A RECURSIVE PROCESS RELATED TO A PARTIZAN VARIATION OF WYTHOFF[1]

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Abstract

WYTHOFF QUEENS is a classical combinatorial game related to very interesting mathematical results. An amazing one is the fact that the $\mathcal{P}$-positions are given by $(\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor)$ and $(\lfloor \varphi^2 n \rfloor, \lfloor \varphi n \rfloor)$ where $\varphi = \frac{1+\sqrt{5}}{2}$. In this paper, we analyze a different version where one player (Left) plays with a chess bishop and the other (Right) plays with a chess knight. The new game (call it CHESSFIGHTS) lacks a Beatty sequence structure in the $\mathcal{P}$-positions as in WYTHOFF QUEENS. However, it is possible to formulate and prove some general results of a general recursive law which is a particular case of a PARTIZAN SUBTRACTION game. [3]

[1]Research in *Play mathematics - A Workshop in Combinatorial Game Theory*, Center for Mathematics and Fundamental Applications, Lisbon, November, 2010.
http://ptmat.fc.ul.pt/arquivo/docs/seminarios/confwork/2010/Mini_workshop.pdf

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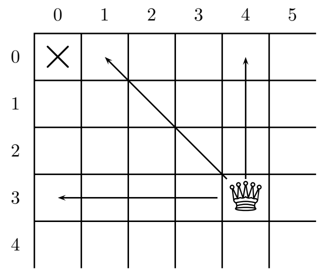
1. Introduction

WYTHOFF QUEENS is played on a quarter-infinite chessboard, extending downwards and to the right. A chess queen is placed in some cell of the board. On each turn, a player moves the queen as in chess, except that the queen can only move left, up, or diagonally up-left. The player who moves the queen to the corner $(0, 0)$ wins.

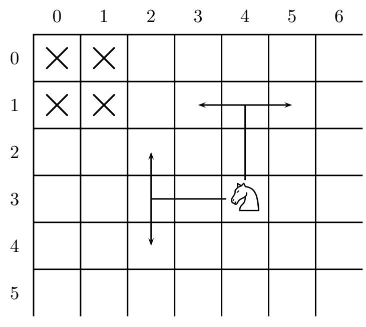


We can also interpret WYTHOFF QUEENS as a pile game. There are two piles of stones and, on each turn, a player either removes an arbitrary number of stones from one pile, or the same number of stones from both piles. The player who makes the last move wins.

A nice result about WYTHOFF QUEENS is the following one (first proved in [6]): The $\mathcal{P}$-positions of WYTHOFF QUEENS are given by $(\lfloor \varphi n \rfloor, \lfloor \varphi^2 n \rfloor)$ and $(\lfloor \varphi^2 n \rfloor, \lfloor \varphi n \rfloor)$ where $\varphi = \frac{1+\sqrt{5}}{2}$.

There are some variations of the game. One very interesting, analyzed in [2] (page 56), is the game WHITE KNIGHT. In this variation, instead of a queen, the players move a chess knight. The legal moves are the following (row x and column y):

$(x, y) \to (x - 1, y - 2)$ or $(x, y) \to (x + 1, y - 2)$ or $(x, y) \to (x - 2, y - 1)$ or $(x, y) \to (x - 2, y + 1)$

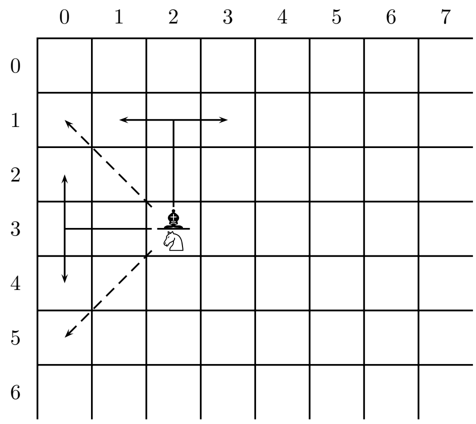


We consider a variation of WYTHOFF QUEENS, the game CHESSFIGHTS. The rules of this variation are the following ones:

— The board is as in WYTHOFF QUEENS and WHITE KNIGHT;

— Right plays with the knight as in WHITE KNIGHT;

— Left plays with the bishop: $(x, y) \to (x - i, y - i)$ or $(x, y) \to (x + i, y - i)$ (in the first case, we must have $x - i \geqslant 0 \wedge y - i \geqslant 0$ and, in the second case, we must have $x + i \geqslant 0 \wedge y - i \geqslant 0$, in other words, the move must be made inside the board).



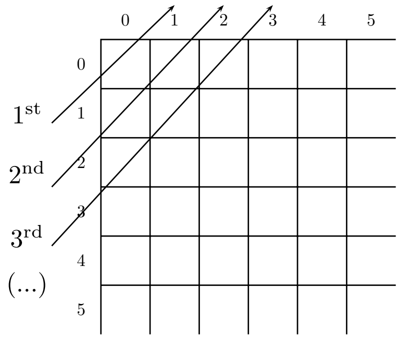
CHESSFIGHTS is a *partizan* game. For ease, the game with the piece in the cell (x, y) will be represented by the pair (x, y).

The game converges to the end because, after two moves, $(x, y) \mapsto (x', y') \mapsto (x'', y'')$, we have $x'' + y'' < x + y$.

2. Some Theorems of CHESSFIGHTS

The *options* of a game are all those positions which can be reached in one move. In combinatorial game theory, games can be expressed recursively as $G = \{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L$ are the Left options and $\mathcal{G}^R$ are the Right options of G. The *followers* of G are all the games that can be reached by all the possible sequences of moves from G (this is the usual notation of [3], [2], and [1]).

In the particular case of CHESSFIGHTS, we can compute the values of the cells (or, rather, the games corresponding to the placement of a single piece in a cell). The best way to do it is to choose a diagonal path:



With this procedure, we get an organized table (the following example corresponds to 9×9):

0	1	$\{1\mid 0\}$	$\frac{1}{2}$	1	$\{1\mid \uparrow\}$	$\frac{1}{2}$	1	$\{1\mid \uparrow 3*\}$
0	1	$\{1\mid 0\}$	$\uparrow$	1	$\{1\mid \frac{1}{2}, \{1\mid *\}\}$	$\uparrow 3*$	1	$\{1\mid \frac{1}{2}\}$
0	$*$	$\{1\mid 0\}$	$\{1\mid *\}$	$\Uparrow$	$\{1\mid \uparrow\}$	$\{1\mid 1, \{1\mid *2\}\}$	$\uparrow 3$	$\{1\mid \uparrow 3*\}$
0	$*$	$\uparrow *$	$\{1\mid *\}$	$\{1\mid *2\}$	$\Uparrow *$	$\{1\mid \Uparrow, \{1\mid *2\}\}$	$\{1\mid \{1\mid \uparrow\}, \uparrow 3*\}$	$\uparrow 3*3$
0	$*$	$*2$	$\uparrow$	$\{1\mid *2\}$	$\{1\mid \uparrow\}$	$\Uparrow *2$	$\{1\mid \Uparrow *, \{1\mid\mid 0\mid *, *2\}\}$	$\{1\mid \uparrow 3, \{1\mid \uparrow *3\}\}$
0	$*$	$*2$	$\uparrow$	$\uparrow *3$	$\{1\mid\mid 0\mid *, *2\}$	$\{1\mid \uparrow *3\}$	$\{0\mid\mid 0\mid *, *2\}$	$\{1\mid \Uparrow *2\}$
0	$*$	$*2$	$\{0\mid *, *2\}$	$\uparrow *3$	$\{0\mid\mid 0\mid *, *2\}$	$\{1\mid \uparrow *3\}$	$\{1\mid\mid\mid 0\mid\mid 0\mid *, *2\}$	$\{0\mid\mid 0\mid *2, \{0\mid *, *2\}\}$
0	$*$	$*2$	$\{0\mid *, *2\}$	$\uparrow *3$	$\{0\mid\mid 0\mid *, *2\}$	$\{0\mid\mid 0\mid *2, \{0\mid *, *2\}\}$	$\{1\mid\mid\mid 0\mid\mid 0\mid *, *2\}$	$\{1\mid\mid\mid 0\mid\mid 0\mid *2, \{0\mid *, *2\}\}$
0	$*$	$*2$	$\{0\mid *, *2\}$	$\{0\mid *2, \{0\mid *, *2\}\}$	$\{0\mid\mid 0\mid *, *2\}$	$\{0\mid\mid 0\mid *2, \{0\mid *, *2\}\}$	$\{0\mid\mid 0\mid *, *2\}$	$\{1\mid\mid\mid 0\mid\mid 0\mid *2, \{0\mid *, *2\}\}$

The same table just with the reduced canonical forms:

0	1	$\{1\mid 0\}$	$\frac{1}{2}$	1	$\{1\mid 0\}$	$\frac{1}{2}$	1	$\{1\mid 0\}$
0	1	$\{1\mid 0\}$	0	1	$\{1\mid \frac{1}{2}\}$	0	1	$\{1\mid \frac{1}{2}\}$
0	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0	$\{1\mid 0\}$	1	0	$\{1\mid 0\}$
0	0	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0
0	0	0	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0	$\{1\mid 0\}$	$\{1\mid 0\}$
0	0	0	0	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0	$\{1\mid 0\}$
0	0	0	0	0	0	$\{1\mid 0\}$	$\{1\mid 0\}$	0
0	0	0	0	0	0	0	$\{1\mid 0\}$	$\{1\mid 0\}$
0	0	0	0	0	0	0	0	$\{1\mid 0\}$

A visual inspection of the table allows us to guess some patterns. In fact, it is possible to prove some results.

Proposition 1. $(x, 0) = 0$.

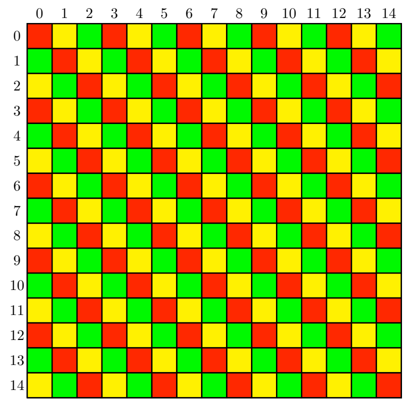
Proof. Left has no options. Right has no options (cases $(0, 0)$ and $(1, 0)$) or Right has just one option to $(x-2, 1)$. If so, Left plays to $(x-1, 0)$ and, by induction, Right loses. □

In the next results, it is important to consider the following groups of cells:

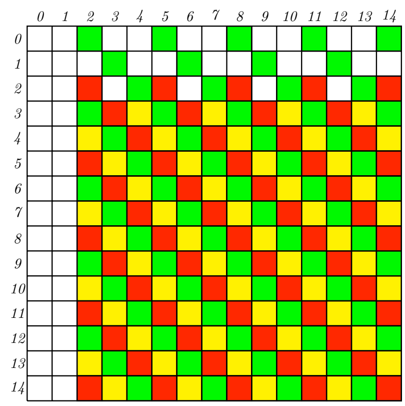
— Red $\longrightarrow (x, y) : y - x \equiv 0 \pmod 3$

— Yellow $\longrightarrow (x, y) : y - x \equiv 1 \pmod 3$

— Green $\longrightarrow (x, y) : y - x \equiv 2 \pmod 3$



Lemma 1. *From the games in the following region (call it $\mathfrak{R}$),*



Right to move, has a strategy allowing, at all times, if the sub-position is still not zero, a Right move to a green cell or a Right move to zero.

Proof. Let us analyze all the possible sub-positions (a, b) (Right moving).

— If $b = 0$ then the position $(a, b) = 0$ (Proposition 1).

— If $(a, b) \in \mathfrak{R}$ is green ($b - a \equiv 2 \pmod 3$) then Right moves to $(a + 1, b - 2)$. We can see that $(a + 1, b - 2)$ remains green because $(b - 2) - (a + 1) \equiv 2 \pmod 3$.

— If $(a, b) \in \mathfrak{R}$ is red ($b - a \equiv 0 \pmod 3$) then Right moves to $(a - 1, b - 2)$. We can see that $(a - 1, b - 2)$ turns green because $(b - 2) - (a - 1) \equiv 2 \pmod 3$.

— If $(a,b) \in \mathfrak{R}$ is yellow ($b-a \equiv 1 \pmod 3$) then Right moves $(a-2,b-1)$. We can see that $(a-2,b-1)$ turns green because $(b-1)-(a-2) \equiv 2 \pmod 3$.

— The only possible Left moves to $(a,b) \notin \mathfrak{R}$ are $(a,0)$ (item 1) and $(a,1) \wedge a > 1$ (in this case, the Right option to $(a-2,b-1) = 0$ is available). The moves indicated in the previous items never allow other options $(a,b) \notin \mathfrak{R}$ for Left.

□

Proposition 2. $(0,3k+1) = 1$ $(k \geqslant 0)$ *and* $(0,3k) = \frac{1}{2}$ $(k \geqslant 1)$.

Proof. Let us prove that $(0,3k+1) = 1$ $(k \geqslant 0)$.
The base case $(0,1) = 1$ is calculated by hand. We want to prove that, for $k \geqslant 1$, $(0,3k+1) + \{\,|\,0\} = 0$, i.e., $(0,3k+1) + \{\,|\,0\}$ is in $\mathcal{P}$.

If Right plays to $(0,3k+1)$, Left replies to $(3k+1,0) = 0$ (Proposition 1).

If Right plays to $(1,3k-1) + \{\,|\,0\}$, Left replies to
$(0,3k-2) + \{\,|\,0\} = (0,3(k-1)+1) + \{\,|\,0\} = 1 - 1$ (induction).

So, if Right plays, Right loses.

If Left plays first to $(a,b) + \{\,|\,0\}$ then $(a,b) \in \mathfrak{R}$ or $(a,b) = (a,0)$ or $(a,b) = (a,1) \wedge a > 1$. The last two cases are trivial. For the first case, Right just plays in (a,b) with the strategy of the Lemma 1 eventually ending in $0 - 1$. So, playing first, Left loses.

Let us prove that $(0,3k) = \frac{1}{2}$ $(k \geqslant 1)$. The base case $(0,3) = \frac{1}{2}$ is calculated by hand. We want to prove that, for $k > 1$, $(0,3k) + \{-1|\,0\} = 0$, i.e., $(0,3k) + \{-1|\,0\}$ is in $\mathcal{P}$.

If Right plays to $(0,3k)$, Left replies to $(3k,0) = 0$ (Proposition 1).

If Right plays to $(1,3k-2) + \{-1|\,0\}$, Left replies to
$(0,3k-3) + \{-1|\,0\} = (0,3(k-1)) + \{-1\,|\,0\} = \frac{1}{2} - \frac{1}{2}$ (induction).

So, if Right plays, Right loses.

If Left plays first to $(1,3k-1) + \{-1\,|\,0\}$, Right replies to
$(0,3k-3) + \{-1\,|\,0\} = (0,3(k-1)) + \{-1\,|\,0\} = \frac{1}{2} - \frac{1}{2}$ (induction).

If Left plays to $(a,b) + \{-1\,|\,0\}$ with $a > 1$ then $(a,b) \in \mathfrak{R}$ or $(a,b) = (a,0)$ or $(a,b) = (a,1) \wedge a > 1$. The last two cases are trivial. For the first case, Right just plays in (a,b) with the strategy of the Lemma 1 eventually ending in $0 - \frac{1}{2}$. So, playing first, Left loses. □

The next proposition is a useful inequality. With this result it will be possible to make some arguments of domination and reversibility.

We will write $\underline{(x,y)}$ to represent the game (x,y), but Left playing with the Knight and Right with the Bishop. We have $-(x,y) = \underline{(x,y)}$. This is a nice tool to perform proofs on the board with two different pieces. Also, we call *principal diagonal* to the set of cells such that $x = y$.

Lemma 2. *If* $k \geqslant 2$ *and* $x' > y - k$ *then* $(x, y) + \underline{(x', y - k)} \not\leqslant 0$ *(if the second component is below the principal diagonal and the components are separated by more than one column, Left wins playing first).*

Proof. If $y - k = 0$ then $\underline{(x', y - k)} = 0$ (Proposition 1). So, Left plays in the other component to $(x + y, 0) + \underline{(x', 0)}$ going to zero.

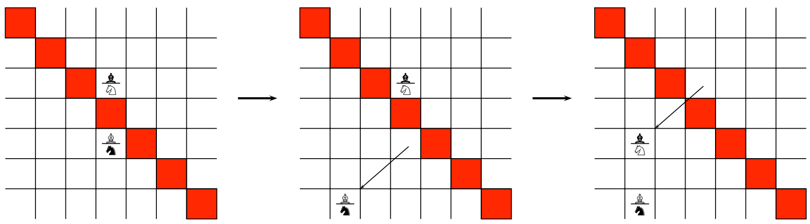
If $y - k = 1$, Left moves to $(x, y) + \underline{(x' - 2, 0)}$ which is equal to (x, y) (Proposition 1). Following, after a move by Right in (x, y), Left moves this component to the column 0.

If $y - k > 1$, Left moves to $(x, y) + \underline{(x' + 1, y - k - 2)}$. Following, all the possible moves by Right maintain the Lemma conditions. So, by induction, Left wins. □

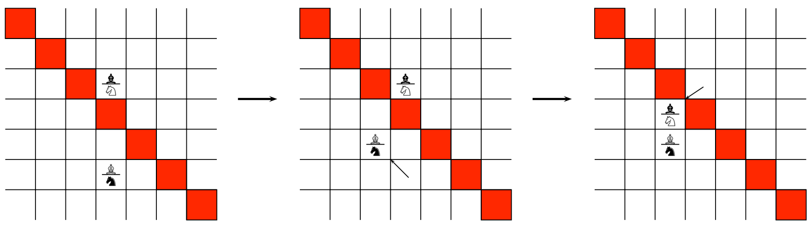
Proposition 3. *If* $x > y$ *then* $(x - k, y) \geqslant (x, y)$ *($k \geqslant 0$, positions inside the board).*

Proof. We want to prove that, if $x > y$, $(x - k, y) - (x, y) \geqslant 0$. So, we want to prove that Right loses playing first in the game $(x - k, y) + \underline{(x, y)}$. We will analyze all the Right options (consider the principal diagonal, red cells such that $x = y$).

— Right plays to $(x - k, y) + \underline{(x + i, y - i)}$.
Left moves to $(x - k + i, y - i) + \underline{(x + i, y - i)}$ and, by induction, Left wins.

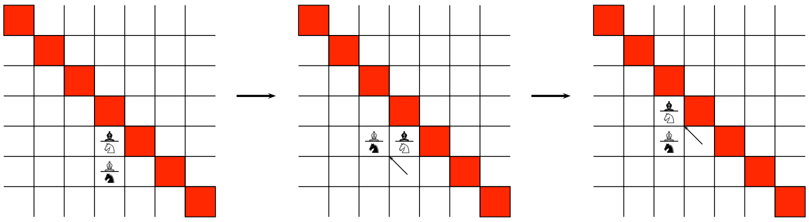


— Right plays to $(x - k, y) + \underline{(x - 1, y - 1)}$ (and k¿1).
Left moves to $(x - k + 1, y - 1) + \underline{(x - 1, y - 1)}$ and, by induction, Left wins.

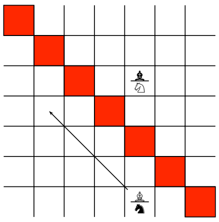


— Right plays to $(x - k, y) + \underline{(x - 1, y - 1)}$ (and $k \leqslant 1$).
Left moves to $(x - k - 1, y - 1) + \underline{(x - 1, y - 1)}$ (available) and, by induction,

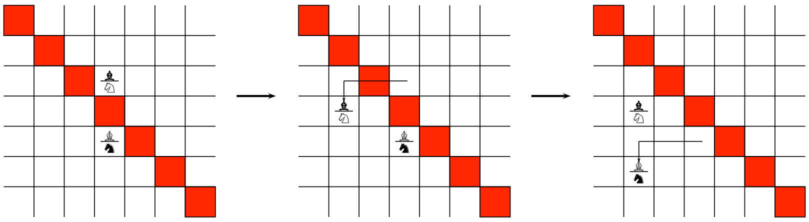
Left wins.



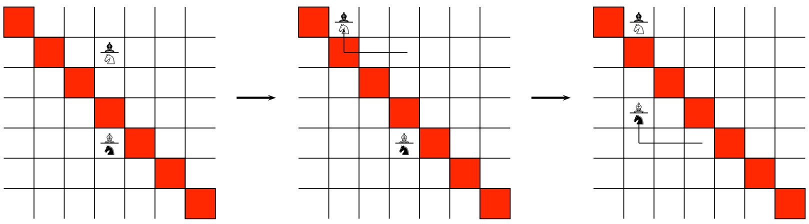
— Right plays to $(x-k, y) + \underline{(x-i, y-i)}(i > 1)$.
By Lemma 2, Left wins.



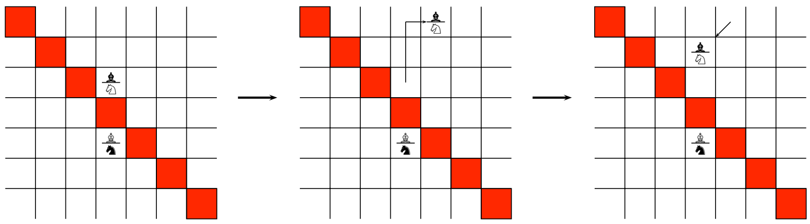
— Right plays to $(x-k+1, y-2) + \underline{(x, y)}$.
Left moves to $(x-k+1, y-2) + \underline{(x+1, y-2)}$ and, by induction, Left wins.



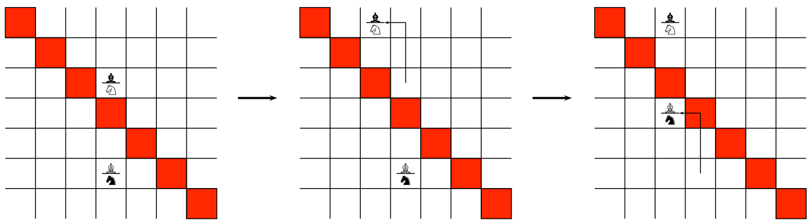
— Right plays to $(x-k-1, y-2) + \underline{(x, y)}$
Left moves to $(x-k-1, y-2) + \underline{(x-1, y-2)}$ and, by induction, Left wins.



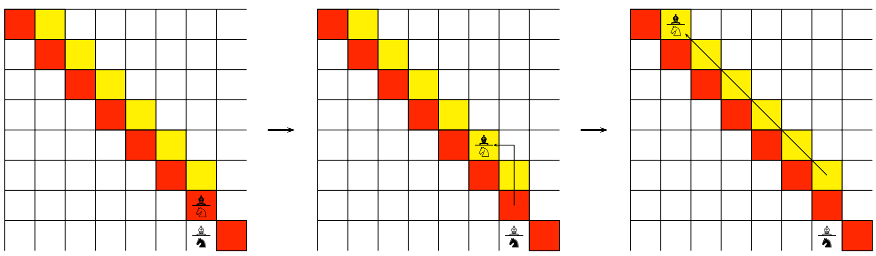
— Right plays to $(x-k-2, y+1) + \underline{(x, y)}$
Left moves to $(x-k-1, y) + \underline{(x, y)}$ and, by induction, Left wins.



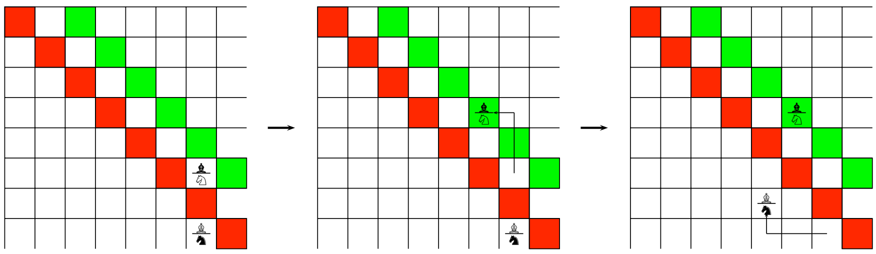
— Right plays to $(x-k-2, y-1) + \underline{(x,y)}$ and $(x-1 > y$ or $k=0)$.
Left moves to $(x-k-2, y-1) + \underline{(x-2, y-1)}$ and, by induction, Left wins.



— Right plays to $(x-k-2, y-1) + \underline{(x,y)}$ and, using the previous notation, $(x-k-2, y-1)$ is a red or a yellow cell.
Left moves to $(0, y-x+k+1) + \underline{(x,y)}$ and, because $(0, y-x+k+1) = 1$ or $(0, y-x+k+1) = \frac{1}{2}$ (Proposition 2), Left wins maintaining the second component below the principal diagonal.



— Right plays to $(x-k-2, y-1) + \underline{(x,y)}$ and $(x-k-2, y-1)$ is a green cell.
Left moves to $(x-k-2, y-1) + \underline{(x-1, y-2)}$ and, if Right wants to avoid the induction, must move to $(x-k-4, y-2) + \underline{(x-1, y-2)}$. After this pair of moves, $(x-k-4, y-2)$ turns red or yellow and Left chooses the strategy of the previous item.



□

Proposition 4. *If $x \geqslant 2$ then $(x, 1) = *$.*

Proof. We can calculate by hand $(2, 1) = *$. Now we prove the theorem by induction in x. The Left options of $(x, 1)$ are 0 (Proposition 1). The Right options are $(x-2, 0) = 0$ and $(x-2, 2)$. Against a Right's move to $(x-2, 2)$, Left can immediately reply to $(x-1, 1)$. By Proposition 3, $(x-1, 1) \geqslant (x, 1)$. So, by reversibility, the Right option $(x-2, 2)$ can be replaced by Right options of $(x-1, 1)$. But, by induction, $(x-1, 1) = *$ and $(x-2, 2)$ can be replaced by 0. □

Lemma 3. *If $x > y$ then $1 \geqslant (x, y)$.*

Proof. Let us analyze $1 + \underline{(x, y)}$ to see that Right, playing first, loses. Against a Right move (if he has one)

— To $1 + \underline{(x', 0)}$. In that case, the game turned $1 + 0$.

— To $1 + \underline{(x', 1)}$. In that case, the game turned $1*$.

— To $1 + \underline{(x', k)}$ $(k \geqslant 2)$. In that case, Left answers to $1 + \underline{(x'+1, k-2)}$ reaching the same kind of position as before.

In all cases, Left wins. □

Lemma 4. *If $x > y$ then $(x-2, y+1) \geqslant (x+1, y-2)$.*

Proof. Let us analyze $(x-2, y+1) + \underline{(x+1, y-2)}$ to see that Right, playing first, loses. If Right plays in the component $(x-2, y+1)$, Left replies in the same component to the column $y-2$ and wins (Proposition 3).

If Right plays to $(x-2, y+1) + \underline{(x+1-i, y-2-i)}$, Left replies to $(x-2-i, y+1-i) + \underline{(x+1-i, y-2-i)}$ maintaining the situation. If the Left answer was not available, that was because Right's move was to $(x-2, y+1) + \underline{(k, 1)}$ $(k \geqslant 2)$ or to $(x-2, y+1) + \underline{(k, 0)}$ $(k \geqslant 1)$. Against the first, Left moves the component $(x-2, y+1)$ to the column 1 and against the second, Left moves the component $(x-2, y+1)$ to the column 0.

In both cases, Left wins. □

Theorem 1. *The games (x, y) for $x > y$ are all-small.*

Proof. Let us consider $y \geqslant 2$ (the cases $y = 0$ and $y = 1$ are already known). By induction, Left options are all-small. Right has 4 options. By induction, $(x+1, y-2)$ and $(x-1, y-2)$ are all-small.

— Right option to $(x-2, y-1)$.
If $(x-2, y-1)$ is not in the principal diagonal, by induction, $(x-2, y-1)$ is all-small.
If $(x-2, y-1)$ is in the principal diagonal, Left can answer to $(1, 1) = 1$. By Lemma 3, $1 \geqslant (x, y)$. So, the Right option is reversible to $\emptyset$.

— Right option to $(x-2, y+1)$.
By Lemma 4 $(x+1, y-2)$ dominates $(x-2, y+1)$. Because we are thinking for columns with index $y \geqslant 2$, $(x+1, y-2)$ is available.

□

3. The General Recursive Process

As we saw in the previous section, the Right option $(x-2, y+1)$ is dominated (see Theorem 1). For the sensible options, the column number is decreased by one or two. This strongly motivates the analysis of the recursion

$$g(n) = \{g(0), \ldots g(n-1) \,|\, g(n-1), g(n-2)\}.$$

This is a special case of a partizan subtraction game (see [4]). The first elements of the sequence are

0	1	2	3	4
0	$*$	$*2$	$\{0\|*, *2\}$	$\{0\|*2, \{0\|*, *2\}\}$

5	6
$\{0\|\{0\|*, *2\}, \{0\|*2, \{0\|*, *2\}\}\}$	$\{0\|\{0\|*2, \{0\|*, *2\}\}, \{0\|\{0\|*, *2\}, \{0\|*2, \{0\|*, *2\}\}\}\}$

We can generalize the recursive law for similar chess knights (capable of making "larger" moves):

$$g_k(n) = \{g_k(0), \ldots g_k(n-1) \,|\, g_k(n-k), g_k(n-2k)\} \ (n \geqslant 0).$$

There is no problem with the $g_k(i)$ not previously defined. The empty set is available for the construction of the games.

For impartial subtraction games, it is well-known that SUBTRACTION$(ms_1, \ldots, ms_k)$ is the $m-$plicate of SUBTRACTION$(s_1, \ldots, s_k)$ ([2], page 98 and a proof in [5], page 36). We will prove that the general g_k is also a kind of "dilation" of g_1. Just for intuition, we list the first elements of $g_2(n)$ and $g_3(n)$:

0	1	2	3	4	5	6
0	1	$\{1\|0\}$	$1*$	$\{1, 1*\|0, \{1\|0\}\}$	$1*2$	$\{1\|\{1\|0\}, \{1, 1*\|0, \{1\|0\}\}\}$

7	8
$\{1\|1*, 1*2\}$	$\{1\|\{1, 1*\|0, \{1\|0\}\}, \{1\|\{1\|0\}, \{1, 1*\|0, \{1\|0\}\}\}\}$

0	1	2	3	4	5	6	7
0	1	2	$\{2\|0\}$	$\{2\|1\}$	$2*$	$\{2, 2*\|0, \{2\|0\}\}$	$\{2, 2*\|1, \{2\|1\}\}$

8	9	10
$2*2$	$\{2\|\{2\|0\}, \{2, 2*\|0, \{2\|0\}\}\}$	$\{2\|\{2\|1\}, \{2, 2*\|1, \{2\|1\}\}\}$

We start with a result about the left options of $g_k(n)$.

Lemma 5. *For $k \geqslant 1$, we have*

$$g_k(n) = \{g_k(0), \ldots, g_k(n-1) \mid g_k(n-k), g_k(n-2k)\}$$

$$= \begin{cases} n & n \leqslant k-1 \\ \{k-1, (k-1)* \mid g_k(n-k), g_k(n-2k)\} & 2k \leqslant n \leqslant 3k-1 \\ \{k-1 \mid g_k(n-k), g_k(n-2k)\} & \textit{other cases} \end{cases} .$$

Proof. `Case (a)` $n \leqslant k-1$. By definition,

$$g_k(0) = \{\,|\,\} = 0$$
$$g_k(1) = \{g_k(0)\,|\,\} = \{0\,|\,\} = 1$$
$$(\ldots)$$
$$g_k(k-1) = \{g_k(k-2)\,|\,\} = \{k-2\,|\,\} = k-1.$$

`Case (b)` $k \leqslant n \leqslant 2k-1$. We already know that $g_k(0) = 0$, $g_k(1) = 1, \ldots$, $g_k(k-1) = k-1$. Therefore, by definition (and domination),

$$g_k(k) = \{k-1 \mid 0\}$$

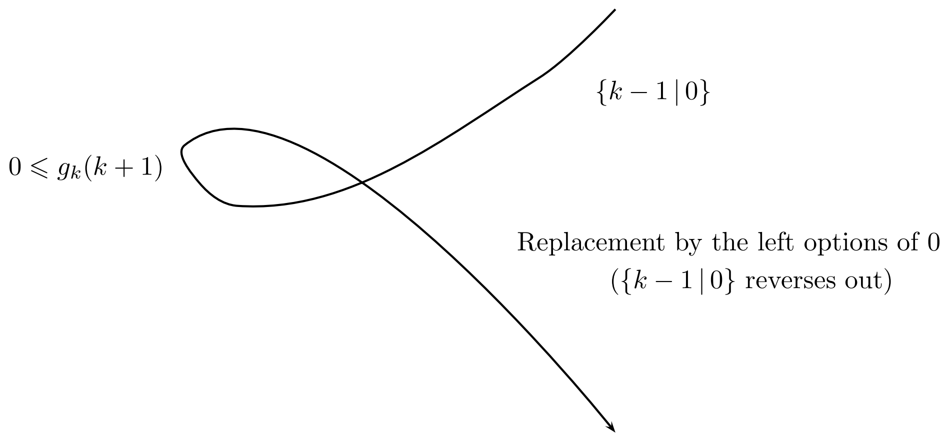
$$g_k(k+1) = \{k-1, \{k-1 \mid 0\} \mid 1\}$$

$$g_k(k+2) = \{k-1, \{k-1 \mid 0\}, \{k-1, \{k-1 \mid 0\} \mid 1\} \mid 2\}$$

$$(\vdots).$$

We can use reversibility arguments:

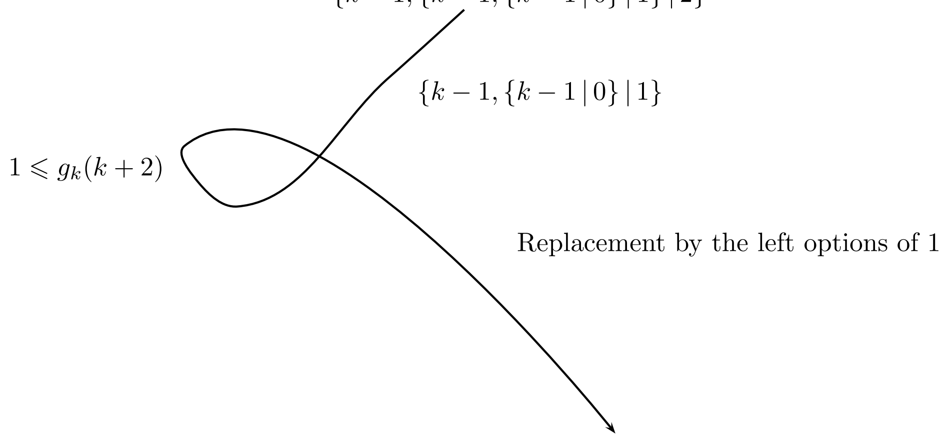
$$g_k(k+1) = \{k-1, \{k-1 \,|\, 0\} \,|\, 1\}$$



$$g_k(k+1) = \{k-1 \,|\, 1\}$$

Similarly,

$$\begin{aligned} g_k(k+2) \quad &= \{k-1, \{k-1 \,|\, 0\}, \{k-1, \{k-1 \,|\, 0\} \,|\, 1\} \,|\, 2\} \\ &= \{k-1, \{k-1, \{k-1 \,|\, 0\} \,|\, 1\} \,|\, 2\} \end{aligned}$$



$$g_k(k+2) = \{k-1, 0 \,|\, 2\} = \{k-1 \,|\, 2\}$$

In general, for $0 \leqslant j \leqslant k-1$,

$$g_k(k+j) = \{k-1, g_k(k), g_k(k+1), \ldots, g_k(k+j-1) \,|\, j\}$$

and
$g_k(k)$ reverses out through 0;
$g_k(k+1)$ reverses through 1 to 0 which is dominated by $k-1$;

$(\vdots)$
$g_k(k+j-1)$ reverses through $j-1$ to $j-2$ which is dominated by $k-1$.

The reversibility effects are justified by the inequality

$$\{k-1, g_k(k), g_k(k+1), \ldots,\ g_k(k+j-1) \mid j\} \geqslant j-1$$

We can conclude that the property is true for $k \leqslant n \leqslant 2k-1$.
`Case (c)` $2k \leqslant n \leqslant 3k-1$. We have,

$$g_k(2k) = \{k-1, (k-1)* \mid 0, \{k-1 \mid 0\}\}$$

$$g_k(2k+1) = \{k-1, (k-1)*, g_k(2k) \mid 1, \{k-1 \mid 1\}\}$$

$$g_k(2k+2) = \{k-1, (k-1)*, g_k(2k), g_k(2k+1) \mid 2, \{k-1 \mid 2\}\}$$

$$(...)$$

As the previous cases, it is easy to check that only the left options $k-1$ and $(k-1)*$ don't reverse. In fact, in general, for $0 \leqslant j \leqslant k-1$,

$$g_k(2k+j) = \{k-1, (k-1)*, g_k(2k), g_k(2k+1), \ldots,\ g_k(2k+j-1) \mid j, \{k-1 \mid j\}\}$$

and
$g_k(2k)$ reverses out through 0;
$g_k(2k+1)$ reverses through 1 to 0 which is dominated by $k-1$;
$(\vdots)$
$g_k(2k+j-1)$ reverses through $j-1$ to $j-2$ which is dominated by $k-1$.

The reversibility effects are justified by the inequality

$$\{k-1, (k-1)*, g_k(2k), g_k(2k+1), \ldots,\ g_k(2k+j-1) \mid j, \{k-1 \mid j\}\} \geqslant j-1.$$

We can conclude that the property is true for $2k \leqslant n \leqslant 3k-1$.
`Case d)` Other cases. In the other cases, also $(k-1)*$ reverses. This is true because, in these cases, we have

$$\{k-1, (k-1)* \mid g_k(n-k), g_k(n-2k)\} \geqslant k-1.$$

We can see that, in the game

$$\{k-1, (k-1)* \mid g_k(n-k), g_k(n-2k)\} + 1 - k,$$

if Right begins, Right loses. This happens because the Left option $k-1$ is available in the games $g_k(n-k)$ and $g_k(n-2k)$. □

Now, we are able to prove a kind of "dilation" theorem.

Theorem 2. *Consider $n \geqslant 0$ and $k \geqslant 1$.*

1. *If $n \leqslant k-1$, $g_k(n) = n$.*

2. *If $n > k-1$, we obtain $g_k(n)$ from $g_1(n)$ as indicated: consider $i \in \{0, \ldots, k-1\}$ such that $n \equiv i \pmod{k}$. Let G be the game $g_1\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$ (the form of the game according to its initial definition) and J the game constructed from G executing the following:*

   (a) Add $k-1$ to the games G^L, G^{RL}, G^{RRL},...

   (b) Add i to the games G^R, G^{RR}, G^{RRR},... not affected by the first step.

   We have $g_k(n) = J$.

Proof. The theorem is compatible with Lemma 5 because adding $k-1$ to the Left options of the game $g_1\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$ generates exactly the same Left options for $g_k(n)$ indicated in the Lemma 5. So, we just have to analyze the Right options.

Just the induction step is non-trivial. Consider the game

$$g_k(n+1) = \{g_k(0), \ldots, g_k(n) \mid g_k(n+1-k), g_k(n+1-2k)\}.$$

By induction, we have to add $k-1$ and i in the games $g_1\left(\left\lfloor \frac{n+1-k}{k} \right\rfloor\right)$ and $g_1\left(\left\lfloor \frac{n+1-2k}{k} \right\rfloor\right)$ where $n+1-2k \equiv n+1-k \equiv i \pmod{k}$.

But this is exactly the same as adding $k-1$ and i in the Right options of $g_1\left(\left\lfloor \frac{n+1}{k} \right\rfloor\right)$. This is true because the Right options of $g_1\left(\left\lfloor \frac{n+1}{k} \right\rfloor\right)$ are $g_1\left(\left\lfloor \frac{n+1}{k} \right\rfloor - 1\right)$ $= g_1\left(\left\lfloor \frac{n+1-k}{k} \right\rfloor\right)$ and $g_1\left(\left\lfloor \frac{n+1}{k} \right\rfloor - 2\right) = g_1\left(\left\lfloor \frac{n+1-2k}{k} \right\rfloor\right)$ and $n+1 \equiv i \pmod{k}$. □

References

[1] M. H. Albert, R. J. Nowakowski, D. Wolfe, Lessons in Play: An Introduction to Combinatorial Game Theory, A. K. Peters, 2007.

[2] E. R. Berlekamp, J. Conway, R. Guy, Winning Ways. Academic Press, London, 1982.

[3] J. Conway, On Numbers and Games, Academic Press, 1976.

[4] A. S. Fraenkel, A. Kotzig. "Partizan Octal Games. Partizan Subtraction Games," *International Journal of Game Theory*, 1987.

[5] C. Santos, Some Notes on Impartial Games and Nim Dimension, PhD Thesis, University of Lisbon, 2010.

[6] W. A. Wythoff. A modification of the game of Nim, *Niew Archief voor Wiskunde*, 1907.