H(\mathbf{A})$, can be written as:

$$\frac{1}{P_H(\mathbf{A})}\frac{\partial P_H(\mathbf{A})}{\partial A_k} = H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - \frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right] \qquad (12)$$

In anticipation of combining (12) and (10), we introduce new definitions for parametric interest rate risk measures.

Definition 1 *The parametric duration of a bond is a measure of first-order sensitivity of bond prices to changes in interest rates as given by modifications in parameters A_k ($k = 0, ..., 5$). For bond l, the parametric duration is denoted $D^{(l)}(k, \mathbf{A})$, and is defined, for $B_0^l(\mathbf{A}) \neq 0$, as follows:*

$$\begin{aligned}D^{(l)}(k, \mathbf{A}) &= -\frac{1}{B_0^l(\mathbf{A})}\frac{\partial B_0^l(\mathbf{A})}{\partial A_k}\\ &= \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} tc_{lt}e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]. \qquad (13)\end{aligned}$$

Definition 2 *Let $w_l = \frac{n_l B_0^l(\mathbf{A})}{P_0(\mathbf{A})}$ denote the percentage of portfolio invested in bond l, such that $\sum_{l=1}^{L} w_l = 1$. The parametric duration of a bond portfolio is a measure of first-order sensitivity of a bond portfolio to changes in interest rates as given by modifications in parameters A_k ($k = 0, ..., 5$). It is calculated as the weighted average of the parametric durations of the bonds making up the*

portfolio, the weights being the shares of each bond in the portfolio. Denoted $D^{(P)}(k, \mathbf{A})$, it is defined, for $P_0(\mathbf{A}) \neq 0$, as follows:

$$
\begin{aligned}
D^{(P)}(k, \mathbf{A}) &= -\frac{1}{P_0(\mathbf{A})} \frac{\partial P_0(\mathbf{A})}{\partial A_k} \\
&= \sum_{l=1}^{L} w_l D^{(l)}(k, \mathbf{A}).
\end{aligned}
\tag{14}
$$

Each equation in (13) represents a bond's interest rate risk measure for a particular type of shift in the yield curve. For instance, the first element, $D^{(l)}(0, \mathbf{A})$, is defined as:

$$
D^{(l)}(0, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t} \right]
\tag{15}
$$

and corresponds to the traditional Fisher-Weil duration measure. It is defined as the weighted average of the times of payment of all the cashflows generated by the bond, the weights being the shares of the bond's cashflows in the bond's present value, and captures the sensitivity of bond returns to changes in the consol factor a_0, i.e., the responsiveness of bond returns to height shifts in the term structure of interest rates. The second element, $D^{(l)}(1, \mathbf{A})$, is defined as:

$$
D^{(l)}(1, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right) a_4 \right]
\tag{16}
$$

and captures the sensitivity of bond returns to changes in parameter a_1, that is, to changes in the slope of the yield curve. The third, $D^{(l)}(2, \mathbf{A})$, and fourth, $D^{(l)}(3, \mathbf{A})$, elements of the duration vector summarize the sensitivity of bond returns to changes in the curvature parameters a_2 and a_3, and are defined as:

$$
D^{(l)}(2, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] a_4 \right\}
\tag{17}
$$

and

$$
D^{(l)}(3, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right] a_5 \right\}
\tag{18}
$$

respectively. Finally, The fourth, $D^{(l)}(4, \mathbf{A})$, and fifth, $D^{(l)}(5, \mathbf{A})$, elements of the duration vector summarize the sensitivity of bond returns to changes in the location parameters a_4 and a_5, and are defined as

12

$$D^{(l)}(4, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[\left(\frac{a_1}{t} + \frac{a_2}{t} \right) (1 - e^{-\frac{t}{a_4}}) - \left(\frac{a_1}{a_4} + \frac{a_2}{a_4} \right) e^{-\frac{t}{a_4}} - a_2 \frac{t}{a_4^2} e^{-\frac{t}{a_4}} \right] \right\} \tag{19}$$

and

$$D^{(l)}(5, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[\frac{a_3}{t} \left(1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} - \frac{t^2}{a_5^2} \right) \right) \right] \right\} \tag{20}$$

respectively. Taking this into account, the generic element of the gradient vector (12) can be simplified to

$$\frac{1}{P_H(\mathbf{A})} \frac{\partial P_H(\mathbf{A})}{\partial A_k} = H \frac{\partial r(H, \mathbf{A})}{\partial A_k} - D^{(P)}(k, \mathbf{A}) \tag{21}$$

Let us now address first-order conditions for bond portfolio immunization. For simplicity of exposition, we assume that parameters a_4 and a_5 are fixed at prespecified values.[11] We know from standard optimization theory that if a function partial differentiable has an extremum at an interior point, then all first-order derivatives are required to be zero.[12] In other words, setting the gradient vector equal to zero is a necessary (but clearly insufficient) condition for an interior local minimum. From (21) this is equivalent to a fourth-dimensional vector of the form:

$$D^{(P)}(k, \mathbf{A}) = H \frac{\partial r(H, \mathbf{A})}{\partial A_k} \qquad (k = 0, ..., 3). \tag{22}$$

Each of the conditions in (22) defines an immunization condition for a different type of yield curve shift. For instance, selecting a bond portfolio such that its $D^{(P)}(0, \mathbf{A})$ is set equal to the planning horizon H protects the investment against a parallel shift in the yield curve. In other words, the traditional approach to immunization can be considered, to some extend, a particular case of the parametric model. Similarly, immunization against slope shifts is attained if the condition $D^{(P)}(1, \mathbf{A}) = a_4 \left[1 - \exp(-\frac{H}{a_4}) \right]$ is fulfilled. Finally, appropriate protection against changes in the curvature of the term structure is obtained by choosing a portfolio's composition such that $D^{(P)}(2, \mathbf{A}) = a_4 \left[1 - e^{-\frac{H}{a_4}} \left(1 + \frac{H}{a_4} \right) \right]$ and $D^{(P)}(3, \mathbf{A}) = a_5 \left[1 - e^{-\frac{H}{a_5}} \left(1 + \frac{H}{a_5} \right) \right]$.

---

[11] The approach can easily be expanded to admit changes in the location of the humps of the forward curve.

[12] See, for example, Apostol (1969).

To sum up, from equation (22) two implications follow immediately. First, the vector of parametric duration measures is determined only by the structure of the bond portfolio and can therefore be controlled by the portfolio manager. Second, given that convexity conditions are respected and a sufficient number of bonds are available (i.e. $L \geq 4$ or $L \geq 5$, if we include the initial self-financing constraint), complete immunization against interest rate changes (both parallel and non-parallel) can be achieved by selecting a bond portfolio such that all of the first-order immunization constraints are satisfied. Note that the investor can always adopt a more active role in the immunization strategy by choosing, deliberately, to satisfy only some of the conditions in (22). He can, for example, use the principal components analysis to select those interest rate shifts that are more likely, or account most for the volatility of the yield curve and then engage in the appropriate immunization strategy. Alternatively, investors may try to obtain a yield pick-up and at the same time to be risk-neutral against to changes in the level and/or the yield curve by engaging in butterfly trades.

In those cases where there is more than one bond portfolio satisfying all of the immunization constraints, a particular objective function might be considered. For example, Chambers et al. (1988) argue that an acceptable portfolio construction criterion would be to minimize the sum of squared weights, i.e., $Min \sum_{l=1}^{L} w_l^2$. According to them, this will lead to a diversified portfolio that minimizes the impact of unsystematic risk caused by transitory pricing errors.

Finally, note that similar to Prisman and Shores (1988), except for the trivial case where a single zero coupon bond maturing on the planning horizon composes the portfolio[13], the solution to the immunization constraints given in equation (22) requires short positions in some bonds, i.e., any immunized portfolio must have both positive and negative cash flows. The non-monotone nature of the cash flow structure makes the existence of local minima at $\mathbf{A} = \mathbf{a}$ more problematic. In particular, we will see below that 'most' first-order immunized portfolios yield a horizon value which is not locally convex with respect to perturbations in the yield curve parameters.

3.2   Second-Order Conditions

We know from standard optimization theory that setting the gradient vector $\nabla P_H(\mathbf{A})$ equal to zero is a necessary, but insufficient, condition for a minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$. Let us now

---

[13]Paradoxically, the existence of such a bond would mean that the immunization strategy is unnecessary.

address second-order conditions and their implications for portfolio construction. For a local minimum of $P_H(\mathbf{A})$ at $\mathbf{A} = \mathbf{a}$, second-order conditions stipulate that to equations (22) we must add those corresponding to a positive semidefinite Hessian matrix for $P_H(\mathbf{A})$. The generic element of the Hessian matrix, $\gamma_{km}(\mathbf{A}) = \frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m}$, is derived from (11) by taking the partial derivative with respect to A_m $(m = 0, ..., 3)$.[14]

$$
\begin{aligned}
\gamma_{km}(\mathbf{A}) &= \frac{\partial}{\partial A_m} \left\{ \sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]} \left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - t\frac{\partial r(t,\mathbf{A})}{\partial A_k} \right] \right\} \\
&= \sum_{t=1}^{N} c_t e^{[r(H,\mathbf{A})H - r(t,\mathbf{A})t]} \left[H\frac{\partial r(H,\mathbf{A})}{\partial A_k} - t\frac{\partial r(t,\mathbf{A})}{\partial A_k} \right] \left[H\frac{\partial r(H,\mathbf{A})}{\partial A_m} - t\frac{\partial r(t,\mathbf{A})}{\partial A_m} \right] \quad (23)
\end{aligned}
$$

To simplify notation, let:

$$
q_t = c_t \exp\left[r(H,\mathbf{A})H - r(t,\mathbf{A})t \right] \qquad (t = 1, ..., N) \qquad (24)
$$

represent the cash flow received from portfolio at time t expressed in future value. From (23), $\gamma_{km}(\mathbf{A})$ is then:

$$
\begin{aligned}
\gamma_{km}(\mathbf{A}) = \sum_{t=1}^{N} q_t &\left[H^2 \frac{\partial r(H,\mathbf{A})}{\partial A_k} \frac{\partial r(H,\mathbf{A})}{\partial A_m} - H\frac{\partial r(H,\mathbf{A})}{\partial A_k} t\frac{\partial r(t,\mathbf{A})}{\partial A_m} - \right.\\
&\left. - H\frac{\partial r(H,\mathbf{A})}{\partial A_m} t\frac{\partial r(t,\mathbf{A})}{\partial A_k} + t^2 \frac{\partial r(t,\mathbf{A})}{\partial A_k} \frac{\partial r(t,\mathbf{A})}{\partial A_m} \right] \qquad (k,m = 0, ...,3)
\end{aligned}
$$

or the equivalent:

$$
\begin{aligned}
\gamma_{km}(\mathbf{A}) = H^2 &\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H,\mathbf{A})}{\partial A_m} \right) \sum_{t=1}^{N} q_t - H\frac{\partial r(H,\mathbf{A})}{\partial A_k} \sum_{t=1}^{N} tq_t \frac{\partial r(t,\mathbf{A})}{\partial A_m} - \\
&- H\frac{\partial r(H,\mathbf{A})}{\partial A_m} \sum_{t=1}^{N} tq_t \frac{\partial r(t,\mathbf{A})}{\partial A_k} + \sum_{t=1}^{N} t^2 q_t \frac{\partial r(t,\mathbf{A})}{\partial A_k} \frac{\partial r(t,\mathbf{A})}{\partial A_m} \qquad (k,m = 0, ...,3) \ (25)
\end{aligned}
$$

Dividing both term in (23) by $P_H(\mathbf{A})$ we obtain:

---

[14]In Equation (23), we have made use of the fact that all second-order cross partial derivatives are zero, i.e., $\frac{\partial}{\partial A_m} \left(\frac{\partial^2 r(\cdot,\mathbf{A})}{\partial A_k} \right) = 0$, $m = 0, ..., 3$.

$$\frac{1}{P_H(\mathbf{A})}\frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} = H^2\left(\frac{\partial r(H,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(H,\mathbf{A})}{\partial A_m}\right) - H\frac{\partial r(H,\mathbf{A})}{\partial A_k}\left\{\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right]\right\} -$$
$$-H\frac{\partial r(H,\mathbf{A})}{\partial A_m}\left\{\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} tc_t e^{-r(t,\mathbf{A})t}\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right]\right\} +$$
$$+\frac{1}{P_0(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right] \tag{26}$$

where in (26), we have made use of the fact that $\sum_{t=1}^{N} q_t = P_H(\mathbf{A}) = P_0(\mathbf{A})e^{r(H,\mathbf{A})H}$. We are now in a condition to introduce the essential definitions of parametric convexity of a bond and of a bond portfolio.

Definition 3 *The parametric convexity of a bond is a measure of second-order sensitivity of bond prices to changes in interest rates as given by modifications in parameters A_k and A_m $(k, m = 0, ..., 3)$. For bond l, the parametric convexity is denoted $C^{(l)}(k, m, \mathbf{A})$, and is equal, for $B_0^l(\mathbf{A}) \neq 0$, to:*

$$\begin{aligned} C^{(l)}(k,m,\mathbf{A}) &= \frac{1}{B_0^l(\mathbf{A})}\frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_k \partial A_m} \\ &= \frac{1}{B_0^l(\mathbf{A})}\left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t}\left(\frac{\partial r(t,\mathbf{A})}{\partial A_k}\right)\left(\frac{\partial r(t,\mathbf{A})}{\partial A_m}\right)\right] \end{aligned} \tag{27}$$

Definition 4 *Let $w_l = \frac{n_l B_0^l(\mathbf{A})}{P_0(\mathbf{A})}$ denote the percentage of portfolio invested in bond l, such that $\sum_{l=1}^{L} w_l = 1$. The parametric convexity of a bond portfolio is a measure of second-order sensitivity of a bond portfolio to changes in interest rates as given by modifications in parameters A_k and A_m $(k, m = 0, ..., 3)$. It is calculated as the weighted average of the parametric convexities of the bonds making up the portfolio, the weights being the shares of each bond in the portfolio. Denoted $C^{(P)}(k, m, \mathbf{A})$, is equal, for $P_0(\mathbf{A}) \neq 0$, to:*

$$\begin{aligned} C^{(P)}(k,m,\mathbf{A}) &= \frac{1}{P_0(\mathbf{A})}\frac{\partial^2 P_0(\mathbf{A})}{\partial A_k \partial A_m} \\ &= \sum_{l=1}^{L} w_l C^{(l)}(k,m,\mathbf{A}) \end{aligned} \tag{28}$$

To simplify notation, let $C_{k,m}^{(l)}(\mathbf{A}) = C^{(l)}(k, m, \mathbf{A})$. Each equation in (27) measures second-order effects for a particular type of shift in the term structure. For instance, the equation for $C_{0,0}^{(l)}(\mathbf{A})$ is defined as:

$$C_{0,0}^{(l)}(\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_0^2} = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t} \right]. \tag{29}$$

Surprisingly, or not, the parametric model provides a second-order sensitivity measure of the bond's price to changes in the level coefficient of the yield curve that is similar to the traditional (continuously compounded) definition of convexity.[15] We can then conclude, once again, that the traditional approach to immunization can be considered a particular case of the parametric model. Second-order effects caused by shifts in the slope parameter a_1 only can now be quantified by using:

$$C_{1,1}^{(l)}(\mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right)^2 a_4^2 \right],$$

and so on. The complete set of definitions can be found in the **Appendix**. Substituting (28) and (14) in (26) yields:

$$
\begin{aligned}
\frac{1}{P_H(\mathbf{A})} \frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} &= H^2 \left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H,\mathbf{A})}{\partial A_m} \right) - H \left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) D^{(P)}(k, \mathbf{A}) - \\
&\quad - H \left(\frac{\partial r(H,\mathbf{A})}{\partial A_m} \right) D^{(P)}(m, \mathbf{A}) + C^{(P)}(k, m, \mathbf{A})
\end{aligned}
\tag{30}
$$

From the first order conditions (22) for bond portfolio immunization, we know that:

$$D^{(P)}(k, \mathbf{A}) = H \frac{\partial r(H,\mathbf{A})}{\partial A_k} \qquad (k = 0, ..., 3) \tag{31}$$

which is also valid when k is replaced by m ($m = 0, ..., 3$). Substituting this expression in (30), the generic element of the Hessian matrix at $\mathbf{A} = \mathbf{a}$ becomes ($k, m = 0, ..., 3$):

$$
\begin{aligned}
\frac{1}{P_H(\mathbf{A})} \frac{\partial^2 P_H(\mathbf{A})}{\partial A_k \partial A_m} &= H^2 \left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H,\mathbf{A})}{\partial A_m} \right) - 2H^2 \left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) \left(H \frac{\partial r(H,\mathbf{A})}{\partial A_m} \right) + C^{(P)}(k, m, \mathbf{A}) \\
&= C^{(P)}(k, m, \mathbf{A}) - H^2 \left(\frac{\partial r(H,\mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H,\mathbf{A})}{\partial A_m} \right)
\end{aligned}
\tag{32}
$$

---

[15]See, for example, Lacey and Nawalkha (1993).

Let ω_{km} denote the difference:

$$\omega_{km}(\mathbf{A}) = C^{(P)}(k, m, \mathbf{A}) - H^2 \left(\frac{\partial r(H, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H, \mathbf{A})}{\partial A_m} \right) \qquad (k, m = 0, ..., 3) \qquad (33)$$

Each element in (33) has a clear interpretation, since it defines the difference between the parametric convexity of a bond portfolio and the sensitivity of the perfect immunization asset (i.e. of a zero coupon maturing on the horizon date) to changes in the yield curve shape parameters.[16] That is, each element in (33) represents the extent to which second-order interest rate risk measures deviate from the target. This is not surprising since from equation (32), we observe that all elements $\gamma_{km}(\mathbf{A})$ of the Hessian matrix $\nabla^2 P_H(\mathbf{A})$ are those of the matrix $\Pi = [\omega_{km}]_{k,m=0}^3$. This means that the discussion of the positive semidefiniteness of $\nabla^2 P_H(\mathbf{A})$ reduces to that of the symmetric matrix Π. At least two alternative methodologies can be used to determine the sign definiteness of the Hessian matrix: The **determinantal test approach** and the **eigenvalue test approach**. We will show how both can be used in the context of immunization.

3.2.1 The Determinantal Test Approach

Let us focus first on the use of the determinantal test approach. Let Π be a square $(n \times n)$ symmetric matrix of the form:

$$\Pi = \begin{bmatrix} \omega_{00} & \omega_{01} & ... & \omega_{0k} \\ \omega_{10} & \omega_{11} & ... & \omega_{1k} \\ ... & ... & & ... \\ \omega_{k0} & \omega_{k1} & ... & \omega_{kk} \end{bmatrix}, \; \omega_{ij} = \omega_{ji}, \; i \neq j$$

with $n = 4$. The jth order leading principal minors of the matrix Π, denoted D_j $(j = 1, ..., 4)$, are the determinants of the submatrices formed by deleting the entries in the last $n - j$ rows and columns of Π. Given Π, we may also define the jth order principal minors of Π, denoted $|D_j|$, as the determinants of the submatrices formed by deleting the entries in the $n - j$ rows and the corresponding $n - j$ columns of Π. Following these definitions, the criteria for semidefiniteness

---

[16]Note also that we can interpret $w_{km}(\mathbf{A})$ as a sort of "generalized variance" since its expression is analogous to the formula $Var(X) = E(X^2) - (E(X))^2$.

requires that for Π to be positive semidefinite, all of its principal minors of order j must be non-negative, i.e., $|D_j| \geq 0$.[17] Let us consider now the implications of this result for bond portfolio immunization. The first-order principal minors of Π, $|D_1^i|$ $(i = 1, ..., 4)$ are:

$$\left|D_1^1\right| = \omega_{00} \quad \text{and} \quad \left|D_1^2\right| = \omega_{11} \quad \text{and} \quad \left|D_1^3\right| = \omega_{22} \quad \text{and} \quad \left|D_1^4\right| = \omega_{33}, \tag{34}$$

which must all be positive or zero. From the definitions of ω_{00}, ω_{11}, ω_{22} and ω_{33} above, we can observe that its sign is determined by the portfolio structure and cannot, unfortunately, be determined without ambiguity. The task is even more difficult when we recall that matching first-order conditions requires short positions in some bonds. Consequently, since the positive definiteness of $\nabla^2 P_H(\mathbf{A})$ cannot be guaranteed by first-order conditions, we are forced to conclude that setting the gradient vector $\nabla P_H(\mathbf{A})$ equal to zero is not sufficient to protect the investment against changes in the yield curve. This means that second-order conditions play an important role in the immunization problem and need to be addressed in a convenient way.

To ensure the positive semidefiniteness of Π, we need then to impose certain restrictions on the portfolio's composition. Assume, for instance, that $\omega_{00} = \omega_{11} = \omega_{22} = 0$. The second-order principal minors of Π, $|D_2^i|$ $(i = 1, ..., 6)$, are defined as:

$$\left|D_2^1\right| = \begin{vmatrix} \omega_{00} & \omega_{01} \\ \omega_{10} & \omega_{11} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{01} \\ \omega_{01} & 0 \end{vmatrix} \quad \text{and} \quad \left|D_2^2\right| = \begin{vmatrix} \omega_{00} & \omega_{02} \\ \omega_{20} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{02} \\ \omega_{02} & 0 \end{vmatrix}$$

$$\left|D_2^3\right| = \begin{vmatrix} \omega_{00} & \omega_{03} \\ \omega_{30} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{03} \\ \omega_{03} & \omega_{33} \end{vmatrix} \quad \text{and} \quad \left|D_2^4\right| = \begin{vmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{12} \\ \omega_{12} & 0 \end{vmatrix}$$

$$\left|D_2^5\right| = \begin{vmatrix} \omega_{11} & \omega_{13} \\ \omega_{31} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{13} \\ \omega_{13} & \omega_{33} \end{vmatrix} \quad \text{and} \quad \left|D_2^6\right| = \begin{vmatrix} \omega_{22} & \omega_{23} \\ \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & \omega_{23} \\ \omega_{23} & \omega_{33} \end{vmatrix}. \tag{35}$$

From (35), we observe that the determinants $|D_2^1|$, $|D_2^2|$ and $|D_2^3|$ are equal to $-(\omega_{01})^2$, $-(\omega_{02})^2$ and $-(\omega_{03})^2$, respectively, which are all negative, thus violating the conditions for positive semi-definiteness. For these minors to be positive or zero ω_{01}, ω_{02} and ω_{03} must all be set equal to zero. Similarly, from (35) we note that the values of $|D_2^4|$, $|D_2^5|$ and $|D_2^6|$ are all negative and equal to $-(\omega_{12})^2$, $-(\omega_{13})^2$ and $-(\omega_{23})^2$, respectively. Using the same argument, in order to ensure the positive semidefiniteness of Π, we need to select a bond portfolio such that the entries ω_{12},

[17]See Takayama (1990) and references therein for an extensive discussion of the determinantal test for second-order necessary conditions for a minimum.

ω_{13} and ω_{23} are all equal to zero. Let us now turn to the third-order principal minors of Π, $|D_3^i|$ $(i = 1, ..., 4)$. They can be written as:

$$
\left| D_3^1 \right| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}
$$

$$
\left| D_3^2 \right| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{13} \\ \omega_{30} & \omega_{31} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}
$$

$$
\left| D_3^3 \right| = \begin{vmatrix} \omega_{00} & \omega_{02} & \omega_{03} \\ \omega_{20} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}
$$

$$
\left| D_3^4 \right| = \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_{33} \end{vmatrix}, \tag{36}
$$

and, as can be seen above, their values are all equal to zero. Finally, by definition, the fourth-order principal minor of Π, $|D_4|$, is equal to the determinant of Π. Therefore, we have

$$
\left| D_4 \right| = \begin{vmatrix} \omega_{00} & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{33} \end{vmatrix}, \tag{37}
$$

which is also equal to zero. To sum up, in order to guarantee the positive semidefiniteness of Π, we need to select a bond portfolio such that all entries ω_{km} are equal to zero, except one, equal to $\omega_{33} = C^{(P)}(3, 3, \mathbf{A}) - \left\{ a_5 \left[1 - e^{-\frac{H}{a_2}} \left(1 + \frac{H}{a_5} \right) \right] \right\}^2$, which must be set to an arbitrary positive value U. Accordingly, whereas first-order conditions for bond portfolio immunization imply the following $k + 1$ $(k = 0, ..., 3)$ restrictions:

$$
D^{(P)}(k, \mathbf{A}) = H \frac{\partial r(H, \mathbf{A})}{\partial A_k},
$$

second-order conditions entail the subsequent equations

$$C^{(P)}(k, m, \mathbf{A}) = \begin{cases} H^2 \left(\frac{\partial r(H, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H, \mathbf{A})}{\partial A_m} \right) + U, & k = m = 3 \\ H^2 \left(\frac{\partial r(H, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(H, \mathbf{A})}{\partial A_m} \right), & \text{other cases} \end{cases}, \tag{38}$$

to which the self-financing constraint may be added. The solution to the above immunization problem requires a considerable number of different bonds ($L \geq 14$ or $L \geq 15$, if we include the initial self-financing constraint) in the portfolio. Given that a sufficient number of bonds are available, it is theoretically possible to immunize a bond portfolio against both parallel and non-parallel interest rate shifts. Standard optimization techniques may be used to determine the immunizing portfolio. Let us now come back to the Hessian matrix. From (38), it reduces to:

$$\frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})} = \begin{bmatrix} 0 & \dots & & 0 \\ \dots & & & \dots \\ & & 0 & 0 \\ 0 & \dots & 0 & U \end{bmatrix}. \tag{39}$$

The associated quadratic form is then:

$$d\mathbf{A}^T \begin{bmatrix} 0 & \dots & & 0 \\ \dots & & & \dots \\ & & 0 & 0 \\ 0 & \dots & 0 & U \end{bmatrix} d\mathbf{A} = U \cdot (dA_3)^2. \tag{40}$$

Taking into account both first-order and second-order conditions for immunization, the percentage change in the terminal value of the bond portfolio can be expressed in the following manner:

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{2} U \cdot (dA_3)^2 + R_2^*(\mathbf{a}, d\mathbf{A}), \tag{41}$$

where $R_2^*(\mathbf{a}, d\mathbf{A})$ represents again the remaining terms of the Taylor series. Given that by definition $R_2^*(\mathbf{a}, d\mathbf{A}) \to 0$ as $d\mathbf{A} \to 0$, we can always choose a value U such that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} > 0$, i.e., we can always choose a value U such that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ is convex at $\mathbf{a}$, whatever the magnitude of the displacement of $\mathbf{A}$, $d\mathbf{A}$.

3.2.2 The Eigenvalue Test Approach

As we mentioned before, the solution to the immunization problem requires a considerable number of different bonds in the portfolio. While for large investment banks this is not a major problem, since they usually hold and manage many different bonds in several markets, for small investors based in emerging markets, this may pose a serious obstacle when it comes to implement the strategy. In these cases, the investor may opt to select a bond portfolio that matches first-order conditions for immunization and then evaluate the sufficiency of these conditions on a particular basis, using an alternative test to determine the sign-definiteness of the quadratic form: the Eigenvalue Test. Recall that at a stationary point we have $\frac{\nabla P_H(\mathbf{A})}{P_H(\mathbf{A})} = 0$, which means that the Taylor expansion in (10) reduces to:

$$\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})} = \frac{1}{2}d\mathbf{A}^T \cdot \frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})} \cdot d\mathbf{A} + R_2^*(\mathbf{a}, d\mathbf{A}) \tag{42}$$

Let $\mathbf{S} = \frac{\nabla^2 P_H(\mathbf{A})}{P_H(\mathbf{A})}$ be a $n \times n$ symmetric matrix. From standard linear algebra, we know that because $\mathbf{S}$ is symmetric, is has real eigenvalues, $\{\lambda_n\}$, and n independent unit eigenvectors, $\{\nu_n\}$, which are mutually orthogonal. Let $\mathbf{V}$ denote the a $n \times n$ matrix with $\{\nu_n\}$ as column vectors. By construction, $\mathbf{V}$ is an orthogonal matrix, $\mathbf{V}^T = \mathbf{V}^{-1}$. Changing coordinates to the $\{\nu_n\}$ basis, let $d\mathbf{A} = \mathbf{V}\mathbf{y}$. Substituting into (42), we obtain:

$$\begin{aligned} d\mathbf{A}^T \mathbf{S} d\mathbf{A} &= \mathbf{y}^T (\mathbf{V}^T \mathbf{S} \mathbf{V}) \mathbf{y} \\ &= \sum_{n=1}^{4} \lambda_n y_n^2, \end{aligned} \tag{43}$$

We also know that $\mathbf{S}$ is positive-definite (resp. negative definite) iff all its eigenvalues are positive (resp. negative). In other words, if $\mathbf{S}$ is positive-definite (resp. negative definite), we can conclude that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ has a minimum (resp. maximum) at the stationary point. In Section 3.2.1 we were able to conclude that unless additional restrictions on portfolio structure are imposed, we cannot guarantee that the hessian matrix is positive semidefinite. As a result, the possibility of getting both positive and negative eigenvalues from the spectral decomposition of matrix $\mathbf{S}$ cannot be disregarded. In other words, the possibility of obtaining negative eigenvalues means

that for certain 'directions' (interest rate shifts) the portfolio's horizon value will not be convex at $\mathbf{A} = \mathbf{a}$ and the investor is thus exposed to interest rate risk.

Taking this into account, the solution to the immunization problem must be evaluated on a particular basis. For this, we now propose a three-step procedure to find bond portfolios that satisfy both first-order and second-order immunization conditions.

Step 1 - Select a bond portfolio that matches the gradient conditions for immunization, as defined in (22);

Step 2 - Calculate the eigenvalues of $\mathbf{S}$ in order to assess if first-order conditions are sufficient to guarantee that $\frac{dP_H(\mathbf{A})}{P_H(\mathbf{A})}$ has a minimum at the stationary point derived. First-order conditions will be sufficient iff all of the eigenvalues of the hessian matrix are positive;

Step 3 - If first-order conditions are not sufficient, i.e., if not all of the eigenvalues of the Hessian matrix are positive, we recommend a type of "second-best" strategy. Since there is usually more than one bond portfolio satisfying first-order conditions, repeat Steps 1 and 2 for all of the candidate solutions and select the bond portfolio that most closely matches the conditions for a minimum. Since negative eigenvalues represent yield curve displacements for which the portfolio's horizon value is not convex, we think that a reasonable criterion for selecting an acceptable portfolio will be to minimize the impact of those yield curve directions. In this sense, we recommend a choose of candidate solution for which the sum of the absolute value of the negative eigenvalues is minimum, i.e., the one for which the quantity $\sum_{\lambda_n < 0} |\lambda_n|$ is minimum. To implement the procedure, standard optimization algorithms may be used.

4   Interest Rate Sensitivity Of Bond Risk Measures

In this section, we derive a simple expression for the sensitivity of parametric durations to changes in term structure shape parameters. Portfolio managers are often required to maintain target levels of interest rate risk exposure, both for assets and liabilities. From standard duration theory, we know that the duration of a bond changes as time passes, not only because the bond approaches maturity, but mainly due to changes in the yield curve. In volatile interest rate environments, interest rate risk measures can change rapidly as a result of modifications in the shape of the term structure of interest rates. For portfolio managers, this is a subject of major interest since maintaining the portfolio exposure up to a desired level requires frequent portfolio rebalancing.

To do so, it is of great interest to understand how interest rate risk measures themselves change with modifications in the yield curve.

The sensitivity of a bond's duration to changes in the bond's yield to maturity has been extensively analyzed in the literature (e.g. Bierwag, 1987). In spite of this, it is well known that the usefulness of this analysis is limited when yield curves are not flat and non-parallel term structure shifts may occur. In this section, we extend previous research by investigating the sensitivity of parametric duration measures to a wider a range of yield curve movements.

Consider again the definition of parametric duration presented in (13):

$$D^{(l)}(k, \mathbf{A}) = \frac{1}{B_0^l(\mathbf{A})} \sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \qquad (k = 0, ..., 3). \tag{44}$$

Differentiating with respect to A_m $(m = 0, ..., 3)$ yields:

$$
\begin{aligned}
\frac{\partial D(k, \mathbf{A})}{\partial A_m} &= \frac{\partial}{\partial A_m} \left\{ \frac{1}{B_0^l(\mathbf{A})} \sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right\} \\
&= \frac{1}{B_0^l(\mathbf{A})^2} \left\{ \frac{\partial}{\partial A_m} \left[\sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] B_0^l(\mathbf{A}) - \left[\sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] \frac{\partial B_0^l(\mathbf{A})}{\partial A_m} \right\} \\
&= \frac{1}{B_0^l(\mathbf{A})^2} \left\{ - \left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t} \left(\frac{\partial r(t, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(t, \mathbf{A})}{\partial A_m} \right) \right] B_0^l(\mathbf{A}) - \right. \\
&\quad \left. - B_0^l(\mathbf{A})^2 \left[\frac{1}{B_0^l(\mathbf{A})} \sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] \frac{1}{B_0^l(\mathbf{A})} \frac{\partial B_0^l(\mathbf{A})}{\partial A_m} \right\} \\
&= -\frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_t e^{-r(t,\mathbf{A})t} \left(\frac{\partial r(t, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(t, \mathbf{A})}{\partial A_m} \right) \right] + \\
&\quad + \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t c_t e^{-r(t,\mathbf{A})t} \frac{\partial r(t, \mathbf{A})}{\partial A_k} \right] \left(-\frac{1}{B_0^l(\mathbf{A})} \frac{\partial B_0^l(\mathbf{A})}{\partial A_m} \right)
\end{aligned}
\tag{45}
$$

$$\tag{46}$$

Substituting the definitions of parametric duration and parametric convexity in equation (46) yields:

$$\frac{\partial D(k, \mathbf{A})}{\partial A_m} = D(k, \mathbf{A}) D(m, \mathbf{A}) - C(k, m, \mathbf{A}) \qquad (k, m = 0, ..., 3) \tag{47}$$

24

Equation (47) provides a general expression for the sensitivity of interest rate risk measures (parametric duration measures) to changes in interest rates as given by modifications in yield curve parameters. For any combination of term structure shifts the sensitivity of parametric duration is computed as a product of two duration measures minus the corresponding parametric convexity. To have a broader understanding of the significance of equation (47), consider the following cases of interest.

Case 1: Let $k = 0$ and $m = 0$. From (47) we have:

$$\frac{\partial D(0, \mathbf{A})}{\partial A_0} = [D(0, \mathbf{A})]^2 - C_{0,0}(\mathbf{A}) \tag{48}$$

Therefore, the sensitivity of traditional Fisher-Weil duration to changes in the level of the yield curve is equal to duration squared minus the traditional convexity measure. Note also that if gradient conditions for immunization against shifts in A_0 are satisfied (i.e., if $D(0, \mathbf{A}) = H$), the sensitivity $\frac{\partial D(0, \mathbf{A})}{\partial A_0}$ can be written as the negative of the popular M-squared dispersion measure (M^2) proposed by Fisher and Weil (1983, 1984), i.e.,

$$\frac{\partial D(0, \mathbf{A})}{\partial A_0} = -\left[C_{0,0}(\mathbf{A}) - (D(0, \mathbf{A}))^2 \right] = -M^2 \tag{49}$$

Case 2: Let $k = 0$ and $m = 1$. Then:

$$\frac{\partial D(0, \mathbf{A})}{\partial A_1} = D(0, \mathbf{A})D(1, \mathbf{A}) - C_{0,1}(\mathbf{A}) \tag{50}$$

Hence, the sensitivity of traditional Fisher-Weil duration to changes in the slope parameter of the yield curve is equal to the product of duration and $D(1, \mathbf{A})$ minus $C_{0,1}(\mathbf{A})$. Generalizing the above examples, we can estimate the combined effects produced by changes in the term structure level, slope and curvature on interest rate risk measures using the concept of total differential:

$$
\begin{aligned}
\Delta D(k, \mathbf{A}) &\approx \sum_{m=0}^{3} \frac{\partial D(k, \mathbf{A})}{\partial A_m} \Delta A_m \\
&\approx \sum_{m=0}^{3} \left[D(k, \mathbf{A})D(m, \mathbf{A}) - C(k, m, \mathbf{A}) \right] \Delta A_m
\end{aligned}
\tag{51}
$$

5   Conclusion

Traditionally, the study of the interest-rate sensitivity of the price of a portfolio of assets or liabilities has been performed using single factor models from which simple expressions for duration and convexity have been derived. In general, the ability of such models to predict price sensitivity or to achieve immunization is dependent on the validity of yield curve assumptions. In this sense, the classical duration analysis can greatly understate price sensitivity when non-parallel term structure shifts occur.

In this paper, we have developed a general multivariate duration and convexity analysis that does not depend on previous statements as to the way in which the yield curve moves. Differently, the model links interest rate risk factors to the parameters of the Svensson specification of the yield curve and is valid in virtually all yield curve environments. The model extends classical duration and convexity analysis to include yield curve shifts that are not parallel. The concepts of parametric duration and parametric convexity provide, in this context, natural first-order and second-order sensitivity measures of bond or bond portfolio prices to changes in interest rates. Moreover, the interest rate risk measures derived quantify the sensitivity of the portfolio to yield curve shifts that have an economic 'meaning', namely, changes in the level, slope and curvature of the yield curve.

Contrary to most interest rate risk models we emphasize the importance of second-order conditions for bond portfolio immunization. In concrete, we show that it is impossible to achieve immunization simply by meeting first-order conditions and that the key to successful immunization will be to build up a portfolio such that the gradient of its future value is zero, and such that its Hessian matrix is positive semidefinite. We present two alternative methods to determine the sign definiteness of the Hessian matrix: the determinantal test and the eigenvalue test, emphasizing the advantages and shortcomings of both methods. Finally, we analyse the sensitivity of parametric interest rate risk measures to changes in term structure shape parameters, offering fixed-income portfolio managers a powerful new tool with which to assess the combined effects of changes in the term structure level, slope and curvature on interest rate risk measures.

Future research should investigate the empirical performance of the parametric model when compared with that obtained with alternative single- and multiple-factor duration matching strategies.

Appendix: Formulae for Parametric Convexity

First recall that

$$r(t, \mathbf{A}) = a_0 + a_1 \frac{a_4}{t} \left(1 - e^{-\frac{t}{a_4}} \right) + a_2 \frac{a_4}{t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] + a_3 \frac{a_5}{t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right]. \quad (52)$$

The general expression for the parametric convexity of a bond, $C^{(l)}(k, m, \mathbf{A})$, is given by

$$
\begin{aligned}
C^{(l)}(k, m, \mathbf{A}) &= \frac{1}{B_0^l(\mathbf{A})} \frac{\partial^2 B_0^l(\mathbf{A})}{\partial A_k A_m} \\
&= \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t} \left(\frac{\partial r(t, \mathbf{A})}{\partial A_k} \right) \left(\frac{\partial r(t, \mathbf{A})}{\partial A_m} \right) \right], \quad k, m = 0, ..., 3. \quad (53)
\end{aligned}
$$

Differentiating equation (52) with respect to A_k ($k = 0, ..., 3$) and substituting in (53) we obtain the following complete set of formulas for parametric convexity:

k	m	$C^{(l)}(k, m, \mathbf{A})$
0	0	$C_{0,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t^2 c_{lt} e^{-r(t,\mathbf{A})t} \right]$
0	1	$C_{0,1}^{(l)} = C_{1,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right) a_4 \right]$
0	2	$C_{0,2}^{(l)} = C_{2,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] a_4 \right\}$
0	3	$C_{0,3}^{(l)} = C_{3,0}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} t c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right] a_5 \right\}$
1	1	$C_{1,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left[\sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right)^2 a_4^2 \right]$
1	2	$C_{1,2}^{(l)} = C_{2,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} a_4^2 \left(1 - e^{-\frac{t}{a_4}} \right) \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] \right\}$
1	3	$C_{1,3}^{(l)} = C_{3,1}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left(1 - e^{-\frac{t}{a_4}} \right) a_4 \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right] a_5 \right\}$
2	2	$C_{2,2}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} a_4^2 \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right]^2 \right\}$
2	3	$C_{2,3}^{(l)} = C_{3,2}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_4}} \left(1 + \frac{t}{a_4} \right) \right] a_4 \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right] a_5 \right\}$
3	3	$C_{3,3}^{(l)} = \frac{1}{B_0^l(\mathbf{A})} \left\{ \sum_{t=1}^{N} c_{lt} e^{-r(t,\mathbf{A})t} \left[1 - e^{-\frac{t}{a_5}} \left(1 + \frac{t}{a_5} \right) \right]^2 a_5^2 \right\}$

Table 1: Formulae for Parametric Convexity

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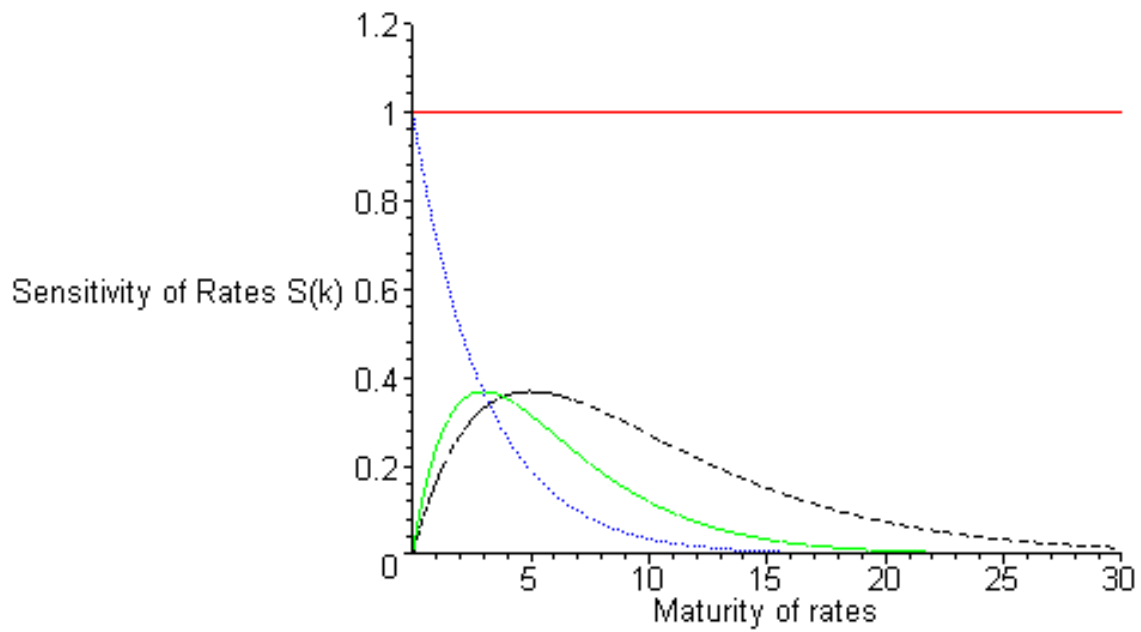
Figure 1: Sensitivity of forward rates to the parameters of the Svensson mathematical function; these sensitivities are obtained by fixing parameter values equal to $a_4 = 3$ and $a_5 = 5$.