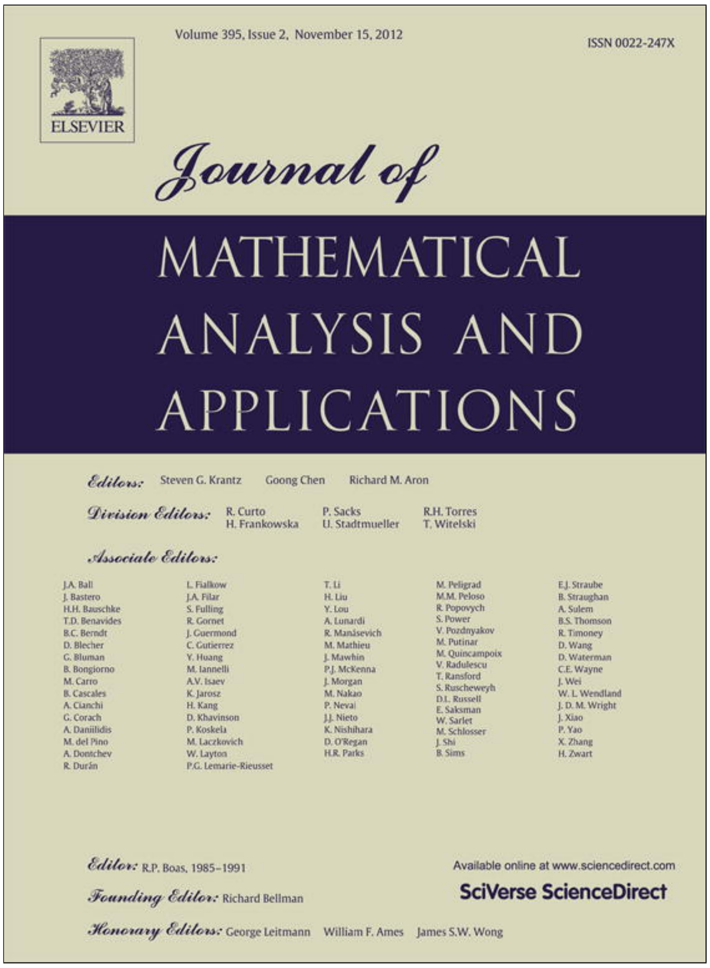
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On higher order fully periodic boundary value problems

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ABSTRACT

In this paper we present sufficient conditions for the existence of periodic solutions of the higher order fully differential equation

$$u^{(n)}(x) = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)),$$

with $n \geq 3$, $x \in [a, b]$ and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ a continuous function verifying a Nagumo-type growth condition.

A new type of lower and upper solutions, eventually non-ordered, allows us to obtain, not only the existence, but also some qualitative properties on the solution. The last section contains two examples to stress the application to both cases of n odd and n even.

1. Introduction

In this work we consider the higher order periodic boundary value problem composed by the fully differential equation

$$u^{(n)}(x) = f\left(x, u(x), u'(x), \ldots, u^{(n-1)}(x)\right) \tag{1}$$

for $n \geq 3$, $x \in I := [a, b]$, and $f : I \times \mathbb{R}^n \to \mathbb{R}$ a continuous function and the periodic boundary conditions

$$u^{(i)}(a) = u^{(i)}(b), \quad i = 0, 1, \ldots, n-1. \tag{2}$$

Higher order periodic boundary value problems have been studied by several authors in the last decades, using different types of arguments and techniques, as it can be seen in [1–3] for variational methods, in [4–17], for first and higher order equations and in [18–20] for a linear or quasi-linear nth order periodic problem. A fully nonlinear differential equation of higher order as in (1) was studied in some works, such as, for instance, [21], for f a bounded and periodic function verifying different assumptions for n even or odd. Moreover, in [22], the nonlinear part f of (1) must verify the following assumptions:

(A_1) There are continuous functions $e(x)$ and $g_i(x, y)$, $i = 0, \ldots, n-1$, such that

$$|f(x, y_0, \ldots, y_{n-1})| \leq e(x) + \sum_{i=0}^{n-1} g_i(x, y_i)$$

with

$$\lim_{|y|\to\infty} \sup_{x\in[0,1]} \frac{|g_i(x, y)|}{|y|} = r_i \geq 0, \quad i = 0, 1, \ldots, n-1.$$

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(A_2) There is a constant $M > 0$ such that, for $x \in [0, 1]$,

$$f(x, y_0, 0, \ldots, 0) > 0, \quad \text{for } y_0 > M,$$

and

$$f(x, y_0, 0, \ldots, 0) < 0, \quad \text{for } y_0 < -M.$$

(A_3) There are real numbers $L \geq 0$, $\alpha > 0$ and $a_i \geq 0$, $i = 1, \ldots, n-1$, such that

$$|f(x, y_0, \ldots, y_{n-1})| \geq \alpha |y_0| - \sum_{i=1}^{n-1} a_i |y_i| - L,$$

for every $x \in [0, 1]$ and $(y_0, \ldots, y_{n-1}) \in \mathbb{R}^n$.

The arguments followed in this paper allow more general nonlinearities, namely, f does not need to have a sublinear growth in $y_0, \ldots, y_{n-1}$ (as in (A_1)) or change sign (as in (A_2)). In fact, condition (10) in our main result (see Theorem 4) refers an, eventually, opposite monotony to (A_2) and improves the existent results in the literature for periodic higher order boundary value problems. In short, our technique is based on lower and upper solutions not necessarily ordered, in the topological degree theory, like it was suggested, for example, in [23,24], and has the following key points:

- A Nagumo-type condition on the nonlinearity, useful to obtain an *a priori* estimation for the $(n-1)$th derivative and to define an open and bounded set where the topological degree is well defined.
- A new kind of definition of lower and upper solutions, required to deal with the absence of a definite order for lower and upper functions and their derivatives up to the $(n-3)$th order. We remark that with such functions it is only required boundary data for the derivatives of order $n-2$ and $n-1$. Therefore the set of admissible functions for lower and upper solutions is more general.
- An adequate auxiliary and perturbed problem, where the truncations and the homotopy are extended to some mixed boundary conditions, allowing an invertible linear operator and the evaluation of the Leray–Schauder degree.

The last section contains two examples to emphasize that these results cover some cases in the literature where it is needed to particularize if n is odd and/or even.

2. Definitions and *a priori* bounds

It is introduced in this section, a Nagumo-type growth condition, initially presented in [25], and now useful to obtain an *a priori* estimate for the $(n-1)$th derivative.

Definition 1. A continuous function $f : I \times \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the Nagumo type condition in

$$E = \left\{ (x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : \gamma_i(x) \leq y_i \leq \Gamma_i(x), i = 0, 1, \ldots, n-2 \right\}, \tag{3}$$

with $\gamma_i(x)$ and $\Gamma_i(x)$ continuous functions such that,

$$\gamma_i(x) \leq \Gamma_i(x), \quad \text{for } i = 0, 1, \ldots, n-2 \text{ and every } x \in I, \tag{4}$$

if there exists a real continuous function $h_E : [0, +\infty[\to]0, +\infty[$ such that

$$|f(x, y_0, \ldots, y_{n-1})| \leq h_E(|y_{n-1}|), \quad \text{for every } (x, y_0, \ldots, y_{n-1}) \in E, \tag{5}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{6}$$

In the following we denote

$$\|w\|_\infty := \sup_{x \in I} |w(x)|.$$

The *a priori* bound is given by the next lemma:

Lemma 2 (*[24, Lemma 1]*). *Consider $\gamma_i, \Gamma_i \in \mathcal{C}(I, \mathbb{R})$, for $i = 0, 1, \ldots, n-1$, such that (4) holds and E is defined by (3). Assume that, for some $k > 0$, there is $h_E \in \mathcal{C}([0, +\infty[, [k, +\infty[)$, such that*

$$\int_\eta^{+\infty} \frac{s}{h_E(s)} ds > \max_{x \in I} \Gamma_{n-2}(x) - \min_{x \in I} \gamma_{n-2}(x),$$

where $\eta \geq 0$ *is given by*

$$\eta := \max\left\{\frac{\Gamma_{n-2}(b) - \gamma_{n-2}(a)}{b-a}, \frac{\Gamma_{n-2}(a) - \gamma_{n-2}(b)}{b-a}\right\}.$$

Then there exists $R > 0$ *(depending on* γ_{n-2}, Γ_{n-2} *and* h_E*) such that for every continuous function* $f : E \to \mathbb{R}$ *verifying* (5) *and for every* $u(x)$ *solution of* (1) *such that*

$$\gamma_i(x) \leq u^{(i)}(x) \leq \Gamma_i(x), \quad i = 0, 1, \ldots, n-2,$$

for every $x \in I$*, we have*

$$\left\| u^{(n-1)} \right\|_\infty < R.$$

In this paper, upper and lower solutions, not necessarily ordered, are used. In fact, by applying adequate auxiliary functions not only we get some order to define the branches where the solution and its derivatives are localized, but also we can deal with more general boundary conditions.

Definition 3. The function $\alpha \in \mathcal{C}^n(I)$ is a lower solution of problem (1)–(2) if

(i) $\alpha^{(n)}(x) \geq f\left(x, \alpha_0(x), \alpha_1(x), \ldots, \alpha_{n-3}(x), \alpha^{(n-2)}(x), \alpha^{(n-1)}(x)\right)$
where

$$\alpha_i(x) := \alpha^{(i)}(x) - \sum_{j=i}^{n-3} \left\|\alpha^{(j)}\right\|_\infty (x-a)^{j-i}, \quad i = 0, \ldots, n-3. \tag{7}$$

(ii) $\alpha^{(n-1)}(a) \geq \alpha^{(n-1)}(b), \alpha^{(n-2)}(a) = \alpha^{(n-2)}(b)$.
The function $\beta \in \mathcal{C}^n(I)$ is an upper solution of problem (1)–(2) if

(iii) $\beta^{(n)}(x) \leq f\left(x, \beta_0(x), \beta_1(x), \ldots, \beta_{n-3}(x), \beta^{(n-2)}(x), \beta^{(n-1)}(x)\right)$
where

$$\beta_i(x) := \beta^{(i)}(x) + \sum_{j=i}^{n-3} \left\|\beta^{(j)}\right\|_\infty (x-a)^{j-i}, \quad i = 0, \ldots, n-3. \tag{8}$$

(iv) $\beta^{(n-1)}(a) \leq \beta^{(n-1)}(b), \beta^{(n-2)}(a) = \beta^{(n-2)}(b)$.

Remark that although α and β are not necessarily ordered, the auxiliary functions α_i and β_i are well ordered for $i = 0, \ldots, n-3$. In fact, by (7) and (8),

$$\alpha_i(x) \leq 0 \leq \beta_i(x), \quad \text{for every } i = 0, \ldots, n-3 \text{ and } x \in I.$$

Moreover, there is no need of data on the values of the lower solution α or the upper solution β and their derivatives until order $(n-3)$ in the boundary. In fact, this is a key point to have more general sets of admissible functions as lower or upper solutions to problem (1)–(2).

3. Existence of periodic solutions

The main theorem provides an existence and location result for problem (1)–(2) in the presence of lower and upper solutions, not necessarily ordered.

Theorem 4. *Assume that* $\alpha, \beta \in \mathcal{C}^n(I)$ *are lower and upper solutions of* (1)–(2) *such that*

$$\alpha^{(n-2)}(x) \leq \beta^{(n-2)}(x), \quad \forall x \in I. \tag{9}$$

Let $f : I \times \mathbb{R}^n \to \mathbb{R}$ *be a continuous function verifying a Nagumo-type condition in*

$$E_* = \left\{ \begin{array}{c} (x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : \alpha_i \leq y_i \leq \beta_i, i = 0, 1, \ldots, n-3, \\ \alpha^{(n-2)} \leq y_{n-2} \leq \beta^{(n-2)} \end{array} \right\}$$

and

$$\begin{aligned} f(x, \alpha_0, \ldots, \alpha_{n-3}, y_{n-2}, y_{n-1}) &\geq f(x, y_0, \ldots, y_{n-3}, y_{n-2}, y_{n-1}) \\ &\geq f(x, \beta_0, \ldots, \beta_{n-3}, y_{n-2}, y_{n-1}) \end{aligned} \tag{10}$$

for fixed $(x, y_{n-2}, y_{n-1}) \in I \times \mathbb{R}^2$ *and* $\alpha_i \leq y_i \leq \beta_i$, $i = 0, 1, \ldots, n-3$.

Then problem (1)–(2) has at least a periodic solution $\mathcal{C}^n(I)$ such that

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x), \quad i = 0, 1, \dots, n-3,$$

and

$$\alpha^{(n-2)}(x) \le u^{(n-2)}(x) \le \beta^{(n-2)}(x),$$

for $x \in I$.

Remark 5. The proof that the solution found is nontrivial can be obtained by the location part of the theorem. For example, if the lower solution is chosen such that $\alpha^{(n-2)}(x) > 0$, or the upper solution such that $\beta^{(n-2)}(x) < 0$, for some $x \in [a, b]$, then the periodic solution of (1)–(2) is nontrivial, as it can be seen forward.

Proof. Consider the homotopic and truncated auxiliary equation

$$u^{(n)}(x) = \lambda f\left(x, \delta_0(x, u(x)), \dots, \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) + u^{(n-2)}(x) - \lambda\delta_{n-2}\left(x, u^{(n-2)}(x)\right) \tag{11}$$

where the continuous functions $\delta_i, \delta_{n-2} : \mathbb{R}^2 \to \mathbb{R}$, $i = 0, \dots, n-3$, are given by

$$\delta_i(x, y_i) = \begin{cases} \beta_i(x), & y_i > \beta_i(x) \\ y_i, & \alpha_i(x) \le y_i \le \beta_i(x) \\ \alpha_i(x), & y_i < \alpha_i(x) \end{cases}$$

with α_i and β_i defined in (7) and (8), respectively,

$$\delta_{n-2}(x, y_{n-2}) = \begin{cases} \beta^{(n-2)}(x), & y_{n-2} > \beta^{(n-2)}(x) \\ y_{n-2}, & \alpha^{(n-2)}(x) \le y_{n-2} \le \beta^{(n-2)}(x) \\ \alpha^{(n-2)}(x), & y_{n-2} < \alpha^{(n-2)}(x) \end{cases}$$

coupled with the boundary conditions

$$\begin{aligned} &u^{(k)}(a) = \lambda\, \eta_k\left(u^{(k)}(b)\right), \quad k = 0, \dots, n-3 \\ &u^{(n-2)}(a) = u^{(n-2)}(b) \\ &u^{(n-1)}(a) = u^{(n-1)}(b) \end{aligned} \tag{12}$$

where the functions $\eta_k : \mathbb{R} \to \mathbb{R}$, $k = 0, \dots, n-3$, are defined by

$$\eta_k\left(u^{(k)}(b)\right) = \begin{cases} \beta_k(a), & u^{(k)}(b) > \beta_k(a) \\ u^{(k)}(b), & \alpha_k(a) \le u^{(k)}(b) \le \beta_k(a) \\ \alpha_k(a), & u^{(k)}(b) < \alpha_k(a). \end{cases} \tag{13}$$

Take $r_{n-2} > 0$ such that, for every $x \in I$

$$-r_{n-2} < \alpha^{(n-2)}(x) \le \beta^{(n-2)}(x) < r_{n-2}, \tag{14}$$

$$f\left(x, \alpha_0(x), \dots, \alpha_{n-3}(x), \alpha^{(n-2)}(x), 0\right) - \alpha^{(n-2)}(x) - r_{n-2} < 0, \tag{15}$$

$$f\left(x, \beta_0(x), \dots, \beta_{n-3}(x), \beta^{(n-2)}(x), 0\right) - \beta^{(n-2)}(x) + r_{n-2} > 0. \tag{16}$$

Step 1: *Every solution of the problem (11)–(12) satisfies in I*

$$\left|u^{(i)}(x)\right| < r_i, \quad i = 0, \dots, n-2$$

independently of $\lambda \in [0, 1]$, with r_{n-2} given as above and

$$r_j = \xi_j + r_{n-2}(b-a)^{n-2-j}, \quad j = 0, \dots, n-3, \tag{17}$$

where

$$\xi_j := \max\left\{\sum_{j=i}^{n-3} \beta_j(a)\,(b-a)^{j-i}, -\sum_{j=i}^{n-3} \alpha_j(a)\,(b-a)^{j-i}\right\}. \tag{18}$$

Let u be a solution of (11)–(12).

Assume, by contradiction, that there exists $x \in I$ such that $\left|u^{(n-2)}(x)\right| \ge r_{n-2}$. Consider the case $u^{(n-2)}(x) \ge r_{n-2}$ and define

$$\max_{x \in I} u^{(n-2)}(x) := u^{(n-2)}(x_0) \quad (\ge r_{n-2} > 0). \tag{19}$$

If $x_0 \in]a, b[$, then $u^{(n-1)}(x_0) = 0$ and $u^{(n)}(x_0) \le 0$. By (10), (14) and (16), for $\lambda \in]0, 1]$ the following contradiction holds

$$\begin{aligned} 0 &\ge u^{(n)}(x_0) \\ &= \lambda f\left(x_0, \delta_0(x_0, u(x_0)), \ldots, \delta_{n-2}\left(x_0, u^{(n-2)}(x_0)\right), u^{(n-1)}(x_0)\right) + u^{(n-2)}(x_0) - \lambda\delta_{n-2}\left(x_0, u^{(n-2)}(x_0)\right) \\ &\ge \lambda\left[f\left(x_0, \beta_0(x_0), \ldots, \beta^{(n-2)}(x_0), 0\right) - \beta^{(n-2)}(x_0) + r_{n-2}\right] > 0. \end{aligned}$$

For $\lambda = 0$ the contradiction results from (19):

$$0 \ge u^{(n)}(x_0) = u^{(n-2)}(x_0) \ge r_{n-2} > 0.$$

If $x_0 = a$ then

$$\max_{x\in I} u^{(n-2)}(x) := u^{(n-2)}(a) \quad (\ge r_{n-2} > 0).$$

By (12),

$$0 \ge u^{(n-1)}(a) = u^{(n-1)}(b) \ge 0,$$

therefore $u^{(n-1)}(a) = 0$ and $u^{(n)}(a) \le 0$. Applying the same technique and computations as above, replacing x_0 by a, a similar contradiction is achieved.

The case $x_0 = b$ is analogous and so $u^{(n-2)}(x) < r_{n-2}$, for every $x \in I$. As the inequality $u^{(n-2)}(x) > -r_{n-2}$, for every $x \in I$, can be proved by the same argument, then

$$\left|u^{(n-2)}(x)\right| < r_{n-2}, \quad \forall x \in I.$$

By integration in $[a, x]$, using (12) and (13) the following relations are obtained

$$\begin{aligned} u^{(n-3)}(x) &< u^{(n-3)}(a) + r_{n-2}(x-a) = \lambda\eta_{n-3}\left(u^{(n-3)}(b)\right) + r_{n-2}(x-a) \\ &\le \lambda\beta_{n-3}(a) + r_{n-2}(b-a) \\ &\le \beta_{n-3}(a) + r_{n-2}(b-a) \le \xi_{n-3} + r_{n-2}(b-a) \end{aligned}$$

and

$$\begin{aligned} u^{(n-3)}(x) &> u^{(n-3)}(a) - r_{n-2}(x-a) \\ &\ge \alpha_{n-3}(a) - r_{n-2}(b-a) \ge -\xi_{n-3} - r_{n-2}(b-a). \end{aligned}$$

Therefore,

$$\left|u^{(n-3)}(x)\right| < r_{n-3}, \quad \forall x \in I,$$

with $r_{n-3} = \xi_{n-3} + r_{n-2}(b-a)$ and ξ_{n-3} given by (18).

Following the same technique, by (17) and (18), we have

$$\left|u^{(j)}(x)\right| < r_j, \quad j = 0, \ldots, n-3,$$

with r_j given by (17).

Step 2: *There exists $R > 0$ such that every solution u of problem* (11)–(12) *satisfies*

$$\left|u^{(n-1)}(x)\right| < R, \quad \forall x \in I,$$

independently of $\lambda \in [0, 1]$.

For r_i, $i = 0, \ldots, n-2$, given in the previous step, consider the set

$$E_1 = \left\{(x, y_0, \ldots, y_{n-1}) \in I \times \mathbb{R}^n : -r_i \le y_i \le r_i, i = 0, 1, \ldots, n-2\right\}$$

and the function $F_\lambda : E_1 \to \mathbb{R}$ given by

$$F_\lambda(x, y_0, \ldots, y_{n-1}) = \lambda f(x, \delta_0(x, y_0), \ldots, \delta_{n-2}(x, y_{n-2}), y_{n-1}) + y_{n-2} - \lambda\delta_{n-2}(x, y_{n-2}). \tag{20}$$

As f satisfies a Nagumo-type condition in E_*, consider the function $h_{E_*} \in \mathcal{C}([0, +\infty[, [k, +\infty[)$ for some $k > 0$, such that (5) and (6) hold with E replaced by E_*. Thus, for $(x, y_0, \ldots, y_{n-1}) \in E_1$, we have, by (11) and (14),

$$F_\lambda(x, y_0, \ldots, y_{n-1}) \le h_{E_*}(|y_{n-1}|) + 2r_{n-2}.$$

For $h_{E_1}(w) := h_{E_*}(w) + 2r_{n-2}$ then

$$\int_0^{+\infty} \frac{s}{h_{E_1}(s)} ds = \int_0^{+\infty} \frac{s}{h_{E_*}(s) + 2r_{n-2}} ds \ge \frac{1}{1 + \frac{2r_{n-2}}{k}} \int_0^{+\infty} \frac{s}{h_{E_*}(s)} ds = +\infty,$$

and so $h_{E_1}(w)$ verifies (6). Therefore, F_λ satisfies the Nagumo condition in E_1 with $h_{E*}(w)$ replaced by $h_{E_1}(w)$, independently of λ.

Defining

$$\gamma_i(x) := -r_i, \qquad \Gamma_i(x) := r_i, \quad i = 0, \ldots, n-2$$

the assumptions of Lemma 2 are satisfied with E replaced by E_1. So there exists $R > 0$, depending only on r_i, $i = 0, \ldots, n-2$, and φ, such that $\left|u^{(n-1)}(x)\right| < R$, for every $x \in I$.

Step 3: *For* $\lambda = 1$ *the problem* (11)–(12) *has a solution* $u_1(x)$.

Consider the operators

$$\mathcal{L} : \mathcal{C}^n(I) \subset \mathcal{C}^{n-1}(I) \to \mathcal{C}(I) \times \mathbb{R}^n$$

and, for $\lambda \in [0, 1]$,

$$\mathcal{N}_\lambda : \mathcal{C}^{n-1}(I) \to \mathcal{C}(I) \times \mathbb{R}^n$$

where

$$\mathcal{L}u = \left(u^{(n)} - u^{(n-2)}, u(a), \ldots, u^{(n-1)}(a)\right)$$

and

$$\mathcal{N}_\lambda u = \begin{pmatrix} \lambda f\left(x, \delta_0(x, u(x)), \ldots, \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) \\ -\lambda\delta_{n-2}\left(x, u^{(n-2)}(x)\right), \\ \lambda\eta_0(u(b)), \ldots, \lambda\eta_{n-3}\left(u^{(n-3)}(b)\right), u^{(n-2)}(b), u^{(n-1)}(b) \end{pmatrix}.$$

As $\mathcal{L}$ has a compact inverse it can be considered the completely continuous operator

$$\mathcal{T}_\lambda : \left(\mathcal{C}^{n-1}(I), \mathbb{R}\right) \to \left(\mathcal{C}^{n-1}(I), \mathbb{R}\right)$$

defined by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

For R given by Step 2, consider the set

$$\Omega = \left\{y \in \mathcal{C}^{n-1}(I) : \left\|y^{(i)}\right\|_\infty < r_i, i = 0, \ldots, n-2, \left\|y^{(n-1)}\right\|_\infty < R\right\}.$$

By Steps 1 and 2, for every u solution of (11)–(12), $u \notin \partial\Omega$ and so the degree $d(\mathcal{I} - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$. By Mawhin [26, Proposition II.9] and the invariance under homotopy

$$\pm 1 = d(\mathcal{I} - \mathcal{T}_0, \Omega, 0) = d(\mathcal{I} - \mathcal{T}_1, \Omega, 0).$$

Thus the equation $\mathcal{T}_1(x) = x$, equivalent to the problem given by the equation

$$u^{(n)}(x) = f\left(x, \delta_0(x, u(x)), \ldots, \delta_{n-2}\left(x, u^{(n-2)}(x)\right), u^{(n-1)}(x)\right) + u^{(n-2)}(x) - \delta_{n-2}\left(x, u^{(n-2)}(x)\right),$$

coupled with the boundary conditions

$$\begin{aligned} &u^{(k)}(a) = \eta_k\left(u^{(k)}(b)\right), \quad k = 0, 1, \ldots, n-3, \\ &u^{(n-2)}(a) = u^{(n-2)}(b) \\ &u^{(n-1)}(a) = u^{(n-1)}(b), \end{aligned}$$

has at least a solution $u_1(x)$ in Ω.

Step 4: $u_1(x)$ *is a solution of* (1)–(2)

This solution $u_1(x)$ is a solution of (1)–(2) if it verifies

$$\begin{aligned} &\alpha^{(n-2)}(x) \le u_1^{(n-2)}(x) \le \beta^{(n-2)}(x), \\ &\alpha_i(x) \le u_1^{(i)}(x) \le \beta_i(x), \quad i = 0, 1, \ldots, n-3, \forall x \in I. \end{aligned} \tag{21}$$

Suppose, by contradiction, that there is $x \in I$ such that

$$\alpha^{(n-2)}(x) > u_1^{(n-2)}(x)$$

and define

$$\min_{x \in I}\left[u_1^{(n-2)}(x) - \alpha^{(n-2)}(x)\right] := u_1^{(n-2)}(x_1) - \alpha^{(n-2)}(x_1) < 0.$$

If $x_1 \in]a, b[$, then $u_1^{(n-1)}(x_1) - \alpha^{(n-1)}(x_1) = 0$ and $u_1^{(n)}(x_1) - \alpha^{(n)}(x_1) \geq 0$. Therefore, by (10) and Definition 3, we obtain the following contradiction

$$\begin{aligned} 0 &\leq u_1^{(n)}(x_1) - \alpha^{(n)}(x_1) \\ &\leq f\left(x_1, \delta_0(x_1, u_1(x_1)), \dots, \delta_{n-3}\left(x_1, u_1^{(n-3)}(x_1)\right), \alpha^{(n-2)}(x_1), \alpha^{(n-1)}(x_1)\right) \\ &\quad + u^{(n-2)}(x_1) - \alpha^{(n-2)}(x_1) - f\left(x_1, \alpha_0(x_1), \dots, \alpha_{n-3}(x_1), \alpha^{(n-2)}(x_1), \alpha^{(n-1)}(x_1)\right) \\ &\leq u^{(n-2)}(x_1) - \alpha^{(n-2)}(x_1) < 0. \end{aligned} \tag{22}$$

If $x_1 = a$ then

$$\min_{x \in I}\left[u_1^{(n-2)}(x) - \alpha^{(n-2)}(x)\right] := u_1^{(n-2)}(a) - \alpha^{(n-2)}(a) < 0.$$

By Definition 3

$$0 \leq u_1^{(n-1)}(a) - \alpha^{(n-1)}(a) \leq u_1^{(n-1)}(b) - \alpha^{(n-1)}(b) \leq 0$$

and, therefore,

$$u_1^{(n-1)}(a) = \alpha^{(n-1)}(a), \qquad u_1^{(n)}(a) \geq \alpha^{(n)}(a).$$

Using similar computations to (22), an analogous contradiction is obtained. For the case where $x_1 = b$ the proof is identical and so

$$\alpha^{(n-2)}(x) \leq u_1^{(n-2)}(x), \quad \forall x \in I.$$

Applying the same arguments, one can verify that $u_1^{(n-2)}(x) \leq \beta^{(n-2)}(x)$, for every $x \in I$, and (21) holds.

Integrating (21) in $[a, x]$, by (7) and (13)

$$\begin{aligned} u_1^{(n-3)}(x) &\geq u_1^{(n-3)}(a) + \alpha^{(n-3)}(x) - \alpha^{(n-3)}(a) \\ &\geq \alpha_{n-3}(a) + \alpha^{(n-3)}(x) - \alpha^{(n-3)}(a) \\ &= \alpha^{(n-3)}(x) \geq \alpha^{(n-3)}(x) - \left\|\alpha^{(n-3)}\right\|_\infty = \alpha_{n-3}(x). \end{aligned}$$

Analogously for the second inequality in (21), by (8) and (13),

$$\begin{aligned} u_1^{(n-3)}(x) &\leq u_1^{(n-3)}(a) + \beta^{(n-3)}(x) - \beta^{(n-3)}(a) \\ &\leq \beta_{n-3}(a) + \beta^{(n-3)}(x) - \beta^{(n-3)}(a) \\ &= \beta^{(n-3)}(x) \leq \beta^{(n-3)}(x) + \left\|\beta^{(n-3)}\right\|_\infty = \beta_{n-3}(x), \end{aligned}$$

and, therefore,

$$\alpha_{n-3}(x) \leq u_1^{(n-3)}(x) \leq \beta_{n-3}(x), \quad \forall x \in I.$$

By integration and using the same technique it can be proved that

$$\alpha_i(x) \leq u_1^{(i)}(x) \leq \beta_i(x), \quad \text{for } i = 0, 1, \dots, n-3 \text{ and } x \in I. \quad \square$$

4. Examples

In the literature nth order periodic boundary value problems with fully differential equations are often considered only for n even or n odd, like it was mentioned before. So, we introduce two examples, including the odd and even cases.

Example 6. Consider the fifth order fully differential equation

$$u^{(v)}(x) = -\arctan(u(x)) - \frac{(u'(x))^3}{7} - \frac{(u''(x))^5}{8} + \frac{(u'''(x))^6}{8} + \left(u^{(iv)}(x) + 12\right)^{\frac{2}{3}} - 620, \tag{23}$$

for $x \in [0, 1]$, with the boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, 2, 3, 4. \tag{24}$$

The functions $\alpha, \beta : [0, 1] \to \mathbb{R}$ given by

$$\alpha(x) = -\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{6} + \frac{5}{2}x^2 + x + 1,$$
$$\beta(x) = \frac{x^5}{5} - \frac{x^4}{2} + 6x^3 + 12x - 1$$

are non-ordered lower and upper solutions, respectively, of problem (23)–(24) verifying (9) for $n = 5$, with the following auxiliary functions

$$\alpha_0(x) = -\frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{6} - \frac{11}{2}x^2 - \frac{13}{2}x - \frac{119}{30},$$
$$\alpha_1(x) = -x^4 + 2x^3 + \frac{x^2}{2} - 3x - \frac{13}{2},$$
$$\alpha_2(x) = -4x^3 + 6x^2 + x - 3,$$

and

$$\beta_0(x) = \frac{x^5}{5} - \frac{x^4}{2} + 6x^3 + 34x^2 + 41x + \frac{157}{10},$$
$$\beta_1(x) = x^4 - 2x^3 + 18x^2 + 34x + 41,$$
$$\beta_2(x) = 4x^3 - 6x^2 + 36x + 34.$$

The function

$$f(x, y_0, \ldots, y_4) = -\arctan y_0 - \frac{(y_1)^3}{7} - \frac{(y_2)^5}{8} + \frac{(y_3)^6}{8} + (y_4 + 12)^{\frac{2}{3}} - 620$$

is continuous, verifies conditions (5) and (6) in

$$E_2 = \left\{(x, y_0, \ldots, y_4) \in [0, 1] \times \mathbb{R}^5 : \alpha_i \le y_i \le \beta_i, i = 0, 1, 2, \alpha''' \le y_3 \le \beta'''\right\}$$

with

$$h_{E_2}(|y_4|) = 4.6 \times 10^7 + \frac{\pi}{2} + (y_4 + 12)^{\frac{2}{3}}$$

and it satisfies (10).

By Theorem 4 there is a nontrivial periodic solution $u(x)$ of problem (23)–(24), such that

$$\alpha_i(x) \le u^{(i)}(x) \le \beta_i(x), \quad \text{for } i = 0, 1, 2,$$
$$-12x^2 + 12x + 1 \le u'''(x) \le 12x^2 - 12x + 36, \quad \text{for } x \in [0, 1].$$

Remark that this solution is a nontrivial periodic one because a constant function cannot be the solution of (23).

Example 7. For $x \in [0, 1]$ consider the sixth order differential equation

$$u^{(vi)}(x) = -(u(x))^3 - \arctan\left(u'(x)\right) - \left(u''(x)\right)^5 - \exp(u'''(x)) + 50\left(u^{(iv)}(x)\right)^{2p+1} + \left|u^{(v)}(x) + 1\right|^{\theta} + 2, \qquad (25)$$

with $p \in \mathbb{N}$ and $0 < \theta \le 2$, along with the boundary conditions

$$u^{(i)}(0) = u^{(i)}(1), \quad i = 0, 1, \ldots, 5. \qquad (26)$$

The functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ given by

$$\alpha(x) = -\frac{x^4}{4!} + 1 \quad \text{and} \quad \beta(x) = \frac{x^4}{4!} - 1$$

are lower and upper solutions, respectively, of problem (25)–(26) verifying (9) for $n = 6$, with the auxiliary functions given by Definition 3

$$\alpha_0(x) = -\frac{x^4}{4!} - x^3 - \frac{x^2}{2} - \frac{x}{6},$$
$$\alpha_1(x) = -\frac{x^3}{6} - x^2 - \frac{x}{2} - \frac{1}{6},$$
$$\alpha_2(x) = -\frac{x^2}{2} - x - \frac{1}{2},$$
$$\alpha_3(x) = -x - 1$$

and

$$\beta_0(x) = \frac{x^4}{4!} + x^3 + \frac{x^2}{2} + \frac{x}{6},$$
$$\beta_1(x) = \frac{x^3}{6} + x^2 + \frac{x}{2} + \frac{1}{6},$$
$$\beta_2(x) = \frac{x^2}{2} + x + \frac{1}{2}$$
$$\beta_3(x) = x + 1.$$

The function

$$f(x, y_0, \ldots, y_5) = -(y_0)^3 - \arctan y_1 - (y_2)^5 - \exp(y_3) + 50\,(y_4)^{2p+1} + |y_5 + 1|^{\theta} + 2$$

is continuous, verifies conditions (5) and (6) in

$$E_3 = \left\{ \begin{array}{c} (x, y_0, \ldots, y_5) \in [0, 1] \times \mathbb{R}^5 : \alpha_i \le y_i \le \beta_i, i = 0, 1, 2, 3 \\ \alpha^{(iv)} \le y_4 \le \beta^{(iv)} \end{array} \right\}$$

with

$$h_{E_3}(|y_5|) = 39 + \frac{\pi}{2} + e^2 + |y_5 + 1|^{\theta}$$

and satisfies (10).

By Theorem 4 there is a nontrivial periodic solution $u(x)$ of problem (25)–(26), such that

$$-\frac{x^4}{4!} - x^3 - \frac{x^2}{2} - \frac{x}{6} \le u(x) \le \frac{x^4}{4!} + x^3 + \frac{x^2}{2} + \frac{x}{6},$$
$$-\frac{x^3}{6} - x^2 - \frac{x}{2} - \frac{1}{6} \le u'(x) \le \frac{x^3}{6} + x^2 + \frac{x}{2} + \frac{1}{6},$$
$$-\frac{x^2}{2} - x - \frac{1}{2} \le u''(x) \le \frac{x^2}{2} + x + \frac{1}{2},$$
$$-x - 1 \le u'''(x) \le x + 1,$$
$$-1 \le u^{(iv)}(x) \le 1, \quad \forall x \in [0, 1].$$

Remark that this solution is nontrivial because the unique constant solution of (25) is not in the set $[\alpha_0, \beta_0]$. Moreover, as in the previous example, the functions α_i, β_i for $i = 0, 1, 2, 3$ are well ordered despite $\alpha(x)$ and $\beta(x)$ are not ordered, for $x \in [0, 1]$.

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