Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Tese de Doutoramento

Coupled Periodic Systems and Synchronization

Sara Isabel Aleixo Perestrelo

Orientador(es) | Feliz Manuel Minhós

Henrique Manuel dos Santos Silveira de Oliveira

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A tese de doutoramento foi objeto de apreciação e discussão pública pelo seguinte júri nomeado pelo Diretor do Instituto de Investigação e Formação Avançada:

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Sistemas Periódicos Acoplados e Sincronização

Resumo

Este trabalho é dedicado ao estudo de sistemas impulsivos generalizados de dois osciladores acoplados usando técnicas de Análise Não Linear, com as quais obtemos resultados para a existência e localização de soluções periódicas, e de Sistemas Dinâmicos, através das quais abordamos o fenómeno da sincronização, emergente no sistema de relógios de Huygens.

Na primeira parte apresentamos resultados de existência e localização para soluções periódicas de sistemas planares não lineares acoplados de: primeira ordem, primeira ordem com impulsos, segunda ordem e segunda ordem com impulsos, sem requisitos de periodicidade para as não linearidades. Os argumentos para a existência baseiam-se no teorema do ponto fixo de Schauder, em variantes da condição de Nagumo, na Teoria do Grau Topológico e em resultados para funções equirreguladas. Os argumentos para a localização baseiam-se numa técnica de sub e sobre soluções, que envolve funções com translacções que não necessitam de estar bem ordenadas.

Na segunda parte apresentamos resultados de sincronização e estabilidade de dois osciladores acoplados que partilham uma interacção mútua discreta, sob o modelo de Andronov, e exploramos a sincronização *master-master* e *master-slave*, mostrando que as órbitas resultantes de perturbações no ciclo limite de um oscilador de Andronov permanecem próximas do ciclo limite original, não perturbado. Obtemos as línguas de Arnold para o acoplamento de Huygens de dois relógios, um com uma frequência próxima de um múltiplo da do outro. Observamos uma relação *master-slave*: enquanto as múltiplas perturbações resultantes dos impulsos do oscilador mais rápido sobre o oscilador mais lento tendem a cancelar-se mutuamente, o único impulso do oscilador lento tem um efeito secular no sistema. Propomos que este padrão de sincronização, emergente desta dinâmica perturbativa de impactos, é prevalente em vários sistemas físicos.

Coupled Periodic Systems and Synchronization

Abstract

This work is dedicated to the study of generalized impulsive systems of two coupled oscillators using techniques from Nonlinear Analysis, under which we obtain existence and localization results for periodic solutions, and techniques from Dynamical Systems, through which we approach the emergence of the phenomenon of synchronization in Huygens' system of clocks.

In the first part we present some existence and localization results for periodic solutions of coupled nonlinear planar systems of: first-order, first-order with impulses, second-order and second-order with impulses, without periodicity requirements for the nonlinearities. The arguments for the existence are based on Schauder's fixed point theorem, variations of the Nagumo condition, the Topological Degree Theory and results on equi-regulated functions. The localization tool is based on a technique of lower and upper solutions, involving functions with translations, that are not required to be well-ordered.

In the second part we present results on synchronization and stability of two coupled oscillators exchanging a mutual discrete interaction, under the Andronov model, and explore both master–master and master-slave synchrony. We show that the orbits resulting from perturbations of a limit cycle of an Andronov oscillator remain close to the original unperturbed limit cycle. We obtain the Arnold tongues for the Huygens coupling of two clocks, one with frequency near a multiple of the other. We observe a pattern of master–slave relationship: while the multiple bursts of the faster oscillator tend to cancel each other out upon impacting the slower oscillator, the single burst of the slower oscillator has a secular effect. We propose that this synchronization pattern, arising from perturbative impacts, is prevalent across various physical systems.

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Prologue

This work, entitled *Coupled Periodic Systems and Synchronization* is a result of my research, as a Mathematics PhD student, on coupled periodic systems and synchronization. The former topic, Coupled Periodic Systems, was explored under the supervision of Professor Feliz Minhós, from the University of Évora, while the latter, Synchronization, was explored under the supervision of Professor Henrique Oliveira, from the Technical University of Lisbon.

I chose to dedicate the last few years studying Mathematics, essentially because of its beauty. There are immense worlds within it that fascinate me. Notwithstanding, I decided to embark on the branch of Analysis, within which the study of coupled periodic systems was performed by employing elegant techniques. At the same time, I had been wanting to pick up where I left my master's work on Synchronization, where I studied the phenomenon observed in the Huygens' famous system of coupled pendulum clocks. Back then, we tackled the problem by applying the Andronov model for an oscillator to a system of two clocks, and using techniques from Dynamical Systems to deal with a discrete coupling interaction between them. That project gave me a lot of satisfaction, with an ingenious theory and robust experimental evidences supporting some historical bold observations from the XVIII century by Sir Huygens.

It was, indeed, a theme I did not want to leave unfinished. So, when I started to engage in the study of coupled systems of ODE's, within the domain of Nonlinear Analysis, some endeavours were made to somehow connect two subjects of different branches of Mathematics. My motivation was mainly driven by the use of completely different techniques, that offered different views of the behaviour of certain systems of coupled oscillators – the Huygens' clocks being a particular case of an impulsive coupled periodic second-order system.

The attempt was made, and I ended up exploring, through two totally different lenses, an impulsive system of two oscillators which exhibit common features, such as the particular case of the Van der Pol oscillator, that we wish to continue exploring in future projects. Publications were made, I learned a little more and developed this dissertation. Rather than looking after aesthetics, I decided this work to be a footprint of my academic journey and, as so, I included both studies in two parts, however disconnected they may seem.

This work is organized in two parts: the first part is dedicated to the study of coupled periodic systems, and it is composed of four chapters, each one presenting techniques for the existence and localization of solutions in first and second order systems, with and without impulses; the second part is dedicated to the study of synchronization, being composed of two chapters describing the model engineered to explain the phenomenon of synchronization observed in the Huygens' clocks, not only at the level of the geometry and stability of the system, but also covering the *master-slave* synchronization type. Finally, each chapter is an adaptation of an academic paper, already published.

Aknowledgements

My thanks go to my supervisors, who serve as an example of work dedication and intellectual honesty, and to all those who supported this project and helped me throughout this journey. A special thanks to my new family: my life partner, my twin babies, my dog and my cat.

To all those who help my mind and soul prosper, I am deeply grateful.

Part I

Coupled periodic systems

Introduction

Nature is a repository of phenomena that can be modelled using Mathematics, according to different levels of complexity. One of the most transversal and useful mathematical tools in this context is ordinary differential equations. These chapters are dedicated to the study of planar systems of coupled nonlinear ordinary differential equations.

The interaction between different variables of different natural systems results, in general, in a complex dynamics that differential equations can help understand. Many fields of research, including exact and social sciences, have been dedicated to the study of these dynamics, namely Thermodynamics [1], Biology [2, 3, 4], Chemistry [5], Ecology [6], Electronics [7, 8], Population Dynamics [9], Neuroscience [10], Celestial Mechanics [11, 12], Physiology [13], Economy [14], just to name a few. The importance of modelling real life mathematically is hard to measure, as it provides prediction tools that allow an understanding of each system of study that transcends the present time.

Whatever the phenomenon, when we think about coupled systems, we look for solutions that represent the evolution of the variables involved. However, due to Nature's high complexity, these systems are usually nonlinear, which hampers the search of those same solutions.

The complexity of nonlinear systems increases when part of their evolution is described by sudden changes, *i.e.*, impulses, [15, 16, 17, 18, 19, 20, 21]. In general, the approach to find solutions in such systems is more delicate and very often special techniques are required. This complexity increases even more when we consider state-dependent impulses [22, 23, 24].

But beyond the interest in finding solutions, it is of particular importance to find *periodic* solutions, since they repeat themselves through time. The search for periodic solutions in nonlinear coupled systems is extensively documented in the literature. However, some approaches consider periodicity of some form in the nonlinearities [25, 26, 16], or nonlinearities without full dependence on all the variables [27, 28]. The range of periodic solutions can be restricted [29, 30], others are just approximate solutions [31, 32, 33]. When dealing with impulsive problems, the impulse functions do not usually depend on the variables and their derivatives, neither are they coupled [34, 35, 36]. Thus, the search for periodic solutions in coupled impulsive nonlinear problems can be specially hard to deal with.

This work aims to fill these gaps in the literature, by looking for periodic solutions in nonlinear coupled systems with full dependence on their variables. When such systems are endowed with impulsive conditions, these are coupled and state-dependent, allowing a more realistic description of phenomena.

We present, in four chapters, the study of first order systems, first order systems with impulses, second order systems and second order systems with impulses. In each problem we prove the existence of at least a periodic solution. The existence tools are based on Schauder's fixed point theorem and the topological degree theory.

In addition, existing solutions are localized with an original variant of the well-known method of upper and lower solutions [37]. In every chapter the definition of upper and lower solutions changes, according to the conditions of the problem.

More specifically, in Chapter 1, we study first order coupled systems. The proof considers a perturbed and truncated problem, and Schauder's fixed point theorem. We prove the existence and localization of a periodic solution. This Chapter is an adaptation of the work [38]. Similar strategy is used in Chapter 2, when adding impulsive conditions to the first order system studied in Chapter 1. Chapter 2 is an adaptation of the work [39]. In Chapter 3 we study second order systems and use the topological degree theory for the existence. The localization follows the previous method. This Chapter is an adaptation of the work [40]. In Chapter 4, we use Green's functions and Schauder's fixed point theorem for the existence of periodic solutions. We provide a real-case application of the model in each chapter.

Chapter 1

First-order differential systems

1.1 Introduction

In this chapter we study the following first-order coupled nonlinear system,

$$\begin{cases} z'(t) = f\left(t, z(t), w(t)\right) \\ w'(t) = g\left(t, z(t), w(t)\right) \end{cases}, \tag{1.1}$$

with $t \in [0,T]$ $(T > 0)$ and $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ L^1-Carathéodory functions, together with the boundary conditions

$$\begin{aligned} z(0) &= z(T), \\ w(0) &= w(T). \end{aligned} \tag{1.2}$$

Periodic solutions are usually found in first-order differential systems. For example, in [26], the authors present sufficient conditions for the solvability of a class of singular Sturm-Liouville equations with periodic boundary value conditions. The problem describes the periodic oscillations of the axis of a satellite in the plane of the elliptic orbit around its center of mass. In another example [16], the authors study the stability of two coupled Andronov oscillators sharing a mutual discrete periodic interaction. The study was motivated by the phenomenon of synchronization observed in Huygens' clocks [41]. In [42], the authors prove the existence of positive periodic solutions in a generalized Nicholson's blowflies model, using Krasnoselskii cone fixed point theorem. In [29], it is considered the dynamics of hematopoietic stem cell populations under some external periodic regulatory factors at the cellular cycle level. The existence of periodic solutions in the respective non-autonomous differential system is typically associated to the presence of some hematological disease.

However, the search for periodic solutions in nonlinear problems is hampered when the nonlinearities do not present any periodicity, and special techniques are sometimes required. To overcome that difficulty, such techniques are presented in [43], where the authors use well-ordered lower and upper solutions to guarantee the existence of solutions of a first-order fully coupled system. In [44], the authors introduce the idea of well-ordered coupled lower and upper solutions, to obtain an existence and localization result for periodic solutions of first-order coupled non-linear systems of n ordinary differential equations.

Motivated by these works, in this chapter we present some novelties: in regards to the standard definition of upper and lower solutions, a new translation technique is applied to allow the lower and upper solutions to not be necessarily well-ordered; a method to provide sufficient conditions to obtain existence and localization results, combining the adequate coupled lower and upper solutions definition with the monotonicity of the nonlinearities.

This chapter is organized as follows. In Section 1.2 we present the definitions and the solvability conditions for problem (1.1), (1.2). In Section 1.3, we present the main results regarding the existence and localization of solutions for the problem, using the lower and upper solutions technique, together with Schauder's Fixed Point Theorem, and a numerical example. In Section 1.4 we adapt the solvability conditions of Section 1.2 to the lack of monotonicity criteria in the nonlinearities. We find coupled lower and upper solutions for the original problem. Finally, in Section 1.5, we present an application to a real case scenario of Population Dynamics followed by a discussion in Section 1.6.

1.2 Definitions

We start by considering the space of continuous functions in $[0,T]$, $(C[0,T])^2$, with the norm $\|(z,w)\| = \max\{\|z\|, \|w\|\}$, with $\|u\| = \max_{t\in[0,T]} |u(t)|$.

A Banach space is a complete vector space B with a norm $\|\cdot\|$. Therefore, $(C[0,T])^2$ is a Banach space.

The nonlinearities f, g in problem (1.1), (1.2) are assumed to be L^1-Carathéodory functions. This means that they follow the conditions established by one of the most fundamental definitions of this work:

Definition 1.2.1 (Carathéodory function)**.** *The function* $h : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$, *is a* L^1-*Carathéodory if it verifies*

(*i*) *for each* $(y_1, y_2) \in \mathbb{R}^2$, $t \mapsto h(t, y_1, y_2)$ *is Lebesgue-measurable in* $[0,T]$;

(*ii*) *for almost every* $t \in [0,T]$, $(y_1, y_2) \mapsto h(t, y_1, y_2)$ *is continuous in* $\mathbb{R}^2$;

(iii) *for each* $L > 0$*, there exists a positive function* $\psi_L \in L^1[0,T]$*, such that, for* $\max\{\|y_i\|, i = 1,2\} < L$*,*

$$|h(t, y_1, y_2)| \leq \psi_L(t), \; a.e. \; t \in [0,T].$$

The importance of the Definition of a Carathéodory function lies on the generalization of the composition of a Lebesgue-measurable function with a continuous function. Such composition is still Lebesgue-measurable, but the Carathéodory definition allows the continuity assumption to be relaxed. A Carathéodory function is, therefore, more general than a continuous function, but still measurable.

Naturally, throughout this work, we make use of this definition, which reveals to be of special importance in Chapters 2 and 4, when dealing with impulsive problems: the nonlinearities are allowed to have discontinuities at the instants of impulse.

An important tool for the localization of periodic solutions in this work is the method of upper and lower solutions. The first reference to this method goes back to [45], using an iterative technique. Since then, other definitions were proposed in the literature to approach different kinds of differential problems, according to the order of differentiability [46, 47], the nonlinear boundary value conditions [37], coupling [48], etc. However, most of these definitions presuppose the existence of a criterion of order, where typically the lower solution is required to be above the upper solution. Other studies provide case-wise definitions according to the ordering of the upper and lower solutions [27]. One differentiating factor in our work is precisely the absence of any order criteria. We present a definition for upper and lower solutions, not necessarily well-ordered. By *solution* we mean $(C^1[0,T])^2$ functions verifying (1.1), (1.2). The order of the upper and lower solutions is recovered using adequate translations, as described in Definition 1.2.2.

Definition 1.2.2. *Consider* C^1*-functions* $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1,2$. *The functions* (α_1, α_2) *are lower solutions of the periodic problem (1.1), (1.2) if*

$$\begin{cases} \alpha_1'(t) \leq f\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \\ \alpha_2'(t) \leq g\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \end{cases},$$

with

$$\alpha_i^0(t) := \alpha_i(t) - \|\alpha_i\|, \; i = 1,2,$$

and

$$\alpha_i(0) \leq \alpha_i(T), \; i = 1,2.$$

The functions (β_1, β_2) *are upper solutions of the periodic problem (1.1), (1.2) if*

$$\begin{cases} \beta_1'(t) \geq f\left(t, \beta_1^0(t), \beta_2^0(t)\right) \\ \beta_2'(t) \geq g\left(t, \beta_1^0(t), \beta_2^0(t)\right) \end{cases},$$

with

$$\beta_i^0(t) := \beta_i(t) + \|\beta_i\|\,,\ i = 1, 2,$$

and

$$\beta_i(0) \geq \beta_i(T),\ i = 1, 2.$$

The existence of a periodic solution is guaranteed by the well-known Schauder's Fixed-Point Theorem (theorem 2.A of [49]). This result was first published in 1930, and it is going to be the existence argument for problems of Chapters 1, 2 and 4.

Theorem 1.2.3. *Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

1.3 Existence and localization theorem

The method presented henceforth provides an existence and localization result for non-trivial periodic solutions of problem (1.1), (1.2). For an existing pair $(z(t), w(t))$ of solutions, each variable is localized in a strip bounded by lower and upper functions. The result is as follows.

Theorem 1.3.1. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (1.1), (1.2), respectively.*
Assume that f is a L^1-Carathéodory function in the set

$$\left\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_1^0(t) \leq y_1 \leq \beta_1^0(t)\right\},$$

g is a L^1-Carathéodory function in

$$\left\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_2^0(t) \leq y_2 \leq \beta_2^0(t)\right\},$$

with

$$f\,(t, y_1, y_2)\ \text{ nondecreasing in } y_2, \tag{1.3}$$

for fixed $t \in [0, T]$ and $y_1 \in \mathbb{R}$, and

$$g\,(t, y_1, y_2)\ \text{ nondecreasing in } y_1,$$

for fixed $t \in [0, T]$ and $y_2 \in \mathbb{R}$. Then problem (1.1), (1.2) has, at least, a solution $(z, w) \in \left(C^1[0, T]\right)^2$ such that

$$\begin{aligned} \alpha_1^0(t) &\leq z(t) \leq \beta_1^0(t), \\ \alpha_2^0(t) &\leq w(t) \leq \beta_2^0(t),\ \text{for all } t \in [0, T]. \end{aligned}$$

We now present a proof of Theorem 1.3.1 in three steps. Since the original problem is non-invertible, as constants are also admitted for solution, we consider a modified problem, perturbed and truncated. The modified problem only admits non-trivial solutions, since the null space only contains the null vector. In the first step we write the integral form of the modified problem. In the second step we prove the operator of the modified problem has a fixed point, using Schauder's fixed point theorem. In the last step we prove by contradiction that the truncated functions must necessarily be reduced to the branch of the corresponding variable. With this argument, the modified problem is reduced to the original problem, and we show both problems are, in fact, equivalent. The proof is as follows.

Proof. For $i = 1, 2$, define the continuous truncated functions $\delta_i^0 : [0, T] \times \mathbb{R} \to \mathbb{R}$, given by

$$\delta_1^0(t,z) = \begin{cases} \alpha_1^0(t) & \text{if} \quad z < \alpha_1^0(t) \\ z & \text{if} \quad \alpha_1^0(t) \le z \le \beta_1^0(t) \\ \beta_1^0(t) & \text{if} \quad z > \beta_1^0(t) \end{cases}$$

and

$$\delta_2^0(t,w) = \begin{cases} \alpha_2^0(t) & \text{if} \quad w < \alpha_2^0(t) \\ w & \text{if} \quad \alpha_2^0(t) \le w \le \beta_2^0(t) \\ \beta_2^0(t) & \text{if} \quad w > \beta_2^0(t) \end{cases}$$

and consider the auxiliary problem composed by

$$\begin{cases} z'(t) + z(t) = f\left(t, \delta_1^0(t,z), \delta_2^0(t,w)\right) + \delta_1^0(t,z) \\ w'(t) + w(t) = g\left(t, \delta_1^0(t,z), \delta_2^0(t,w)\right) + \delta_2^0(t,w) \end{cases}, \tag{1.4}$$

together with the periodic boundary conditions (1.2).

Step 1: *Integral form of (1.4), (1.2).*

The general solution of (1.4), (1.2) is

$$\begin{cases} z(t) = e^{-t}\Big(C_z + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) \\ \\ w(t) = e^{-t}\Big(C_w + \int_0^t e^s q_2(s, \delta_1^0, \delta_2^0)\, ds\Big) \end{cases},$$

with

$$q_1(s, \delta_1^0, \delta_2^0) := f(s, \delta_1^0(s, z(s)), \delta_2^0(s, w(s))) + \delta_1^0(s, z(s)),$$
$$q_2(s, \delta_1^0, \delta_2^0) := g(s, \delta_1^0(s, z(s)), \delta_2^0(s, w(s))) + \delta_2^0(s, w(s)).$$

and, by (1.2),

$$\begin{cases} C_z = \frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \\ C_w = \frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_2(s, \delta_1^0, \delta_2^0)\, ds \end{cases} .$$

Therefore, the integral form of (1.4), (1.2) is

$$\begin{cases} z(t) = e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big) \\ \\ w(t) = e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_2(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_2(s, \delta_1^0, \delta_2^0)\, ds \Big) \end{cases} .$$

We define the operator $T : (C[0,T])^2 \to (C[0,T])^2$ such that

$$T(z,w)(t) = (T_1(z,w)(t), T_2(z,w)(t)),$$

with

$$\begin{cases} T_1(z,w)(t) = e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds \Big) \\ \\ T_2(z,w)(t) = e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_2(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_2(s, \delta_1^0, \delta_2^0)\, ds \Big) \end{cases} .$$

Following Definition 1.2.1, the norm of the operator T is given by

$$\|T(z,w)\| = \max_{i=1,2} \big\{ \|T_1(z,w)\|,\ \|T_2(z,w)\| \big\} =$$
$$\max \big\{ \max_{t\in[0,T]} |T_1(z(t), w(t))|,\ \max_{t\in[0,T]} |T_2(z(t), w(t))| \big\}.$$

Step 2: *T has a fixed point.*

The conditions for Theorem 1.2.3 require the existence of a nonempty, bounded, closed and convex subset $D \subset (C\,[0,T])^2$ such that $TD \subset D$.

As f, g are L^1-Carathéodory functions, by Definition 1.2.1, there are positive $L^1[0,T]$ functions $\psi_{iL}, i = 1,2$, such that

$$\begin{matrix} |f(t, \delta_1^0(t,z), \delta_2^0(t,w))| \le \psi_{1L}(t) \\ |g(t, \delta_1^0(t,z), \delta_2^0(t,w))| \le \psi_{2L}(t) \end{matrix} , \quad a.e.\ t \in [0,T],$$

with

$$L := \max\{|\alpha_1^0(t)|, |\alpha_2^0(t)|, \beta_1^0(t), \beta_2^0(t)\}.$$

Now, consider the closed ball of radius K,

$$D := \{(z,w) \in (C[0,T])^2 : \|(z,w)\| \le K\},$$

with K given by

$$K = \max\left\{ \tfrac{e^T}{1-e^{-T}}\Big(\int_0^T \psi_{1L}(s)\,ds + LT\Big), \tfrac{e^T}{1-e^{-T}}\Big(\int_0^T \psi_{2L}(s)\,ds + LT\Big) \right\}. \quad (1.5)$$

For $t \in [0,T]$,

$$\begin{aligned}
|T_1(z,w)(t)| &= \left| e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}}\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big)\right| \\
&\le \frac{e^{-T}}{1-e^{-T}}\int_0^T |e^s|\,|q_1(s,\delta_1^0,\delta_2^0)|\,ds + \int_0^t |e^s|\,|q_1(s,\delta_1^0,\delta_2^0)|\,ds \\
&\le \frac{e^{T}}{1-e^{-T}}\int_0^T |q_1(s,\delta_1^0,\delta_2^0)|\,ds \\
&\le \frac{e^{T}}{1-e^{-T}}\int_0^T (|f(s,\delta_1^0,\delta_2^0)| + L)\,ds \\
&= \frac{e^{T}}{1-e^{-T}}\Big(\int_0^T |f(s,\delta_1^0,\delta_2^0)|\,ds + LT\Big) \\
&\le \frac{e^{T}}{1-e^{-T}}\Big(\int_0^T \psi_{1L}(s)\,ds + LT\Big) \le K.
\end{aligned}$$

Similarly,

$$|T_2(z,w)(t)| < \frac{e^{T}}{1-e^{-T}}\Big(\int_0^T \psi_{2L}(s)\,ds + LT\Big) \le K, \quad \forall t \in [0,T].$$

Since T_1 and T_2 are uniformly bounded, T is also uniformly bounded.

Now, consider $t_1, t_2 \in [0,T]$ and $t_1 < t_2$, without loss of generality. Then,

$$\begin{aligned}
&\big|T_1(z,w)(t_1) - T_1(z,w)(t_2)\big| \\
&= \Bigg| e^{-t_1}\Big(\frac{e^{-T}}{1-e^{-T}}\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^{t_1} e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big) \\
&\qquad - e^{-t_2}\Big(\frac{e^{-T}}{1-e^{-T}}\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^{t_2} e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big)\Bigg| \\
&= \Bigg| \big(e^{-t_1} - e^{-t_2}\big)\Big(\frac{e^{-T}}{1-e^{-T}}\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^{t_1} e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big) \\
&\qquad - e^{-t_2}\int_{t_1}^{t_2} e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Bigg| \xrightarrow[t_1\to t_2]{} 0,
\end{aligned}$$

proving that T_1 is equicontinuous.

In the same way, it can be proved that $\left|T_2(z(t_1),w(t_1)) - T_2(z(t_2),w(t_2))\right| \xrightarrow[t_1\to t_2]{} 0.$ Therefore, T is equicontinuous.

By (1.5), it is clear that $TD \subset D$, then, by Schauder's Fixed Point Theorem, T has a fixed point $(z^*(t), w^*(t)) \in (C[0,T])^2$, which is a solution of (1.4), (1.2).

Step 3: *The pair $(z^*(t), w^*(t))$, solution of (1.4), (1.2), is a solution of the initial problem, (1.1), (1.2).*

To prove that $(z^*, w^*) \in (C[0,T])^2$ is a solution of the original problem (1.1), (1.2), it is enough to prove that

$$\alpha_1^0(t) \le z^*(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w^*(t) \le \beta_2^0(t), \quad \forall t \in [0,T]. \tag{1.6}$$

In the first case, suppose, by contradiction, that there exists some $t \in [0,T]$ such that

$$z^*(t) < \alpha_1^0(t),$$

and define

$$\min_{t\in[0,T]} (z^* - \alpha_1^0)(t) := z^*(t_0) - \alpha_1^0(t_0) < 0. \tag{1.7}$$

If $t_0 \in]0,T[$, then

$$(z^*)'(t_0) = (\alpha_1^0)'(t_0), \tag{1.8}$$

and, by (1.7), (1.3) and Definition 1.2.2, the following contradiction with (1.8) holds:

$$\begin{aligned}
(z^*)'(t_0) &= f(t_0, \delta_1^0(t_0, z^*(t_0)), \delta_2^0(t_0, w(t_0))) + \delta_1^0(t_0, z^*(t_0)) - z^*(t_0) \\
&= f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) + \alpha_1^0(t_0) - z^*(t_0) \\
&> f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) \\
&\ge f(t_0, \alpha_1^0(t_0), \alpha_2^0(t_0)) \\
&\ge {\alpha_1}'(t_0).
\end{aligned}$$

If $t_0 = 0$, then, by (1.2) and Definition 1.2.2,

$$\begin{aligned}
z^*(0) - \alpha_1^0(0) =& z^*(T) - (\alpha_1(0) - \|\alpha\|) \\
=& z^*(T) - \alpha_1(0) + \|\alpha\| \\
\ge& z^*(T) - \alpha_1(T) + \|\alpha\| \\
=& z^*(T) - (\alpha_1(T) - \|\alpha\|) \\
=& z^*(T) - \alpha_1^0(T).
\end{aligned}$$

Then, by (1.7),

$$z^*(0) - \alpha_1^0(0) = z^*(T) - \alpha_1^0(T),$$

and

$$(z^*)'(T) - (\alpha_1)'(T) \leq 0. \tag{1.9}$$

Therefore, by (1.7), (1.3) and Definition 1.2.2,

$$\begin{aligned}(z^*)'(T) &= f(T, \delta_1^0(T, z^*(T)), \delta_2^0(T, w(T))) + \delta_1^0(T, z^*(T)) - z^*(T)\\ &= f(T, \alpha_1^0(T), \delta_2^0(T, w(T))) + \alpha_1^0(T) - z^*(T)\\ &> f(T, \alpha_1^0(T), \delta_2^0(T, w(T)))\\ &\geq f(T, \alpha_1^0(T), \alpha_2^0(T))\\ &\geq \alpha_1'(T),\end{aligned}$$

which contradicts (1.9).
If $t_0 = T$, we get (1.9), and the previous reasoning applies.

Therefore, $z^*(t) \geq \alpha_1^0(t), \forall t \in [0, T]$. The same arguments can be used to prove the other inequalities of (1.6). □

In the following lines we present an example of the applicability of Theorem 1.3.1.

Example 1.3.2. *Consider the following system, for $t \in [0, 1]$,*

$$\begin{cases} z'(t) = w(t) - (z(t))^3 \\ w'(t) = 1 - t + \arctan(z(t)) - 4\sqrt[3]{w(t)} \end{cases} \tag{1.10}$$

together with the periodic boundary conditions (1.2).
The functions $\alpha_i, \beta_i : [0, 1] \to \mathbb{R}$, $i = 1, 2$, given by

$$\begin{aligned}\alpha_1(t) &= t - 2, \qquad \beta_1(t) = 2 - t^2,\\ \alpha_2(t) &= t^2 - 1, \qquad \beta_2(t) = 1 - t,\end{aligned}$$

are, respectively, lower and upper solutions of problem (1.10), (1.2), according to Definition 1.2.2, with

$$\alpha_1^0(t) = t - 4, \qquad \alpha_2^0(t) = t^2 - 2,$$

and

$$\beta_1^0(t) = 4 - t^2, \qquad \beta_2^0(t) = 2 - t.$$

The above problem is a particular case of (1.1), (1.2), where

$$f(t, z(t), w(t)) = w(t) - (z(t))^3$$

and

$$g(t, z(t), w(t)) = 1 - t + \arctan(z(t)) - 4\sqrt[3]{w(t)}.$$

As the assumptions of Theorem 1.3.1 are verified, then the system (1.10), (1.2) has, at least, a solution $(z^*, w^*) \in (C[0,1])^2$ *such that*

$$\begin{aligned} t - 4 \le z^*(t) \le 4 - t^2, \\ t^2 - 2 \le w^*(t) \le 2 - t, \quad \forall t \in [0,1]. \end{aligned}$$

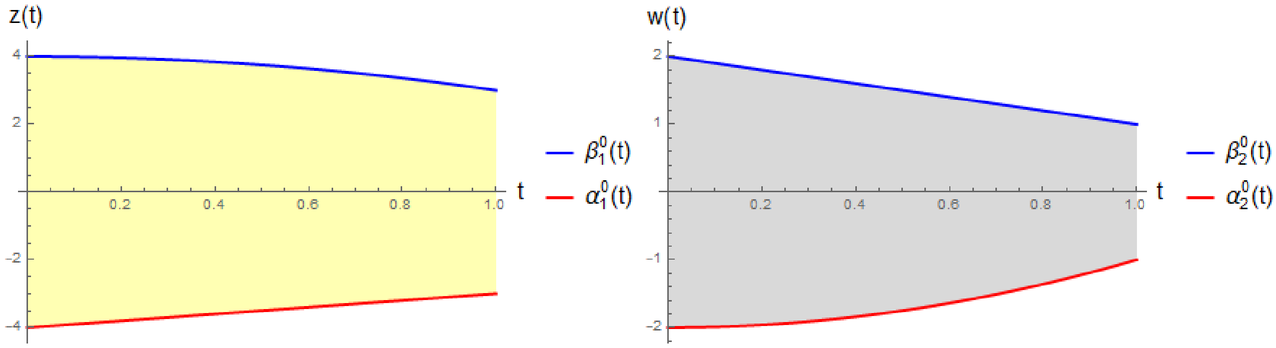


Figure 1.1: (z^*, w^*)-solution localization.

Moreover, this solution is not trivial, since constants are not solutions of (1.10).

1.4 Ordered lower and upper-solutions and monotonicity

In the previous section we verify that the monotonicity of the nonlinearities is the determinant factor for the equivalence relation between the modified problem and the original one.

However, the *nature* of the monotonicity depends on the ordering of the upper and lower solutions. In the previous section, there is no such order requirements, since the well-order is always re-established with adequate translations. This yields a lack of monotonicity requirements on the nonlinearities as well, since they are nondecreasing, regardless of the order of the upper and lower solutions.

To overcome this lack of requirements, in this section we define coupled lower and upper solutions α_i and β_i, conveniently, to show the effective relation between the order of the lower and upper solutions with the monotonicity of the nonlinearities.

Definition 1.4.1. *Consider the C^1-functions $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1,2$. The functions (α_1, α_2) are coupled lower solutions of the periodic problem (1.1), (1.2) if*

$$\begin{cases} \alpha_1'(t) \geq f\left(t, \alpha_1(t), \alpha_2(t)\right) \\ \alpha_2'(t) \leq g\left(t, \beta_1(t), \alpha_2(t)\right) \end{cases},$$

and

$$\begin{aligned} \alpha_1(0) &\geq \alpha_1(T), \\ \alpha_2(0) &\leq \alpha_2(T). \end{aligned}$$

The functions (β_1, β_2) are coupled upper solutions of the periodic problem (1.1), (1.2) if

$$\begin{cases} \beta_1'(t) \leq f\left(t, \beta_1(t), \beta_2(t)\right) \\ \beta_2'(t) \geq g\left(t, \alpha_1(t), \beta_2(t)\right) \end{cases},$$

and

$$\begin{aligned} \beta_1(0) &\leq \beta_1(T), \\ \beta_2(0) &\geq \beta_2(T). \end{aligned}$$

Theorem 1.4.2. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (1.1), (1.2), respectively, according to Definition 1.4.1, such that*

$$\alpha_1(t) \geq \beta_1(t) \text{ and } \alpha_2(t) \leq \beta_2(t), \ \forall t \in [0,T].$$

Assume that f is a L^1-Carathéodory function in the set

$$\left\{(t, y_1, y_2) \in [0,T] \times \mathbb{R}^2 : \beta_1(t) \leq y_1 \leq \alpha_1(t)\right\},$$

g is a L^1-Carathéodory function in

$$\left\{(t, y_1, y_2) \in [0,T] \times \mathbb{R}^2 : \alpha_2(t) \leq y_2 \leq \beta_2(t)\right\},$$

with

$$f\left(t, y_1, y_2\right) \text{ nonincreasing in } y_2, \tag{1.11}$$

for $t \in [0,T]$, $y_1 \in \mathbb{R}$, and

$$g\left(t, y_1, y_2\right) \text{ nondecreasing in } y_1,$$

for $t \in [0,T]$, $y_2 \in \mathbb{R}$. Then problem (1.1), (1.2) has, at least, a solution $(z, w) \in \left(C^1[0,T]\right)^2$ such that

$$\begin{aligned} &\beta_1(t) \leq z(t) \leq \alpha_1(t), \\ &\alpha_2(t) \leq w(t) \leq \beta_2(t), \text{ for all } t \in [0,T]. \end{aligned}$$

Remark 1.4.3. *This method relating the definition of coupled lower and upper-solutions, and the monotonicity properties of the nonlinearities, can be applied to other types of monotonicity.*

Proof. For $i = 1, 2$, define the continuous functions $\delta_i : [0, T] \times \mathbb{R} \to \mathbb{R}$, given by

$$\delta_1(t, z) = \begin{cases} \beta_1(t) & \text{if} \quad z < \beta_1(t) \\ z & \text{if} \quad \beta_1(t) \le z \le \alpha_1(t) \\ \alpha_1(t) & \text{if} \quad z > \alpha_1(t) \end{cases}$$

and

$$\delta_2(t, w) = \begin{cases} \alpha_2(t) & \text{if} \quad w < \alpha_2(t) \\ w & \text{if} \quad \alpha_2(t) \le w \le \beta_2(t) \\ \beta_2(t) & \text{if} \quad w > \beta_2(t) \end{cases}$$

and consider the auxiliary problem composed by

$$\begin{cases} z'(t) + z(t) = f\left(t, \delta_1(t, z), \delta_2(t, w)\right) + \delta_1(t, z) \\ w'(t) + w(t) = g\left(t, \delta_1(t, z), \delta_2(t, w)\right) + \delta_2(t, w) \end{cases}, \tag{1.12}$$

together with the periodic boundary conditions (1.2).

Problem (1.12), (1.2) can be addressed using the arguments for the proof of Theorem 1.3.1. The corresponding operator has a fixed point $(\bar{z}, \bar{w})$.

The delicate step, however, is to show that $(\bar{z}, \bar{w})$ is a solution of (1.1), (1.2), such that

$$\beta 1(t) \le \bar{z}(t) \le \alpha_1(t), \quad \alpha_2(t) \le \bar{w}(t) \le \beta_2(t), \quad \forall t \in [0, T].$$

To prove the case for $\beta_1(t) \le \bar{z}(t), \forall t \in [0, T]$, we suppose, by contradiction, that there exists some $t \in [0, T]$ such that

$$\bar{z}(t) < \beta_1(t),$$

and define

$$\min_{t \in [0,T]} (\bar{z} - \beta_1)(t) := \bar{z}(t_0) - \beta_1(t_0) < 0. \tag{1.13}$$

If $t_0 \in]0, T[$, then

$$\bar{z}'(t_0) = \beta_1'(t_0), \tag{1.14}$$

and, by (1.13), (1.11) and Definition 1.4.1, the following contradiction with (1.14) holds:

$$\begin{aligned} \bar{z}'(t_0) &= f(t_0, \delta_1(t_0, \bar{z}(t_0)), \delta_2(t_0, w(t_0))) + \delta_1(t_0, \bar{z}(t_0)) - \bar{z}(t_0) \\ &= f(t_0, \beta_1(t_0), \delta_2(t_0, w(t_0))) + \beta_1(t_0) - \bar{z}(t_0) \\ &> f(t_0, \beta_1(t_0), \delta_2(t_0, w(t_0))) \\ &\ge f(t_0, \beta_1(t_0), \beta_2(t_0)) \\ &\ge \beta_1'(t_0). \end{aligned}$$

If $t_0 = 0$, then, by (1.2) and Definition 1.4.1,

$$\bar{z}(0) - \beta_1(0) = \bar{z}(T) - \beta_1(0) \geq \bar{z}(T) - \beta_1(T).$$

Then, by (1.13),

$$\bar{z}(0) - \beta_1(0) = \bar{z}(T) - \beta_1(T),$$

and

$$\bar{z}'(T) - \beta_1'(T) \leq 0. \tag{1.15}$$

But by (1.13), (1.11) and Definition 1.4.1,

$$\begin{aligned}\bar{z}'(T) &= f(T, \delta_1(T, \bar{z}(T)), \delta_2(T, w(T))) + \delta_1(T, \bar{z}(T)) - \bar{z}(T)\\ &= f(T, \beta_1(T), \delta_2(T, w(T))) + \beta_1(T) - \bar{z}(T)\\ &> f(T, \beta_1(T), \delta_2(T, w(T)))\\ &\geq f(T, \beta_1(T), \beta_2(T))\\ &\geq \beta_1'(T),\end{aligned}$$

which contradicts (1.15).
If $t_0 = T$, then we get (1.15), and the previous reasoning applies.

Therefore, $\bar{z}(t) \geq \beta_1(t), \forall t \in [0, T]$. The same arguments can be applied to prove the inequality $\bar{z}(t) \leq \alpha_1(t)$.

Using the same technique, one can prove that

$$\alpha_2(t) \leq \bar{w}(t) \leq \beta_2(t), \forall t \in [0, T].$$

□

In the next section we see how can this technique be applied to a real-case scenario.

1.5 Criminal vs non-criminal population dynamics

In [50], the authors present an analysis of the dynamics of the interaction between criminal and non-criminal populations based on the prey-predator Lotka-Volterra models. Motivated by this work, we consider a variant of the constructed model with a logistic growth of the non-criminal population and a law enforcement term, that is,

$$\left\{\begin{array}{c} N'(t) = \mu N(t)\left(1 - \frac{1}{K}N(t)\right) - aN(t)C(t)\\ C'(t) = -\gamma C(t) - l_c C(t) + bN(t)C(t)\end{array}\right. , \tag{1.16}$$

where C and N are, respectively, the criminal and non-criminal population as functions of time, μ is the growth rate of N in the absence of C, a and b are the variation rates of N and C, respectively, due to their interaction, γ is the natural mortality rate of C, K is the carrying capacity for N in the absence of C, and l_c is the measure of enforced law on C.

We note that the interaction term could have different effects in each of the populations, hence, the, eventually, different multiplying factors a and b.

As a numerical example, we consider the parameter set:

$$\begin{array}{c} \mu = 2.5,\ a = 0.1,\ b = 0.01, \\ \gamma = 2.5,\ K = 8,\ l_c = 5. \end{array} \tag{1.17}$$

We choose a normalized period $T = 1$, representing the evolution dynamics in one year, and the periodic boundary conditions

$$\begin{array}{c} N(0) = N(1), \\ C(0) = C(1). \end{array} \tag{1.18}$$

Notice that the problem (1.16), (1.18), with (1.17), is a particular case of (1.1), (1.2) with

$$\begin{array}{c} f\left(t, N(t), C(t)\right) = 2.5N(t)\left(1 - \frac{1}{8}N(t)\right) - 0.1N(t)C(t) \\ g\left(t, N(t), C(t)\right) = -2.5C(t) - 5C(t) + 0.01N(t)C(t), \end{array}$$

and the quadratic functions

$$\begin{array}{rcl} \alpha_1(t) & = & -t^2 + t + 10, \\ \alpha_2(t) & = & -3t^2 + 3t - 1, \\ \beta_1(t) & = & t^2 + 5, \\ \beta_2(t) & = & -4t^2 + t + 4, \end{array}$$

are, respectively, coupled lower and upper solutions of (1.16), (1.18), for the parameters (1.17), according to Definition 1.4.1.

As all the assumptions of Theorem 1.4.2 are verified, then there is at least a solution (N, C) of (1.16), moreover,

$$t^2 + 5 \leq N(t) \leq -t^2 + t + 10,$$

and

$$-3t^2 + 3t - 1 \leq C(t) \leq -4t^2 + t + 4.$$

The only constant solutions (N, C) admissible by problem (1.16) are $(0, 0)$, $(750, 0)$, $(0, -2318.75)$ and $(750, -2318.75)$, which are beyond the lower and upper bounds α_i, β_i. Hence, the region of solutions shown in Figure 1.2 contains only non-trivial solutions.

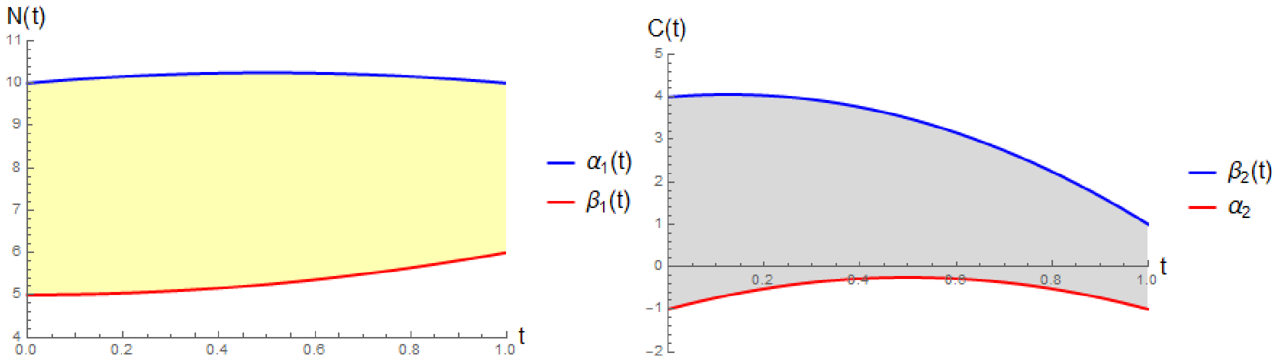


Figure 1.2: (N, C)-solution localization.

1.6 Discussion

In this chapter we started by addressing problem (1.1), (1.2), by presenting a methodology based on the existence of a fixed point for the operator of the system and the localization of the respective periodic solutions.

Since 1930, the scientific literature contains diverse results on the method of upper and lower solutions related to n^{th} order differential problems, n^{th}-sized systems, problems with different boundary conditions, coupled systems, among many others. Some results can be more restrictive than others, in the sense that they require more conditions but their contribution is essential for the understanding of this mathematical subject.

This work aims at contributing to enrich the literature within this subject by presenting sufficient conditions that guarantee the existence of periodic solutions and their localization, with new definitions of lower and upper solutions.

In Definition1.2.2, a translation is applied, such that, regardless of the order relation between the lower and upper solutions, the shifted functions are always well-ordered. As consequence, both the nonlinearities must be nondecreasing in Theorem 1.3.1.

Definition1.4.1 does not apply any translations, yet it presents a coupled relation between lower and upper-solutions and the nonlinearities. This variant implies changes to the monotonicity requirements in 1.4.2, also showing that this technique can be modified to be adapted to several real-case applications.

In real life, some systems experience spikes in the variation of certain intrinsic quantities. Ordinary differential problems can be applied to such cases, but may not be enough to describe their dynamics faithfully. In the next chapter, we explore the conditions for the existence and localization of periodic solutions in a system of ordinary differential equations with impulsive conditions.

Chapter 2

First-order periodic systems with impulses

2.1 Introduction

We study the following first-order coupled nonlinear system,

$$\begin{cases} z'(t) = f(t, z(t), w(t)) \\ w'(t) = g(t, z(t), w(t)) \end{cases}, \tag{2.1}$$

a.e. t in $[0,T], T > 0$, and the $f, g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ L^1-Carathéodory functions, with the periodic boundary conditions

$$\begin{array}{c} z(0) = z(T), \\ w(0) = w(T), \end{array} \tag{2.2}$$

subject to impulses given by

$$\begin{array}{c} \Delta z(t_k) = I_k(t_k, z(t_k), w(t_k)), \\ \Delta w(\tau_l) = J_l(\tau_l, z(\tau_l), w(\tau_l)), \end{array} \tag{2.3}$$

with $k = 1, ..., n-1$, $l = 1, ..., m-1$, $n > 2$, $m > 2$, $\Delta z(t_k) = z(t_k^+) - z(t_k^-)$, $\Delta w(\tau_l) = w(\tau_l^+) - w(\tau_l^-)$, $I_k, J_l \in C([0,T] \times \mathbb{R}^2, \mathbb{R})$, and the time instants t_k, τ_l such that $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and $0 = \tau_0 < \tau_1 < ... < \tau_{m-1} < \tau_m = T$, where

$$u(t_k^-) = \lim_{t \to t_k^-} u(t) \qquad \text{and} \qquad u(t_k^+) = \lim_{t \to t_k^+} u(t).$$

The study of problem (2.1), (2.2), (2.3) is motivated by the complexity of nonlinear phenomena driven by sudden changes, *i.e.*, impulses [15, 16, 17, 18, 19]. This complexity increases even more when one admits the possibility of state-dependent impulses [22, 23, 24] and non-periodic non-linearities, so finding periodic solutions is a challenge.

In this chapter we address impulsive coupled first order problems of the type of (2.1), (2.2), (2.3) through an existence and localization methodology, similar to that of Chapter 2. The strategy consists of considering a perturbed and truncated problem, equivalent to the original problem, and setting it to the conditions of Schauder's fixed point theorem.

Following the translation technique suggested in [38], we obtain the existence and the localization of at least an impulsive periodic solution in the interval $[0, T]$.

In such method, for C^1 lower and upper functions, there is a change of sign in the nonlinearities (see Definition 2.2.5). However, with less regularity (PC^1 lower and upper functions, as in Definition 2.4.1), the change of sign in the nonlinearities can be overcome in the presence of well-ordered lower and upper solutions.

So, for impulsive first-order coupled systems, some novelties are obtained: 1) sufficient conditions for the existence of periodic solutions are given, even when the nonlinearities have no periodicity at all; 2) lower and upper solutions do not need to be well-ordered; 3) the sum of all jumps must be null (so the problem must have more than one instant of impulse); 4) there exist non-negative periodic impulsive solutions despite the eventual change of sign in the nonlinearities.

The main results require some monotonicity relations in the nonlinearities and in the impulsive bounding functions. Furthermore, results on equi-regulated functions [51, 52] are essential to deal with the discontinuities at the instants of impulse. A similar problem is studied in [53, 54, 55], and with similar techniques in [38, 56].

As an example, we apply this method to a variant of a Wilson-Cowan system of strongly coupled neurons [57], using one of the most commonly used activation functions in neural networks [58] as the impulsive function, defined in certain time instants.

We organize this chapter by starting to present the required definitions and auxiliary theorems in Section 2.2. Section 2.3 contains one of the main results, together with the respective proof of the existence and localization of at least one solution of problem (2.1), (2.2), (2.3), together with a numerical example. In Section 2.4 we adapt the solvability conditions of Section 2.3 by allowing the sign of the nonlinearities to remain constant and, for the effect, recovering the order of the upper and lower solutions. In Section 2.5 we present a numerical result by applying our technique to a variant of a Wilson-Cowan system of strongly coupled neurons.

2.2 Definitions and Auxiliary results

We consider the space of piecewise continuous functions in $[0,T]$, $(PC[0,T])^2$. Consequently, there is no guarantee of the existence of a maximum in this interval. Therefore, the norm is defined as $\|(z,w)\| = \max\{\|z\|,\|w\|\}$, with $\|u\| = \sup_{t\in[0,T]}|u(t)|$. Hence, $(PC[0,T])^2$ is a Banach space.

When dealing with functions with discontinuities, it is convenient to work with *regulated functions*, as we shall define below, according to [51]:

Definition 2.2.1. *A function $h : [a,b] \to \mathbb{R}^N$ is regulated if for every $t \in [a,b[$, the right-sided limit $\lim_{\tau\to t^+} h(\tau) = h(t^+)$ exists and is finite, and for every $t \in]a,b]$, the left-sided limit $\lim_{\tau\to t^-} h(\tau) = h(t^-)$ exists and is finite.*

We consider $\mathcal{G}$ as the space of regulated functions,

$$\mathcal{G} = \{u : u(t^-) \in \mathbb{R}, \forall t \in]0,T], u(s^+) \in \mathbb{R}, \forall s \in [0,T[\,\}. \tag{2.4}$$

A main result for regulated functions is given by Theorem 2.2.2:

Theorem 2.2.2. *[52] A given subset B of the space $\mathcal{G}$ of regulated functions is relatively compact if and only if*

- *B is the set of equi-regulated functions, i.e. for every $\epsilon > 0$ there is a division $\xi_0 < \cdots < \xi_p$ of the interval $[0,T]$ such that, for every $v \in B$, $j \in \{1,\ldots,p\}$ and every $t,s \in (\xi_{j-1},\xi_j)$ we have $|v(t)-v(s)| < \epsilon$;*
- *the set $\{v(t) : v \in B\} \subset \mathbb{R}$ is bounded for each $t \in [0,T]$.*

Define the set $D = \{\xi_1,\ldots,\xi_{p-1}\}$ such that $0 = \xi_0 < \xi_1 < \cdots < \xi_{p-1} < \xi_p = T$. Let PC_D be the space of piecewise continuous functions on $[0,T]$, given by

$$PC_D := \{u \in C([0,T]\backslash D) : u(t_k^-) = u(t_k), u(t_k^+) \in \mathbb{R}\}.$$

Define

$$D_z = \{t_1,\ldots,t_n\}$$
$$D_w = \{\tau_1,\ldots,\tau_m\}$$

as the sets of instants of impulse of functions z and w, respectively.

It is clear that $PC_D \subseteq \mathcal{G}$, and that $PC_{D_z}[0,T] \times PC_{D_w}[0,T]$, endowed with the norm $\|\cdot\|$, is a Banach space.

The relation between equi-regulation and compactness on PC_D is based on Corollary 2.2.3:

Corollary 2.2.3. *[52] A subset B of the space PC_D is relatively compact if and only if:*

- *for a given $t \in [0,T]$, the set $\{u(t) : u \in B\} \subset \mathbb{R}$ is bounded;*
- *the set $\{u(t) : u \in B\}$ is equi-regulated.*

The nonlinearities in (2.1) are assumed to be L^1-Carathéodory functions, in accordance to Definition 1.2.1.

Remark 2.2.4. *A general impulsive periodic first order problem can be written, for $f \in L^1[0,T]$, as*

$$\begin{cases} z'(t) = f(t, z(t)), \\ z(0) = z(T) \\ \Delta z(t_k) = I_k(\cdot), k = 1, ..., n. \end{cases}$$

By integration of the differential equation in $[0,t]$, $t \in [0,T]$,

$$z(t) = z(0) + \int_0^t f(s, z(s))\, ds.$$

From the boundary conditions $z(0) = z(T)$, and the impulses $\Delta z(t_k) = I_k(\cdot)$, $k = 1, ..., n$,

$$z(0) = z(0) + \sum_{k=1}^{n} I_k(\cdot) + \int_0^T f(s, z(s))\, ds$$

and

$$\sum_{k=1}^{n} I_k(\cdot) = -\int_0^T f(s, z(s))\, ds.$$

So, in impulsive periodic first-order problems, there is a relation between the sum of the jumps and the mean value of f in $[0,T]$.
If the constant of integration is chosen, such that the mean value in $[0,T]$ is zero then, in this case,

$$\sum_{k=1}^{n} I_k(\cdot) = 0.$$

We localize existing solutions of (2.1), (2.2), (2.3) with bounding functions with translations. Because the upper bound is shifted towards its maximum and the lower bound is shifted towards its minimum, there is no requirement for the order of the upper and lower solutions, as stated by Definition below:

Definition 2.2.5. *Consider the C^1-functions $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1,2$. The functions (α_1, α_2) are lower solutions of the periodic problem (2.1), (2.2), (2.3) if*

$$\begin{cases} \alpha_1'(t) \le f\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \\ \alpha_2'(t) \le g\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) \end{cases}, \tag{2.5}$$

with

$$\alpha_i^0(t) := \alpha_i(t) - \|\alpha_i\|\,,\ i = 1,2, \tag{2.6}$$

with

$$\alpha_i(0) \le \alpha_i(T),\ i = 1,2, \tag{2.7}$$

and

$$\begin{cases} \Delta\alpha_1(t_k) > I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k)) \\ \Delta\alpha_2(\tau_l) > J_l(\tau_l, \alpha_1^0(\tau_l), \alpha_2^0(\tau_l)) \end{cases}. \tag{2.8}$$

The functions (β_1, β_2) are upper solutions of the periodic problem (2.1), (2.2), (2.3) if

$$\begin{cases} \beta_1'(t) \ge f\left(t, \beta_1^0(t), \beta_2^0(t)\right) \\ \beta_2'(t) \ge g\left(t, \beta_1^0(t), \beta_2^0(t)\right) \end{cases}, \tag{2.9}$$

with

$$\beta_i^0(t) := \beta_i(t) + \|\beta_i\|\,,\ i = 1,2,$$

with

$$\beta_i(0) \ge \beta_i(T),\ i = 1,2, \tag{2.10}$$

and

$$\begin{cases} \Delta\beta_1(t_k) < I_k(t_k, \beta_1^0(t_k), \beta_2^0(t_k)) \\ \Delta\beta_2(\tau_l) < J_l(\tau_l, \beta_1^0(\tau_l), \beta_2^0(\tau_l)) \end{cases}.$$

The existence of a periodic solution for problem (2.1), (2.2), (2.3) is guaranteed by Theorem 1.2.3.

2.3 Existence and localization theorem

The main result consists of a theorem that establishes the conditions for the existence and localization of periodic solutions for problem (2.1), (2.2), (2.3). The localization of each variable $z(t)$ or $w(t)$ is achieved by finding functions that verify the conditions stated in Definition 2.2.5. Such functions are lower and upper bounds of a strip that represents the set of possible periodic solutions of the impulsive problem.

Theorem 2.3.1. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (2.1), (2.2), (2.3), respectively.*
Assume that f, g are L^1-Carathéodory functions on the set

$$\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_1^0(t) \leq y_1 \leq \beta_1^0(t),\ \alpha_2^0(t) \leq y_2 \leq \beta_2^0(t)\},$$

with

$$f(t, y_1, \alpha_2^0(t)) \leq f(t, y_1, y_2) \leq f(t, y_1, \beta_2^0(t)), \tag{2.11}$$

for fixed $t \in [0, T]$, $y_1 \in \mathbb{R}$, and $\alpha_2^0(t) \leq y_2 \leq \beta_2^0(t)$, and with

$$g(t, \alpha_1^0(t), y_2) \leq g(t, y_1, y_2) \leq g(t, \beta_1^0(t), y_2),$$

for fixed $t \in [0, T]$, $y_2 \in \mathbb{R}$, and $\alpha_1^0(t) \leq y_1 \leq \beta_1^0(t)$.
Assume that the impulse functions I_k and J_l verify

$$I_k(t_k, y_1, \alpha_2^0) \geq I_k(t_k, y_1, y_2) \geq I_k(t_k, y_1, \beta_2^0), \tag{2.12}$$

for some fixed $k \in \{1, \ldots n-1\}$, $y_1 \in \mathbb{R}$, and $\alpha_2^0(t) \leq y_2 \leq \beta_2^0(t)$, and that

$$\sum_{k=1}^{n-1} I_k(t_k, y_1, y_2) = 0. \tag{2.13}$$

Assume that the impulse functions J_l verify

$$J_l(\tau_l, \alpha_1^0, y_2) \geq J_l(\tau_l, y_1, y_2) \geq J_l(\tau_l, \beta_1^0, y_2),$$

for some fixed $l \in \{1, \ldots m-1\}$, $y_2 \in \mathbb{R}$ and $\alpha_1^0(t) \leq y_1 \leq \beta_1^0(t)$, and that

$$\sum_{l=1}^{m-1} J_l(\tau_l, y_1, y_2) = 0. \tag{2.14}$$

Then, problem (2.1), (2.2), (2.3) has, at least, a solution $(z, w) \in (PC^1[0, T])^2$ such that

$$\alpha_1^0(t) \leq z(t) \leq \beta_1^0(t),$$
$$\alpha_2^0(t) \leq w(t) \leq \beta_2^0(t), \forall t \in [0, T].$$

Remark 2.3.2. *Conditions (2.13) and (2.14) imply, respectively, that $n > 2$ and $m > 2$.*

Like in Chapter 1, we present a proof of Theorem 2.3.1 in three steps, based on a perturbed and truncated version of the original problem (2.1), (2.2), (2.3). In the first step we write the integral form of the modified problem; in the second step we prove that the operator has a fixed point in the terms of Schauder's fixed point theorem; in the third step we localize a solution, consisting of a pair $(z, w) \in (PC[0,T])^2$ within a strip bounded by upper and lower solutions. The proof is as follows.

Proof. For $i = 1, 2$, define the continuous functions $\delta_i^0 : [0,T] \times \mathbb{R}$, given by

$$\delta_1^0(t,z) = \begin{cases} \alpha_1^0(t) & \text{if} \quad z < \alpha_1^0(t) \\ z & \text{if} \quad \alpha_1^0(t) \le z \le \beta_1^0(t) \\ \beta_1^0(t) & \text{if} \quad z > \beta_1^0(t) \end{cases}$$

and

$$\delta_2^0(t,w) = \begin{cases} \alpha_2^0(t) & \text{if} \quad w < \alpha_2^0(t) \\ w & \text{if} \quad \alpha_2^0(t) \le w \le \beta_2^0(t) \\ \beta_2^0(t) & \text{if} \quad w > \beta_2^0(t) \end{cases},$$

and consider the modified problem, perturbed and truncated,

$$\begin{cases} z'(t) + z(t) = f\left(t, \delta_1^0(t,z(t)), \delta_2^0(t,w(t))\right) + \delta_1^0(t,z(t)), & t \neq t_k, \\ w'(t) + w(t) = g\left(t, \delta_1^0(t,z(t)), \delta_2^0(t,w(t))\right) + \delta_2^0(t,w(t)), & t \neq \tau_l, \end{cases} \tag{2.15}$$

together with the boundary conditions (2.2) and the truncated impulse conditions,

$$\begin{array}{c} \Delta z(t_k) = I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))), \\ \Delta w(\tau_l) = J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l))). \end{array} \tag{2.16}$$

Step 1: *Integral form of the problem (2.15), (2.2), (2.16).*

The integral form of (2.15), (2.2) considering the impulses (2.16) is given by

$$\begin{cases} z(t) = \displaystyle\sum_{k:t>t_k} I_k(t_k, \delta_1^0, \delta_2^0) + \\ \qquad + e^{-t}\left(\dfrac{e^{-T}}{1-e^{-T}} \displaystyle\int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\right) \\ w(t) = \displaystyle\sum_{l:t>\tau_l} J_l(\tau_l, \delta_1^0, \delta_2^0) + \\ \qquad + e^{-t}\left(\dfrac{e^{-T}}{1-e^{-T}} \displaystyle\int_0^T e^s q_2(s, \delta_1^0, \delta_2^0)\, ds + \int_0^t e^s q_2(s, \delta_1^0, \delta_2^0)\, ds\right), \end{cases}$$

where

$$I_k(t_k,\delta_1^0,\delta_2^0) = I_k(t_k,\delta_1^0(t_k,z(t_k)),\delta_2^0(t_k,w(t_k))),$$
$$J_l(\tau_l,\delta_1^0,\delta_2^0) = J_l(\tau_l,\delta_1^0(\tau_l,z(\tau_l)),\delta_2^0(\tau_l,w(\tau_l))),$$

and

$$q_1(s,\delta_1^0,\delta_2^0) := f(s,\delta_1^0(s,z(s)),\delta_2^0(s,w(s))) + \delta_1^0(s,z(s)),$$
$$q_2(s,\delta_1^0,\delta_2^0) := g(s,\delta_1^0(s,z(s)),\delta_2^0(s,w(s))) + \delta_2^0(s,w(s)).$$

Define the operator $T : (PC[0,T])^2 \to (PC[0,T])^2$ such that

$$T(z,w)(t) = (T_1(z,w)(t), T_2(z,w)(t)),$$

with

$$\begin{cases} T_1(z,w)(t) = \displaystyle\sum_{k:t>t_k} I_k(t_k,\delta_1^0,\delta_2^0) + \\ \qquad + e^{-t}\Big(\dfrac{e^{-T}}{1-e^{-T}}\displaystyle\int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big) \\ T_2(z,w)(t) = \displaystyle\sum_{l:t>\tau_l} J_l(\tau_l,\delta_1^0,\delta_2^0) + \\ \qquad + e^{-t}\Big(\dfrac{e^{-T}}{1-e^{-T}}\displaystyle\int_0^T e^s q_2(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_2(s,\delta_1^0,\delta_2^0)\,ds\Big). \end{cases} \tag{2.17}$$

The norm of the operator T is given by

$$\|T(z,w)\| = \max\big\{\|T_1(z,w)\|,\ \|T_2(z,w)\|\big\} =$$
$$\max\big\{\sup_{t\in[0,T]}|T_1(z,w)(t)|,\ \sup_{t\in[0,T]}|T_2(z,w)(t)|\big\}.$$

Step 2: *T has a fixed point.*

The conditions for Theorem 1.2.3 require the existence of a nonempty, bounded, closed and convex subset $B \subset (PC[0,T])^2$ such that $TB \subset B$.

As f, g are L^1-Carathéodory functions, by Definition 1.2.1, there are positive $L^1[0,T]$ functions $\psi_{iL}, i = 1,2$, such that

$$\begin{array}{l} |f(t,\delta_1^0(t,z),\delta_2^0(t,w))| \le \psi_{1L}(t) \\ |g(t,\delta_1^0(t,z),\delta_2^0(t,w))| \le \psi_{2L}(t) \end{array}, \quad a.e.\ t\in[0,T]$$

with

$$L := \max\{|\alpha_1^0(t)|, |\alpha_2^0(t)|, \beta_1^0(t), \beta_2^0(t), t\in[0,T]\}.$$

We consider the closed ball of radius K,

$$B := \{(z,w) \in (PC[0,T])^2 : \|(z,w)\| \leq K\}, \tag{2.18}$$

with K given by

$$K = \max \left\{ \begin{array}{l} M_I(n-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{1L}(s)\, ds + LT \Big), \\ M_J(m-1) + \frac{e^T}{1-e^{-T}} \Big(\int_0^T (\psi_{2L}(s)\, ds + LT \Big) \end{array} \right\}, \tag{2.19}$$

and

$$M_I := \max_{1 \leq k \leq n-1} \{ |I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k)))| \},$$
$$M_J := \max_{1 \leq l \leq m-1} \{ |J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l)))| \}.$$

For $t \in [0,T]$,

$$
\begin{aligned}
&|T_1(z,w)(t)| = \\
&= \Big| \sum_{k:t>t_k} I_k(t_k,\delta_1^0,\delta_2^0) + \\
&\qquad + e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big)\Big| \\
&\le \Big| \sum_{k:t>t_k} I_k(t_k,\delta_1^0,\delta_2^0)\Big| + \\
&\qquad + \Big|e^{-t}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big)\Big| \\
&\le \sum_{k:t>t_k} \Big| I_k(t_k,\delta_1^0,\delta_2^0)\Big| + \\
&\qquad + \Big|\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds + \int_0^t e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big)\Big| \\
&\le \sum_{k:t>t_k} M_I + \Big|\frac{1}{1-e^{-T}} \int_0^T e^s q_1(s,\delta_1^0,\delta_2^0)\,ds\Big| \\
&\le M_I(n-1) + \frac{e^T}{1-e^{-T}}\Big|\int_0^T q_1(s,\delta_1^0,\delta_2^0)\,ds\Big| \\
&\le M_I(n-1) + \frac{e^T}{1-e^{-T}}\int_0^T |q_1(s,\delta_1^0,\delta_2^0)|\,ds \\
&\le M_I(n-1) + \frac{e^T}{1-e^{-T}}\Big(\int_0^T |f(s,\delta_1^0,\delta_2^0)|ds + LT\Big) \\
&\le M_I(n-1) + \frac{e^T}{1-e^{-T}}\Big(\int_0^T \psi_{1L}(s)ds + LT\Big).
\end{aligned}
$$

From (2.19), we have

$$|T_1(z,w)(t)| \le M_I(n-1) + \frac{e^T}{1-e^{-T}}\Big(\int_0^T (\psi_{1L}(s)\,ds + LT\Big) \le K, \quad \forall t \in [0,T].$$

Similarly,

$$|T_2(z,w)(t)| \le M_J(m-1) + \frac{e^T}{1-e^{-T}}\Big(\int_0^T (\psi_{2L}(s)\,ds + LT\Big) \le K, \quad \forall t \in [0,T].$$

Since T_1 and T_2 are uniformly bounded, so is T and, by (2.18) and (2.19), $TB \subseteq B$.

Consider a, b, with $a < b$, without loss of generality, and let $[a, b] \subseteq (t_k, t_{k+1})$ for some $k \in \{0, \dots, n-1\}$.

Then,

$$
\begin{aligned}
&|T_1(z,w)(a) - T_1(z,w)(b)| = \\
&= \Bigg| \sum_{k:a>t_k} I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))) + \\
&+ e^{-a}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^a e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) - \\
&- \sum_{k:b>t_k} I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k))) - \\
&- e^{-b}\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \int_0^b e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big)\Bigg| = \\
&= \Bigg|(e^{-a} - e^{-b})\Big(\frac{e^{-T}}{1-e^{-T}} \int_0^T e^s q_1(s, \delta_1^0, \delta_2^0)\, ds + \\
&\quad \int_0^a e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Big) - e^{-b} \int_a^b e^s q_1(s, \delta_1^0, \delta_2^0)\, ds\Bigg| \xrightarrow[a \to b]{} 0,
\end{aligned}
$$

proving that T_1 is equi-regulated.

Similarly, $|T_2(z,w)(a) - T_2(z,w)(b)| \xrightarrow[a \to b]{} 0$. Therefore, T is equi-regulated.

By Corollary 2.2.3, T is relatively compact. Then, by Theorem 1.2.3, T has a fixed point $(z^*(t), w^*(t)) \in (PC[0,T])^2$, which is solution of (2.15), (2.2), (2.16).

Step 3: *The pair $(z^*(t), w^*(t))$, solution of (2.15), (2.2), (2.16), is a solution of the initial problem, (2.1), (2.2), (2.3).*

To prove that $(z^*, w^*) \in (PC[0,T])^2$ is a solution of the original problem (2.1), (2.2), (2.3) it is enough to prove that

$$
\alpha_1^0(t) \le z^*(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w^*(t) \le \beta_2^0(t), \quad \forall t \in [0,T]. \tag{2.20}
$$

In the first inequality, suppose, by contradiction, that there exists $t \in [0,T]$ such that

$$
z^*(t) < \alpha_1^0(t),
$$

and define

$$
\inf_{t \in [0,T]} (z^* - \alpha_1^0)(t) := z^*(t_0) - \alpha_1^0(t_0) < 0. \tag{2.21}
$$

We now consider different cases for t_0.

If $t_0 \in]t_k, t_{k+1}[$ for some $k \in \{0, \ldots, n\}$, then, by (2.21)

$$(z^* - \alpha_1^0)'(t_0) = 0. \tag{2.22}$$

However, by (2.21), (2.11) and (2.5),

$$\begin{aligned}
(z^*)'(t_0) &= f(t_0, \delta_1^0(t_0, z^*(t_0)), \delta_2^0(t_0, w(t_0))) + \delta_1^0(t_0, z^*(t_0)) - z^*(t_0) \\
&= f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) + \alpha_1^0(t_0) - z^*(t_0) \\
&> f(t_0, \alpha_1^0(t_0), \delta_2^0(t_0, w(t_0))) \\
&\geq f(t_0, \alpha_1^0(t_0), \alpha_2^0(t_0)) \\
&\geq {\alpha_1}'(t_0),
\end{aligned}$$

contradicting (2.22).

If $t_0 = t_k$ for some $k \in \{1, \ldots, n-1\}$, then either $t_0 = t_k^+$ or $t_0 = t_k^-$. If $t_0 = t_k^+$, then $t_0 \in]t_k, t_{k+1}[$, so the previous reasoning must be applied. If, instead, $t_0 = t_k^-$, then we consider

$$\min_{t \in [0,T]} (z^* - \alpha_1^0)(t) := z^*(t_0) - \alpha_1^0(t_0) < 0, \tag{2.23}$$

and thus, the impulse on t_k is necessarily non-negative. By (2.23), (2.6), (2.16), (2.12) and (2.8) the following contradiction holds,

$$\begin{aligned}
0 &\leq \Delta(z^* - \alpha_1^0)(t_k) \\
&= I_k(t_k, \delta_1^0(t_k, z^*(t_k)), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1^0(t_k) \\
&= I_k(t_k, \alpha_1^0(t_k), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1^0(t_k) \\
&= I_k(t_k, \alpha_1^0(t_k), \delta_2^0(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\
&\leq I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k)) - \Delta\alpha_1(t_k) < 0.
\end{aligned} \tag{2.24}$$

Finally, if $t_0 = 0$, we can consider (2.23). By (2.2), (2.6) and (2.7), then

$$\begin{aligned}
z^*(0) - \alpha_1^0(0) &= z^*(T) - (\alpha_1(0) - \|\alpha\|) \\
&= z^*(T) - \alpha_1(0) + \|\alpha\| \\
&\geq z^*(T) - \alpha_1(T) + \|\alpha\| \\
&= z^*(T) - (\alpha_1(T) - \|\alpha\|) \\
&= z^*(T) - \alpha_1^0(T).
\end{aligned}$$

Then, by (2.23),

$$z^*(0) - \alpha_1^0(0) = z^*(T) - \alpha_1^0(T),$$

and

$$(z^*)'(T) - (\alpha_1)'(T) \leq 0. \tag{2.25}$$

Then, by (2.23), (2.11) and (2.5), the following contradiction with 2.25 holds,

$$\begin{aligned}(z^*)'(T) &= f(T, \delta_1^0(T, z^*(T)), \delta_2^0(T, w(T))) + \delta_1^0(T, z^*(T)) - z^*(T)\\ &= f(T, \alpha_1^0(T), \delta_2^0(T, w(T))) + \alpha_1^0(T) - z^*(T)\\ &> f(T, \alpha_1^0(T), \delta_2^0(T, w(T)))\\ &\geq f(T, \alpha_1^0(T), \alpha_2^0(T))\\ &\geq \alpha_1'(T).\end{aligned}$$

Therefore, $z^*(t) \geq \alpha_1^0(t), \forall t \in [0, T]$.

The same arguments can be applied to prove the other inequalities in (2.20).

□

Example 2.3.3. *Consider the following system, for $t \in [0, 1]$,*

$$\begin{cases} z'(t) = a_1 z^3(t) + a_2 w(t) + a_3 t, & a_2 > 0,\\ w'(t) = b_1 w^3(t) + b_2 z(t) + b_3 t, & b_2 > 0,\end{cases}$$

together with the periodic boundary conditions (2.2), and the impulse conditions

$$\begin{cases} \Delta z(t_k) = S_{\epsilon_1}(c_1 z(t_k) - c_2 w(t_k) + c_3) - 1/2, & c_2 > 0,\\ \Delta w(\tau_l) = S_{\epsilon_2}(-d_1 z(\tau_l) + d_2 w(\tau_l) + d_3) - 1/2, & d_1 > 0,\end{cases}$$

where $S : \mathbb{R} \to \mathbb{R}^+$ is the sigmoid function, defined as

$$S_\epsilon(x) = \frac{1}{1 + e^{-\epsilon x}}, \qquad \epsilon > 0.$$

As a numerical example, we consider the normalized period, $T = 1$, and the parameter set,

$$\begin{array}{cccc} a_1 = -5 & a_2 = 0.1 & a_3 = 2 & \\ b_1 = -0.5 & b_2 = 1 & b_3 = 10 & \\ c_1 = 6 & c_2 = 2 & c_3 = 1 & \epsilon_1 > 0\\ d_1 = 3 & d_2 = 6 & d_3 = 1 & \epsilon_2 > 0 \end{array}$$

and we shall consider impulses at $t_1 = 1/2$, $t_2 = 3/4$ and $\tau_l = 1/3$, $\tau_2 = 2/3$, that is, we shall consider the problem

$$\begin{cases} z'(t) = -5z^3(t) + 0.1w(t) + 2t\\ w'(t) = -0.5w^3(t) + z(t) + 10t \end{cases}, \tag{2.26}$$

together with the boundary conditions

$$\begin{aligned} z(0) &= z(1) \\ w(0) &= w(1) \end{aligned} \tag{2.27}$$

and the impulses

$$\begin{cases} \Delta x_1(t_1) = S_{\epsilon_1}(6x_1(t_1) - 2x_2(t_1) + 1) - 1/2, & \epsilon_1 > 0 \\ \Delta x_1(t_2) = S_{\epsilon_1}(6x_1(t_2) - 2x_2(t_2) + 1) - 1/2, & \epsilon_1 > 0 \\ \Delta x_2(\tau_1) = S_{\epsilon_2}(-3x_1(\tau_1) + 6x_2(\tau_1) + 1) - 1/2, & \epsilon_2 > 0 \\ \Delta x_2(\tau_2) = S_{\epsilon_2}(-3x_1(\tau_2) + 6x_2(\tau_2) + 1) - 1/2, & \epsilon_2 > 0 \end{cases}, \tag{2.28}$$

It is clear that (2.26), (2.27), (2.28) is a particular case of problem (2.1), (2.2), (2.3), with

$$\begin{cases} f(t, z(t), w(t)) = -5z^3(t) + 0.1w(t) + 2t \\ g(t, z(t), w(t)) = -0.5w^3(t) + z(t) + 10t \end{cases},$$

and

$$\begin{cases} I_1(t_1, x_1(t_1), x_2(t_1)) = S_{\epsilon_1}(6x_1(t_1) - 2x_2(t_1) + 1) - 1/2 \\ I_2(t_2, x_1(t_2), x_2(t_2)) = S_{\epsilon_1}(6x_1(t_2) - 2x_2(t_2) + 1) - 1/2 \\ J_1(\tau_1, x_1(\tau_1), x_2(\tau_1)) = S_{\epsilon_2}(-3x_1(\tau_1) + 6x_2(\tau_1) + 1) - 1/2 \\ J_2(\tau_2, x_1(\tau_2), x_2(\tau_2)) = S_{\epsilon_2}(-3x_1(\tau_2) + 6x_2(\tau_2) + 1) - 1/2 \end{cases}.$$

The functions $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, *given by*

$$\begin{aligned} \alpha_1(t) &= t, & \alpha_2(t) &= 2t - 1, \\ \beta_1(t) &= 1 - t, & \beta_2(t) &= 2 - t. \end{aligned}$$

are, respectively, lower and upper solutions of problem (2.26), (2.27), (2.28), according to Definition 2.2.5, with

$$\begin{aligned} \alpha_1^0(t) &= t - 1, & \alpha_2^0(t) &= 2t - 2, \\ \beta_1^0(t) &= 2 - t, & \beta_2^0(t) &= 4 - t. \end{aligned}$$

We observe that the inequalities required by Definition 2.2.5 are verified on the interval $[0,1]$ *for the nonlinearities,*

$$\begin{aligned} 1 = \alpha_1'(t) &\le f\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) = -5t^3 + 15t^2 - 12.80t + 4.8, \\ 2 = \alpha_2'(t) &\le g\left(t, \alpha_1^0(t), \alpha_2^0(t)\right) = -4t^3 + 12t^2 - t + 3, \\ -1 = \beta_1'(t) &\ge f\left(t, \beta_1^0(t), \beta_2^0(t)\right) = 5t^3 - 30t^2 + 61.90t - 39.6, \\ -1 = \beta_2'(t) &\ge g\left(t, \beta_1^0(t), \beta_2^0(t)\right) = 0.5t^3 - 6t^2 + 33t - 30. \end{aligned}$$

As for the impulses, we notice that, for $\epsilon > 0$, given a function $u(t)$, the following monotonicity relations hold,

$$S_\epsilon(u(t)) - 1/2 > 0 \Rightarrow u(t) > 0,$$
$$S_\epsilon(u(t)) - 1/2 < 0 \Rightarrow u(t) < 0.$$

Therefore, in order to verify the inequalities of Definition 2.2.5, i.e.,

$$\begin{aligned}
0 = \Delta\alpha_1(t_1) &> I_1\left(t_1, \alpha_1^0(t_1), \alpha_2^0(t_1)\right),\\
0 = \Delta\alpha_1(t_2) &> I_2\left(t_2, \alpha_1^0(t_2), \alpha_2^0(t_2)\right),\\
0 = \Delta\alpha_2(\tau_1) &> J_1\left(\tau_1, \alpha_1^0(\tau_1), \alpha_2^0(\tau_1)\right),\\
0 = \Delta\alpha_2(\tau_2) &> J_2\left(\tau_2, \alpha_1^0(\tau_2), \alpha_2^0(\tau_2)\right),
\end{aligned}$$

$$\begin{aligned}
0 = \Delta\beta_1(t_1) &< I_1\left(t_1, \beta_1^0(t_1), \beta_2^0(t_1)\right),\\
0 = \Delta\beta_1(t_2) &< I_2\left(t_2, \beta_1^0(t_2), \beta_2^0(t_2)\right),\\
0 = \Delta\beta_2(\tau_1) &< J_1\left(\tau_1, \beta_1^0(\tau_1), \beta_2^0(\tau_1)\right),\\
0 = \Delta\beta_2(\tau_2) &< J_2\left(\tau_2, \beta_1^0(\tau_2), \beta_2^0(\tau_2)\right),
\end{aligned}$$

we only need to show that, for every $\epsilon_i > 0$,

$$\begin{aligned}
c_1\alpha_1^0(t_1) - c_2\alpha_2^0(t_1) + c_3 &= -2 < 0,\\
c_1\alpha_1^0(t_2) - c_2\alpha_2^0(t_2) + c_3 &= -3/2 < 0,\\
-d_1\alpha_1^0(\tau_1) + d_2\alpha_2^0(\tau_1) + d_3 &= -10/3 < 0,\\
-d_1\alpha_1^0(\tau_2) + d_2\alpha_2^0(\tau_2) + d_3 &= -8/3 < 0,
\end{aligned}$$

$$\begin{aligned}
c_1\beta_1^0(t_1) - c_2\beta_2^0(t_1) + c_3 &= 1/2 > 0,\\
c_1\beta_1^0(t_2) - c_2\beta_2^0(t_2) + c_3 &= 1/4 > 0,\\
-d_1\beta_1^0(\tau_1) + d_2\beta_2^0(\tau_1) + d_3 &= 2 > 0,\\
-d_1\beta_1^0(\tau_2) + d_2\beta_2^0(\tau_2) + d_3 &= 2 > 0.
\end{aligned}$$

As all the assumptions of Theorem 2.3.1 are verified, then there is at least a non-trivial periodic solution (z^*, w^*) of problem (2.26), (2.27), (2.28), moreover,

$$\begin{array}{c} t-1 \le z^*(t) \le 2-t, \\ 2t-2, \le w^*(t) \le 4-t, \end{array}, \forall t \in [0,1].$$

Figure 2.1 shows the range of possible values for existing periodic solutions (z^*, w^*). The strips for $z^*(t)$ (top) and for $w^*(t)$ (bottom) are bounded by the respective upper

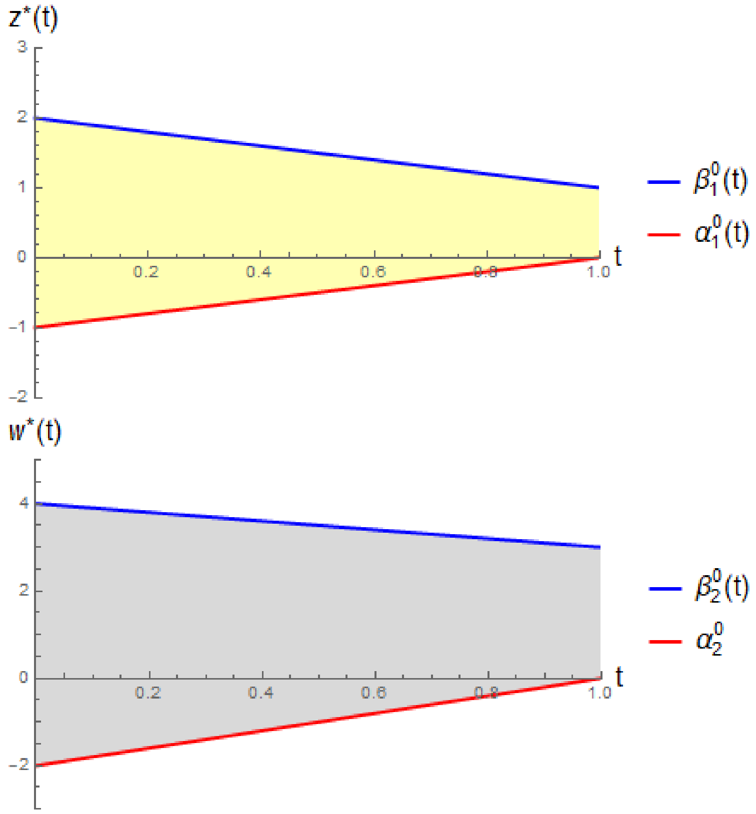


Figure 2.1: (z^*, w^*)-solution localization, in $[0, 1]$.

and lower functions with translations. We note that the functions α_i^0 *are always negative, given that the downward translation is by the amount of its maximum in the interval* $[0, T]$*. Analogously, the functions* β_i^0 *are always positive because their translations by the amount equal to the maximum values in* $[0, T]$*, respectively, is upwards.*

2.4 Ordered lower and upper solutions and sign of nonlinearities

In Section 2.3, Theorem 2.3.1 localizes an existing solution of problem (2.1), (2.2), (2.3) in a strip bounded by upper and lower solutions with translations, $\alpha_i^0(t) \leq 0$ and $\beta_i^0(t) \geq 0$ (see Definition 2.2.5). However, the differential inequalities (2.5) and

(2.9), together with the boundary conditions (2.7) and (2.10), require the change of sign in the nonlinearities.

In this section, it is required less regularity to lower and upper solutions. However, it is necessary to impose an order relation between them, $\alpha_i(t) \leq \beta_i(t), i = 1, 2$, in order to define a method that does not require the sign of the nonlinearities to change.

Definition 2.4.1. *Consider the PC^1-functions $\alpha_i, \beta_i : [0, T] \to \mathbb{R}$, $i = 1, 2$. The functions (α_1, α_2) are lower solutions of the periodic problem (2.1), (2.2), (2.3) if*

$$\begin{cases} \alpha_1'(t) \leq f\left(t, \alpha_1(t), \alpha_2(t)\right) \\ \alpha_2'(t) \leq g\left(t, \alpha_1(t), \alpha_2(t)\right) \end{cases}, \tag{2.29}$$

with

$$\alpha_i(0) \leq \alpha_i(T), \;\; i = 1, 2, \tag{2.30}$$

and

$$\begin{cases} \Delta\alpha_1(t_k) > I_k(t_k, \alpha_1(t_k), \alpha_2(t_k)) \\ \Delta\alpha_2(\tau_l) > J_l(\tau_l, \alpha_1(\tau_l), \alpha_2(\tau_l)) \end{cases}. \tag{2.31}$$

The functions (β_1, β_2) are upper solutions of the periodic problem (2.1), (2.2), (2.3) if

$$\begin{cases} \beta_1'(t) \geq f\left(t, \beta_1(t), \beta_2(t)\right) \\ \beta_2'(t) \geq g\left(t, \beta_1(t), \beta_2(t)\right) \end{cases}, \tag{2.32}$$

with

$$\beta_i(0) \geq \beta_i(T), \;\; i = 1, 2,$$

and

$$\begin{cases} \Delta\beta_1(t_k) < I_k(t_k, \beta_1(t_k), \beta_2(t_k)) \\ \Delta\beta_2(\tau_l) < J_l(\tau_l, \beta_1(\tau_l), \beta_2(\tau_l)) \end{cases}.$$

Theorem 2.4.2. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (2.1), (2.2), (2.3), respectively, according to Definition 2.4.1, such that*

$$\alpha_i(t) \leq \beta_i(t), \quad i = 1, 2, \quad \forall t \in [0, T],$$

Assume that f, g are L^1-Carathéodory functions on the set

$$\{(t, y_1, y_2) \in [0, T] \times \mathbb{R}^2 : \alpha_1(t) \leq y_1 \leq \beta_1(t), \;\; \alpha_2(t) \leq y_2 \leq \beta_2(t)\},$$

with

$$f(t, y_1, \alpha_2(t)) \leq f(t, y_1, y_2) \leq f(t, y_1, \beta_2(t)), \tag{2.33}$$

for fixed $t \in [0, T]$, $y_1 \in \mathbb{R}$, and $\alpha_2(t) \leq y_2 \leq \beta_2(t)$, and with

$$g(t, \alpha_1(t), y_2) \leq g(t, y_1, y_2) \leq g(t, \beta_1(t), y_2),$$

for fixed $t \in [0,T]$, $y_2 \in \mathbb{R}$, *and* $\alpha_1(t) \le y_1 \le \beta_1(t)$.
Assume that the impulse functions I_k *and* J_l *verify*

$$I_k(t_k, y_1, \alpha_2) \ge I_k(t_k, y_1, y_2) \ge I_k(t_k, y_1, \beta_2), \tag{2.34}$$

for some fixed $k \in \{1, \dots n-1\}$, $y_1 \in \mathbb{R}$, *and* $\alpha_2(t) \le y_2 \le \beta_2(t)$, *and that*

$$\sum_{k=1}^{n-1} I_k(t_k, y_1, y_2) = 0.$$

Assume that the impulse functions J_l *verify*

$$J_l(\tau_l, \alpha_1, y_2) \ge J_l(\tau_l, y_1, y_2) \ge J_l(\tau_l, \beta_1, y_2),$$

for some fixed $l \in \{1, \dots m-1\}$, $y_2 \in \mathbb{R}$ *and* $\alpha_1(t) \le y_1 \le \beta_1(t)$, *and that*

$$\sum_{l=1}^{m-1} J_l(\tau_l, y_1, y_2) = 0.$$

Then, problem (2.1), (2.2), (2.3) has, at least, a solution $(z, w) \in (PC^1[0,T]))^2$ *such that*

$$\begin{aligned} &\alpha_1(t) \le z(t) \le \beta_1(t), \\ &\alpha_2(t) \le w(t) \le \beta_2(t), \forall t \in [0,T]. \end{aligned}$$

Proof. For $i = 1, 2$, define the continuous functions $\delta_i : [0,T] \times \mathbb{R}$, given by

$$\delta_1(t,z) = \left\{ \begin{array}{ccc} \alpha_1(t) & \text{if} & z < \alpha_1(t) \\ z & \text{if} & \alpha_1(t) \le z \le \beta_1(t) \\ \beta_1(t) & \text{if} & z > \beta_1(t) \end{array} \right.$$

and

$$\delta_2(t,w) = \left\{ \begin{array}{ccc} \alpha_2(t) & \text{if} & w < \alpha_2(t) \\ w & \text{if} & \alpha_2(t) \le w \le \beta_2(t) \\ \beta_2(t) & \text{if} & w > \beta_2(t) \end{array} \right. ,$$

and consider the modified problem, composed by

$$\left\{ \begin{array}{ll} z'(t) + z(t) = f\left(t, \delta_1(t, z(t)), \delta_2(t, w(t))\right) + \delta_1(t, z(t)), & t \ne t_k, \\ w'(t) + w(t) = g\left(t, \delta_1(t, z(t)), \delta_2(t, w(t))\right) + \delta_2(t, w(t)), & t \ne \tau_l, \end{array} \right. \tag{2.35}$$

together with the boundary conditions (2.2) and the truncated impulse conditions,

$$\begin{aligned}\Delta z(t_k) &= I_k(t_k, \delta_1(t_k, z(t_k)), \delta_2(t_k, w(t_k))),\\ \Delta w(\tau_l) &= J_l(\tau_l, \delta_1(\tau_l, z(\tau_l)), \delta_2(\tau_l, w(\tau_l))).\end{aligned} \tag{2.36}$$

Problem (2.35), (2.2), (2.36) can be addressed using the arguments for the proof of Theorem 2.3.1. The corresponding operator has a fixed point $(\bar{z}, \bar{w})$. However, the new definitions in this section require changes to the third step, *i.e.*, to show that $(\bar{z}, \bar{w})$ is a solution of (2.35), (2.2), (2.36), such that

$$\alpha_1(t) \le \bar{z}(t) \le \beta_1(t), \quad \alpha_2(t) \le \bar{w}(t) \le \beta_2(t), \quad \forall t \in [0, T]. \tag{2.37}$$

In the first inequality, suppose, by contradiction, that there exists $t \in [0, T]$ such that

$$\bar{z}(t) < \alpha_1(t),$$

and define

$$\inf_{t\in[0,T]} (\bar{z} - \alpha_1)(t) := \bar{z}(t_0) - \alpha_1(t_0) < 0. \tag{2.38}$$

We now consider different cases for t_0.

If $t_0 \in]t_k, t_{k+1}[$ for some $k \in \{0, \ldots, n\}$, then, by (2.38)

$$(\bar{z} - \alpha_1)'(t_0) = 0. \tag{2.39}$$

However, by (2.38), (2.33) and (2.29),

$$\begin{aligned}(\bar{z})'(t_0) &= f(t_0, \delta_1(t_0, \bar{z}(t_0)), \delta_2(t_0, w(t_0))) + \delta_1(t_0, \bar{z}(t_0)) - \bar{z}(t_0)\\ &= f(t_0, \alpha_1(t_0), \delta_2(t_0, w(t_0))) + \alpha_1(t_0) - \bar{z}(t_0)\\ &> f(t_0, \alpha_1(t_0), \delta_2(t_0, w(t_0)))\\ &\ge f(t_0, \alpha_1(t_0), \alpha_2(t_0))\\ &\ge {\alpha_1}'(t_0),\end{aligned}$$

contradicting (2.39).

If $t_0 = t_k$ for some $k \in \{1, \ldots, n-1\}$, then either $t_0 = t_k^+$ or $t_0 = t_k^-$. If $t_0 = t_k^+$, then $t_0 \in]t_k, t_{k+1}[$, so the previous reasoning must be applied. If, instead, $t_0 = t_k^-$, then we consider

$$\min_{t\in[0,T]} (\bar{z} - \alpha_1)(t) := \bar{z}(t_0) - \alpha_1(t_0) < 0, \tag{2.40}$$

and thus, the impulse on t_k is necessarily non-negative. By (2.40), (2.36), (2.34) and

(2.31) the following contradiction holds,

$$\begin{aligned} 0 &\leq \Delta(\bar{z}-\alpha_1)(t_k) \\ &= I_k(t_k, \delta_1(t_k, \bar{z}(t_k)), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &= I_k(t_k, \alpha_1(t_k), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &= I_k(t_k, \alpha_1(t_k), \delta_2(t_k, w(t_k))) - \Delta\alpha_1(t_k) \\ &\leq I_k(t_k, \alpha_1(t_k), \alpha_2(t_k)) - \Delta\alpha_1(t_k) < 0. \end{aligned} \tag{2.41}$$

Finally, if $t_0 = 0$, we can consider (2.40). By (2.2) and (2.30), then

$$\bar{z}(0) - \alpha_1(0) = \bar{z}(T) - \alpha_1(0) \geq \bar{z}(T) - \alpha_1(T).$$

Then, by (2.40),

$$\bar{z}(0) - \alpha_1(0) = \bar{z}(T) - \alpha_1(T),$$

and

$$(\bar{z})'(T) - (\alpha_1)'(T) \leq 0. \tag{2.42}$$

Then, by (2.40), (2.33) and (2.29), the following contradiction with 2.42 holds,

$$\begin{aligned} (\bar{z})'(T) &= f(T, \delta_1(T, \bar{z}(T)), \delta_2(T, w(T))) + \delta_1(T, \bar{z}(T)) - \bar{z}(T) \\ &= f(T, \alpha_1(T), \delta_2(T, w(T))) + \alpha_1(T) - \bar{z}(T) \\ &> f(T, \alpha_1(T), \delta_2(T, w(T))) \\ &\geq f(T, \alpha_1(T), \alpha_2(T)) \\ &\geq \alpha_1'(T). \end{aligned}$$

Therefore, $\bar{z}(t) \geq \alpha_1(t), \forall t \in [0, T]$.

The same arguments can be applied to prove the other inequalities in (2.37).

□

2.5 Strongly connected Wilson-Cowan neural oscillators

In the work [57] the authors study the dynamics, synchronization and control of chaos in a system of strongly connected Wilson-Cowan neural oscillators, and present the respective mathematical model,

$$x_i'(t) = -\phi_i x_i(t) + S\Big(\rho_i + \sum_{j=1}^{n} \theta_{ij} x_j\Big), \qquad x_i \in \mathbb{R}, \quad i, j = 1, \ldots, n, \quad \phi_i \geq 0, \tag{2.43}$$

where ϕ_i is the internal decay rate of the i^{th} neuron, ρ_i, θ_{ij} are the input parameters of the activation function $S : \mathbb{R} \to \mathbb{R}^+$ of neuron i, acting on coupled neuron j. From the list of most commonly used activation functions suggested in [58], we choose $S(x) = \tanh(x)$.

Motivated by these works, we adapted (2.43) to a system of two neurons with the following form,

$$\begin{cases} x_1'(t) = -a_1 x_1(t) + e^{a_2 x_2(t)} + a_3 t \\ x_2'(t) = -b_1 x_2(t) + e^{b_2 x_1(t)} + b_3 t \end{cases}, \tag{2.44}$$

with $\alpha_i > 0$, $\beta_i > 0$, $i = 1, 2$, together with the periodic boundary conditions

$$x_i(0) = x_i(T), \qquad i = 1, 2, \tag{2.45}$$

and the impulses given by

$$\begin{cases} \Delta x_1(t_k) = \tanh(c_1 x_1(t_k) - c_2 x_2(t_k) + c_3) \\ \Delta x_2(\tau_l) = \tanh(-d_1 x_1(\tau_l) + d_2 x_2(\tau_l) + d_3) \end{cases}, \tag{2.46}$$

with $c_2 > 0$, $d_1 > 0$.

The quantity x_i denotes the activation state of the i^{th} neuron, a_1, b_1 the respective internal decay rate, a_2, b_2 are the weights of the non-linear components, and $a_3 t, b_3 t$ are the time-dependent external inputs, with $a_3, b_3 \in \mathbb{R}$.

The quantities Δx_i denote the instantaneous jumps of the i^{th} neuron at the respective instants of impulse, modelled by the function S, together with the weights of each variables, where $c_1, d_2, c_3, d_3 \in \mathbb{R}$.

As a numerical example, we consider the normalized period, $T = 1$, and the parameter set,

$$\begin{array}{ccc} a_1 = 0.1 & a_2 = 0.2 & a_3 = -0.5 \\ b_1 = 0.1 & b_2 = 0.1 & b_3 = 0.1 \\ c_1 = 5 & c_2 = 5 & c_3 = -3 \\ d_1 = 0.5 & d_2 = 1.5 & d_3 = -5 \end{array}$$

and we shall consider impulses at $t_1 = 1/3$, $t_2 = 2/3$, and $\tau_l = 1/4$, $\tau_2 = 1/2$, $\tau_3 = 3/4$. In short, we consider the numerical problem

$$\begin{cases} x_1'(t) = -0.1 x_1(t) + e^{0.2 x_2(t)} - 0.5t \\ x_2'(t) = -0.1 x_2(t) + e^{0.1 x_1(t)} + 0.1t \end{cases}, \tag{2.47}$$

together with the boundary conditions

$$x_i(0) = x_i(1), \tag{2.48}$$

and the impulses

$$\left\{\begin{array}{c} \Delta x_1(t_1) = \tanh(5x_1(t_1) - 5x_2(t_1) - 3) \\ \Delta x_1(t_2) = \tanh(5x_1(t_2) - 5x_2(t_2) - 3) \\ \Delta x_2(\tau_1) = \tanh(-0.5x_1(\tau_1) + 1.5x_2(\tau_1) - 5) \\ \Delta x_2(\tau_2) = \tanh(-0.5x_1(\tau_2) + 1.5x_2(\tau_2) - 5) \\ \Delta x_2(\tau_3) = \tanh(-0.5x_1(\tau_3) + 1.5x_2(\tau_3) - 5). \end{array}\right. \tag{2.49}$$

It is clear that (2.47), (2.48), (2.49) is a particular case of problem (2.1), (2.2), (2.3), with

$$\left\{\begin{array}{l} f(t, x_1(t), x_2(t)) = -0.1x_1(t) + e^{0.2x_2(t)} - 0.5t \\ g(t, x_1(t), x_2(t)) = -0.1x_2(t) + e^{0.1x_1(t)} + 0.1t \end{array}\right.,$$

and

$$\left\{\begin{array}{c} I_1(t_1, x_1(t_1), x_2(t_1)) = \tanh(5x_1(t_1) - 5x_2(t_1) - 3) \\ I_2(t_2, x_1(t_2), x_2(t_2)) = \tanh(5x_1(t_2) - 5x_2(t_2) - 3) \\ J_1(\tau_1, x_1(\tau_1), x_2(\tau_1)) = \tanh(-0.5x_1(\tau_1) + 1.5x_2(\tau_1) - 5) \\ J_2(\tau_2, x_1(\tau_2), x_2(\tau_2)) = \tanh(-0.5x_1(\tau_2) + 1.5x_2(\tau_2) - 5) \\ J_3(\tau_3, x_1(\tau_3), x_2(\tau_3)) = \tanh(-0.5x_1(\tau_3) + 1.5x_2(\tau_3) - 5). \end{array}\right.$$

The functions $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, given by

$$\alpha_1(t) = \left\{\begin{array}{ll} -t + \frac{1}{3}, & 0 \le t \le \frac{1}{3} \\ -t + 1, & \frac{1}{3} < t \le \frac{2}{3} \\ -t + \frac{5}{3}, & \frac{2}{3} < t \le 1 \end{array}\right. \qquad \beta_1(t) = \left\{\begin{array}{ll} 4t + 5, & 0 \le t \le \frac{1}{3} \\ 4t + 3, & \frac{1}{3} < t \le \frac{2}{3} \\ 4t + 1, & \frac{2}{3} < t \le 1 \end{array}\right.,$$

$$\alpha_2(t) = \left\{\begin{array}{ll} -t + \frac{1}{4}, & 0 \le t \le \frac{1}{4} \\ -t + \frac{3}{4}, & \frac{1}{4} < t \le \frac{1}{2} \\ -t + \frac{5}{4}, & \frac{1}{2} < t \le \frac{3}{4} \\ -t + \frac{7}{4}, & \frac{3}{4} < t \le 1 \end{array}\right., \qquad \beta_2(t) = \left\{\begin{array}{ll} 3t + 5, & 0 \le t \le \frac{1}{4} \\ 3t + 4, & \frac{1}{4} < t \le \frac{1}{2} \\ 3t + 3, & \frac{1}{2} < t \le \frac{3}{4} \\ 3t + 2, & \frac{3}{4} < t \le 1 \end{array}\right.,$$

are, respectively, lower and upper solutions of problem (2.47), (2.48), (2.49), according to Definition 2.4.1. In fact, the four differential inequalities (equations (2.29), (2.32)) are verified in the interval $[0, 1]$, as shown in Figure 2.3.

As for the impulses, we verify the inequalities of Definition 2.4.1,

$$\begin{aligned}
2/3 = \Delta\alpha_1(t_1) &> I_1\left(t_1, \alpha_1(t_1), \alpha_2(t_1)\right) = -0.999923,\\
2/3 = \Delta\alpha_1(t_2) &> I_2\left(t_2, \alpha_1(t_2), \alpha_2(t_2)\right) = -0.999593,\\
1/2 = \Delta\alpha_2(\tau_1) &> J_1\left(\tau_1, \alpha_1(\tau_1), \alpha_2(\tau_1)\right) = -0.999916,\\
1/2 = \Delta\alpha_2(\tau_2) &> J_2\left(\tau_2, \alpha_1(\tau_2), \alpha_2(\tau_2)\right) = -0.999883,\\
1/2 = \Delta\alpha_2(\tau_3) &> J_3\left(\tau_3, \alpha_1(\tau_3), \alpha_2(\tau_3)\right) = -0.999837,
\end{aligned}$$

$$\begin{aligned}
-2 = \Delta\beta_1(t_1) &< I_1\left(t_1, \beta_1(t_1), \beta_2(t_1)\right) = 0.998694,\\
-2 = \Delta\beta_1(t_2) &< I_2\left(t_2, \beta_1(t_2), \beta_2(t_2)\right) = 0.321513,\\
-1 = \Delta\beta_2(\tau_1) &< J_1\left(\tau_1, \beta_1(\tau_1), \beta_2(\tau_1)\right) = 0.5546,\\
-1 = \Delta\beta_2(\tau_2) &< J_2\left(\tau_2, \beta_1(\tau_2), \beta_2(\tau_2)\right) = 0.635149,\\
-1 = \Delta\beta_2(\tau_3) &< J_3\left(\tau_3, \beta_1(\tau_3), \beta_2(\tau_3)\right) = 0.703906,
\end{aligned}$$

As all the assumptions of Theorem 2.4.2 are verified, then there is at least a non-trivial non-negative periodic solution (x_1^*, x_2^*) of problem (2.47), (2.48), (2.49), moreover,

$$\begin{array}{c} \alpha_1(t) \leq x_1^*(t) \leq \beta_1(t), \\ \alpha_2(t), \leq x_2^*(t) \leq \beta_2(t), \end{array}, \forall t \in [0,1].$$

Figure 2.2 exhibits the range of possible values for existing piecewise periodic solutions (x_1^*, x_2^*). Now under the application of Theorem 2.4.2, the strips for $x_1^*(t)$ (top) and for $x_2^*(t)$ (bottom) are bounded by piecewise upper and lower functions, which provide some freedom to overcome the constraints of Definition 2.4.1. Also, the bounding functions α_i β_i can be either positive or negative, as they are not constructed under any translation.

Figure 2.3 shows, graphically, the relation between the lower and upper functions and the non-linearities of problem (2.44), (2.45), (2.46). We observe that these relations, for the particular case of this application, meet the requirements of Definition 2.4.1.

2.6 Discussion

Like in Chapter 1, in this chapter we approach a first-order differential problem with periodic boundary conditions, but with impulses. Although the method of upper and lower solutions has been widely studied in the last decades, our study explores a new definition of upper and lower solutions that can be applied to impulsive periodic

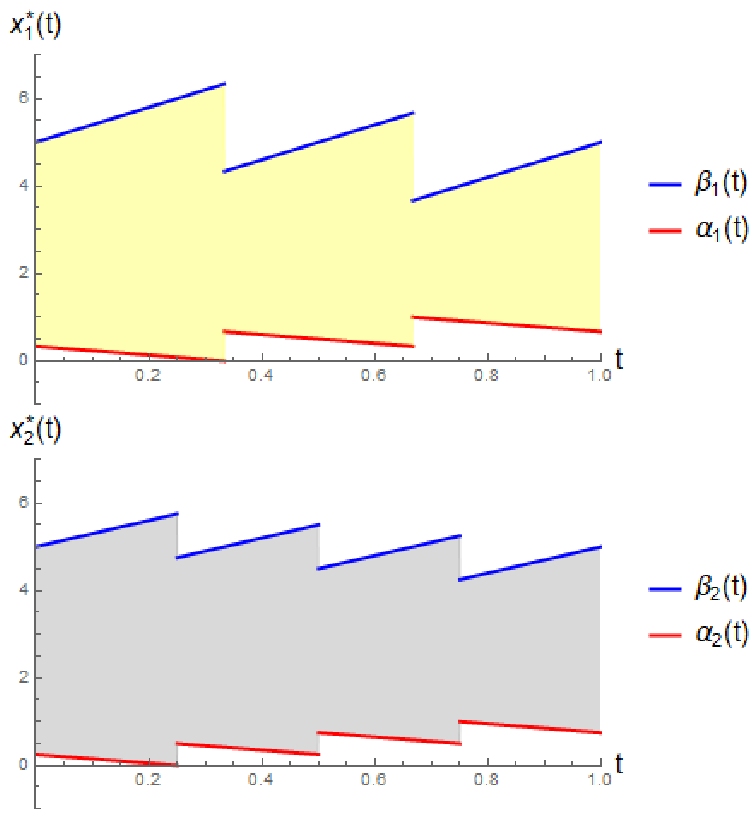


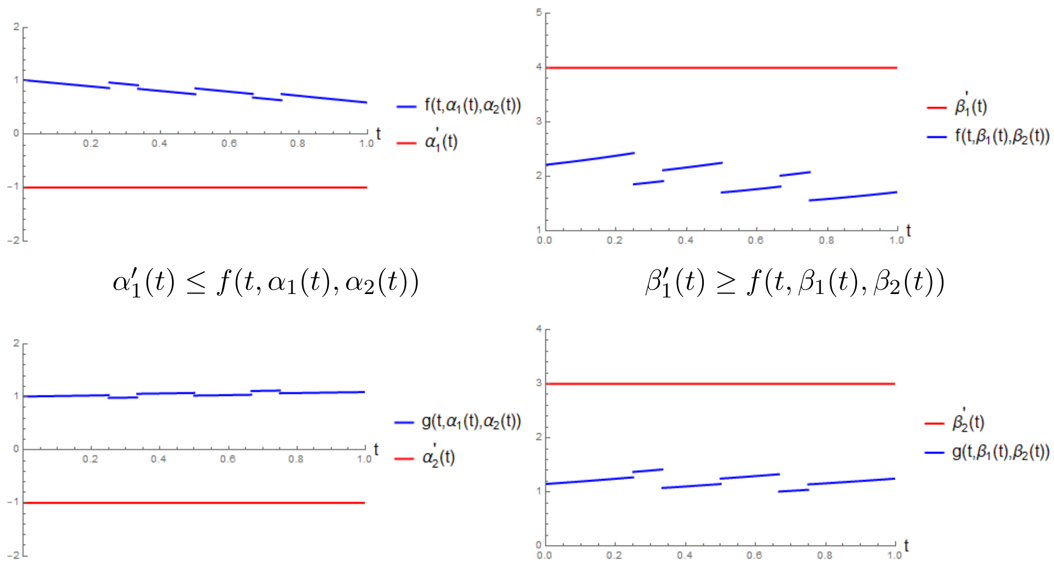
Figure 2.2: (x_1^*, x_2^*)-solution localization, in $[0, 1]$.

first-order ordinary differential systems. In the same fashion as in the previous Chapter, these upper and lower solutions, without any order requirements, are a useful tool to localize existing solutions to problem (2.1), (2.2), (2.3).

The first result, in Section 2.3, obligates the nonlinearities to change sign in the interval $[0, T]$. We overcome that restriction by requiring less regularity to the lower and upper functions in Section 2.4, but recovering a relation of order between them.

It is important to notice that the impulses are state-dependent, which is an attempt to approximate this model to the impulsive dynamics of real-case scenarios. Sudden variations in specific quantities of certain systems are most likely driven by the variation of all intervening variables. For this reason we found it wise to consider such complete dependence. The Impulse functions are, therefore, also subject to the truncated functions.

In the next chapter we study periodic second-order differential systems without impulses. Although the method used so far is suitable, we approach the existence issue using the Topological Degree Theory, for the sake of diversity of mathematical



$\alpha_1'(t) \leq f(t, \alpha_1(t), \alpha_2(t))$ $\beta_1'(t) \geq f(t, \beta_1(t), \beta_2(t))$

$\alpha_2'(t) \leq g(t, \alpha_1(t), \alpha_2(t))$ $\beta_2'(t) \geq g(t, \beta_1(t), \beta_2(t))$

Figure 2.3: Relation between the nonlinearities and the lower and upper solutions.

tools in this work.

Chapter 3

Second-order periodic systems

3.1 Introduction

We study the following second-order non-linear coupled system,

$$\begin{cases} z''(t) = f\left(t, z(t), w(t), z'(t), w'(t)\right) \\ w''(t) = g\left(t, z(t), w(t), z'(t), w'(t)\right) \end{cases}, \tag{3.1}$$

for $t \in [0, T]$, $T > 0$, with continuous functions $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$, and the periodic boundary conditions

$$\begin{array}{ll} z(0) = z(T), & z'(0) = z'(T), \\ w(0) = w(T), & w'(0) = w'(T). \end{array} \tag{3.2}$$

The solvability of second-order complete systems has been studied by several authors with different types of boundary conditions. As an example, we mention [59, 60, 61] for non-linear boundary conditions, and [62, 27, 25] for the periodic case.

The most common argument for obtaining periodic solutions is based on the assumptions of periodicity in the non-linearities. As an example, we refer to the work [25], where the following second-order differential equation is studied,

$$\chi''(t) + \Psi(\chi(t))\,\chi'(t) + \phi(t)\chi^m(t) - \frac{a(t)}{\chi^\mu(t)} + \frac{b(t)}{\chi^y(t)} = 0, \tag{3.3}$$

where $\Psi \in C((0, +\infty), \mathbb{R})$, ϕ, a and b are T-periodic, and in $L([0, T], \mathbb{R})$, m, μ and y are constants with $m \geq 0$ and $\mu \geq y > 0$. Here, asymptotic arguments are determinant to obtain periodic solutions.

The above problem, (3.3), requires periodicity for the non-linearities. In [27], the authors use the method of lower and upper solutions to study the second-order differential problem,

$$\begin{cases} \ddot{x}(t) = \Theta(t, x), \\ x(0) = x(T), \\ \dot{x}(0) = \dot{x}(T), \end{cases} \tag{3.4}$$

without requiring periodicity for the non-linearities. However, the function Θ does not depend on the first derivative, thus making problem (3.4) a particular case of problem (3.1), (3.2). Moreover, the obtained results require an order relation (for the well and inverse order cases) between lower and upper solutions.

To overcome the above constraints, and motivated by previous works existent in the literature, (see, for instance, [63, 44, 64, 65]), in this chapter we apply a methodology for finding periodic solutions in generalized second-order coupled systems. Sufficient conditions are given to prove the existence of at least a periodic solution for system (3.1), (3.2), without assuming periodicity in the non-linearities. Similarly to the previous chapters, using our technique of lower and upper solutions, the lack of order is overcome with adequate translations, thus the set of admissible functions for possible lower and upper solutions has a wider range.

The strategy employed consists of proving the existence of at least a periodic solution $(z(t), w(t))$ in a perturbed and truncated problem, applying the Topological Degree Theory. We ensure the control of the first derivatives with a variant of the Nagumo condition. Lastly, we localize existing solutions in a strip bounded by lower and upper functions.

This Chapter is organized as follows. Section 3.2 introduces all the required definitions and lemmas. In Section 3.3 the existence and localization result is formulated as the main theorem, together with the respective proof in four claims. A numerical example shows the applicability of the main theorem. In Section 3.4 we apply our methodology to the problem of two coupled Van der Pol oscillators with forcing terms. As all the requisites of the main theorem in Section 3.3 are verified, we suggest possible localizing functions for the existing periodic solutions of this problem. We provide some discussion of the results in Section 3.5.

3.2 Definitions

Consider the Banach space $X := C^1[0, T]$, equipped with the norm

$$\|u\|_X := \max\{\|u\|, \|u'\|\}, \qquad \|u\| := \max_{t\in[0,T]}\{|u(t)|\},$$

and the vectorial space $X^2 := (C^1[0, T])^2$, with the norm

$$\|(u, v)\|_{X^2} = \max\{\|u\|_X, \|v\|_X\}.$$

For a second-order problem like (1.1), (1.2), a Carathéodory function is defined by the following conditions:

Definition 3.2.1. *A function is $h : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ is L^1-Carathéodory if*

i) for each $(z_0, w_0, z_1, w_1) \in \mathbb{R}^4$, $t \mapsto h(t, z_0, w_0, z_1, w_1)$ is L^1-measurable in $[0, T]$;

ii) for a.e. $t \in [0, T]$, $(z_0, w_0, z_1, w_1) \mapsto h(t, z_0, w_0, z_1, w_1)$ is continuous in $\mathbb{R}^4$;

iii) for each $\rho > 0$, there exists a positive function $\phi_\rho \in L^1([0, T])$ such that, for $\max\{|z_0|, |w_0|, |z_1|, |w_1|\} < \rho$,

$$|h(t, z_0, w_0, z_1, w_1)| \leq \phi_\rho(t), \quad a.e.\ t \in [0, T].$$

In second-order differential equations, specially considering the third clause of Definition 3.2.1, we need to guarantee that the dependence on the first derivatives is controlled. We do so by applying a Nagumo-type condition, introduced for the first time in [66].

Definition 3.2.2. *Consider C^1 continuous functions $\gamma_i, \Gamma_i : [0, T] \to \mathbb{R}, i = 1, 2$, and the set*

$$S = \{(t, z_0, w_0, z_1, w_1) \in [0, T] \times \mathbb{R}^4 : \gamma_1(t) \leq z_0 \leq \Gamma_1(t), \gamma_2(t) \leq w_0 \leq \Gamma_2(t)\}.$$

The continuous functions $f, g : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(t), \Gamma_1(t)]$ and $[\gamma_2(t), \Gamma_2(t)]$, for all $t \in [0, T]$, if there exist continuous functions $\varphi, \psi : [0, +\infty[\to]0, +\infty[$ verifying

$$\int_0^{+\infty} \frac{ds}{\varphi(|s|)} = +\infty, \qquad \int_0^{+\infty} \frac{ds}{\psi(|s|)} = +\infty, \tag{3.5}$$

such that

$$\begin{aligned} |f(t, z_0, w_0, z_1, w_1)| &\leq \varphi(|z_1|), \quad \forall (t, z_0, w_0, z_1, w_1) \in S, \\ |g(t, z_0, w_0, z_1, w_1)| &\leq \psi(|w_1|), \quad \forall (t, z_0, w_0, z_1, w_1) \in S. \end{aligned} \tag{3.6}$$

In Definition 3.2.2, for each element of the set S, the variables z_0, w_0 are bounded by time-dependent functions $\gamma_i, \Gamma_i, i = \{1, 2\}$, in the interval $[0, T]$. While analysing the behaviour of the nonlinearities within the strip defined by such functions, *i.e.*, the intervals $[\gamma_1(t), \Gamma_1(t)]$ and $[\gamma_2(t), \Gamma_2(t)]$, it is important to guarantee that the dependence on the variables z_1, w_1 does not diverge therein.

The functions ϕ, ψ ensure such control. However, note that to guarantee the divergence of the integrals, the relations in equation (3.6) require ϕ, ψ to be linear. In Chapter 4 we provide another variant of the Nagumo condition that changes the nature of ϕ and ψ.

We provide *a priori* estimates for the first derivatives through the following lemma.

Lemma 3.2.3. *Suppose that the continuous functions $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(t), \Gamma_1(t)]$ and $[\gamma_2(t), \Gamma_2(t)]$, for all $t \in [0,T]$.*

Then, for every solution $(z(t), w(t)) \in X^2$ of (3.1), (3.2) verifying

$$\gamma_1(t) \le z(t) \le \Gamma_1(t), \quad \textit{and} \quad \gamma_2(t) \le w(t) \le \Gamma_2(t), \quad \forall t \in [0,T], \tag{3.7}$$

there are $N_1, N_2 > 0$ such that

$$\|z'\| \le N_1, \qquad \|w'\| \le N_2. \tag{3.8}$$

Proof. Let $(z(t), w(t))$ be a solution of (3.1), (3.2) verifying (3.7).

By (3.2) and by Rolle's Theorem, there exists $t_0 \in [0,T]$ such that $z'(t_0) = 0$.

Consider $N_i > 0, i = 1,2$ such that

$$\int_0^{N_1} \frac{\mathrm{d}s}{\varphi(|s|)} > T, \qquad \int_0^{N_2} \frac{\mathrm{d}s}{\psi(|s|)} > T, \tag{3.9}$$

and assume, without loss of generality, there exist $t_1, t_2 \in [0,T]$ such that $z'(t_1) \le 0$ and $z'(t_2) > 0$.

By continuity of $z'(t)$, there exists $t_3 \in [t_1, t_2]$ such that $z'(t_3) = 0$. Using a convenient change of variables, and by (3.1) and (3.6),

$$\int_{z'(t_3)}^{z'(t_2)} \frac{\mathrm{d}s}{\varphi(|s|)} = \int_{t_3}^{t_2} \frac{z''(t)}{\varphi(|z'(t)|)}\mathrm{d}t \le \int_0^T \frac{|z''(t)|}{\varphi(|z'(t)|)}\mathrm{d}t$$
$$= \int_0^T \frac{|f(t, z(t), w(t), z'(t), w'(t))|}{\varphi(|z'(t)|)}\mathrm{d}t \le \int_0^T \frac{\varphi(|z'(t)|)}{\varphi(|z'(t)|)}\mathrm{d}t = T.$$

By (3.9),

$$\int_{z'(t_3)}^{z'(t_2)} \frac{\mathrm{d}s}{\varphi(|s|)} = \int_0^{N_1} \frac{\mathrm{d}s}{\varphi(|s|)} \le T < \int_0^{N_1} \frac{\mathrm{d}s}{\varphi(|s|)},$$

and, therefore, $z'(t_2) < N_1$. As t_2 is chosen arbitrarily, then $z'(t) < N_1$, for all $t \in [0,T]$.

The case where $z'(t_1) > 0$ and $z'(t_2) \le 0$ follows similar arguments. Therefore, $\|z'\| \le N_1$, for all $t \in [0,T]$. Likewise, $\|w'\| \le N_2$, for all $t \in [0,T]$.

Remark that the constant N_1 depends only on γ_i, Γ_i, φ and T. In the same way, N_2 depends only on γ_i, Γ_i, ψ and T.

□

We present below the definition of upper and lower solutions.

Definition 3.2.4. *The pair of real functions* $(\alpha_1, \alpha_2) \in (C^1[0,T])^2$ *is a lower solution of the periodic problem (3.1), (3.2) if*

$$\begin{aligned} \alpha_1''(t) &\geq f(t, \alpha_1^0(t), \alpha_2^0(t), \alpha_1'(t), w_1), \forall w_1 \in \mathbb{R}, \\ \alpha_2''(t) &\geq g(t, \alpha_1^0(t), \alpha_2^0(t), z_1, \alpha_2'(t)), \forall z_1 \in \mathbb{R}, \end{aligned}$$

with

$$\alpha_i^0(t) := \alpha_i(t) - \|\alpha_i\|,$$

and

$$\alpha_i(0) = \alpha_i(T), \qquad \alpha_i'(0) \geq \alpha_i'(T).$$

The pair of real functions $(\beta_1, \beta_2) \in (C^1[0,T])^2$ *is an upper solution of the periodic problem (3.1), (3.2) if*

$$\begin{aligned} \beta_1''(t) &\leq f(t, \beta_1^0(t), \beta_2^0(t), \beta_1'(t), w_1), \forall w_1 \in \mathbb{R}, \\ \beta_2''(t) &\leq g(t, \beta_1^0(t), \beta_2^0(t), z_1, \beta_2'(t)), \forall z_1 \in \mathbb{R}, \end{aligned} \tag{3.10}$$

with

$$\beta_i^0(t) := \beta_i(t) + \|\beta_i\|, \tag{3.11}$$

and

$$\beta_i(0) = \beta_i(T), \qquad \beta_i'(0) \leq \beta_i'(T). \tag{3.12}$$

3.3 Main result

In a similar fashion to the previous chapters, the following theorem establishes the conditions for the existence and localization of periodic solutions of problem (3.1), (3.2), using the method of lower and upper solutions. However, there needs to be aditional conditions regarding the first derivatives. According to Lemma 3.2.3, and the definition of norm, the quantities z', w' are bounded in modulo by positive estimates N_1 and N_2, respectively, which correspond to the maximum and minimum values in the interval $[0,T]$.

Theorem 3.3.1. *Let* $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ *be continuous functions. Suppose* (α_1, α_2), (β_1, β_2) *are lower and upper solutions of problem (3.1), (3.2), as in Definition 3.2.4.*

Suppose f, g *satisfy a Nagumo-type condition, according to Definition 3.2.2 relative to the intervals* $[\alpha_1^0(t), \beta_1^0(t)]$ *and* $[\alpha_2^0(t), \beta_2^0(t)]$*, for all* $t \in [0,T]$*, with*

$$f(t, z_0, w_0, z_1, w_1) \text{ non-increasing in } w_0, \text{ for } t \in [0,T], z_0 \in \mathbb{R} \text{ fixed,} \tag{3.13}$$

$$\min\left\{\min_{t\in[0,T]}\alpha_1'(t),\min_{t\in[0,T]}\beta_1'(t)\right\}\le z_1\le\max\left\{\max_{t\in[0,T]}\alpha_1'(t),\max_{t\in[0,T]}\beta_1'(t)\right\},$$

and

$$g(t,z_0,w_0,z_1,w_1)\ \textit{non-increasing in } z_0,\ \textit{for } t\in[0,T], w_0\in\mathbb{R}\ \textit{fixed},$$

$$\min\left\{\min_{t\in[0,T]}\alpha_2'(t),\min_{t\in[0,T]}\beta_2'(t)\right\}\le w_1\le\max\left\{\max_{t\in[0,T]}\alpha_2'(t),\max_{t\in[0,T]}\beta_2'(t)\right\}.$$

Then, there exists at least one pair $(z(t),w(t))\in(C^2[0,T])^2$, solution of problem (3.1), (3.2) and, moreover,

$$\alpha_1^0(t)\le z(t)\le\beta_1^0(t),\qquad \alpha_1^0(t)\le w(t)\le\beta_1^0(t),\qquad \forall t\in[0,T].$$

We now present a proof of Theorem 3.3.1 in four claims. The first three claims are dedicated to proving the existence of a periodic solution of problem (3.1), (3.2), using the topological degree theory.

We start by considering a modified version of the original problem, with truncated functions and a homotopy perturbation. The first claim shows that every pair $(z(t),w(t))$ that is a solution of the modified problem is bounded by a positive integer, independently of the homotopy parameters. In the second claim, the same reasoning is applied to the first derivatives. The first two claims set the problem to the conditions of the topological degree theory, which is applied in the third claim. By homotopy invariance, we guarantee the existence of a periodic solution to the modified problem.

The fourth claim is dedicated to the localization of an existing solution, using the method of lower and upper solutions. The equivalence between the original and modified problems is shown by proving that only the main branch of the truncated functions does not lead to a contradiction.

Proof. Define the truncated functions $\delta_i:[0,T]\times\mathbb{R}\to\mathbb{R}$,

$$\delta_1(t,z)=\begin{cases}\beta_1^0(t), & z>\beta_1^0(t)\\ z, & \alpha_1^0(t)\le z\le\beta_1^0(t)\\ \alpha_1^0(t), & z<\alpha_1^0(t)\end{cases},\qquad \delta_2(t,w)=\begin{cases}\beta_2^0(t), & w>\beta_2^0(t)\\ w, & \alpha_2^0(t)\le w\le\beta_2^0(t)\\ \alpha_2^0(t), & w<\alpha_2^0(t)\end{cases}.$$

For $\lambda,\mu\in[0,1]$, consider the truncated, perturbed and homotopic auxiliary problem

$$\begin{cases}z''(t)-z(t)=\lambda\, f(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t))-\lambda\,\delta_1(t,z(t))\\ w''(t)-w(t)=\mu\, g(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t))-\mu\,\delta_2(t,w(t))\end{cases},\tag{3.14}$$

for all $t\in[0,T]$, with the boundary conditions (3.2).

Consider $r_i > 0, i = 1, 2$, such that, for all $\lambda, \mu \in [0, 1]$, every $t \in [0, T]$, and every $N_1^*, N_2^* > 0$ given by (3.8),

$$-r_i < \alpha_i^0(t) \le \beta_i^0(t) < r_i, \tag{3.15}$$

with

$$\begin{array}{ll} \lambda\, f(t, \beta_1^0(t), \beta_2^0(t), 0, w'(t)) - \beta_1^0(t) + r_1 > 0, & \text{for } \|w'\| \le N_2^*, \\ \mu\, g(t, \beta_1^0(t), \beta_2^0(t), z'(t), 0) - \beta_2^0(t) + r_2 > 0, & \text{for } \|z'\| \le N_1^*, \\ \lambda\, f(t, \alpha_1^0(t), \alpha_2^0(t), 0, w'(t)) - \alpha_1^0(t) - r_1 < 0, & \text{for } \|w'\| \le N_2^*, \\ \mu\, g(t, \alpha_1^0(t), \alpha_2^0(t), z'(t), 0) - \alpha_2^0(t) - r_2 < 0, & \text{for } \|z'\| \le N_1^*. \end{array} \tag{3.16}$$

Claim 1. *Every solution of (3.14), (3.2) verifies $|z(t)| < r_1$ and $|w(t)| < r_2$, for all $t \in [0, T]$, independently of $\lambda, \mu \in [0, 1]$.*

Assume, by contradiction, that there exist $\lambda \in [0, 1]$, a pair $(z(t), w(t))$, solution of problem (3.14), (3.2), and $t \in [0, T]$ such that $|z(t)| \ge r_1$. If $z(t) \ge r_1$, define

$$\max_{t \in [0,T]} z(t) := z(t_0).$$

If $t_0 \in]0, T[$ and $\lambda \in]0, 1]$, then $z'(t_0) = 0$ and $z''(t_0) \le 0$. By (3.15), (3.13) and (3.16), the following contradiction holds:

$$\begin{aligned} 0 \ge z''(t_0) &= \lambda\, f(t_0, \delta_1(t_0, z(t_0)), \delta_2(t_0, w(t_0)), z'(t_0), w'(t_0)) - \lambda\, \delta_1(t_0, z(t_0)) + z(t_0) \\ &= \lambda\, f(t_0, \beta_1^0(t_0), \delta_2(t_0, w(t_0)), 0, w'(t_0)) - \lambda\, \beta_1^0(t_0) + z(t_0) \\ &\ge \lambda\, f(t_0, \beta_1^0(t_0), \delta_2(t_0, w(t_0)), 0, w'(t_0)) - \beta_1^0(t_0) + z(t_0) \\ &\ge \lambda\, f(t_0, \beta_1^0(t_0), \delta_2(t_0, w(t_0)), 0, w'(t_0)) - \beta_1^0(t_0) + r_1 > 0 \end{aligned}$$

If $t_0 = 0$ or $t_0 = T$, then, by (3.2),

$$0 \ge z'(0) = z'(T) \ge 0.$$

So, $z'(0) = z'(T) = 0$, $z''(0) \le 0$ and $z''(T) \le 0$. Therefore, the arguments follow the previous case.

If $\lambda = 0$, we obtain the contradiction

$$0 \ge z''(t_0) = z(t_0) \ge r_1 > 0.$$

Then, $z(t) < r_1, \forall t \in [0, T]$, regardless of λ.

The same arguments can be made to prove that $z(t) > -r_1$. Therefore, $|z(t)| < r_1$, for $t \in [0, T]$, independently of λ.

Likewise, $|w(t)| < r_2$, for $t \in [0,T]$, independently of μ.

Claim 2. *For every solution $(z(t), w(t))$ of (3.14), (3.2), there are $N_1^*, N_2^* > 0$ such that $|z'(t)| < N_1^*$ and $|w'(t)| < N_2^*$, $\forall t \in [0,T]$, independently of $\lambda, \mu \in [0,1]$.*

Define the continuous functions

$$F_\lambda(t, z_0, w_0, z_1, w_1)) := \lambda\, f(t, \delta_1(t, z_0), \delta_2(t, w_0), z_1, w_1) - \lambda\, \delta_1(t, z_0) + z_0,$$
$$G_\mu(t, z_0, w_0, z_1, w_1) := \mu\, f(t, \delta_1(t, z_0), \delta_2(t, w_0), z_1, w_1) - \mu\, \delta_2(t, w_0) + w_0,$$

and, as $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\alpha_1^0(t), \beta_1^0(t)]$ and $[\alpha_2^0(t), \beta_2^0(t)]$, then,

$$\begin{aligned}
&|F_\lambda(t, z(t), w(t), z'(t), w'(t))| \\
&\qquad \le |f(t, \delta_1(t, z(t)), \delta_2(t, w(t)), z'(t), w'(t))| + |\delta_1(t, z(t))| + |z(t)| \\
&\qquad < \varphi(|z'|) + 2r_1, \\
&|G_\mu(t, z(t), w(t), z'(t), w'(t))| \\
&\qquad \le |f(t, \delta_1(t, z(t)), \delta_2(t, w(t)), z'(t), w'(t))| + |\delta_2(t, w(t))| + |w(t)| \\
&\qquad < \psi(|w'|) + 2r_2.
\end{aligned}$$

Therefore, for continuous positive functions $\varphi^*, \psi^* : [0, +\infty[\to]0, +\infty[$ given by

$$\varphi^*(|z'|) := \varphi(|z'|) + 2r_1, \qquad \psi^*(|w'|) := \psi(|w'|) + 2r_2,$$

F_λ and G_μ satisfy a Nagumo-type condition in

$$E = \{(t, z, w, z', w') \in [0,T] \times \mathbb{R}^4 : |z| < r_1, |w| < r_2\},$$

as, by (3.5), we have

$$\int_0^{+\infty} \frac{\mathrm{d}s}{\varphi^*(|z'|)} = \int_0^{+\infty} \frac{\mathrm{d}s}{\varphi(|z'|) + 2r_1} = +\infty,$$
$$\int_0^{+\infty} \frac{\mathrm{d}s}{\psi^*(|w'|)} = \int_0^{+\infty} \frac{\mathrm{d}s}{\psi(|w'|) + 2r_2} = +\infty.$$

Therefore, by Lemma 3.2.3, there are $N_1^*, N_2^* > 0$ such that $|z'(t)| < N_1^*$ and $|w'(t)| < N_2^*$, $\forall t \in [0,T]$, independently of $\lambda, \mu \in [0,1]$.

Claim 3. *Problem (3.14), (3.2), for $\lambda = \mu = 1$, has at least one solution $(z(t), w(t))$.*

Define the operators

$$\mathcal{L} : (C^2[0,T])^2 \subset (C^1[0,T])^2 \to (C[0,T])^2 \times \mathbb{R}^4,$$

given by

$$\mathcal{L}(z,w) := (z''(t)-z(t), w''(t)-w(t), z(0), w(0), z'(0), w'(0)),$$

and

$$\mathcal{N}_{(\lambda,\mu)} : (C^1[0,T])^2 \to (C[0,T])^2 \times \mathbb{R}^4,$$

given by

$$\mathcal{N}_{(\lambda,\mu)}(z,w) := (Z_\lambda(t), W_\mu(t), z(T), w(T), z'(T), w'(T)),$$

where

$$\begin{aligned} Z_\lambda(t) &:= \lambda\, f(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t)) - \lambda\,\delta_1(t,z(t)),\\ W_\mu(t) &:= \mu\, f(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t)) - \mu\,\delta_2(t,w(t)). \end{aligned}$$

We define the continuous operator

$$\mathcal{T} : (C^1[0,T])^2 \to (C^1[0,T])^2,$$

given by

$$\mathcal{T}_{(\lambda,\mu)}(z,w) := \mathcal{L}^{-1}\mathcal{N}_{(\lambda,\mu)}(z,w).$$

As the equation $(z,w) = \mathcal{T}_{(0,0)}(z,w)$, that is, the homogeneous system

$$\begin{cases} z''(t) - z(t) = 0 \\ w''(t) - w(t) = 0, \end{cases}$$

with the periodic conditions (3.2), admits only the null solution, then $\mathcal{L}^{-1}$ is well-defined.

For $M := \max\{r_1, r_2, N_1^*, N_2^*\}$, consider the set

$$\Omega = \{(z,w) \in (C^1([0,T]))^2 : \|(z,w)\|_{X^2} < M\}.$$

By Claims 1 and 2, for all $\lambda,\mu \in [0,1]$, the degree $d(I - \mathcal{T}_{(0,0)}, \Omega, 0)$ is well defined and, by homotopy invariance,

$$d(I - \mathcal{T}_{(0,0)}, \Omega, 0) = d(I - \mathcal{T}_{(1,1)}, \Omega, 0), \tag{3.17}$$

and, by degree theory,

$$d(I - \mathcal{T}_{(0,0)}, \Omega, 0) = \pm 1.$$

By (3.17), $d(I - \mathcal{T}_{(1,1)}, \Omega, 0) = \pm 1$. Therefore, the equation $(z,w) = \mathcal{T}_{(1,1)}(z,w)$ has at least one solution, that is, the auxiliary problem with $\lambda = \mu = 1$,

$$\begin{cases} z''(t) - z(t) = f(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t)) - \delta_1(t,z(t)) \\ w''(t) - w(t) = g(t,\delta_1(t,z(t)),\delta_2(t,w(t)),z'(t),w'(t)) - \delta_2(t,w(t)), \end{cases}$$

together with the boundary conditions (3.2), has at least one solution $(z_*(t), w_*(t))$.

Claim 4. *The pair $(z_*(t), w_*(t)) \in X^2$, solution of the auxiliary problem (3.14), (3.2), for $\lambda = \mu = 1$, is also a solution of the original problem (3.1), (3.2).*

This Claim is proven if the solution $(z_*(t), w_*(t))$ satisfies

$$\alpha_1^0(t) \le z_*(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w_*(t) \le \beta_2^0(t), \quad \forall t \in [0,T]. \tag{3.18}$$

Suppose, by contradiction, that there is $t \in [0,T]$ such that $z_*(t) > \beta_1^0(t)$, and define

$$\max_{t\in[0,T]} \{z_*(t) - \beta_1^0(t)\} := z_*(t_0) - \beta_1^0(t_0) > 0. \tag{3.19}$$

If $t_0 \in]0,T[$, then $(z_* - \beta_1^0)'(t_0) = 0$ and $(z_* - \beta_1^0)''(t_0) \le 0$. Then, by (3.14), (3.19), (3.13) and (3.10), the following contradiction holds,

$$\begin{aligned}
0 &\ge z_*''(t_0) - (\beta_1^0)''(t_0) \\
&= f(t_0, \delta_1(t_0, z_*(t_0)), \delta_2(t_0, w(t_0)), z_*'(t_0), w'(t_0)) - \delta_1(t_0, z_*(t_0)) + z_*(t_0) - \beta_1''(t_0) \\
&= f(t_0, \beta_1^0(t_0), \delta_2(t_0, w(t_0)), \beta_1'(t_0), w'(t_0)) - \beta_1^0(t_0) + z_*(t_0) - \beta_1''(t_0) \\
&> f(t_0, \beta_1^0(t_0), \delta_2(t_0, w(t_0)), \beta_1'(t_0), w'(t_0)) - \beta_1''(t_0) \\
&\ge f(t_0, \beta_1^0(t_0), \beta_2^0(t_0), \beta_1'(t_0), w'(t_0)) - \beta_1''(t_0) \ge 0.
\end{aligned}$$

If $t_0 = 0$ or $t_0 = T$, then $(z_* - \beta_1^0)'(0) \le 0$ and $(z_* - \beta_1^0)'(T) \ge 0$. By (3.11), (3.2) and (3.12),

$$\begin{aligned}
0 \ge (z_* - \beta_1^0)'(0) &= (z_* - \beta_1)'(0) = z_*'(T) - \beta_1'(0) \\
&\ge z_*'(T) - \beta_1'(T) = (z_* - \beta_1^0)'(T) \ge 0,
\end{aligned}$$

so,

$$(z_* - \beta_1^0)'(0) = (z_* - \beta_1^0)'(T) = 0,$$

and $(z_* - \beta_1^0)''(t_0) \le 0$. Therefore, we can apply the previous arguments to obtain a similar contradiction, and so, $z_*(t) \le \beta_1^0(t)$.

Similar arguments can be used to prove the other inequalities in (3.18). □

Example 3.3.2. *Consider the following system, for $t \in [0,1]$,*

$$\begin{cases} z''(t) = 2z^3(t) - w(t) + 3z'(t) - \dfrac{2}{1 + (w'(t))^2} - 10t, \\ w''(t) = -z(t) + 10w^3(t) - e^{-(z'(t))^2} - 3w'(t) - 12t, \end{cases} \tag{3.20}$$

together with the periodic boundary conditions (3.2).

The functions $\alpha_i, \beta_i : [0,1] \to \mathbb{R}$, $i = 1, 2$, *given by*

$$\alpha_1(t) = 1 + 2t^2 - 2t^3, \qquad \beta_1(t) = 6/5 - 2t^2 + 2t^3,$$
$$\alpha_2(t) = 2t - 2t^2, \qquad \beta_2(t) = 1 - 3t + 3t^2,$$

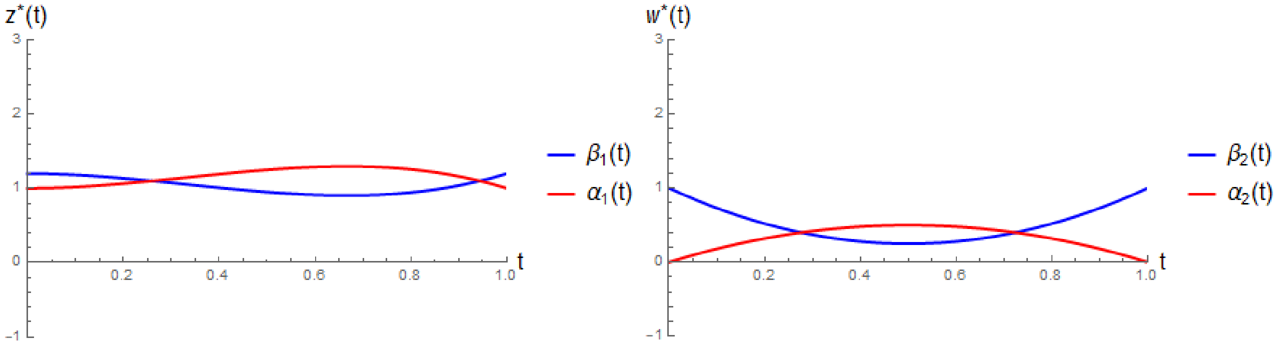


Figure 3.1: Orderless α_i, β_i functions, with $i = 1, 2$.

are, respectively, lower and upper solutions of problem (3.20), (3.2), according to Definition 3.2.4, with

$$\alpha_1^0(t) = -8/27 + 2t^2 - 2t^3, \qquad \beta_1^0(t) = 12/5 - 2t^2 + 2t^3,$$
$$\alpha_2^0(t) = -1/2 + 2t - 2t^2, \qquad \beta_2^0(t) = 2 - 3t + 3t^2.$$

Remark that the lower and upper functions, shown in Fig. 3.1, are not ordered, as it is usual in the literature.

The above problem is a particular case of (3.1), (3.2), with $T = 1$, *and where the non-linearities, according to Definition 3.2.2, satisfy a Nagumo-type condition in the set*

$$\widetilde{S} = \left\{ \begin{array}{l} (t, z_0, w_0, z_1, w_1) \in [0,1] \times \mathbb{R}^4 : \\ \qquad -8/27 + 2t^2 - 2t^3 \leq z_0 \leq 12/5 - 2t^2 + 2t^3, \\ \qquad -1/2 + 2t - 2t^2 \leq w_0 \leq 2 - 3t + 3t^2 \end{array} \right\},$$

such that

$$\begin{aligned}
|f(t,z,w,z',w')| &\leq 2\,|z|^3 + w + 3\,|z'| + \left|\frac{2}{1+(w')^2}\right| + 10\,|t| \\
&\leq 2\times\frac{12}{5} + 2 + 3\,|z'| + 2 + 10 \\
&= \frac{94}{5} + 3\,|z'| := \widetilde{\varphi}(|z'|), \\
|g(t,z,w,z',w')| &\leq |z| + 10\,|w|^3 + \left|e^{(z')^2}\right| + 3\,|w'| + 12\,|t| \\
&\leq \frac{12}{5} + 10\times 2^3 + 1 + 3\,|w'| + 12 \\
&= \frac{477}{5} + 3\,|w'| := \widetilde{\psi}(|w'|).
\end{aligned}$$

It is clear that functions $\widetilde{\varphi}$ and $\widetilde{\psi}$ satisfy (3.5).

As the assumptions of Theorem 3.3.1 are verified, then the system (3.20), (3.2) has, at least, a solution $(z^*(t), w^*(t)) \in (C^2[0,1])^2$ such that

$$\begin{aligned}
-8/27 + 2t^2 - 2t^3 \leq z^*(t) \leq 12/5 - 2t^2 + 2t^3, \\
-1/2 + 2t - 2t^2 \leq w^*(t) \leq 2 - 3t + 3t^2, \quad \forall t \in [0,1],
\end{aligned}$$

as shown in Fig. 3.2.

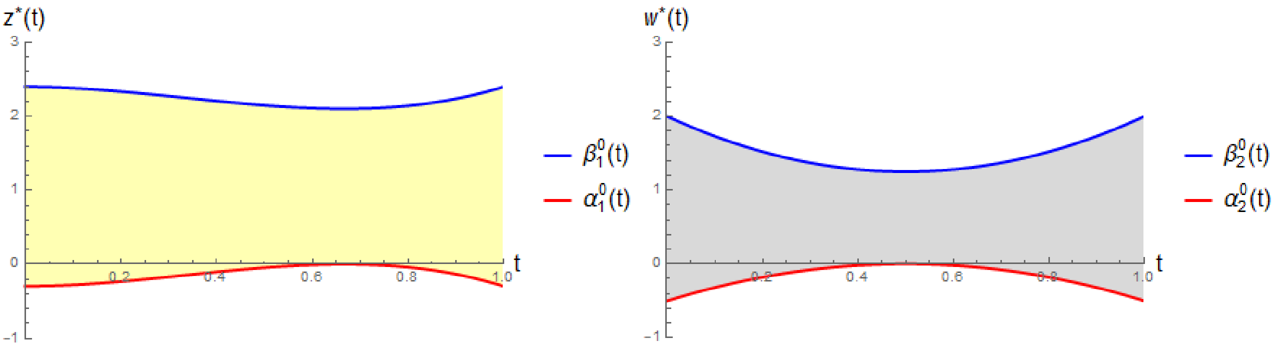


Figure 3.2: Shifted functions, $\alpha_i^0, \beta_i^0, i = 1,2$, localizing the solution pair $(z^*(t), w^*(t))$.

3.4 Coupled forced Van der Pol oscillators

The equation for the damped harmonic motion is

$$x''(t) + \mu x'(t) + x(t) = 0.$$

Balthazar Van der Pol (1889–1959) modified the damped harmonic oscillator by considering a negative quadratic term for the friction term to obtain self-sustained oscillations. This modification resulted in the Van der Pol oscillator [67, 68], which can be represented by the following equation,

$$x''(t) - \epsilon(1 - x^2(t))x'(t) + x(t) = 0,$$

where $x(t)$ is the time-dependent variable and $-\epsilon(1-x^2(t))$ is the non-linear damping term, with $\epsilon > 0$.

A variant of this problem can be thought of, by coupling two Van der Pol oscillators. We consider the following system, for $t \in [0, T]$,

$$\begin{cases} z''(t) = z'(t)(A_1 - B_1 z^2(t)) - C_1 z(t) + D_1 \tanh(E_1 z(t) - F_1 w(t)) + G_1 \cos(t) \\ w''(t) = w'(t)(A_2 - B_2 w^2(t)) - C_2 w(t) + D_2 \arctan(E_2 w(t) - F_2 z(t)) + G_2 \cos(t) \end{cases} \tag{3.21}$$

with $A_i, B_i, C_i, D_i, F_i > 0$ and $E_i, G_i \in \mathbb{R}$, $i = 1, 2$, together with the periodic boundary conditions (3.2).

The terms $D_1 \tanh(E_1 z(t) - F_1 w(t))$ and $D_2 \arctan(E_2 w(t) - F_2 z(t))$ correspond, respectively, to each coupling and $G_i \cos(t)$ is a time-dependent periodic forcing. We chose $T = 1$ and the parameter set

$$\begin{array}{lllllll} A_1 = 1 & B_1 = 0.5 & C_1 = 1 & D_1 = 6 & E_1 = 3 & F_1 = 2 & G_1 = 1, \\ A_2 = 1 & B_2 = 0.5 & C_2 = 1 & D_2 = 8 & E_2 = 2 & F_2 = 1 & G_2 = -1. \end{array}$$

The functions $\alpha_i, \beta_i : [0, 1] \to \mathbb{R}$, $i = 1, 2$, given by

$$\begin{array}{ll} \alpha_1(t) = -1 + t - t^2, & \beta_1(t) = 1 - t^2/2 + t^3/2, \\ \alpha_2(t) = -3/4 + t - t^2, & \beta_2(t) = 1 - t^2 + t^3, \end{array}$$

are, respectively, lower and upper solutions of problem (3.21), (3.2), with the numerical values (3.4) according to Definition 3.2.4, with

$$\begin{array}{ll} \alpha_1^0(t) = -5/4 + t - t^2, & \beta_1^0(t) = 2 - t^2/2 + t^3/2, \\ \alpha_2^0(t) = -1 + t - t^2, & \beta_2^0(t) = 2 - t^2 + t^3. \end{array}$$

The functions f and g verify the monotonicity requirements and the Nagumo-type condition of Definition 3.2.2 in the set

$$S^* = \left\{ \begin{array}{l} (t, z_0, w_0, z_1, w_1) \in [0, 1] \times \mathbb{R}^4 : \\ \qquad -5/4 + t - t^2 \le z_0 \le 2 - t^2/2 + t^3/2, \\ \qquad -1 + t - t^2 \le w_0 \le 2 - t^2 + t^3 \end{array} \right\},$$

with

$$f(t, z_0, w_0, z_1, w_1) \leq |z_1| \left(1 + 0.5|z_0|^2\right) + |z_0| + 6|\tanh(3z_0 - 2w_0)| + |\cos(t)| \\ \leq 3|z_1| + 9 := \varphi^*(|z_1|),$$
$$g(t, z_0, w_0, z_1, w_1) \leq |w_1| \left(1 + 0.5|w_0|^2\right) + |w_0| + 8|\arctan(2w_0 - z_0)| + |\cos(t)| \\ \leq 3|w_1| + 3 + 4\pi := \psi^*(|w_1|).$$

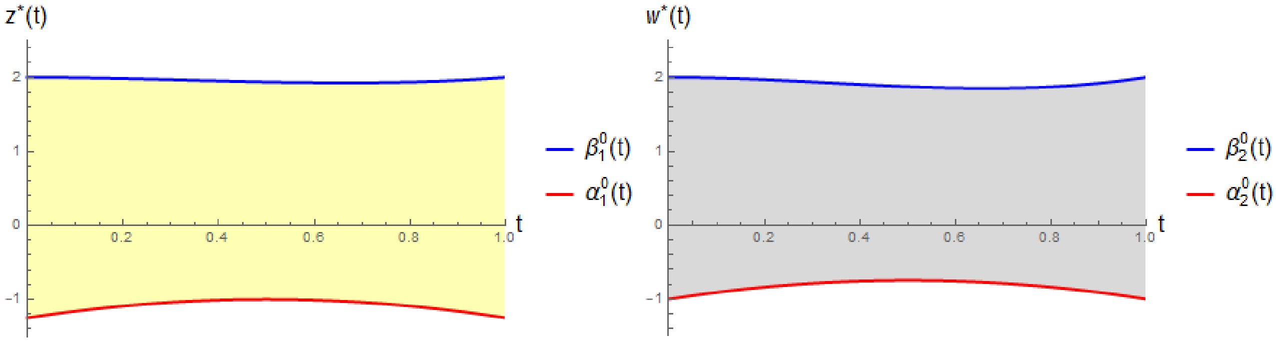


Figure 3.3: Shifted functions, $\alpha_i^0, \beta_i^0, i = 1, 2$, localizing the solution pair $(z^*(t), w^*(t))$.

As all the assumptions of Theorem 3.3.1 are verified, then there is at least a periodic solution for the system of two coupled forced Van der Pol oscillators (3.21) with the periodic boundary conditions (3.2), and the parameter set of values (3.4). Moreover, remark that, from (3.21) and (3.4), this solution is a non-trivial one.

Figure 3.3 shows the shifted functions that localize existing periodic solutions in the system (3.21), (3.2), with parameter values (3.4).

3.5 Discussion

In contrast to the previous chapters, this chapter is dedicated to the study of a general second-order differential coupled system, with dependencies on both variables, z, w, their derivatives, z', w', and on time, t.

Sufficient conditions are presented to prove the existence of periodic solutions for problem (3.1), (3.2) without assuming any type of periodicity in the non-linearities. This is achieved by constructing the adequate arguments to apply the topological degree theory in the first three claims. Similarly to the previous chapters, a modified problem is considered, by applying a perturbation to the original problem. This is motivated by the fact that the homogeneous original problem admits constant solutions, whereas the kernel of the perturbed problem has only the zero vector.

The existence argument stated by the topological degree theory requires that the dependencies of the nonlinearities are bounded by some positive real numbers, independently of the homotopy parameters. This independence allows to establish an equivalence between the homotopy perturbed problem and its homogeneous version. The latter is proved to have a periodic solution. By homotopy invariance, so has the homotopic problem.

The fourth claim is dedicated to the localization of an existing periodic solution, using the method of upper and lower solutions without order requirements, according to Definition 3.2.4. Visualizations of this method are provided by Example 3.3.2 and an application to a system of coupled forced Van der Pol oscillators, in Section 3.4.

When applying Theorem 3.3.1 to a specific second-order system, one must verify that the non-linearities satisfy the Nagumo-type condition of Definition 3.2.2, as well as the required monotonicities. Note that Theorem 3.3.1 does not guarantee the existence of a non-trivial solution. That guarantee arises with the nature of the system in study, *i.e.*, whether if it allows for constant solutions or not.

Since a homotopy is a continuous deformation between two continuous functions, by using the topological degree theory the nonlinearities are required to be continuous. For this reason, exceptionally, this chapter deals with a system that cannot relate to the definition of a Carathéodory function. In the next chapter we study an impulsive second-order differential system, therefore, we do not use the topological degree theory, but rather recover a similar strategy to that presented in the previous chapters, although with different techniques.

Chapter 4

Second-order periodic systems with impulses

4.1 Introduction

We study the following second-order non-linear coupled system,

$$\begin{cases} z''(t) = f\left(t, z(t), w(t), z'(t), w'(t)\right) \\ w''(t) = g\left(t, z(t), w(t), z'(t), w'(t)\right) \end{cases}, \tag{4.1}$$

a.e. t in $[0,T], T > 0$, and the $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ L^1-Carathéodory functions, with the periodic boundary conditions

$$\begin{array}{ll} z(0) = z(T), & z'(0) = z'(T), \\ w(0) = w(T), & w'(0) = w'(T), \end{array} \tag{4.2}$$

subject the impulse conditions, given by the generalized functions

$$\begin{array}{ll} \Delta z(t_k) = I_k(t_k, z(t_k), w(t_k), z'(t_k)), & k = 1, ..., n-1, n > 2, \\ \Delta w(\tau_l) = J_l(\tau_l, z(\tau_l), w(\tau_l), w'(\tau_l)), & l = 1, ..., m-1, m > 2, \end{array} \tag{4.3}$$

with $I_k, J_l \in C([0,T] \times \mathbb{R}^3, \mathbb{R})$, with jumps given by

$$\begin{array}{c} \Delta z(t_k) = z(t_k^+) - z(t_k^-), \\ \Delta w(\tau_l) = w(\tau_l^+) - w(\tau_l^-), \end{array}$$

and the time instants t_k, τ_l such that $0 = t_0 < t_1 < ... < t_{n-1} < t_n = T$ and $0 = \tau_0 < \tau_1 < ... < \tau_{m-1} < \tau_m = T$, with the left and right limits defined as

$$u(t_k^{\pm}) = \lim_{t \to t_k^{\pm}} u(t).$$

There are several cases where the dynamics of natural phenomena is impulse-driven and can be modelled by second-order differential systems [16, 20, 21]. In such systems, finding periodic solutions is of special importance, but also a difficult challenge.

In this chapter we present a result for the existence and localization of periodic solutions in impulsive second-order nonlinear systems, using a variant of the method of upper and lower solutions, [37]. The greatest advantage of our approach is that there are no periodicity requirements for the nonlinearities and no order requirements for the upper and lower solutions.

Different techniques have been employed to find periodic solutions in second order impulsive problems. For example, in [30], the authors establish the conditions for the existence of a positive periodic solution in a family of scalar periodic delay differential equations with linear impulses. In another example, [69], the existence of periodic solutions for a class of second order nonlinear random impulse differential equations is explored, using the theory of topological degree, the coincidence degree, and Mawhin's continuity theorem.

In [36], the authors explore the existence of positive periodic solutions in the system

$$\begin{cases} -u''(t)+\rho^2u(t)=f(t,u(t),u'(t)), & t\in J', \\ -\Delta u(t)|_{t=t_k}=I_k(u(t_k)), & k=1,2,...,m, \\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi), \end{cases} \tag{4.4}$$

where $f:[0,2\pi]\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is continuous, $\mathbb{R}^+=[0,+\infty[$, $J=[0,2\pi]$, $\rho>0$, $J'=J\backslash\{t_1,t_2,...,t_m\}$. The existence result is guaranteed by some inequality conditions on f and the spectral radius condition of linear operator, by the theory of fixed point index.

In [35] the authors study a system of nonlinear second-order and impulsive differential equations with periodic boundaries

$$\begin{cases} u''(t)-\lambda^2u(t)=-f_1(t,u(t),v(t),\theta), & t\in\mathcal{J}=[0,2\pi], t\neq t_k, k\in[1,m]_{\mathbb{N}}, \\ v''(t)-\lambda^2v(t)=-f_2(t,u(t),v(t),\theta), & t\in\mathcal{J}, t\neq t_k, k\in[1,m]_{\mathbb{N}}, \\ u(t_k^+)-u(t_k^-)=I_k^1(u(t_k^-),v(t_k^-)), & k\in[1,m]_{\mathbb{N}}, \\ v(t_k^+)-v(t_k^-)=I_k^2(u(t_k^-),v(t_k^-)), & k\in[1,m]_{\mathbb{N}}, \\ u'(t_k^+)-u'(t_k^-)=\bar{I}_k^1(u(t_k^-),v(t_k^-)), & k\in[1,m]_{\mathbb{N}}, \\ v'(t_k^+)-v'(t_k^-)=\bar{I}_k^2(u(t_k^-),v(t_k^-)), & k\in[1,m]_{\mathbb{N}}, \\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi), \\ v(0)=v(2\pi), \quad v'(0)=v'(2\pi), \end{cases} \tag{4.5}$$

where $[n,m]_{\mathbb{N}} = \{n, n+1, ..., m\}$, for all $n,m \in \mathbb{N}, \lambda \in \mathbb{R}^*$, and θ is a real parameter, $f_1, f_2 \in C^0(\mathcal{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are given functions, $I_k^i, \bar{I}_k^i \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $t_k \in \mathcal{J}, 0 = t_0 < t_1 < ... < t_m < t_{m+1} = 2\pi$, and $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h), u(t_k^-) = \lim_{h \to 0^+} u(t_k - h)$ are the right and left limits of u at t_k. The authors provide sufficient conditions for the existence of solutions, using Schaefer's fixed point theorem.

In [34] the authors consider the existence and uniqueness of the solution for linear nonhomogeneous T-periodic systems

$$u''(t) = Au(t) + g(t), \quad t \geq 0, \quad t \neq t_i, \quad i \in \mathbb{N}, \tag{4.6}$$

and semilinear T-periodic systems

$$u''(t) = Au(t) + f(t, u(t)), \quad t \geq 0, \quad t \neq t_i, \quad i \in \mathbb{N}, \tag{4.7}$$

with impulsive conditions

$$\begin{aligned} u(t_i^+) - u(t_i^-) &= B_1 u(t_i^-), \quad i \in \mathbb{N}, \\ u'(t_i^+) - u'(t_i^-) &= B_2 u'(t_i^-), \quad i \in \mathbb{N}, \end{aligned} \tag{4.8}$$

and boundary conditions

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{4.9}$$

where $A, B_1, B_2 \in \mathbb{R}^{n \times n}$, and satisfy $AB_1 = B_1 A, AB_2 = B_2 A, B_1 B_2 = B_2 B_1$; $g : \mathbb{R} \to \mathbb{R}^n$ is continuous and T-periodic, $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and T-periodic with respect to t. $0 = t_0 < t_1 < ... < t_k < ...$; and t_i satisfies $t_{i+m+1} = t_i + T$ for all $i \in \mathbb{Z}$ and some natural m.

Systems (4.4) and (4.5) and problems (4.6), (4.8) and (4.7), (4.8) are a particular case of the more generalized system (4.1), (4.2), (4.3), where the nonlinearities, as well as the impulse functions, have complete dependence on the variables and their derivatives. We present a methodology for the existence and localization of nontrivial periodic solutions (either positive or negative) in impulsive second-order differential systems, even when there is no evidence of periodicity in the nonlinearities.

We construct the operator of the system using Green's functions, and Shauder's fixed point theorem guarantees the existence of periodic solutions, via regulated functions [51, 52], provided a Nagumo-type condition to control first derivatives growths. The localization of an existing solution is given by a variant of the method of upper and lower solutions: using adequate translations, we show that the lower and upper functions may be either well-ordered or not well ordered. The consequence of this lack of order requirement is that we increase the range of pairs for the upper and lower functions considered for the localization part.

The lack of periodicity requirements in the nonlinearities, as well as the lack of order requirements for the upper and lower solutions, constitute an added value to our approach. Only a few monotonicity requirements are made to the nonlinearities and to the impulsive functions.

Some similar techniques have also been employed in other works, see [63, 44, 64, 65]. Using the same type of arguments, we provide the existence and localization of periodic solutions, where the methodology used does not require periodicity for the right-hand side of the system, nor the impulse functions, and neither do the bounding solutions require order [38, 39, 40].

This chapter is organized as follows. In Section 4.2 we introduce some essential definitions and preliminary results. In Section 4.3 we present the main result on the existence and localization of periodic solutions for problem (4.1), (4.2), (4.3), and provide a proof in three steps. Section 4.4 contains an application of the main result to a system of dissipative Liénard-type equations with state-dependent impulses.

4.2 Definitions and preliminary results

Consider the Banach space of piecewise continuous functions $X := PC^1[0,T]$, equipped with the norm

$$\|u\|_X := \max\{\|u\|, \|u'\|\}, \qquad \|u\| := \sup_{t\in[0,T]} \{|u(t)|\},$$

and the vectorial space $X^2 := (PC^1[0,T])^2$, with the norm

$$\|(u,v)\|_{X^2} = \max\{\|u\|_X, \|v\|_X\}.$$

Consider the space of regulated functions, $\mathcal{G}$, defined as is equation (2.4), and define the set $D = \{\xi_1, \dots, \xi_{p-1}\}$ such that $0 = \xi_0 < \xi_1 < \cdots < \xi_{p-1} < \xi_p = T$, according to Theorem 2.2.2. Let PC_D be the space of piecewise continuous functions on $[0,T]$, given by

$$PC_D := \{u \in C([0,T]\backslash D) : u(t_k^-) = u(t_k), u(t_k^+) \in \mathbb{R}\}.$$

Define

$$D_z = \{t_1, \dots, t_n\}$$
$$D_w = \{\tau_1, \dots, \tau_m\}$$

as the sets of instants of impulse of functions z and w, respectively.

The set PC_D is a subset of the space of regulated functions $\mathcal{G}$, establishing left-continuity, therefore, the set $PC_{D_z}[0,T] \times PC_{D_w}[0,T]$, endowed with the norm $\|\cdot\|$, is a Banach space.

Similarly to the case studied in Chapter 2, the relation between equi-regulation and compactness on PC_D is provided by Corollary 2.2.3.

The nonlinearities in (4.1) are assumed to be L^1-Carathéodory functions, according to Definition 3.2.1.

We present below a variation of a Nagumo condition, whose purpose is to control the first derivatives of the second-order differential problem.

Definition 4.2.1. *Consider PC^1 continuous functions $\gamma_i, \Gamma_i : [0,T] \to \mathbb{R}, i = 1,2,$ and the set*

$$S = \{(t, z_0, w_0, z_1, w_1) \in [0,T] \times \mathbb{R}^4 : \gamma_1(t) \le z_0 \le \Gamma_1(t), \gamma_2(t) \le w_0 \le \Gamma_2(t)\}.$$

The Carathéodory functions $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(t), \Gamma_1(t)]$ and $[\gamma_2(t), \Gamma_2(t)]$, for all $t \in [0,T]$, if there exist piecewise continuous functions $\varphi, \psi : [0,+\infty[\to]0,+\infty[$ verifying

$$\int_0^{+\infty} \frac{s}{\varphi(|s|)} ds = +\infty, \qquad \int_0^{+\infty} \frac{s}{\psi(|s|)} ds = +\infty, \tag{4.10}$$

such that

$$|f(t, z_0, w_0, z_1, w_1)| \le \varphi(|z_1|), \quad \forall (t, z_0, w_0, z_1, w_1) \in S,$$
$$|g(t, z_0, w_0, z_1, w_1)| \le \psi(|w_1|), \quad \forall (t, z_0, w_0, z_1, w_1) \in S.$$

Note that this type of Nagumo condition differs from Definition 3.2.2 in the definition of functions φ, ψ. While in Chapter 3 the functions φ, ψ are sub-linear, equation (4.10) requires them to be sub-quadratic.

The following lemma provides *a priori* estimates for the first derivatives. The arguments are similar to the proof of Lemma 3.2.3.

Lemma 4.2.2. *Suppose that the Carathéodory functions $f, g : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy a Nagumo-type condition relative to the intervals $[\gamma_1(t), \Gamma_1(t)]$ and $[\gamma_2(t), \Gamma_2(t)]$, for all $t \in [0,T]$.*

Then, for every solution $(z(t), w(t)) \in X^2$ of (4.1), (4.2), (4.3) verifying

$$\gamma_1(t) \le z(t) \le \Gamma_1(t), \quad \text{and} \quad \gamma_2(t) \le w(t) \le \Gamma_2(t), \quad \forall t \in [0,T], \tag{4.11}$$

there are $N_1, N_2 > 0$ such that

$$\|z'\| \le N_1, \qquad \|w'\| \le N_2.$$

Proof. Let $(z(t), w(t))$ be a solution of (4.1), (4.2), (4.3) verifying (4.11).

Define

$$\eta_1 = \max\left\{\frac{\Gamma_1(t_{k+1}) - \gamma_1(t_k)}{t_{k+1} - t_k}, \frac{\Gamma_1(t_k) - \gamma_1(t_{k+1})}{t_{k+1} - t_k} \quad k = 0, 1, ..., n-1\right\},$$

$$\eta_2 = \max\left\{\frac{\Gamma_2(\tau_{l+1}) - \gamma_2(\tau_l)}{\tau_{l+1} - \tau_l}, \frac{\Gamma_2(\tau_l) - \gamma_2(\tau_{l+1})}{\tau_{l+1} - \tau_l} \quad l = 0, 1, ..., m-1\right\},$$

and $N_1 > \eta_1$, $N_2 > \eta_2$ such that

$$\int_{\eta_1}^{N_1} \frac{s}{\varphi(|s|)} ds \geq \max_{t\in[0,T]} \Gamma_1(t) - \min_{t\in[0,T]} \gamma_1(t),$$

$$\int_{\eta_2}^{N_2} \frac{s}{\psi(|s|)} ds \geq \max_{t\in[0,T]} \Gamma_2(t) - \min_{t\in[0,T]} \gamma_2(t).$$

We first consider, by contradiction, that $|z'(t)| > \eta_1, \forall t \in [t_k, t_{k+1}]$. For the positive modulo branch, $z'(t) > \eta_1$, we have the following contradiction,

$$\begin{aligned}\Gamma_1(t_{k+1}) - \gamma_1(t_k) &\geq z(t_{k+1}) - z(t_k) = \int_{t_k}^{t_{k+1}} z'(t)dt > \int_{t_k}^{t_{k+1}} \eta_1 dt \\ &= \eta_1(t_{k+1} - t_k) \geq \frac{\Gamma_1(t_{k+1}) - \gamma_1(t_k)}{t_{k+1} - t_k}(t_{k+1} - t_k) = \Gamma_1(t_{k+1}) - \gamma_1(t_k).\end{aligned}$$

For the negative modulo branch, $z'(t) < -\eta_1$, we have the following contradiction,

$$\begin{aligned}\Gamma_1(t_k) - \gamma_1(t_{k+1}) &= -(\gamma_1(t_{k+1}) - \Gamma_1(t_k)) \geq -(z(t_{k+1}) - z(t_k)) \\ &= \int_{t_k}^{t_{k+1}} -z'(t)dt > \int_{t_k}^{t_{k+1}} \eta_1 dt = \eta_1(t_{k+1} - t_k) \\ &\geq \frac{\Gamma_1(t_k) - \gamma_1(t_{k+1})}{t_{k+1} - t_k}(t_{k+1} - t_k) = \Gamma_1(t_k) - \gamma_1(t_{k+1}).\end{aligned}$$

Therefore, $|z'(t)| > \eta_1$ leads to a contradiction.

We then consider the case $|z'(t)| \leq \eta_1, \forall t \in [t_k, t_{k+1}]$. In this case, we establish $N_1 := \eta_1$ and the proof is complete.

Now, consider the case where we have $|z'(t)| > \eta_1$ for some $t \in [t_k, t_{k+1}]$. For the positive modulo branch, we consider t_a, t_b, $t_a < t_b$, and $I_k = [t_a, t_b] \subseteq]t_k, t_{k+1}[$ such that

$$z'(t) \geq 0, \forall t \in I_k \ \wedge \ z'(t_a) = \eta_1 \ \wedge \ z'(t) > \eta_1, \forall t \in I_k \setminus \{t_a\}.$$

Then,

$$\begin{aligned}\int_{z'(t_a)}^{z'(t_b)} \frac{s}{\varphi(|s|)} ds &= \int_{t_a}^{t_b} \frac{z'(t)}{\varphi(|z'(t)|)} z''(t)dt = \int_{t_a}^{t_b} \frac{f(t,z,w,z',w')}{\varphi(|z'(t)|)} z'(t)dt \\ &\leq \int_{t_a}^{t_b} z'(t)dt = z(t_b) - z(t_a) \leq \max_{t\in[0,T]} \Gamma_1(t) - \min_{t\in[0,T]} \gamma_1(t) \\ &\leq \int_{\eta_1}^{N_1} \frac{s}{\varphi(|s|)} ds,\end{aligned}$$

so,

$$\int_{z'(t_0)}^{z'(t_1)} \frac{s}{\varphi(|s|)} ds \leq \int_{\eta_1}^{N_1} \frac{s}{\varphi(|s|)} ds.$$

Since we set $\eta_1 = z'(t_a)$, then, by comparison, we conclude that $z'(t_b) \leq N_1$. As t_b was chosen randomly in $[t_k, t_{k+1}]$, then we may conclude $z'(t) \leq N_1$, without loss of generality.

For the negative modulo branch, we consider $I_k = [t_a, t_b]$ such that

$$z'(t) \leq 0, \forall t \in I_k \ \wedge\ z'(t_a) = -\eta_1 \ \wedge\ -z'(t) > \eta_1, \forall t \in I_k \setminus \{t_a\}.$$

Then,

$$\begin{aligned}\int_{z'(t_b)}^{z'(t_a)} \frac{s}{\varphi(|s|)} ds &= \int_{t_b}^{t_a} \frac{z'(t)}{\varphi(|z'(t)|)} z''(t)dt = \int_{t_b}^{t_a} \frac{f(t,z,w,z',w')}{\varphi(|z'(t)|)} z'(t)dt \\ &\leq \int_{t_b}^{t_a} z'(t)dt = z(t_a) - z(t_b) \leq \max_{t\in[0,T]} \Gamma_1(t) - \min_{t\in[0,T]} \gamma_1(t) \\ &\leq \int_{\eta_1^k}^{N_1^k} \frac{s}{\varphi(|s|)} ds,\end{aligned}$$

so,

$$\int_{z'(t_b)}^{z'(t_a)} \frac{s}{\varphi(|s|)} ds = -\int_{z'(t_a)}^{z'(t_b)} \frac{s}{\varphi(|s|)} ds = \int_{-z'(t_a)}^{-z'(t_b)} \frac{s}{\varphi(|s|)} ds \leq \int_{\eta_1}^{N_1} \frac{s}{\varphi(|s|)} ds.$$

Since we set $\eta_1 = -z'(t_a)$, then, by comparison, we conclude that $-z'(t_b) \leq N_1 \Leftrightarrow z'(t_b) \geq -N_1$. As t_b was chosen randomly in $[t_k, t_{k+1}]$, then we may conclude $z'(t) \geq -N_1$, without loss of generality.

We conclude that $|z'(t)| \leq N_1, \forall t \in [0,T]$.

Likewise, $|w'(t)| \leq N_2, \forall t \in [0,T]$.

$\square$

Lemma 4.2.3. *The solution for the linear problem, for* $p, q \in PC^1[0,T]$,

$$\left\{ \begin{array}{c} z''(t) - z(t) = p(t) \\ w''(t) - w(t) = q(t) \end{array} \right. ,$$

with conditions (4.2), (4.3), is given by

$$z(t) = \sum_{k:t>t_k} I_k(t_k, z(t_k), w(t_k), z'(t_k)) + \int_0^T G(t,s)p(s)ds,$$

$$w(t) = \sum_{l:t>\tau_l} J_l(\tau_l, z(\tau_l), w(\tau_l), w'(\tau_l)) + \int_0^T G(t,s)q(s)ds,$$

and the Green function $G(t,s)$ *given by*

$$G(t,s) = \left\{ \begin{array}{ll} \frac{1}{2(e^{-T}-1)} e^{t-s} + \frac{1}{2(1-e^{T})} e^{s-t}, & 0 \le t \le s, \\ \frac{1}{2(1-e^{T})} e^{t-s} + \frac{1}{2(e^{-T}-1)} e^{s-t}, & s \le t \le T. \end{array} \right.$$

Definition 4.2.4. *Consider the* C^1*-functions* $\alpha_i, \beta_i : [0,T] \to \mathbb{R}$, $i = 1,2$. *The functions* (α_1, α_2) *are lower solutions of the periodic problem (4.1), (4.2), (4.3) if*

$$\begin{array}{c} \alpha_1''(t) \ge f(t, \alpha_1^0(t), \alpha_2^0(t), \alpha_1'(t), w_1), \quad \forall w_1 \in \mathbb{R}, \\ \alpha_2''(t) \ge g\left(t, \alpha_1^0(t), \alpha_2^0(t), z_1, \alpha_2'(t)\right), \quad \forall z_1 \in \mathbb{R}, \end{array} \tag{4.12}$$

with

$$\alpha_i^0(t) := \alpha_i(t) - \|\alpha_i\|, \quad i = 1,2, \tag{4.13}$$

with

$$\alpha_i(0) = \alpha_i(T), \quad \alpha_i'(0) \ge \alpha_i'(T), \quad i = 1,2,$$

and

$$\begin{array}{c} \Delta\alpha_1(t_k) > I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k), \alpha_1'(t_k)) \\ \Delta\alpha_2(\tau_l) > J_l(\tau_l, \alpha_1^0(\tau_l), \alpha_2^0(\tau_l), \alpha_2'(\tau_l)) \end{array} \tag{4.14}$$

The functions (β_1, β_2) *are upper solutions of the periodic problem (4.1), (4.2), (4.3) if*

$$\begin{array}{c} \beta_1''(t) \le f\left(t, \beta_1^0(t), \beta_2^0(t), \beta_1'(t), w_1\right), \quad \forall w_1 \in \mathbb{R}, \\ \beta_2''(t) \le g\left(t, \beta_1^0(t), \beta_2^0(t), z_1, \beta_2'(t)\right), \quad \forall z_1 \in \mathbb{R}, \end{array} ,$$

with

$$\beta_i^0(t) := \beta_i(t) + \|\beta_i\|, \; i = 1,2, \tag{4.15}$$

with

$$\beta_i(0) = \beta_i(T), \quad \beta_i'(0) \leq \beta_i'(T), \quad i = 1, 2, \tag{4.16}$$

and

$$\begin{aligned} \Delta\beta_1(t_k) &< I_k(t_k, \beta_1^0(t_k), \beta_2^0(t_k), \beta_1'(t_k)) \\ \Delta\beta_2(\tau_l) &< J_l(\tau_l, \beta_1^0(\tau_l), \beta_2^0(\tau_l), \beta_2'(\tau_l)) \end{aligned}$$

We ensure the compactness with Corollary 2.2.3, whereas the existence is assured by Schauder's Fixed Point Theorem 1.2.3.

Forward, the following lemma, of [70], will be required to guarantee the existence of first derivatives of the truncated functions at almost every point of the interval $[0, T]$.

Lemma 4.2.5. *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x, u) = max\{v, min\{u, w\}\}.$$

Then, for each $u \in C^1(I)$ the next two properties hold:

(a) $\frac{d}{dx}q(x, u(x))$ exists for a.e. $x \in I$.

(b) If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$ then

$$\frac{d}{dx}q(x, u_m(x)) \to \frac{d}{dx}q(x, u(x)) \text{ for a.e. } x \in I.$$

4.3 Existence and localization theorem

In this section we present an existence and localization result for periodic solutions of problem (4.1), (4.2), (4.3). We now combine the techniques used in the previous chapters, by working with a second-order differential system with impulses. As mentioned in the previous chapter, the Topological Degree Theory is not suitable for impulsive conditions, since a homotopy, *i.e.*, a continuous deformation, is involved. Nonetheless, important tools like the Nagumo condition and the localization of a pair $(z(t), w(t))$ using the method of lower and upper solutions are still applied. The result is as follows.

Theorem 4.3.1. *Let (α_1, α_2) and (β_1, β_2) be lower and upper solutions of (4.1), (4.2), (4.3), respectively.*
Assume that f, g are L^1-Carathéodory functions on the set

$$\{(t, z_0, w_0, z_1, w_1) \in [0, T] \times \mathbb{R}^4 : \alpha_1^0(t) \leq z_0 \leq \beta_1^0(t),\ \alpha_2^0(t) \leq w_0 \leq \beta_2^0(t)\},$$

with

$$f(t, z_0, \alpha_2^0(t), z_1, w_1) \geq f(t, z_0, w_0, z_1, w_1) \geq f(t, z_0, \beta_2^0(t), z_1, w_1), \tag{4.17}$$

for fixed $t \in [0, T]$, $z_0, z_1, w_1 \in \mathbb{R}$, *and* $\alpha_2^0(t) \leq w_0 \leq \beta_2^0(t)$, *and with*

$$g(t, \alpha_1^0(t), w_0, z_1, w_1) \geq g(t, z_0, w_0, z_1, w_1) \geq g(t, \beta_1^0(t), w_0, z_1, w_1),$$

for fixed $t \in [0, T]$, $w_0, z_1, w_1 \in \mathbb{R}$, *and* $\alpha_1^0(t) \leq z_0 \leq \beta_1^0(t)$.

Assume that the impulse functions I_k *and* J_l *verify*

$$I_k(t_k, z_0, \alpha_2^0(t_k), z_1) \geq I_k(t_k, z_0, w_0, z_1) \geq I_k(t_k, z_0, \beta_2^0(t_k), z_1), \tag{4.18}$$

for some fixed $k \in \{1, \dots n-1\}$, $z_0 \in \mathbb{R}$, *and* $\alpha_2^0(t) \leq w_0 \leq \beta_2^0(t)$, *and that*

$$\sum_{k=1}^{n-1} I_k(t_k, \alpha_1^0(t_k), \alpha_2^0(t_k), \alpha_1'(t_k)) = 0, \tag{4.19}$$

Assume that the impulse functions J_l *verify*

$$J_l(\tau_l, \alpha_1^0(\tau_l), w_0, w_1) \geq J_l(\tau_l, z_0, w_0, w_1) \geq J_l(\tau_l, \beta_1^0(\tau_l), w_0, w_1),$$

for some fixed $l \in \{1, \dots m-1\}$, $w_0 \in \mathbb{R}$ *and* $\alpha_1^0(t) \leq z_0 \leq \beta_1^0(t)$, *and that*

$$\sum_{l=1}^{m-1} J_l(\tau_l, \alpha_1^0(\tau_l), \alpha_2^0(\tau_l), \alpha_2'(\tau_l)) = 0 \tag{4.20}$$

Then, problem (4.1), (4.2), (4.3) has, at least, a solution $(z, w) \in (PC^1[0, T])^2$ *such that*

$$\begin{aligned} &\alpha_1^0(t) \leq z(t) \leq \beta_1^0(t), \\ &\alpha_2^0(t) \leq w(t) \leq \beta_2^0(t), \forall t \in [0, T]. \end{aligned}$$

The following proof is developed in three steps, where the two first steps are dedicated to the existence arguments of a periodic solution of problem (4.1), (4.2), (4.3). Contrary to the previous chapter, we are now based on the definition of a Carathéodory function to deal with the discontinuities. Therefore, results on regulated functions are required. Also, a variant of the Nagumo condition is used to control the first derivatives.

Consistently, truncated functions are considered to a perturbed problem, that we address using Green's functions. This modified problem is guaranteed to have a solution under the conditions of Schauder's Fixed Point Theorem, as long as the operator is bounded and equi-regulated.

In the last step, by means of the same localization tool, we prove that the modified problem is equivalent to the original problem, by showing that the system variables z, w are only valid at the middle branch of the truncated functions.

The proof is as follows.

Proof. Define the truncated functions $\delta_i : [0,T] \times \mathbb{R} \to \mathbb{R}$,

$$\delta_1^0(t,z) = \begin{cases} \beta_1^0(t), & z > \beta_1^0(t) \\ z, & \alpha_1^0(t) \le z \le \beta_1^0(t) \\ \alpha_1^0(t), & z < \alpha_1^0(t) \end{cases}, \qquad \delta_2^0(t,w) = \begin{cases} \beta_2^0(t), & w > \beta_2^0(t) \\ w, & \alpha_2^0(t) \le w \le \beta_2^0(t) \\ \alpha_2^0(t), & w < \alpha_2^0(t) \end{cases}.$$

Given the periodic boundary conditions (4.2), the solution for the homogeneous problem of (4.1),(4.2) also admits constant solutions. However, we are looking for non-trivial solutions. Hence, we consider the following modified (perturbed and truncated) problem,

$$\begin{cases} z''(t) - z(t) = f\left(t, \delta_1^0(t,z(t)), \delta_2^0(t,w(t)), \frac{d}{dt}(\delta_1^0(t,z(t)), w'(t)\right) - \delta_1^0(t,z(t)) \\ w''(t) - w(t) = g\left(t, \delta_1^0(t,z(t)), \delta_2^0(t,w(t)), z'(t), \frac{d}{dt}(\delta_2^0(t,w(t))\right) - \delta_2^0(t,w(t)) \end{cases}, \tag{4.21}$$

for $a.e.\ t \in [0,T]$, together with periodic boundary conditions (4.2), and the truncated impulses, given by

$$\begin{aligned} \Delta z(t_k) &= I_k(t_k, \delta_1^0(t_k,z(t_k)), \delta_2^0(t_k,w(t_k)), \tfrac{d}{dt}(\delta_1^0(t_k,z(t_k)))), && k = 1,...,n-1, n > 2, \\ \Delta w(\tau_l) &= J_l(\tau_l, \delta_1^0(\tau_l,z(\tau_l)), \delta_2^0(\tau_l,w(\tau_l)), \tfrac{d}{dt}(\delta_2^0(\tau_l,w(\tau_l))), && l = 1,...,m-1, m > 2. \end{aligned} \tag{4.22}$$

Define the operators $T : X^2 \to X^2$ and $T_i : X \to X,\ i = 1,2$, such that

$$T(z,w) = (T_1(z,w), T_2(z,w)),$$

where

$$\begin{aligned} T_1(z,w)(t) = & \sum_{k:t>t_k} I_k(t_k, \delta_1^0(t_k,z(t_k)), \delta_2^0(t_k,w(t_k)), \frac{d}{dt}(\delta_1^0(t_k,z(t_k))) \\ & + \int_0^T G(t,s)p(s)ds, \\ T_2(z,w)(t) = & \sum_{l:t>\tau_l} J_l(\tau_l, \delta_1^0(\tau_l,z(\tau_l)), \delta_2^0(\tau_l,w(\tau_l)), \frac{d}{dt}(\delta_2^0(\tau_l,w(\tau_l)))) \\ & + \int_0^T G(t,s)q(s)ds, \end{aligned}$$

where $p(s), q(s)$ are given by

$$p(s) = f(s, \delta_1^0(s, z(s)), \delta_2^0(s, w(s)), \frac{d}{ds}(\delta_1^0(s, z(s)), w'(s)) - \delta_1^0(s, z(s)),$$
$$q(s) = g(s, \delta_1^0(s, z(s)), \delta_2^0(s, w(s)), z'(s), \frac{d}{ds}(\delta_2^0(s, w(s))) - \delta_2^0(s, w(s)).$$

Note that $p, q \in L^1$, the impulse discontinuities are finite and lose effect in the sum. The discontinuity in the Green's function diagonal also loses is effect under integration. Therefore, T is well-defined in X^2.

Any fixed point of the operator T is a solution of the modified problem (4.21), (4.2), (4.22). In the following steps, we prove that T has a fixed point.

Step 1. TB is uniformly bounded in $B \subset X$.

Since f, g are L^1-Carathéodory functions, by Definition 3.2.1, there are positive $L^1[0, T]$ functions $\phi_{i\rho}$, $i = 1, 2$, such that

$$\left.\begin{array}{l} \left|f(t, \delta_1^0(t, z(t)), \delta_2^0(t, w(t)), \frac{d}{dt}(\delta_1^0(t, z(t)), w'(t))\right| \leq \phi_{1\rho}(t) \\ \left|g(t, \delta_1^0(t, z(t)), \delta_2^0(t, w(t)), z'(t), \frac{d}{dt}(\delta_2^0(t, w(t)))\right| \leq \phi_{2\rho}(t) \end{array}\right\} , a.e.\ t \in [0, T],$$

with

$$\rho := \max_{t\in[0,T]} \{|\alpha_1(t)|, |\alpha_2(t)|, |\beta_1(t)|, |\beta_2(t)|\}$$

We consider the closed ball of radius K,

$$B := \left\{(z, w) \in X^2 : \|(z, w)\| \leq K\right\}, \tag{4.23}$$

where

$$K = \max \left\{ \begin{array}{l} (n-1)M_I + 2e^T \left(\int_0^T \phi_{1\rho}(s)ds + LT \right), \\ (m-1)M_J + 2e^T \left(\int_0^T \phi_{2\rho}(s)ds + LT \right) \end{array} \right\}, \tag{4.24}$$

and

$$M_I := \max_{k=1,\dots,n-1} \{I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k)), \frac{d}{dt}(\delta_1^0(t_k, z(t_k)))\},$$
$$M_J := \max_{l=1,\dots,m-1} \{J_l(\tau_l, \delta_1^0(\tau_l, z(\tau_l)), \delta_2^0(\tau_l, w(\tau_l)), \frac{d}{dt}(\delta_2^0(\tau_l, w(\tau_l))))\},$$

For ease of writing, from now on we will note

$$I_k(t_k) := I_k(t_k, \delta_1^0(t_k, z(t_k)), \delta_2^0(t_k, w(t_k)), \frac{d}{dt}(\delta_1^0(t_k, z(t_k))).$$

Then, for $t \in [0, T]$,

$$\begin{aligned}
|T_1(z,w)(t)| &\leq \sum_{k:t<t_k} |I_k(t_k)| + \int_0^T |G(t,s)p(s)|\,ds \\
&\leq \sum_{k:t<t_k} M_I + \int_0^t \left|\frac{e^{t-s}}{2(e^{-T}-1)} + \frac{e^{s-t}}{2(1-e^T)}\right| |p(s)|\,ds \\
&\qquad + \int_t^T \left|\frac{e^{t-s}}{2(1-e^T)} + \frac{e^{s-t}}{2(e^{-T}-1)}\right| |p(s)|\,ds \\
&\leq (n-1)M_I + \int_0^t \left|\frac{e^{|t-s|}}{2(e^{-T}-1)} + \frac{e^{-|t-s|}}{2(1-e^T)}\right| |p(s)|\,ds \\
&\qquad + \int_t^T \left|\frac{e^{-|s-t|}}{2(1-e^T)} + \frac{e^{|s-t|}}{2(e^{-T}-1)}\right| |p(s)|\,ds \\
&\leq (n-1)M_I + \int_0^t \left|\frac{e^{T}}{2(e^{-T}-1)} + \frac{e^{0}}{2(1-e^T)}\right| |p(s)|\,ds \\
&\qquad + \int_t^T \left|\frac{e^{0}}{2(1-e^T)} + \frac{e^{T}}{2(e^{-T}-1)}\right| |p(s)|\,ds \\
&= (n-1)M_I + \left|\frac{e^T + e^{-T}}{2(e^{-T}-1)}\right| \int_0^T \left|f(s, \delta_1^0(s,z(s)), \delta_2^0(s,w(s)), z', w') - \delta_1^0(s,z(s))\right| ds \\
&\leq (n-1)M_I + \left|\frac{e^T + e^{-T}}{2(e^{-T}-1)}\right| \int_0^T (\phi_{1\rho}(s) + \beta_1(s))\,ds \\
&\leq (n-1)M_I + \left|\frac{e^T + e^{-T}}{2(e^{-T}-1)}\right| \int_0^T \phi_{1\rho}(s)ds + LT \leq K.
\end{aligned}$$

Analogously,

$$|T_2(z,w)(t)| \leq (m-1)M_J + 2e^T \left(\int_0^T \phi_{2\rho}(s)ds + LT\right) \leq K.$$

Thus. T_1, T_2 are uniformly bounded. Therefore, T is uniformly bounded, and, by (4.23) and (4.24), $TB \subset B$.

Step 2. T is equiregulated.

Consider a, b, $a < b$, without loss of generality, and let $[a, b] \subseteq]t_k, t_{k+1}[$ for some $k \in \{0, ..., n-1\}$. Then,

$$|T_1(z,w)(a) - T_1(z,w)(b)| = \Bigg| \sum_{k:a>t_k} I_k(t_k) + \int_0^T G(a,s)p(s)ds$$
$$- \sum_{k:b>t_k} I_k(t_k) - \int_0^T G(a,s)p(s)ds \Bigg|$$
$$= \Bigg| \sum_{k:a>t_k} I_k(t_k) - \sum_{k:b>t_k} I_k(t_k) \int_0^T (G(a,s) - G(b,s))p(s)ds \Bigg| \xrightarrow[a \to b]{} 0.$$

So, T_1 is equi-regulated. Similarly, $|T_2(z,w)(a) - T_2(z,w)(b)| \xrightarrow[a \to b]{} 0$. Therefore, T is equi-regulated.

By Corollary 2.2.3, T is relatively compact.

By Theorem 1.2.3, T has a fixed point $(z^*(t), w^*(t)) \in (PC[0,T])^2$, which is a solution of (4.21), (4.2), (4.22).

Step 3: *The pair $(z^*(t), w^*(t))$, solution of (4.21), (4.2), (4.22), is a solution of the initial problem, (4.1), (4.2), (4.3).*

We prove that $(z^*, w^*) \in (PC[0,T])^2$ is a solution of the original problem (4.1), (4.2), (4.3) by showing that

$$\alpha_1^0(t) \le z^*(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w^*(t) \le \beta_2^0(t), \quad \forall t \in [0,T]. \tag{4.25}$$

To prove $z^*(t) \le \beta_1^0(t)$, suppose, by contradiction, that there exists $t \in [0,T]$ such that

$$z^*(t) > \beta_1^0(t),$$

and define

$$\sup_{t \in [0,T]} (z^* - \beta_1^0)(t) := z^*(t_0) - \beta_1^0(t_0) > 0. \tag{4.26}$$

We then consider different cases for t_0, as follows.

If $t_0 \in]t_k, t_{k+1}[$ for some $k \in \{0, \ldots, n\}$, then, by (4.26)

$$(z^* - \beta_1^0)'(t_0) = 0, \quad \text{and} \quad (z^* - \beta_1^0)''(t_0) \le 0. \tag{4.27}$$

However, by (4.26), (4.17) and (4.12),

$$\begin{aligned}
0 &\geq (z^*)''(t_0) - (\beta_1^0)''(t_0) \\
&= f(t_0, \delta_1^0(t_0, z^*(t_0)), \delta_2^0(t_0, w^*(t_0)), \frac{d}{dt}(\delta_1^0(t, z^*(t_0)), (w^*)'(t_0)) - \delta_1^0(t_0, z^*(t_0)) \\
&+ z^*(t_0) - (\beta_1^0)''(t_0) \\
&= f(t_0, \beta_1^0(t_0), \delta_2^0(t_0, w(t_0)), (\beta_1^0)'(t_0), (w^*)'(t_0)) - \beta_1^0(t_0) + z^*(t_0) - (\beta_1^0)''(t_0) \\
&> f(t_0, \beta_1^0(t_0), \delta_2^0(t_0, w(t_0)), (\beta_1^0)'(t_0), (w^*)'(t)) - (\beta_1^0)''(t_0) \\
&\geq f(t_0, \beta_1^0(t_0), \beta_2^0(t_0), (\beta_1^0)', (w^*)'(t)) - (\beta_1^0)''(t_0) \geq 0,
\end{aligned}$$

which contradicts (4.27).

If $t_0 = 0$ or $t_0 = T$, then $(z^* - \beta_1^0)'(0) \leq 0$ and $(z^* - \beta_1^0)'(T) \geq 0$. By (4.15), (4.2) and (4.16),

$$\begin{aligned}
0 &\geq (z^* - \beta_1^0)'(0) = (z^* - \beta_1)'(0) = (z^*)'(T) - \beta_1'(0) \\
&\geq (z^*)'(T) - \beta_1'(T) = (z^* - \beta_1^0)'(T) \geq 0,
\end{aligned}$$

so,

$$(z^* - \beta_1^0)'(0) = (z^* - \beta_1^0)'(T) = 0,$$

and $(z^* - \beta_1^0)''(t_0) \leq 0$. Therefore, previous arguments are applied to obtain a similar contradiction, and so, $z * (t) \leq \beta_1^0(t)$.

If $t_0 = t_k$ for some $k \in \{1, \ldots, n-1\}$, then either $t_0 = t_k^+$ or $t_0 = t_k^-$. If $t_0 = t_k^+$, then $t_0 \in]t_k, t_{k+1}[$, so the previous reasoning must be applied. If, instead, $t_0 = t_k^-$, then we consider

$$\max_{t \in [0,T]} (z^* - \beta_1^0)(t) := z^*(t_0) - \beta_1^0(t_0) > 0, \tag{4.28}$$

and thus, the impulse on t_k is necessarily non-positive. By (4.28), (4.22), (4.13), (4.18) and (4.14) the following contradiction holds,

$$\begin{aligned}
0 &\geq \Delta(z^* - \beta_1^0)(t_k) \\
&= I_k(t_k, \delta_1^0(t_k, z^*(t_k)), \delta_2^0(t_k, w(t_k)), \frac{d}{dt}(\delta_1^0(t_k, z^*(t_k)))) - \Delta\beta_1^0(t_k) \\
&= I_k(t_k, \beta_1^0(t_k), \delta_2^0(t_k, w(t_k)), (\beta_1^0)'(t_k)) - \Delta\beta_1^0(t_k) \\
&= I_k(t_k, \beta_1^0(t_k), \delta_2^0(t_k, w(t_k)), \beta_1'(t_k)) - \Delta\beta_1(t_k) \\
&\geq I_k(t_k, \beta_1^0(t_k), \beta_2^0(t_k), \beta_1'(t_k)) - \Delta\beta_1(t_k) > 0.
\end{aligned}$$

Therefore, $z^*(t) \leq \beta_1^0(t), \forall t \in [0, T]$. The same arguments can be applied to prove the other inequalities in (4.25).

□

4.4 Dissipative Liénard-type system with state-dependent impulses

In the work [71] the authors investigate the existence of exact harmonic solutions of certain modified Emden-type equations. One of the equations considered by the authors is a dissipative Liénard-type equation:

$$x'' + \alpha x x' + \beta x + \gamma x(a^2 - cx^2)^q = 0, \tag{4.29}$$

where $\alpha, \beta, \gamma, a, c, q$ are arbitrary constants. For certain values of these parameters, the resulting equation (4.29) refer to problems widely investigated in the literature [72, 73].

Furthermore, since the beginning of the current century, some works have been exploring the Liénard-type equations with state-dependent impulses, see [74, 75, 76, **?**], to which our model could be suitable.

Inspired by these works, as an application to the existence and localization result presented in Section 4.3, in this Section we consider a system of two equations, each one a variation of equation (4.29), together with periodic boundary conditions and state-dependent impulses:

$$\begin{cases} \Delta z(t_k) = A_1 t_k + A_2 z(t_k) + A_3 w(t_k) + A_4 z'(t_k), k = 1, 2, \\ \Delta w(\tau_l) = B_1 \tau_l + B_2 z(\tau_l) + B_3 w(\tau_l) + B_4 w'(\tau_l), \tau = 1, 2, 3, 4. \end{cases} \tag{4.30}$$

We consider the set of values $\alpha = -3, \beta = -8, \gamma = 0.5, a = 2, c = 0.5, q = 1/2$, with additional interaction terms and explicit time dependence for equation (4.29). For conditions (4.30), we consider the values $A_1 = -5, A_2 = 3, A_3 = -1, A_4 = 1, B_1 = -2, B_2 = -1, B_3 = 3, B_4 = 0$. We consider the instants of impulse $t_1 = 1/3, t_2 = 2/3$ for variable z and $\tau_1 = 1/5, \tau_2 = 2/5, \tau_3 = 3/5, \tau_4 = 4/5$ for variable w in the normalized time interval $[0, 1]$.

Thus, the numerical problem is as follows

$$\begin{cases} z''(t) = 3zz' + 8z - 0.5z(2^2 - 0.5z)^{1/2} - 10t - 0.5 \arctan w + 0.1 \arctan w' \\ w''(t) = 3ww' + 8w - 0.5w(2^2 - 0.5w)^{1/2} - 10t - 0.5 \arctan z + 0.1 \arctan z' \end{cases}, \tag{4.31}$$

with the periodic boundary conditions

$$\begin{array}{ll} z(0) = z(1), & z'(0) = z'(1), \\ w(0) = w(1), & w'(0) = w'(1), \end{array} \tag{4.32}$$

and the impulsive conditions

$$\begin{cases} \Delta z(t_k) = -5t_k + 3z(t_k) - w(t_k) + z'(t_k), & k = 1, 2, \\ \Delta w(\tau_l) = -2\tau_l - z(\tau_l) + 3w(\tau_l), & l = 1, 2, 3, 4. \end{cases} \tag{4.33}$$

This numerical problem is a particular case of (4.1), (4.2), (4.3), with

$$f(t,z(t),w(t),z'(t),w'(t)) = \\ = 3zz' + 8z - \frac{1}{2}z\sqrt{4-0.5z} - 10t - \frac{1}{2}\arctan w + 0.1\arctan w',$$

$$g(t,z(t),w(t),z'(t),w'(t)) = \\ = 3ww' + 8w - \frac{1}{2}w\sqrt{4-0.5w} - 10t - \frac{1}{2}\arctan z + 0.1\arctan z',$$

$$I_k(t_k,z(t_k),w(t_k),z'(t_k)) = -5t_k + 3z(t_k) - w(t_k) + z'(t_k), \quad k=1,2,$$
$$J_l(\tau_l,z(\tau_l),w(\tau_l),w'(\tau_l)) = -2\tau_l - z(\tau_l) + 3w(\tau_l), \quad l=1,2,3,4.$$

We can verify equations (4.31) and (4.33) satisfy monotony requirements of Theorem 4.3.1.

Consider now the following piecewise functions $\alpha_i, \beta_i : [0,T] \to \mathbb{R}, i=1,2,$

$$\alpha_1(t) = \begin{cases} t-t^2 & 0\le t\le 1/3 \\ 1+2t^2-2t^3 & 1/3<t\le 2/3 \\ t-t^2 & 2/3<t\le 1 \end{cases} \qquad \beta_1(t) = \begin{cases} \frac{6}{5}-2t^2+2t^3 & 0\le t\le 1/3 \\ 1-t+t^2 & 1/3<t\le 2/3 \\ \frac{6}{5}-2t^2+2t^3 & 2/3<t\le 1 \end{cases}$$

$$\alpha_2(t) = \begin{cases} 1+2t^2-2t^3 & 0\le t\le 1/5 \\ t-t^2 & 1/5<t\le 2/5 \\ 1+2t^2-2t^3 & 2/5<t\le 3/5 \\ t-t^2 & 3/5<t\le 4/5 \\ 1+2t^2-2t^3 & 4/5<t\le 1 \end{cases} \qquad \beta_2(t) = \begin{cases} \frac{6}{5}-2t^2+2t^3 & 0\le t\le 1/5 \\ 1-t+t^2 & 1/5<t\le 2/5 \\ \frac{6}{5}-2t^2+2t^3 & 2/5<t\le 3/5 \\ 1-t+t^2 & 3/5<t\le 4/5 \\ \frac{6}{5}-2t^2+2t^3 & 4/5<t\le 1, \end{cases}$$

represented in Figure 4.1.

The piecewise shifted functions $\alpha_i^0, \beta_i^0 : [0,T] \to \mathbb{R}, i=1,2,$ given by

$$\alpha_1^0(t) = \begin{cases} \frac{1}{4}+t-t^2 & 0\le t\le 1/3 \\ -\frac{8}{27}+2t^2-2t^3 & 1/3<t\le 2/3 \\ \frac{1}{4}+t-t^2 & 2/3<t\le 1 \end{cases} \qquad \beta_1^0(t) = \begin{cases} \frac{12}{5}-2t^2+2t^3 & 0\le t\le 1/3 \\ \frac{7}{4}-t+t^2 & 1/3<t\le 2/3 \\ \frac{12}{5}-2t^2+2t^3 & 2/3<t\le 1 \end{cases}$$

$$\alpha_2^0(t) = \begin{cases} -\frac{8}{27}+2t^2-2t^3 & 0\le t\le 1/5 \\ \frac{1}{4}+t-t^2 & 1/5<t\le 2/5 \\ -\frac{8}{27}+2t^2-2t^3 & 2/5<t\le 3/5 \\ \frac{1}{4}+t-t^2 & 3/5<t\le 4/5 \\ -\frac{8}{27}+2t^2-2t^3 & 4/5<t\le 1 \end{cases} \qquad \beta_2^0(t) = \begin{cases} \frac{12}{5}-2t^2+2t^3 & 0\le t\le 1/5 \\ \frac{7}{4}-t+t^2 & 1/5<t\le 2/5 \\ \frac{12}{5}-2t^2+2t^3 & 2/5<t\le 3/5 \\ \frac{7}{4}-t+t^2 & 3/5<t\le 4/5 \\ \frac{12}{5}-2t^2+2t^3 & 4/5<t\le 1 \end{cases}$$

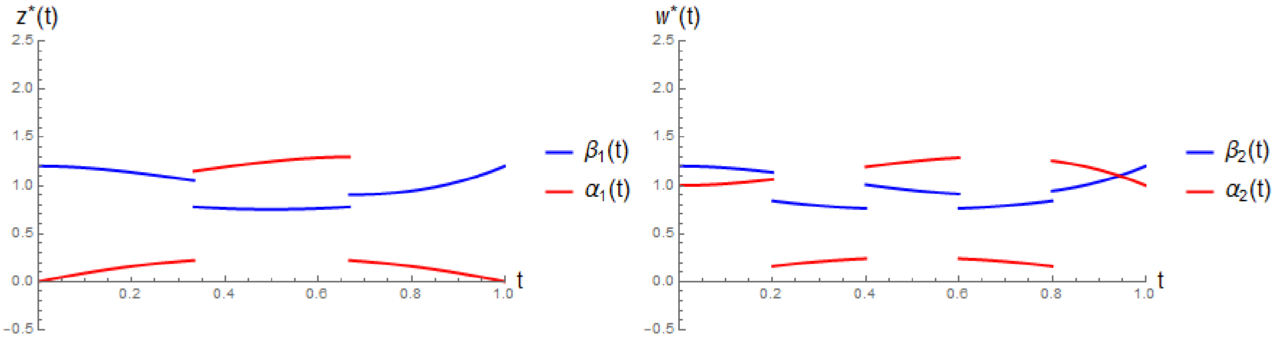


Figure 4.1: The lower solutions (α_1, α_2) and the upper solutions (β_1, β_2) are not ordered.

are, respectively, lower and upper solutions of the problem (4.31), (4.32), (4.33), according to Definition 4.2.4.

The nonlinearities f and g satisfy a Nagumo-type condition, according to Definition 4.2.1, in the set

$$S = \{(t, z(t), w(t), z'(t), w'(t)) \in [0,1] \times \mathbb{R}^4 : \\ \alpha_1^0(t) \le z(t) \le \beta_1^0(t), \quad \alpha_2^0(t) \le w(t) \le \beta_1^0(t)\},$$

such that

$$\begin{aligned}
|f(t,z,w,z',w')| &\le 3|z||z'| + 8|z| + \frac{1}{2}|z|\sqrt{4-0.5|z|} + 10|t| + \\
&\frac{1}{2}|\arctan(w)| + 0.1|\arctan(w')| \\
&\le 3 \times \frac{12}{5}|z'| + 8 \times \frac{12}{5} + \frac{1}{2} \times \frac{12}{5}\sqrt{4-0.5\times 0} + 10 \times 1 + (\frac{1}{2}+0.1)\frac{\pi}{2} \\
&= \frac{36}{5}|z'| + \frac{158}{5} + \frac{3\pi}{10} := \varphi(|z'|), \\
|g(t,z,w,z',w')| &\le 3|w||w'| + 8|w| + \frac{1}{2}|w|\sqrt{4-0.5|w|} + 10|t| + \\
&\frac{1}{2}|\arctan(z)| + 0.1|\arctan(z')| \\
&\le 3 \times \frac{12}{5}|w'| + 8 \times \frac{12}{5} + \frac{1}{2} \times \frac{12}{5}\sqrt{4-0.5\times 0} + 10 \times 1 + (\frac{1}{2}+0.1)\frac{\pi}{2} \\
&= \frac{36}{5}|w'| + \frac{158}{5} + \frac{3\pi}{10} := \psi(|w'|).
\end{aligned}$$

It is clear that functions φ and ψ satisfy (4.10).

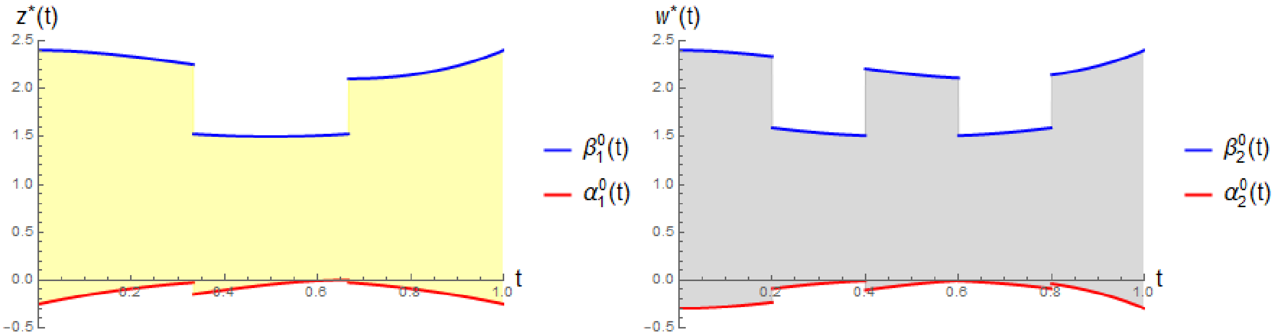


Figure 4.2: Localization of a pair $(z^*(t), w^*(t))$ by shifted functions, $\alpha_i^0, \beta_i^0, i = 1, 2$.

We verify the impulse conditions for each instant of impulse:

$$\Delta\alpha_1^0(t_1) = -0.12037 > -1.38889 = I_1(t_1, \alpha_1^0(t_1), \alpha_2^0(t_1), \alpha_1'(t_1))$$
$$\Delta\beta_1^0(t_1) = 0.724074 < 2.89444 = I_1(t_1, \beta_1^0(t_1), \beta_2^0(t_1), \beta_1'(t_1))$$

$$\Delta\alpha_1^0(t_2) = -0.0277778 > -3.30556 = I_2(t_2, \alpha_1^0(t_2), \alpha_2^0(t_2), \alpha_1'(t_2))$$
$$\Delta\beta_1^0(t_2) = -0.575926 < 0.0555556 = I_2(t_2, \beta_1^0(t_2), \beta_2^0(t_2), \beta_1'(t_2))$$

$$\Delta\alpha_2^0(\tau_1) = 0.142296 > -1.00689 = J_1(\tau_1, \alpha_1^0(\tau_1), \alpha_2^0(\tau_1), \alpha_2'(\tau_1))$$
$$\Delta\beta_2^0(\tau_1) = -0.746 < 4.272 = J_1(\tau_1, \beta_1^0(\tau_1), \beta_2^0(\tau_1), \beta_2'(\tau_1))$$

$$\Delta\alpha_2^0(\tau_2) = -0.0942963 > -0.725704 = J_2(\tau_2, \alpha_1^0(\tau_2), \alpha_2^0(\tau_2), \alpha_2'(\tau_2))$$
$$\Delta\beta_2^0(\tau_2) = 0.698 < 2.22 = J_2(\tau_2, \beta_1^0(\tau_2), \beta_2^0(\tau_2), \beta_2'(\tau_2))$$

$$\Delta\alpha_2^0(\tau_3) = -0.0017037 > -1.21659 = J_3(\tau_3, \alpha_1^0(\tau_3), \alpha_2^0(\tau_3), \alpha_2'(\tau_3))$$
$$\Delta\beta_2^0(\tau_3) = -0.602 < 3.626 = J_3(\tau_3, \beta_1^0(\tau_3), \beta_2^0(\tau_3), \beta_2'(\tau_3))$$

$$\Delta\alpha_2^0(\tau_4) = 0.0497037 > -1.78 = J_4(\tau_4, \alpha_1^0(\tau_4), \alpha_2^0(\tau_4), \alpha_2'(\tau_4))$$
$$\Delta\beta_2^0(\tau_4) = 0.554 < 1.026 = J_4(\tau_4, \beta_1^0(\tau_4), \beta_2^0(\tau_4), \beta_2'(\tau_4))$$

Once verified all the assumptions of Theorem 4.3.1, we conclude that there is a pair (z^*, w^*) that is a periodic solution of problem (4.31), (4.32), (4.33), such that

$$\alpha_1^0(t) \le z^*(t) \le \beta_1^0(t)$$
$$\alpha_2^0(t) \le w^*(t) \le \beta_2^0(t), \quad \forall t \in [0, 1].$$

Figure 4.2 shows the region delimited by the lower and upper functions α_1^0, β_1^0 for variable z (region in yellow), and for variable w, functions α_2^0, β_2^0 (region in gray). At least a periodic solution exists within these bounds, and, from (4.31), this periodic solution is nontrivial, meaning that it is not a constant.

4.5 Discussion

The literature on second-order differential problems, periodic solutions and impulsive problems is extensive. However, it is filtered down to few results when we combine these conditions to the search of periodic solutions in impulsive second-order differential systems. As described in Section 4.1, different and interesting techniques are applied, but always restricting the results either for a positive periodic solution, either for lack of dependence in the system variables or the impulse functions, either for considering a less general system, etc. Our contribution focus on the attempt to approach an impulsive problem as much general and with as few requirements as possible.

As described throughout this chapter, the operator of the system is constructed using Green's functions, and the control of the first derivatives is ensured by a Nagumo-type condition. The existence of at least a non-trivial periodic solution for system (4.1), (4.2), (4.3) is guaranteed using results for regulated functions [51, 52] and Shauder's fixed point theorem. The proof that a function is bounded and equi-regulated is equivalent to the Arzelà-Ascoli theorem, but for regulated functions. This proof allows to guarantee the conditions for Shauder's theorem.

To localize an existence solution, we used a variant of the method of upper and lower solutions [37] with translations, but, again, without order requirements. We may thus consider a wider range of functions α_i and β_i, in accordance to Definition 4.2.4, to localize an existing pair (z^*, w^*).

When apply our methodology to a dissipative Liénard-type system with state dependent impulses, in Section 4.4, it is essential to verify that both the nonlinearities as well as the impulse functions meet the monotonicity criteria required by Theorem 4.3.1. Piecewise upper and lower functions are found to delimit a strip that localizes each variable. Naturally, the requirements must be met at every instant of impulse. Given that all the requirements are met, we then conclude that a periodic solution exists and it is localized as shown in Figure 4.2.

Part II

Synchronization

Introduction

Synchronization is a remarkable phenomenon observed in dynamical systems, characterized by their tendency to coordinate their actions over time. Due to its interdisciplinary nature, crossing the fields of dynamical systems and actual physics, synchronization between oscillators has been called universal [77, 78] and appears frequently in nature [79, 80, 77].

The earliest documented account of synchronization dates back to 1666 when Christiaan Huygens corresponded with De Sluse and others, reporting the synchronous behaviour of two pendulum clocks hanging on the same wall beam in his house [81]. This historical observation laid the foundation for the exploration of synchronization in dynamical systems and initiated a journey of discovery that has since evolved into a rich and multidisciplinary research field.

The pioneering work of Huygens in the 17$^{\text{th}}$ century marked the inception of the scientific investigation of synchronization. His meticulous observations and correspondence with fellow scholars laid the groundwork for subsequent studies on the topic. The initial interest in synchronization primarily stemmed from its practical implications in the field of timekeeping, as accurate and synchronized clocks were of paramount importance in navigation, astronomy, and various other scientific endeavours of the time.

Huygens' findings, while instrumental, also raised intriguing questions about the underlying mechanisms and mathematical principles governing synchronization. This catalysed the interest of mathematicians and physicists in the subsequent centuries, leading to the development of mathematical models and theories to explain the phenomenon. In particular, Huygens' work paved the way for the formulation of phase synchronization concepts that have become fundamental in the study of synchronization [77].

When it comes to synchronization of clocks, in the late 20th century and the first years of this century, numerous works appeared on this theme [82, 83, 84, 85, 86, 87, 88, 89], by considering momentum conservation in the clocks-beam system. These works consider the phenomenon classically with macroscopic degrees of freedom connecting the different oscillators producing the synchronization. On the other hand, every time pendulums appear in the literature, most authors consider viscous damping instead of dry friction (Coulomb friction). We call to that system the *classical model*.

Despite the non-linear nature of systems with dry friction and macroscopic exchange of momentum, which complicates the computations, it is more realistic for the study of metal mechanical devices, where wheels and rods slide against each other, enriching the mathematical study.

There was a clear lack of studies of synchronization via small periodic impact perturbations between the oscillators of coupled systems with dry friction, that we call the *perturbative model*. This is a phenomenon that is deeply related to impulsive systems, studied previously in this dissertation.

This perspective was initiated by Andronov and its collaborators, who studied the isolated oscillator, by presenting one of the most fruitful theories describing the working mechanism of a pendulum clock in the work [90]. Later on, inspired by Andronov's model for the isolated oscillator, the mathematicians Vassalo Pereira and Ralph Abraham [91, 92, 93] studied two coupled oscillators.

These works laid the foundations for the 2015 paper [94], where the authors pursuing this approach of Huygens Synchronization, obtained promising results, confirming experimentally the perturbative model.

Further developments on this model led to the work [16], where the authors prove the stability of a system of coupled Huygens oscillators. This system is composed of two pendulum clocks with similar frequencies, under the perturbative model (Chapter 5).

As a continuation of these works, a recently published article [95] revisits the case of nearly equal frequencies for the two oscillators and then proceeds to explore the scenario where the frequency of one oscillator is close to a non-trivial positive integer multiple of the other (Chapter 6).

In several fields of Science, in many instances, the oscillator with the smaller frequency dictates the behaviour of the remaining oscillators. This setting of unequal frequencies leads to a particular type of synchronization, known as *master-slave* synchronization. This concept refers to the process where one system (the master) drives the dynamics of another system (the slave), forcing the slave to synchronize with the master under specific conditions. A *master-slave* relation means that the pace, i.e., the frequency of one of the oscillators, dictates the frequency of both oscillators in the final coupled, phase-locked state of the system. This is a consequence of the fact that the overall effect of the fast clock in the system has significantly less influence than that of the slow clock. Apart from the pendulums case, master-slave synchronization is a widely studied phenomenon in various scientific and engineering fields, including mathematics [96], electrical engineering [97], and biology [98].

In neuroscience, for example, oscillations in the brain often occur at different frequencies (e.g., gamma, delta, alpha, theta waves) [99]. When different neuronal populations or brain regions synchronize, it is frequently observed that slower oscillations (e.g., delta, theta, and alpha waves) tend to dominate synchronization dynamics over faster ones (such as gamma waves). For instance, theta-gamma coupling is a well-documented phenomenon in both the hippocampus and cortex. In this process, the slower theta waves (3-12 Hz) modulate the amplitude and timing of the faster gamma waves (25-160 Hz) [100]. This coupling is important for various cognitive functions, such as memory encoding and retrieval, spatial navigation, and attention [101]. A recent study also exhibits this trait in rats [102].

Another example can be found in communication systems, where phase-locked loops (PLLs) are used to synchronize signals [103]. When two oscillators are coupled in such systems, with one operating at a frequency and the other at a multiple of that frequency plus a small perturbation, the slower oscillator (typically the reference signal) dictates the overall synchronization. PLLs in wireless communications or signal processing systems maintain synchronization by adjusting the faster oscillator (voltage-controlled oscillator) to match the phase and frequency of the slower reference signal [104].

As another instance, in organisms, circadian rhythms (24-hour cycles) often synchronize other biological processes, including faster oscillatory systems such as hormone release or cellular metabolic cycles [105]. The slower circadian clock tends to impose its timing on these faster processes. The master clock in the brain (the suprachiasmatic nucleus) synchronizes peripheral clocks in organs like the liver and heart, which operate on faster metabolic cycles. Despite the faster cycles, the slower circadian rhythm remains the governing oscillator [106, 107, 108].

These examples demonstrate that synchronization in unequal oscillators is often dominated by the slower oscillator across a wide range of physical, biological, and engineering systems. As discussed above, there is abundant literature on this subject, and numerous mathematical models have been applied to study this type of synchronization. The primary scientific motivation for investigating this phenomenon is the potential universality of synchronization behaviour, where the slowest of the coexisting oscillators dictates the overall dynamics. The applications of the results presented in this work are broad, both theoretically and across the aforementioned fields of research.

Physically, master-slave synchronization involves coordinating two or more systems to operate in unison. The master system acts as the leader, dictating the desired behaviour, while the slave system adjusts its parameters or states to match the master's behaviour. This synchronization is pervasive in various applications, as seen above.

We dedicate this Part of the work to the phenomenon of Synchronization in Huygens' system of coupled clocks. In Chapter 5 we study the stability of this system through the view of the perturbative model, when two similar clocks share a mutual discrete interaction. This Chapter is an adaptation of the work [16].

In Chapter 6 we explore what happens when one of the clocks has approximately a multiple frequency of the other, interpreting the results from the perspective of the *master-slave* concept. This Chapter is an adaptation of the work [95].

Chapter 5

Stability of coupled Huygens oscillators

5.1 Introduction

Clocks are mathematically defined as isochronous oscillators, i.e., dynamical systems with limit cycles exhibiting always the same period, even when the system is slightly perturbed.

This Chapter is dedicated to the study of the stability of the orbits of clocks under external or reciprocal interaction. Here we prove the stability of the limit cycles, previously assumed by the authors in the work [94, 109].

Along this Chapter, for basic and classical definitions and notions related to synchronization, like phase and frequency, we refer to [77] and, for fundamental concepts concerning general theory of discrete and continuous dynamical systems, we refer to [110, 111, 112].

This chapter is organized as follows. In Section 5.2 we present the Andronov model for the pendulum clock [90] and the synchronization theory for nearly identical clocks [94]. In Section 5.3 we discuss the oscillation amplitude for clocks locked in phase. There is a slight change in the amplitude for locked oscillators relative to the isolated or non-coupled oscillators. We prove that the orbits resulting from perturbations of two locked Andronov Oscillators with limit cycles remain close to the original unperturbed limit cycle under small perturbations.

In Section 5.4, concerning the coupled system of two clocks, we deal with the more difficult problem of non locked systems. We then consider a variable, bounded and limited, perturbation to the coupled system. We find an open and invariant neighbourhood in the Poincaré section, containing v_f that attracts all the orbits.

In particular, when we have a number $N > 2$ of isolated Andronov clocks put in interaction at instant $t = 0$, exchanging small perturbations between them, the resulting perturbed orbits in the product phase space remain close to the original torus of the set of isolated Andronov clocks.

Finally, in Section 5.5 we obtain the Arnold tongues for the Huygens coupling of two clocks, one with frequency ω and the other with frequency $N\omega + \varepsilon$, where N is a positive integer and $\varepsilon > 0$ is a small real number.

5.2 Andronov model for two interacting clocks

5.2.1 Andronov model for the isolated clock

The model presented in this section starts from the Andronov model [90] of the phase space limit cycle of isolated pendulum clocks. In [90] the authors prove the existence and stability of the limit cycle. More information on the pendulum working system may be found in [113].

Here we briefly present the results of [94] where the authors assume the exchange of single *impacts* between the two clocks at a specific point of the limit cycle. Two coupling states are obtained, near phase and near phase opposition, the latter being asymptotically stable.

Using the angular coordinate q, the differential equation governing the motion of the pendulum clock is

$$\ddot{q}(t) + \mu \ \text{sign} \ \dot{q}(t) + \omega^2 q(t) = 0, \tag{5.1}$$

where $\mu > 0$ is the dry friction coefficient, ω is the natural angular frequency of the pendulum and sign (x) is the usual signal function.

In [90] was considered that, in each cycle, at the equilibrium position, a fixed amount of normalized kinetic energy $h^2/2$ is given by the escape mechanism to the pendulum to compensate the loss of kinetic energy due to dry friction. We call this transfer of kinetic energy a *kick*. We set the origin such that the kick is given when $q = -\mu/\omega^2$, which is very close to 0. The phase portrait is shown in Fig. 5.1.

Considering initial conditions $q\,(t = 0) = -\mu/\omega^2$ and $\dot{q}\,(t = 0) = v_0$, we draw a Poincaré section ([114] vol. II, page 268) or [111] as the half line $q = -\left(\mu/\omega^2\right)^+$ and $\dot{q} > 0$ [90]. The symbol $+$ refers to the fact that we are considering that the section is taken immediately after the kick. There is a loss of velocity $-4\mu/\omega$ due to friction during a complete cycle. Considering

$$v_n = \dot{q}\left(\frac{2n\pi}{\omega}^+\right), \tag{5.2}$$

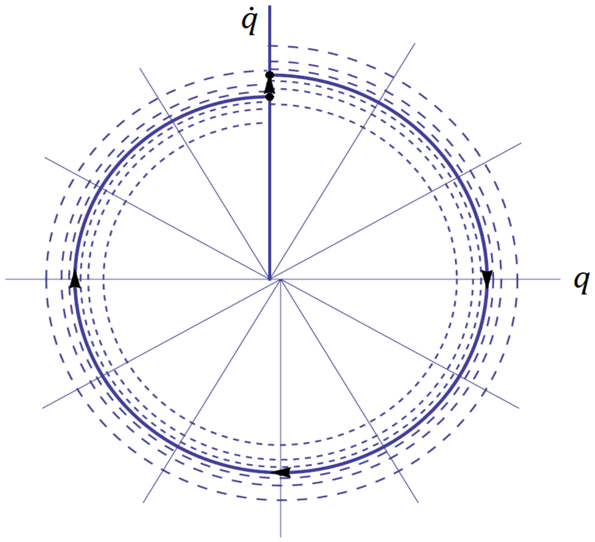


Figure 5.1: Limit cycle of an isolated clock represented as a solid curve in the phase space. The horizontal axis represents the angular position and the vertical axis represents the velocity. We use normalized coordinates to get arcs of circles.

the velocity at the Poincaré section in each cycle. One obtains [90] the return map at this section

$$v_{n+1} = P(v_n) = \sqrt{\left(v_n - \frac{4\mu}{\omega}\right)^2 + h^2}, \tag{5.3}$$

which has the asymptotically stable fixed point

$$v_f = \frac{h^2\omega}{8\mu} + \frac{2\mu}{\omega}. \tag{5.4}$$

The fixed point (5.4) attracts initial conditions v_0 in the interval

$$\left(\frac{4\mu}{\omega}, +\infty\right). \tag{5.5}$$

5.2.2 Model for two Andronov clocks under mutual interaction

Hypotheses

In this subsection we continue to present the results of [94]. We consider two pendulum clocks suspended on the same wall. When one clock receives the kick, the impact propagates in the wall, slightly perturbing the second clock. The perturbation is assumed to be instantaneous since the time of travel of mechanical waves (related to the inverse of the speed of sound) in the wall between the clocks is assumed to be very small compared to the period. The interaction was studied geometrically and qualitatively by Abraham [92, 93].

Consider two oscillators indexed by $i = 1, 2$. Each oscillator satisfies the differential equation

$$\ddot{q}_i + \mu_i \text{ sign } \dot{q}_i + \omega_i^2 q_i = -\alpha F(q_j), \text{ for } i, j = 1, 2,\ i \neq j, \tag{5.6}$$

when $q_i = -\mu_i/\omega_i^2$, the kinetic energy of each oscillator is increased by the fixed amount $h_i^2/2$ as in the *Andronov model*. The coupling term is the normalized force $-\alpha F(q_j)$ with acceleration dimensions, where F is the interaction function and α a very small positive constant. We consider that the effect of the interaction function F is to produce an increment $-\alpha$ in the velocity of each clock leaving the position invariant when the other is struck by the energy kick[1].

The sectional solutions of the differential equation (5.6) are obtainable when the clocks do not suffer kicks. To treat the effect of the kicks we construct a discrete dynamical system for the phase difference. The idea is similar to the construction of a Poincaré section. If there exists an attracting fixed point for that dynamical system, the phase locking occurs. The details of the construction were presented in [94].

Our assumptions for the phenomenon of synchronization of non identical clocks are

1. Dry friction.

2. The pendulums have natural angular frequencies ω_1 and ω_2 near each other, with $\omega_1 = \omega + \varepsilon$ and $\omega_2 = \omega - \varepsilon$, where $\varepsilon \geq 0$ is a small parameter. Since the clocks have the same construction, the energy dissipated at each cycle of the two clocks is the same, $h_1^2/2 = h_2^2/2 = h^2/2$. The friction coefficient is the same for both clocks, $\mu_1 = \mu_2 = \mu$.

3. The perturbation in the momentum is always in the vertical direction in the phase space [92, 93].

4. The perturbation imposes a discontinuity in the momentum but not a discontinuity in the dynamic variable [92, 93]. This assumption follows the lines of Andronov Model [90] and the ideas of [92, 93] where a fast impact is ideally

[1] We can consider that the interaction function is the Dirac delta distribution $\delta\left(q_j + \mu_j/\omega_j^2\right)$, giving exactly the same result.

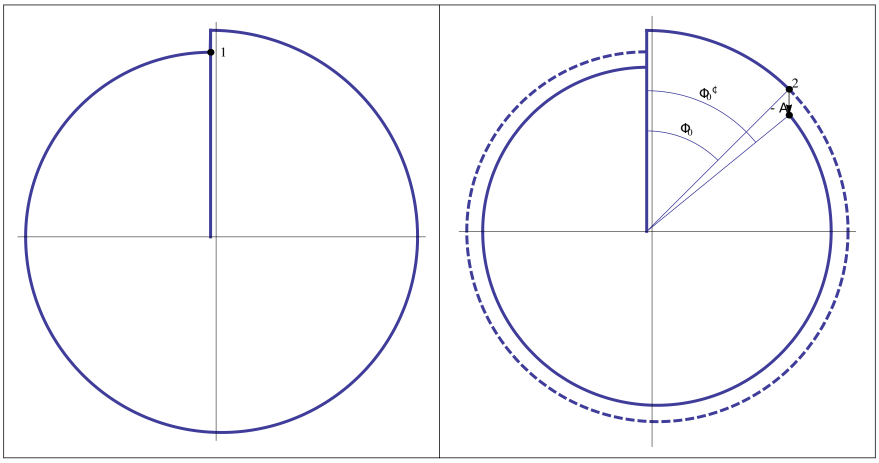


Figure 5.2: Interaction of clock 1 on clock 2 at $t = 0$. We see the original limit cycle, before interaction, and the new one in solid and the original limit cycle in dashed. Note that the value of α and of h are greatly exaggerated to provide a clear view. The effect of the perturbation is secular and cumulative.

represented by an instantaneous jump in the momentum, which, in fact, is a simplification of the real situation.

5. The perturbative interaction is instantaneous.

6. The interaction is symmetric, the coupling has the same constant α when the clock 1 acts on clock 2 and conversely. In our model we assume that α is very small[2].

Naturally, a system that follows these assumptions cannot be globally continuous, nor differentiable. Nevertheless, the system used by us is sectionally continuous and differentiable.

To prove phase locking we solve sectionally the differential equations (5.6) with the two small interactions. Then, we construct an iteration, giving a discrete dynamical system, taking into account the two interactions per cycle seen in Fig. 5.2 and 5.3. After that, we compute the phase difference when clock 1 returns to the initial position. The secular repetition of perturbations leads the system to near phase opposition as we can see by the geometrical analysis of Fig. 5.2 and 5.3.

[2]We are not interested in the particular form of the wave that propagates in the wall, which can be extremely complicated. We are only interested in the effective result of the perturbation in the movement of each clock in the phase space.

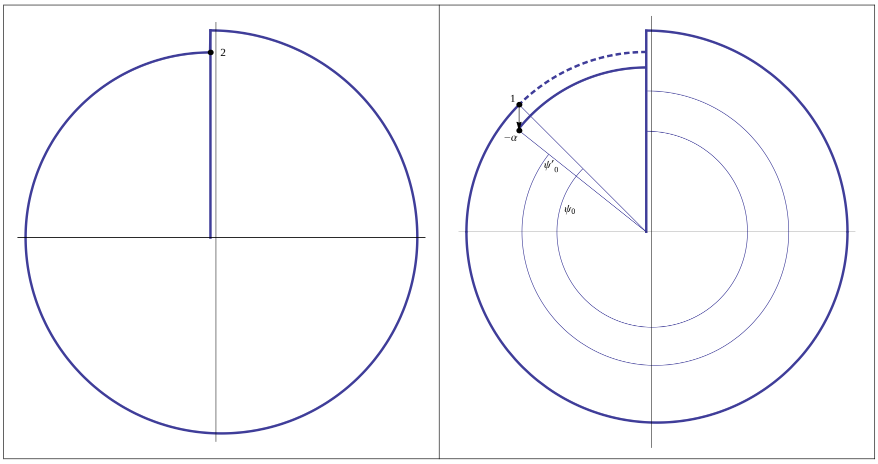


Figure 5.3: Second interaction. Interaction of clock 2 on clock 1 when clock 2 reaches its impact position. All the features are similar to the Fig. 5.2.

Generalizing the above discussion for any cycle $n+1$ relative to the previous cycle n we obtain the map for the phase difference between the two clocks for $n \geq 0$. We denote this map Ω. Naturally,

$$\phi_{n+1} = \Omega\left(\phi_n\right). \tag{5.7}$$

Now, Ω is a map from the interval $[0, 2\pi]$ to itself. Despite the apparent complexity of Ω see [94] or section 5.5 at the end of this Chapter, this map is relatively manageable. Ω has a stable fixed point and a unstable one for small parameters α and ε. The value of the stable fixed point ϕ_f is near to π and the unstable is near 0. The phase difference is asymptotic to the solution ϕ_f. Knowing this value, it is possible to prove the existence and stability of the limit cycle of each clock in interaction and the final asymptotic frequency ω_f. Under this model we can say that Huygens sympathy occurs.

Simplified iterative scheme

We consider the Taylor polynomial of Ω in first order of α and ε. From [94] we have the map

$$\phi_{n+1} = \Omega\left(\phi_n\right) = \phi_n + 4\pi \frac{-\varepsilon}{\omega+\varepsilon} + \frac{2\alpha \sin \phi_0}{\gamma\left(\phi_0, \omega\right)} + h.o.t, \tag{5.8}$$

We define the map

$$\Xi\left(x\right) = x + 4\pi \frac{-\varepsilon}{\omega+\varepsilon} + \frac{2\alpha \sin x}{\gamma\left(x, \omega\right)}. \tag{5.9}$$

We get in first order of α and ε the difference equation for the phase difference

$$\phi_{n+1} = \Xi\left(\phi_n\right) \tag{5.10}$$

There are two fixed points ϕ_f^s and ϕ_f^u of Ξ in the interval $[0, 2\pi]$

$$\phi_f^s = \pi - \arcsin\frac{\pi h^2 \varepsilon}{4\alpha\mu},\ \phi_f^u = \arcsin\frac{\pi h^2 \varepsilon}{4\alpha\mu}. \tag{5.11}$$

The derivative of Ξ at the fixed point ϕ_f^s must be $|\Xi'\left(x_f\right)| < 1$ to have stability and the condition about the argument of the function arcsin gives

$$\frac{\pi h^2}{4\mu}\varepsilon < \alpha < \frac{h^2\omega}{8\mu},\ \text{i.e.,}\ \frac{h^2}{8\mu}\pi\Delta\omega < \alpha < \frac{h^2}{8\mu}\omega. \tag{5.12}$$

Therefore, the limit of the phase difference is, in first order,

$$\phi_f^s = \pi - \arcsin\frac{\pi h^2 \varepsilon}{4\alpha\mu}, \tag{5.13}$$

which is very near to π when the natural frequencies of both clocks are very near, i.e., small ε.

5.3 Stability for two clocks in locked state

In this section, we see what happens to the amplitude when two Andronov clocks interact at locked (synchronized) states at phase opposition or very near phase opposition. The amplitude of oscillation increases slightly. We considered that the perturbation is constant at every oscillation, which is correct if the clocks remain at a synchronized locked state. If the perturbation is small enough the conclusion is that there exists a fixed point near the original one for each clock. Since the frequencies are the same at the locked state, that perturbation occurs once per cycle.

We recall that the difference equation (5.3)

$$v_{n+1} = P(v_n) = \sqrt{\left(v_n - A\right)^2 + h^2}, \qquad A = \frac{4\mu}{\omega}, \tag{5.14}$$

has the attractive fixed point

$$v_f = \frac{A^2 + h^2}{2A}, \tag{5.15}$$

already obtained in equation (5.4), which attracts initial conditions v_0 in the interval $\left]A/\omega, +\infty\right[$.

We define the amplitude of the movement of the pendulum for the isolated Andronov clock as exactly this value v_f, which is the maximum of the angular velocity.

The amplitude of the oscillation in the case of two clocks increases slightly in the final state.

We focus our analysis in the velocity at $q = -\mu^+/\omega^2$ and $\dot{q} > 0$, i.e., at the Poincaré section.

We can work each clock separately because the changes in the period do not affect the Poincaré map. Since the mutual interaction affects each clock when they are in phase opposition, the equation for the Poincaré map for each generic oscillator is now

$$v_{n+1} = \sqrt{(v_n - A + \alpha)^2 + h^2}. \tag{5.16}$$

This equation has the asymptotically stable fixed point

$$v^* = \frac{(A-\alpha)^2 + h^2}{2(A-\alpha)}. \tag{5.17}$$

We note that v^* is the maximum of the amplitude of the oscillation for the mutual interacting oscillators. It is slightly greater than v_f of the isolated clock. More precisely, in first order in α, and recalling that $h > 4\mu/\omega = A$, we get

$$v^* = v_f + \frac{1}{2}\left(\frac{h^2}{A^2} - 1\right)\alpha, \qquad A = \frac{4\mu}{\omega}. \tag{5.18}$$

5.4 Stability for two non-locked clocks

Admitting phase locking, the stability of the amplitudes is clear. However, if we do not assume phase locking, the perturbation between two clocks strongly depends on the phase difference at which it occurs. For instance, at the instant of the impulse, if the phase difference between both clocks is π, the affected clock receives an increment of $-\alpha$ in its negative velocity and the radius of its limit cycle has an increment of α. If, by contrast, the phase difference is zero, then the positive velocity receives the same increment of -α reducing its absolute value and the radius has a variation of $-\alpha$. See Fig. 5.2 to better understand this reasoning.

To cope with this situation, without establishing an extremely complicated iteration relating phase and amplitude, we define a perturbation function γ, function of the mutual phase difference of the two clocks

$$-\alpha \leq \gamma(\phi_n) \leq \alpha. \tag{5.19}$$

Knowing some properties of γ we can prove some stability results without knowing its exact functional dependence on the phase difference. Now, we do not know the exact nature of the ω-limit, but we can establish the existence of a trapping closed interval I in the sense that the ω-limit of the phase dynamics must be contained in I.

The endpoints of I are respectively

$$\frac{(A+\alpha)^2+h^2}{2(A+\alpha)} \text{ and } \frac{(A-\alpha)^2+h^2}{2(A-\alpha)}. \tag{5.20}$$

We consider clocks $C1$ and $C2$. We do not know exactly the new actual map P_γ that governs the iteration of $C1$ (a similar map Q_γ will act on $C2$). We know that this map depends on the function γ. The iteration on the Poincaré section of $C1$ is given by

$$x_{n+1}^I = P_{\gamma(\phi_n)}\left(x_n^I\right), \tag{5.21}$$

analogously for $C2$,

$$x_{n+1}^{II} = Q_{\gamma(\phi_n)}\left(x_n^{II}\right). \tag{5.22}$$

For Andronov clocks, P and Q, we have the expressions

$$\begin{aligned} P_{\gamma(\phi_n)}\left(x_n^I\right) &= \sqrt{\left(x_n^I - A + \gamma\left(\phi_n\right)\right)^2 + h^2}, \\ Q_{\gamma(\phi_n)}\left(x_n^{II}\right) &= \sqrt{\left(x_n^{II} - A + \gamma\left(2\pi - \phi_n\right)\right)^2 + h^2}, \end{aligned} \tag{5.23}$$

$\forall_{n\geq 0}\ \forall_{x\geq(A+\alpha)/\min(\omega^I,\omega^{II})}$, which have a small extra term inside the square root compared with the original Poincaré maps for the isolated oscillators,

$$P\left(x_n\right) = \sqrt{\left(x_n - A\right)^2 + h^2}. \tag{5.24}$$

Naturally, we have two strictly increasing functions,

$$M\left(x\right) = P_\alpha\left(x\right) = \sqrt{(x-A+\alpha)^2+h^2} \tag{5.25}$$

and

$$m\left(x\right) = P_{-\alpha}\left(x\right) = \sqrt{(x-A-\alpha)^2+h^2}. \tag{5.26}$$

Both M and m satisfy the fixed point theorem, that is, have the asymptotically stable fixed points, respectively

$$\frac{(A-\alpha)^2+h^2}{2(A-\alpha)} \tag{5.27}$$

and

$$\frac{(A+\alpha)^2+h^2}{2(A+\alpha)}. \tag{5.28}$$

We consider only the study of clock $C1$, the study of $C2$ is similar. The orbit on the Poincaré section of $C1$ is denoted by

$$\{x_n\}_{n=0,1,2,\dots}. \tag{5.29}$$

We drop the superscript I for the sake of notational simplicity. We do not present the family $P_{\gamma(n)} = P_n$ of functions $P_n : \mathbb{R}^+ \to \mathbb{R}^+$. We consider only the fact

$$m(x) \le P_n(x) \le M(x), \ \forall_{n\ge 0}, \ \forall_{x \ge (A+\alpha)/\min(\omega^I,\omega^{II})}. \tag{5.30}$$

We do not consider initial conditions less than $(A+\alpha)/\min\left(\omega^I,\omega^{II}\right)$ to assure that the clock movement will persist in any circumstances.

The process of iteration on the right starts for initial conditions

$$x_0 > \frac{(A-\alpha)^2 + h^2}{2(A-\alpha)}. \tag{5.31}$$

Now, we adopt the notation that x_n are the iterations of P_n and y_n are the iterations of M. We consider the initial condition $x_0 = y_0$, by equation 5.30, we write

$$x_1 = P_0(x_0) \le M(x_0) = y_1. \tag{5.32}$$

Consequently, in the next iteration, by equation 5.32 plus the strictly increasing property of M, we have

$$x_2 = P_1(x_1) \le M(x_1) \le M(y_1) = y_2, \tag{5.33}$$

and, by induction, we see that

$$x_{n+1} = P_n(x_n) \le M(x_n) \le M(y_n) = y_{n+1}. \tag{5.34}$$

Therefore, we apply the limit superior at both sides of 5.34. We get

$$\limsup_{n\to+\infty} P_n(x_n) \le \limsup_{n\to+\infty} M(y_n). \tag{5.35}$$

The limit superior of M is the same as its limit. By 5.27, we write

$$\limsup_{n\to+\infty} x_n \le \frac{(A-\alpha)^2 + h^2}{2(A-\alpha)}. \tag{5.36}$$

Now, we consider the iterations on the left of

$$x_0 < \frac{(A+\alpha)^2 + h^2}{2(A+\alpha)}. \tag{5.37}$$

Here, y_n are the iterates of m. Considering $x_0 = y_0$, once again, by equation 5.30, we have

$$x_1 = m(x_0) \leq P_0(x_0) = y_1. \tag{5.38}$$

By similar arguments used from 5.32 to 5.34, it is true that

$$y_{n+1} = m(y_n) \leq m\left(x_n\right) \leq P_n\left(x_n\right) = x_{n+1}. \tag{5.39}$$

Now we apply the limit inferior of both members of the previous inequality, that is,

$$\liminf_{n\to+\infty} m(y_n) \leq \liminf_{n\to+\infty} x_n, \tag{5.40}$$

and, once again by 5.28, we have

$$\frac{(A+\alpha)^2+h^2}{2(A+\alpha)} \leq \liminf_{n\to+\infty} x_n. \tag{5.41}$$

It is easy to see, using similar arguments, that when the initial conditions lie between

$$\frac{(A+\alpha)^2+h^2}{2(A+\alpha)} \text{ and } \frac{(A-\alpha)^2+h^2}{2(A-\alpha)}, \tag{5.42}$$

the orbits x_n of P_n cannot cross the fixed points of m and M and will remain between those two points.

From equations 5.36 and 5.41, we know that, from a certain order $n > k$, x_n becomes confined to a certain interval, i.e.,

$$\begin{gathered} \forall\delta > 0,\ \exists k \in \mathbb{N} : \forall n > k, \\ x_n \in \left]\frac{(A+\alpha)^2+h^2}{2(A+\alpha)} - \delta, \frac{(A-\alpha)^2+h^2}{2(A-\alpha)} + \delta\right[, \end{gathered} \tag{5.43}$$

meaning that the closed interval

$$\left[\frac{(A+\alpha)^2+h^2}{2(A+\alpha)}, \frac{(A-\alpha)^2+h^2}{2(A-\alpha)}\right], \tag{5.44}$$

will trap the dynamics of P_n, as we wanted to show. The actual Poincaré map is therefore inside a band delimited by the maps m and M, as depicted in Fig. 5.4. In particular, the coupling of two Andronov clocks corresponds to a small periodic perturbation to each one of the clocks satisfying the hypotheses of this Section.

Even in the case were there is no phase locking, the coupling of two Andronov clocks can only produce a small perturbation to the original two-dimensional torus of the two isolated clocks in their Cartesian product phase space. The final state under perturbation is changed to a blurred torus. The trajectories governed by the dynamics of P_n lie in the neighbourhood of the original trajectories, so, the blurred torus will also be in the neighbourhood of the original 2-dimensional unperturbed torus. The system of two coupled clocks does not break apart.

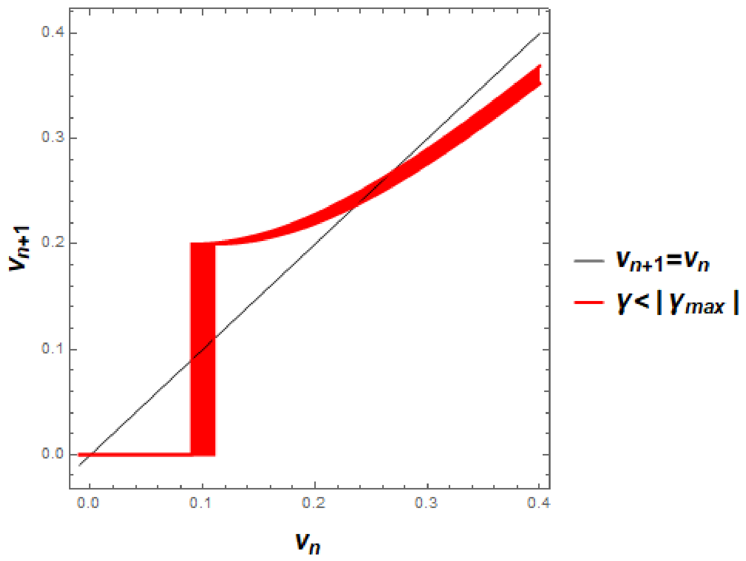


Figure 5.4: The red area is delimited by upper and lower functions m and M,respectively, enclosing all the possible graphs of the functions in the family P_n.

5.4.1 Generalization for N clocks

The generalization for $N > 2$ of the result on stability of the limit cycles does not depend on the particular number of clocks, but only on the fact that the perturbations are small enough.

Each clock receives a finite number of impacts per cycle. Consequently, for a finite number of clocks under small impactive perturbations, the perturbed orbits will remain close to the original non perturbed fixed points given that the sum of all the perturbations is kept small enough. The proof is a simple generalization of the two clocks case.

The existence of phase locking is immaterial concerning stability of the limit cycles of the original differential equations.

5.5 Arnold tongues

In this final section we exemplify our work with Arnold tongues showing some examples obtained numerically.

In [94] we defined the map ς

$$\varsigma(\phi^i, \omega^i, \alpha^i) = \arctan\left(\sin(\phi^i), \cos(\phi^i) - \frac{\alpha^i}{\gamma(\phi^i, \omega^i)}\right), \quad i = 1, 2, \tag{5.45}$$

where ϕ^i and ω^i are the phase and frequency of the clock i, and γ is defined in [94].

We defined also $r(x, d) = (2\pi - x)d$, with two possible frequency relationships:

$$d = \frac{\omega_1}{\omega_2} = d_1 \text{ and } d = \frac{1}{d_1} = d_2. \tag{5.46}$$

For two clocks with near frequencies we obtained in [94] that the phase difference is the composition of four maps

$$\phi_{n+1} = r\Big(\varsigma\Big(r\big(\varsigma(\phi_n, \omega_2, \alpha), d_1\big), \omega_1, \alpha\Big), d_2\Big). \tag{5.47}$$

As in section 5.2.2 in equation (5.7) we can write this equation in a more compact form

$$\phi_{n+1} = \Omega_{1:1}(\phi_n), \tag{5.48}$$

where the index 1 : 1 represents that the frequencies of the two clocks are very near, i.e., in a relation of 1 to 1. When the hypotheses of subsection 5.2.2 and conditions (5.12) are fulfilled we find one stable fixed point, ϕ_f^s, and one unstable fixed point, ϕ_f^u, whose values, in first order approximation, are

$$\phi_f^s = \pi + \arcsin\left(\frac{\pi h^2 \varepsilon}{8\alpha\mu}\right) \text{ and } \phi_f^u = -\arcsin\left(\frac{\pi h^2 \varepsilon}{8\alpha\mu}\right) \tag{5.49}$$

where now we have the frequency ω_1 of clock 1 and the frequency $\omega_2 = \omega_1 + \varepsilon$ of clock 2.

When natural angular frequencies are very near, that is, for small values of ε, we get $\phi_f^s \approx \pi$ and $\phi_f^u \approx 0$. Our interest lies essentially on the stable fixed point, whose value indicates that the system of two coupled clocks with similar natural oscillation frequencies tend to synchronize in phase opposition. In Fig. 5.5 we can see the effect that the parameters α and ε have on the map $\Omega_{1:1}$.

We can construct a similar map composition for frequency relation 2 : 1, where the fastest clock completes around 2 turns for each turn of the reference clock, that is, $\omega_2 = 2\,\omega_1 + \epsilon$. The phase difference is measured always when the slowest clock returns to the vertical position, i.e., when the representative point in phase space for the slowest clock returns to the Poincaré section.

These maps can be constructed for n : 1 frequency relations. The case 2 : 1 the discrete dynamical system governing the phase difference is given by the composition of six maps,

$$\Omega_{2:1}(\phi_n) =$$
$$2\pi + \tfrac{\omega_2}{\omega_1}\bigg(2\pi - \sigma\bigg(\sigma\Big(\tfrac{\omega_1}{\omega_2}\big(2\pi - \sigma(\phi_n, \omega_2, \alpha)\big), \omega_1, \alpha\Big) + 2\pi\tfrac{\omega_1}{\omega_2}, \omega_1, \alpha\bigg)\bigg). \tag{5.50}$$

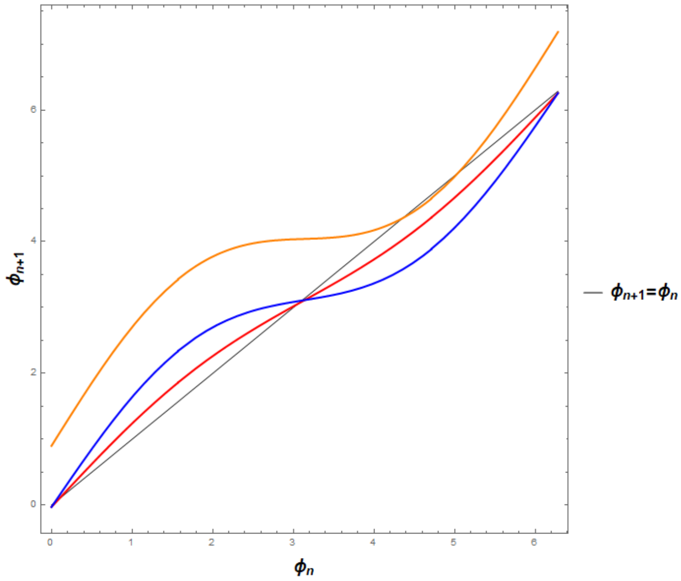


Figure 5.5: Dynamics of the phase difference for two clocks. An increase in α increases the amplitude of oscillations in the curve, while a smaller α corresponds to a softer curve (blue and red curves, respectively). An increase ε, shifts the curve (yellow), but for a sufficiently large α, we can still find fixed points.

The values for the fixed points found for the map $\Omega_{2:1}$ are

$$\phi_f^s = \pi + \arcsin\left(\frac{h^2\pi\epsilon}{2\alpha\mu}\right) \text{ and } \phi_f^u = -\arcsin\left(\frac{h^2\pi\epsilon}{2\alpha\mu}\right), \tag{5.51}$$

which differs from the values found for the fixed points of the map $\Omega_{1:1}$ by a factor of 1/4 in the argument of the arcsin function. For $\varepsilon = 0$, the fixed points coincide in first order for maps $\Omega_{1:1}$ and $\Omega_{2:1}$. However, when ε increases, the shape of $\Omega_{2:1}$ near the fixed point is much more sensitive to ε as we can see in Fig. 5.5, we observe a shift of the intersection $\phi_{n+1} = \phi$ to the right more rapidly for map $\Omega_{2:1}$ than for $\Omega_{1:1}$.

The complexity of these maps increase with the multiples N of the frequency. We cope with this performing linear expansions and simplifications on the model to analyse the question of stability.

As in section 5.2.2 we define the linear approximation of $\Omega_{N:1}$ as

$$\phi_{n+1} = \Xi_{N:1}(\phi), \tag{5.52}$$

where

$$\Xi_{N:1}(\phi) = \phi + (N-1)2\pi + \frac{\varepsilon}{\omega}2\pi + \alpha\frac{\sin\phi}{\gamma(\phi, N\omega+\varepsilon)} - \alpha\sum_{i=1}^{N}\frac{\sin(r(\phi,N))}{\gamma(r(\phi,N),\omega)}, \tag{5.53}$$

and where

$$r(\phi, N) = \frac{2N\pi - \phi}{N\omega + \varepsilon}\,\omega. \tag{5.54}$$

We now want to obtain ε and α, i.e., frequency difference and coupling strength, such that the the maps $\Xi_{N:1}$ have exactly one fixed point (left semi-stable and right semi-unstable). The curves connecting those pairs (ϵ, α) of parameters in the two-dimensional parameter space are the *Arnold tongues* separating the region where the fixed points are attracting from the region where the fixed points are repelling [77], thus separating synchronization and non synchronization regions in the parameter space.

We perform a graphical analysis of the maps $\Xi_{N:1}$. As seen in Fig. 5.5, every change in ε corresponds to a shift of the dynamics upwards or downwards, while a change in α changes the shape of the curve. Graphically this was already observed for the map $\Omega_{1:1}$. However, due to the increasing complexity of the maps with N, the changes in the parameters α and ε lead to very complicated effects. For instance, for maps $\Omega_{2:1}$ and $\Omega_{3:1}$, any change in α also affects the response with ε and *vice versa*.

Algorithm

To produce the graphs with the Arnold tongues we begin to find two fixed points for each map for fixed values of ε. Then, gradually, we reduce the value of α, at a certain point the two fixed points join together to become one semi-stable semi-unstable fixed point, i.e., we find numerically the saddle-node bifurcation point. We then follow the same procedure incrementing ϵ by small amounts.

The resulting Arnold tongues for $N = 1, 2, 3$ are shown in Figs. 5.6, 5.7 and 5.8.

Effect of friction on Arnold tongues

We consider now the dependence of the Arnold tongues with μ, the dry friction coefficient. Considering equation 5.3, if we increase the action of the dry friction, the radius of the limit cycle decreases greatly along the limit cycle.

Consequently, in order to maintain the stability of the oscillations (which means to guarantee the existence of the limit cycle), a higher value of h must be given by the system.

Thus, more kinetic energy is necessary to compensate the energy loss in the the gears of the physical clocks. A higher value of h implies necessarily a higher value of α.

We use the values of the fixed points fulfilling the saddle-node conditions to study the relation between μ and α and its effect in the Arnold tongues

$$\phi_f^s = \phi_f^u \Leftrightarrow \alpha_t = \frac{h^2}{2\mu}\epsilon, \tag{5.55}$$

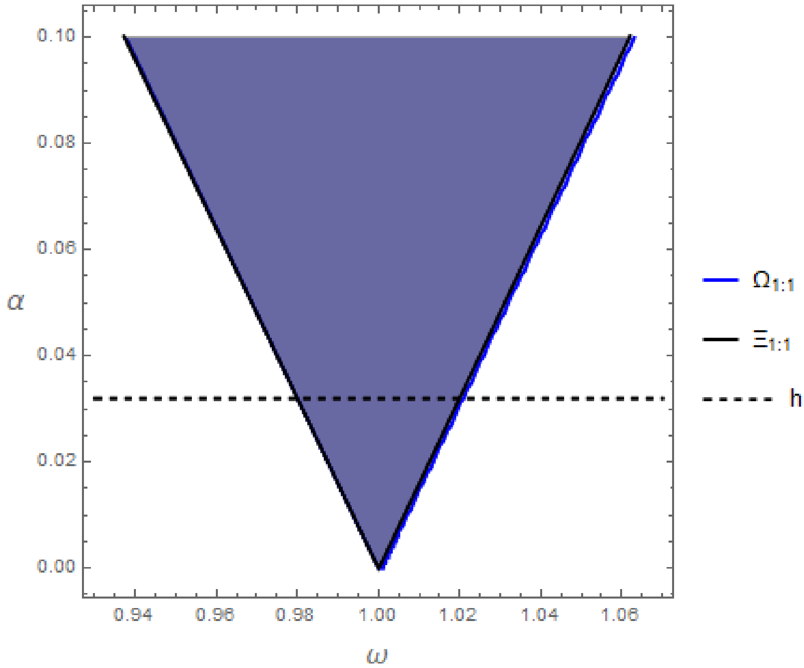


Figure 5.6: Comparison between Arnold tongues of map without approximations $\Omega_{1:1}$ (blue) with its the linearised map $\Xi_{1:1}$ (black). We find a very reasonable match, such that both tongues are practically overlapped.

which implies that, when we increase the dry friction coefficient, the value of α_t that marks the stability threshold decreases. Thus, the absolute value of the slope of the Arnold tongues, at any value of N fixed, is smaller for a higher value of μ. This is valid near the origin, that is, for small values of α and ϵ, since the relation $\phi_f^s = \phi_f^u$ (5.55) is based on the first order approximation values ϕ_f^s and ϕ_f^u, computed in section 5.2.

In Fig. 5.9 we observe that a higher value of μ provides a larger width to the Arnold tongues. The effect of the friction, through this analysis, is to increase the stability of the system, becoming easier for both clocks to synchronize.

5.6 Discussion

In Section 5.2 we provided a description of the Andronov model for the isolated clock, and for two Andronov clocks under a mutual discrete interaction. The map of

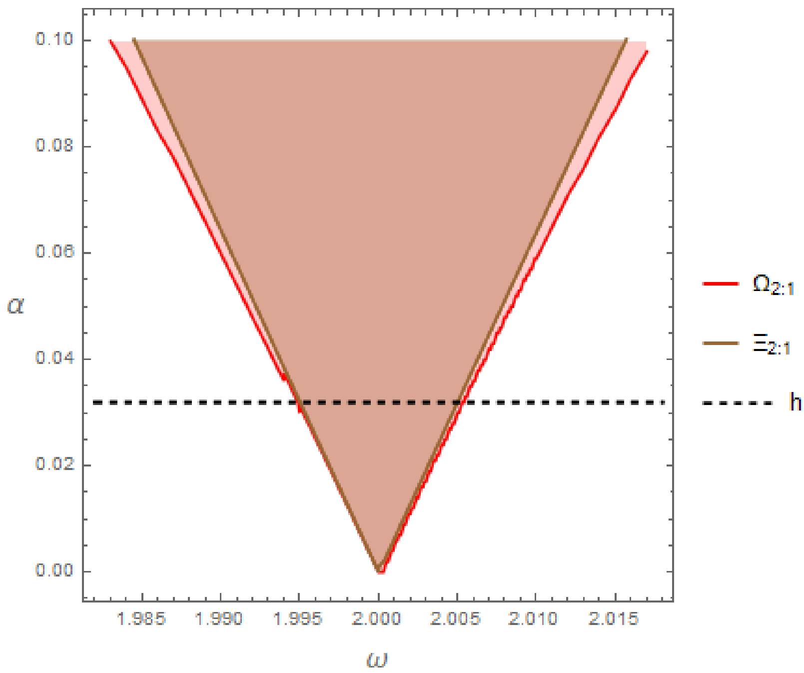


Figure 5.7: Comparison between Arnold tongue of original map $\Omega_{2:1}$ (red) with its linearised approximation $\Xi_{2:1}$ (brown). We observe a very good match for $\alpha < h$, and much more for the negative slope of the tongue, corresponding to $\omega_2 = 2\,\omega - \varepsilon$.

the phase difference between the two clocks exhibits a stable fixed point, near π, and an unstable fixed point, near 0, coherently to what is observed in reality, *i.e.*, phase opposition synchronization.

In Section 5.3, under phase locking, we analysed the velocity of each clock at the Poincaré section. The form of the velocity map for the coupled system shows that the amplitude of the limit cycle increases slightly when compared to that of the isolated clock. Moreover, the orbits resulting from perturbations of two locked Andronov oscillators with limit cycles remain close to the original unperturbed limit cycle under small perturbations.

Once assured the stability for the coupled system in phase locking under small perturbations, in Section 5.4 we study the non-locked regime of oscillations. In such situation, one cannot guarantee a constant perturbation throughout the cycles, as one clock perceives different intensities of the other clock's kick, according to the phase the former is at. Thus, we considered a bounded and limited variable perturbation to the coupled system, and found an open and invariant neighbourhood in the Poincaré section, containing v_f that attracts all the orbits.

To summarize the results presented in this Chapter, in the case where Hypotheses

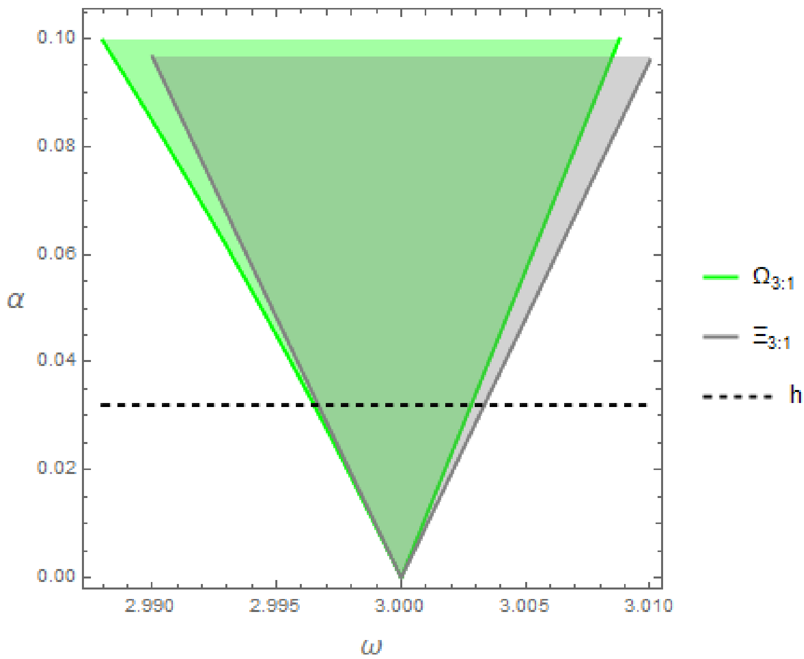


Figure 5.8: Comparison between Arnold tongue of original map $\Omega_{3:1}$ (green) with its linearisation $\Xi_{3:1}$ (gray). We observe a very good match for small α, being better for the negative slope of the tongue, corresponding to $\omega_3 = 3\,\omega - \varepsilon$.

of subsection 5.2.2 and conditions (5.12) are fulfilled, the theory of synchronization developed in [94, 109] can be applied in three steps:

1. We assure the stability of the limit cycles under perturbation, as proved in Section 5.4.

2. We prove phase locking as we did in [94]. This can be done knowing that the radii of the limit cycles are very near to the radii of the original isolated clocks.

3. Having established the existence of phase locking, we can close the reasoning using the result of section 5.3 to prove the existence of a closed curve in the phase space of the coupled system.

As the phenomenon of Huygens' synchronization is justified on the basis of the presented model, in the next Chapter we present two interesting Theorems as a result of further developments.

As we will see, when the frequency of the two clocks are, respectively, ω and $N\omega + \varepsilon$, with $N > 1$, the clocks will synchronize, respectively, at ω and $N\omega$, as the

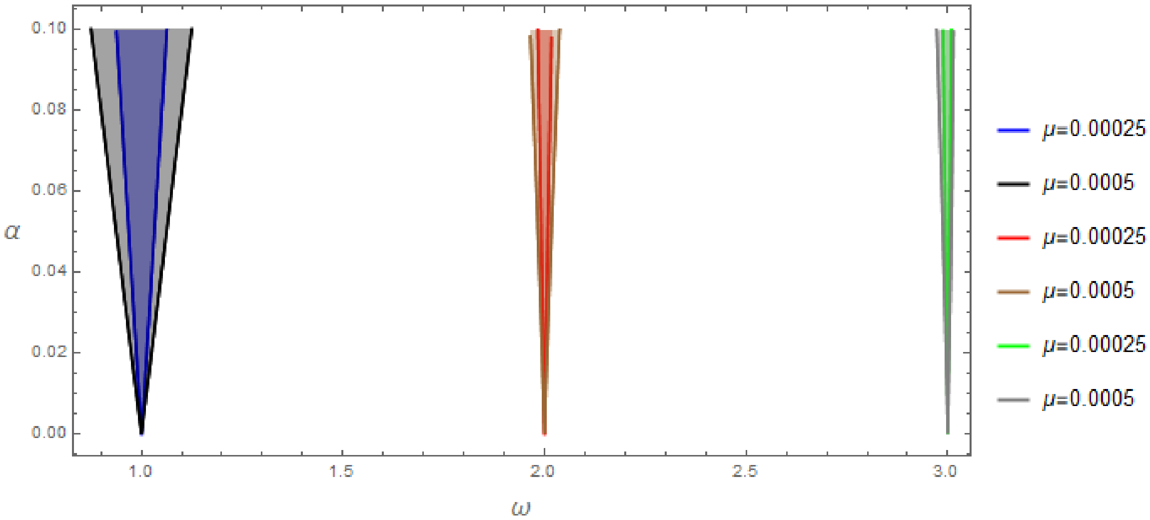


Figure 5.9: Comparison of Arnold tongues for different values of dry friction. We chose the values $\mu = 0.00025$ (1 : 1 (blue), 2 : 1 (red) and 3 : 1 (green)) and $\mu = 0.0005$ (1 : 1 (black), 2 : 1 (brown) and 3 : 1 (gray)) the adequate ones to compare the width at half height.

effect of ε disappears. In such scenario, we say that the slowest clock is the master and the fastest clock is the slave[3].

[3]For the concept of master and slave synchronization see [77].

Chapter 6

The interaction slow-clock-fast-clock is master-slave

6.1 Introduction

In this Chapter we make a precision in the definition of clock for the purpose of the proofs of the results presented below. A clock is an isochronous oscillator whose the amplitude of oscillations is independent of the initial conditions [90], i.e., its phase portrait is an attracting limit cycle, and the system has a very stable period even when subjected to small perturbations, on the other hand there is a field of isochrones in the vicinity of the limit cycle [115].

The synchronization behaviour between two oscillators, one operating at a frequency ω and the other at a frequency near a multiple of the first one, $n\omega$, where $n > 1$, is investigated. It is observed that such systems tend to exhibit a consistent pattern of master-slave relationship. This phenomenon is geometrically elucidated by considering the interaction dynamics: while the multiple bursts of the faster oscillator tend to cancel each other out upon impacting the slower oscillator, the single burst of the slower oscillator on the faster has a secular effect. We propose that this synchronization pattern, arising from perturbative impacts, is prevalent across various physical systems. Further experimental validation of these findings could contribute to a deeper understanding of synchronization phenomena and their implications in natural and technological domains.

In this Chapter we investigate the synchronization and stability of mutually coupled oscillators through small impacts and explore both master-master and master-slave synchrony. We start by introducing a simple model for the synchronization of two oscillators, described by two planar ODEs with limit cycles that are circles. In this model, whenever one of the representative points reaches a particular phase mod 2π, the other oscillator receives a kick (which we will make precise later) in its velocity.

When the mutual influence of symmetric oscillators is reciprocal, the type of synchrony generally obtained is called *master-master* or symmetric [77]. This is the typical situation of identical clocks perturbing each other, as seen in [94, 16].

We organize this Chapter as follows. In Section 6.2 we prove that when two oscillators with circular limit cycles interact with impulsive coupling at the same frequency, the resulting synchronization is *master-master*, as implied in [94]. The more interesting situation arises when one of the oscillators has a multiple frequency of the other, resulting in *master-slave* synchronization, a phenomenon that has not, to our knowledge, been explicitly noted in the literature. While the proof is not intricate, the consequences of this synchronization can be far-reaching, as discussed at the beginning of Part II, in this case the slow oscillator governs the coupled system giving a possible explanation for the above mentioned phenomena across different research areas.

In section 6.3 we explore what happens when the limit cycle is not exactly circular, but approximates a circle, as in the case of pendulum clocks on the same fixed wall, using the Andronov model. In this case, synchronization between clocks with related frequencies also results in master-slave dynamics under general and natural assumptions.

In section 6.4 we present simulations for the case of circular limit cycles and the Van der Pol equation [116, 117, 118]. These simulations corroborate the mathematical findings outlined in the theoretical sections.

6.2 Circular limit cycle

6.2.1 Case of near frequencies

We consider two planar ordinary differential equations with a circle of radius 1. Without loss of generality, any radius can be re-parametrized to 1 through a change of variables. We refer to this limit cycle as a *limit circle*. The theory presented in [94, 16] can be applied to this case, as in those works, the limit cycles were considered to be sectionally circular.

In the first case, we examine two almost identical angular frequencies: a real value $\omega > 0$ for oscillator 1 and another real value $\omega + \varepsilon > 0$ for oscillator 2. Here, the absolute value of ε is small, but it can be either positive or negative. We consider the scenario where a perturbation is applied to the reciprocal oscillator when the phase of each oscillator is $0 \mod 2\pi$, i.e., at the burst/fire point of each clock. This perturbation is assumed to consistently occur in the same vertical direction and is represented by a small real number. The sign of α will determine phase or anti-phase synchrony, respectively.

One example of such a system is the well-known differential equation, which has the feature of circular limit cycles [119, 117]

$$\begin{cases} \dot{x}_1 = x_1 + \omega y_1 - x_1(x_1^2 + y_1^2) \\ \dot{y}_1 = -\omega x_1 + y_1 - y_1(x_1^2 + y_1^2) \end{cases}$$
$$\begin{cases} \dot{x}_2 = x_2 + (\omega + \varepsilon) y_2 - x_2(x_2^2 + y_2^2) \\ \dot{y}_2 = -(\omega + \varepsilon) x_2 + y_2 - y_2(x_2^2 + y_2^2) \end{cases} \tag{6.1}$$

with initial conditions $x_1(0) = 0$, $y_1 = 1$, $x_2(0) = x_2^0$, $y_2(0) = y_2^0$. The origin is a repeller, near the origin the system has a surplus of energy, and far from the origin, the system is dissipative. It is a simple exercise to obtain the attractive limit cycle with a radius of 1.

Another example is the well known Van der Pol equation [116] for values of the parameter μ near zero.

Definition 6.2.1. *Burst or fire points, for the first (respectively second) planar equation we define its burst or fire point whenever a particular value* $x_1(t_j) = a$ *for* $y_1(t_j) > 0$ *is attained (respectively* $x_2(t_i) = a$ *for* $y_2(t_i) > 0$*), naturally, since we have two periodic oscillators, we will have* $i, j = 0, 1, \ldots$ *. For simplicity we will make* $a = 0$ *in the sequence, other choices can be made, but are equivalent under change of variables. The burst/fire points happen when the representative points cross the positive semi-axis* yy *for each planar differential equation.*

Whenever the dynamical system crosses the positive yy semi-axis the other equation receives a perturbation, i.e., a burst, on the yy coordinate, as we can see bellow in equation 6.2.

In actual experiments, we can construct a control system that fires perturbations on the other planar equation when the representative point crosses a critical line or fulfils some concrete condition, that is basically the idea of fire point.

The interaction is constructed such that:

$$\begin{cases} x_1(t^+) &= x_1(t) \\ y_1(t^+) &= y_1(t) + \alpha, \end{cases}, \quad \text{whenever } x_2(t) = 0 \text{ and } y_2 > 0, \textit{and}$$
$$\begin{cases} x_2(t^+) &= x_2(t) \\ y_2(t^+) &= y_2(t) + \alpha, \end{cases}, \quad \text{whenever } x_1(t) = 0 \text{ and } y_1 > 0. \tag{6.2}$$

Naturally, at $t = 0^+$, we have the first burst, since clock 1 is at its burst/fire position, and therefore, clock 2 suffers a small perturbation in its velocity.

Similar to [94], we construct a discrete dynamical system for the phase difference of the two oscillators. In the case of circular limit cycles, this is a modification of the process studied in [94]. We consider the phase of each clock and the respective difference, fixing the phase difference of clock 1 relative to clock 2.

Let $\Phi_1(t)$ and $\Phi_2(t)$ denote the phases of clock 1 and 2, respectively. What is the new phase of clock 2 when it experiences the perturbation as clock 1 crosses the burst/fire point?

As derived in [94], we have

$$\Phi_2(t^+) = \sigma(\Phi_2(t), \alpha), \tag{6.3}$$

where

$$\sigma(\phi, \alpha) = \arctan\left(\frac{\sin(\phi)}{\cos(\phi) + \alpha}\right) \tag{6.4}$$

represents the phase correction, as function of the phase difference $\phi = \Phi_2(t) - \Phi_1(t)$, and the perturbation α.

Expanding this function in powers of α, we get

$$\sigma(\phi, \alpha) = \phi + \sum_{k \geq 1} \frac{(-1)^k \sin n\phi}{k} \alpha^k \tag{6.5}$$

$$= \phi - \alpha \sin \phi + O(\alpha^2). \tag{6.6}$$

Considering a complete cycle of clock 1, we have four steps:

1. the first impact;

2. a drift of the two clocks until clock 2 reaches the critical burst/fire point;

3. clock 1 suffers a perturbation;

4. final drift until clock 1 completes its first turn.

For every complete cycle of clock 1, similar to [94], we obtain a final phase difference ϕ_{n+1}:

$$\phi_{n+1} = \phi_n - 2\pi\frac{\varepsilon}{\omega} - 2\sin(\phi_n)\alpha + O(\alpha^2) \mod 2\pi, \tag{6.7}$$

given an initial phase difference ϕ_0. We note that the dynamical system given by 6.7 exhibits phase locking irrespective of this initial condition if ε is small enough [94, 16].

It is interesting to note that the phase drift of the two clocks under a complete turn of clock 1 is given by the term $-\frac{2\pi\varepsilon}{\omega}$, while the term $-2\sin(\phi_n)\alpha$ accounts for the mutual interaction between clock 1 and clock 2, where the factor of 2 represents the symmetrical nature of the interaction.

In [16], we studied the stability of a system representing the pendulum clock, which has roughly the same properties of the pure circular limit cycle, including Arnold tongues for near multiples of frequencies.

6.2.2 Case of multiple frequencies

We consider here two planar ordinary differential equations (ODEs) describing a circle of radius 1. For notational simplicity, we designate the angular frequency of oscillator 1 as $\omega_1 = 1$ and that of oscillator 2 as $\omega_2 = N + \varepsilon > 0$, where $N > 1$ is an integer. Once again, the absolute value of ε is small and can be either positive or negative. We examine the same scenario in which a perturbation is applied to the reciprocal oscillator when the phase of each oscillator is $0 \mod 2\pi$. This perturbation, akin to the previous case, is now represented by two small real numbers: α_1 and α_2, where α_i denotes the perturbation of clock j by clock i, $i \neq j$, in a neighbourhood of 0.

Systems (6.1) now with two different frequencies, one per planar equation, and normalized are described as follows:

$$\begin{cases} \dot{x}_1 = x_1 + y_1 - x_1(x_1^2 + y_1^2) \\ \dot{y}_1 = -x_1 + y_1 - y_1(x_1^2 + y_1^2) \end{cases}$$
$$\begin{cases} \dot{x}_2 = x_2 + (N+\varepsilon)y_2 - x_2(x_2^2 + y_2^2) \\ \dot{y}_2 = -(N+\varepsilon)x_2 + y_2 - y_2(x_2^2 + y_2^2) \end{cases} \tag{6.8}$$

with initial conditions $x_1(0) = 0$, $y_1(0) = 1$, $x_2(0) = x_2^0$, and $y_2(0) = y_2^0$.

Analogously to (6.2), and with the same definition of burst points, the interaction between these oscillators is defined as

$$\begin{cases} x_1(t^+) &= x_1(t) \\ y_1(t^+) &= y_1(t) + \alpha_1, \end{cases}, \quad \text{whenever } x_2(t) = 0 \text{ and } y_2 > 0, \textit{and}$$
$$\begin{cases} x_2(t^+) &= x_2(t) \\ y_2(t^+) &= y_2(t) + \alpha_2, \end{cases}, \quad \text{whenever } x_1(t) = 0 \text{ and } y_1 > 0,$$

where naturally, at $t = 0^+$, the first burst occurs: $y_2(0^+) = y_2^0 + \alpha_1$ and $x_2(0^+) = x_2^0$.

We formulate a discrete dynamical system to describe the phase difference between the two oscillators. Given that one turn of the slower clock corresponds approximately to N turns of the faster clock, we define $\Phi_1(t)$ and $\Phi_2(t)$ as the phases of clock 1 and 2 respectively. We aim to determine the new phase of clock 2 when it experiences a perturbation as clock 1 crosses the burst/fire point.

At the first burst, the phase of clock 2 is updated according to the correction (6.3).

Expanding function (6.3) in powers of α_1, we obtain the term describing the interaction of the slower clock on the faster one:

$$\sigma_1(\phi, \alpha_1) = \phi + \sum_{k\geq 1} \frac{(-1)^k \sin k\phi}{k} \alpha_1^k \tag{6.9}$$

$$= \phi - \alpha_1 \sin\phi + O(\alpha_1^2). \tag{6.10}$$

The interaction of the faster clock with the slower one is more intricate, as for each turn of the slower clock, the faster clock completes N cycles. The effect of such interaction is described by the sum:

$$\sigma_2(\phi, \alpha_2) = \phi + \sum_{m=0}^{N-1} \sum_{k\geq 1} \frac{(-1)^k \sin k\left(\phi + \frac{2\pi m}{N}\right)}{k} \alpha_2^k, \tag{6.11}$$

where $\frac{2\pi m}{N}$ is the increment to the phase difference before the next m^{th} kick.

For every complete cycle of clock 1, the final phase difference ϕ_{n+1}, accounting the sum of all kicks shared between both clocks is given by

$$\phi_{n+1} = \phi_n - 2\pi\varepsilon + \sum_{k\geq 1} \frac{(-1)^k \sin k\phi_n}{k} \alpha_1^k + \\ + \sum_{m=0}^{N-1} \sum_{k\geq 1} \frac{(-1)^k \sin k\left(\phi_n + \frac{2\pi m}{N}\right)}{k} \alpha_2^k \mod 2\pi.$$

It is noteworthy that the phase drift of the two clocks under a complete turn of clock 1 is given by the term $2\pi\varepsilon$. As the phenomenon loses symmetry with multiples of frequencies, the interaction between the two clocks manifests in two distinct terms. Without the interaction term, the phase difference between the clocks would diverge.

In [16], we explored the stability of a system representing the pendulum clock, which shares similar characteristics with a pure circular limit cycle, including Arnold tongues for near multiples of frequencies.

Theorem 6.2.2. *The relationship between two limit cycle clocks, one with angular frequency $\omega_1 = \omega$ and the other with angular frequency $\omega_2 = N\omega_1 + \varepsilon$ for $N > 1$, that share a constant interaction α_i, is always master-slave. The slower clock dictates the final locked state.*

Proof. It suffices to prove that

$$\sum_{m=0}^{N-1} \sin k\left(\phi + \frac{2\pi m}{N}\right) = 0, \tag{6.12}$$

for all positive integers k and $N > 1$. In other words, the interaction of the faster clock on the slower one sums to zero.

The identity (6.12) is a classic result, nevertheless, we prove the result using some simple trigonometric formulae

$$\sum_{m=0}^{N-1} \sin\left(k\phi + k\frac{2\pi m}{N}\right) = \sin k\phi \sum_{m=0}^{N-1} \cos\frac{2\pi km}{N} + \cos k\phi \sum_{m=0}^{N-1} \sin\frac{2\pi km}{N},$$

we consider the first sum and we use the formula for the sum of a geometric sequence

$$\begin{aligned}\sum_{m=0}^{N-1} \cos\frac{2\pi k}{N}m &= \frac{\sum_{m=0}^{N-1}\left(e^{i\frac{2\pi k}{N}}\right)^m + \sum_{m=0}^{N-1}\left(e^{-i\frac{2\pi k}{N}}\right)^m}{2}\\ &= \frac{1}{2}\frac{1-e^{i2\pi k}}{1-e^{i\frac{2\pi k}{N}}} + \frac{1}{2}\frac{1-e^{-i2\pi k}}{1-e^{-i\frac{2\pi k}{N}}}\\ &= 0.\end{aligned}$$

The second sum follows easily as well by the same reasoning

$$\sum_{m=0}^{N-1} \sin\frac{2\pi k}{N}m = \frac{\sum_{m=0}^{N-1} e^{i\frac{2\pi k}{N}m} - \sum_{m=0}^{N-1} e^{-i\frac{2\pi k}{N}m}}{2i} = 0$$

□

We drop now the subscript in α_1, since there the action of clock 2 on clock 1 is not effective.

Theorem 6.2.3. *There is phase locking between two limit cycle clocks, one with angular frequency* 1 *(or* ω*) and the other with angular frequency* $N+\varepsilon$ *(or* $N\omega+\varepsilon$*) for* $N > 1$*. When* $\alpha < 0$*, synchronization results in nearly zero phase difference (in-phase synchronization), and when* $\alpha > 0$*, synchronization approaches* π *(anti-phase synchronization). These results hold for an appropriate neighbourhood of the origin for* ε*.*

Proof. Consider the difference equation

$$\begin{aligned}\phi_{n+1} &= \phi_n + 2\pi\varepsilon + \sum_{n\geq 1}\frac{(-1)^n \sin n\phi}{n}\alpha^n & (6.13)\\ &= \phi_n + G(\phi_n,\varepsilon). & (6.14)\end{aligned}$$

When $\varepsilon = 0$, the equation

$$G(\phi,\varepsilon) = 0$$

has two solutions: 0 and π. It can be observed that there are two implicit solutions $\phi_0(\varepsilon)$ and $\phi_\pi(\varepsilon)$ in the neighborhood of 0 and π, respectively, which are also fixed points of equation (6.14), since

$$\left.\frac{\partial G(\phi,\varepsilon)}{\partial \phi}\right|_{\phi=0} = -\frac{\alpha}{1+\alpha}$$

and

$$\left.\frac{\partial G(\phi,\varepsilon)}{\partial \phi}\right|_{\phi=\pi} = \frac{\alpha}{1+\alpha}.$$

The stability of these fixed points is determined by the derivative of equation (6.14), $\phi + G(\phi,\varepsilon)$, at the solutions $\phi_0(\varepsilon)$ and $\phi_\pi(\varepsilon)$, these derivatives are close to $1 - \frac{\alpha}{1+\alpha}$ and $1 + \frac{\alpha}{1+\alpha}$ respectively.

When $\alpha < 0$, the fixed point $\phi_0(\varepsilon)$ is unstable (a repeller), and $\phi_\pi(\varepsilon)$ is asymptotically stable (an attractor). Conversely, when $\alpha > 0$, $\phi_0(\varepsilon)$ becomes asymptotically stable (an attractor), and $\phi_\pi(\varepsilon)$ becomes unstable (a repeller). □

6.3 The case of pendulum clocks

6.3.1 Andronov model for a pendulum clock

As outlined in Section 6.1, the second part of this study is founded upon the Andronov model for the isolated pendulum clock. In the book by Andronov et al. [90], the authors establish the existence and stability of the limit cycle for the motion in this dynamical system. Additionally, a comprehensive review of this theory along with an analysis of Huygens synchronization involving two Andronov clocks with similar frequencies can be found in [94, 16].

A more detailed description of this model can be found in subsection 5.2.1. However, we present some key features to be considered in the following sections.

The governing equation for the motion of this system is defined by

$$\ddot{q}(t) + \mu \operatorname{sign} \dot{q}(t) + \omega^2 q(t) = 0, \tag{6.15}$$

with initial conditions

$$q(0) = -\frac{\mu}{\omega^2}, \quad \dot{q}(0) = v_0, \tag{6.16}$$

where q represents the angular coordinate, $\mu > 0$ denotes the dry friction coefficient, and ω is the natural angular frequency of the pendulum.

A kick per cycle occurs at the uppermost vertical position of the limit cycle, where a constant kinetic energy $h^2/2$ is imparted to the pendulum, such that $v_1^2/2 - v_0^2/2 = h^2/2$, where v_0 and v_1 are the velocities before and after the kick, respectively, given whenever $q(t) = -\mu/\omega$ and $\dot{q}(t) > 0$. The friction is assumed to be dry (constant friction) due to the nature of the phenomenon, where dissipation mainly occurs in gearwheels with solid friction between metallic components of the mechanism. Consequently, the magnitude of the friction force remains independent of velocity. However, this modelling approach results in a fundamentally non-continuous nature of the ODEs used to represent the phenomenon, as previously seen in section 5.2.

The limit cycle of this system, depicted in Figure 5.1, is obtained from [90] and represented as a Poincaré section, specifically a half line $q = (-\mu/\omega)^+$ and $\dot{q} > 0$, where the notation "+" indicates that the section is established immediately after the impulse. A loss in velocity of $4\mu/\omega$ due to friction occurs during a complete cycle, which is compensated by a single kinetic energy kick per cycle.

The return map for this system, as analyzed in [94, 16], is given by

$$v_{n+1} = P(v_n) = \sqrt{\left(v_n - \frac{4\mu}{\omega}\right)^2 + h^2}, \tag{6.17}$$

where the map (6.17) possesses an asymptotically stable fixed point at

$$v^* = \frac{h^2\omega}{8\mu} + \frac{2\mu}{\omega}, \tag{6.18}$$

attracting initial conditions v_0 in the interval $(4\mu/\omega, +\infty)$.

6.3.2 Andronov Model for Two Coupled Oscillators

In [94, 16], the authors examine two coupled Huygens' clocks, nearly identical, described by the following equations:

$$\ddot{q}_i(t) + \mu \operatorname{sign} \dot{q}_i(t) + \omega_i^2 q_i(t) = 0, \quad i, j = 1, 2, \tag{6.19}$$

with each clock receiving the same amount of energy as the isolated Andronov clock, as described in the previous subsection 6.3.1. By changing variables, the system is transformed into

$$\dot{x}_1 = y_1, \tag{6.20}$$
$$\dot{y}_1 = -\omega_1 x_1 - \mu y_1, \tag{6.21}$$
$$\dot{x}_2 = y_2, \tag{6.22}$$
$$\dot{y}_2 = -\omega_2 x_2 - \mu y_2, \tag{6.23}$$

with initial conditions $x_1(0) = 0$, $y_1(0) = 1$, $x_2(0) = x_2^0$, and $y_2(0) = y_2^0$.

The interaction between these oscillators is defined similarly to subsection 6.2.1, with adjustments made for clocks with different but close angular velocities, $\omega_1 \neq \omega_2$. All other parameters are assumed to be equal for the two clocks.

We define

$$A = \frac{\omega_j h^2}{8\mu}, \quad \delta = \frac{2\mu}{\omega_j}, \tag{6.24}$$

where A and δ are constants related to the dynamics of the system. The limit cycle of the Andronov clock is circular, with a radius $A + \delta$ when the phase is between 0 and $\pi/2$, A when the phase is between $\pi/2$ and $3\pi/2$, and $A - \delta$ when the phase is between $3\pi/2$ and 2π. The effect of the perturbation on the phase of the limit cycle of each clock is described by the map

$$\sigma(\phi, \alpha) = \arctan\left(\frac{\sin(\phi)}{\cos(\phi) + \frac{\alpha}{A+\Delta(\phi)}}\right), \tag{6.25}$$

where $A + \Delta(\phi)$ represents the radius of each circular section of the limit cycle, with δ indicating a variation from this mean.

$$\Delta(\phi) = \begin{cases} \delta, & 0 \leq \phi < \pi/2, \\ 0, & \pi/2 \leq \phi < 3\pi/2, \\ -\delta, & 3\pi/2 \leq \phi < 2\pi. \end{cases} \tag{6.26}$$

The configuration of this problem is depicted in Figure 5.1, where the deviation from the mean radius has been exaggerated for visibility purposes.

In [94], a detailed construction of the map describing the phase difference measured whenever the slowest clock (with frequency ω) crosses its Poincaré section for the system of coupled Huygens' clocks with nearly identical frequencies ω and $\omega + \varepsilon$ was presented. This treatment involves the composition of four effects: one perturbation of the slow clock on the fast clock, one drift, the perturbation of the fast clock on the slow clock, and another drift until the slow clock completes a reference turn. The map combining all these effects was termed [94]

$$\phi_{n+1} = \Omega(\phi_n), \tag{6.27}$$

where ϕ_n refers to the phase difference between clocks 1 and 2 at the n^{th} cycle, with reference to the cycles of the slowest clock. In both works [94, 16], the map (6.27) is expanded to the form

$$\Omega(\phi) = \Xi(\phi) + O(\alpha^2) = \phi - \frac{2\pi\varepsilon}{\omega} + \frac{2\sin(\phi)}{A}\alpha + O(\alpha^2, \alpha\delta, \delta^2). \tag{6.28}$$

It is worth noting that α represents a very small perturbation, and ε is the difference in frequencies of the two clocks, both of which must be sufficiently small. In [94], it was determined that a realistic α satisfies the inequality $\alpha < 10^{-3}$ and that the detuning of the clocks was on the order of 10^{-4}, resulting in a delay of one clock relative to the other on the order of seconds per day. Additionally, [90] established that $h > \frac{4\mu}{\omega}$ is necessary for the movement of the clock to persist, implying that $\delta < A$. Furthermore, $\delta \ll A$, as the velocity loss along the limit cycle due to damping is $\frac{4\mu}{\omega} = 2\delta$, while the mean radius of this limit cycle is A. By physical considerations, as demonstrated in [94], δ must also be very small compared to the mean radius A. Consequently, the linear approximation $\Xi(\phi)$ of $\Omega(\phi)$ is highly accurate. Thus, the influence of the clocks on each other is reciprocal, a type of synchrony generally referred to as *split-split* or *master-master*, which is typical of identical clocks perturbing each other.

6.3.3 Multiple frequencies case

We are now interested in the asymmetric situation where one clock has a frequency nearly a multiple of the other. The study of the function $\sigma(\phi, \alpha)$ is the main point of the problem; this function describes the effect of each individual perturbation of one clock on the other in the new setup of nearly multiple frequencies. Similar functions to (6.5) and (6.9) were obtained for perfect circular limit cycles. We derive σ by expanding a series in terms of α, which is close to zero, sectionally in each continuity lap, yielding:

$$\sigma(\phi, \alpha) = \phi + \sum_{k\geq 1} \frac{(-1)^k \sin k\phi}{k(A + \Delta(\phi))^k} \alpha^k. \tag{6.29}$$

This expression is almost identical to the previous ones, with the correction in the denominator accounting for the deviation from the perfect circular limit cycle.

Now, let's consider clocks 1 and 2 as the slow and fast clocks, respectively, with frequencies $\omega_1 = \omega$ and $\omega_2 = N\omega + \varepsilon$, where $|\varepsilon|$ is small. For simplicity, we set $\omega = 1$, and we work with $\omega_1 = 1$ and $\omega_2 = N + \varepsilon$ in the remaining of this Chapter. Note that whenever the slow clock turns one time, the fast clock turns N times. Again we have two different interaction coefficients α_1 and α_2. Consequently, one kick given to the fast clock per cycle of the slow clock corresponds to N kicks given to the slow clock by the fast clock.

Moreover, as seen in the previous section, δ/A is assumed to be very small, in the same order[1] of α_1 and α_2.

We state the following theorem:

[1] Both are assumed to be smaller than 10^{-3}.

Theorem 6.3.1. *The relation between two coupled Andronov clocks for $N:1, N>1$, is master-slave.*

Proof. The dynamical system developed in a convergent series in powers of α in the laps of continuity of Δ for the phase difference is

$$\begin{aligned}\phi_{n+1} &= \phi_n - 2\pi\varepsilon + \sum_{k\geq 1} \frac{(-1)^k \sin k\phi_n}{k(A+\Delta(\phi_n))^k}\alpha_1^k \\ &\quad + \sum_{m=0}^{N-1}\sum_{k\geq 1} \frac{(-1)^k \sin k\left(\phi_n + \frac{2\pi m}{N}\right)}{k(A+\Delta(\phi_n+\frac{2\pi m}{N}))^k}\alpha_2^k \mod 2\pi \\ &= \phi_n + \Gamma_N\left(\phi_n, \alpha, \varepsilon, \mu\right).\end{aligned}$$

It suffices to prove that

$$\Gamma_N\left(\phi_n, \alpha, \varepsilon, \mu\right) \text{ is first order in } \alpha_1 \text{ and } \varepsilon \tag{6.30}$$
$$\text{and second order in all the other parameters} \tag{6.31}$$

for all positive integers n and $N>1$. In other words, the interaction of the faster clock on the slower one sums to zero in order one.

We have to study

$$g\left(\phi\right) = -\sum_{m=0}^{N-1} \frac{\sin\left(\phi_n + \frac{2\pi m}{N}\right)}{(A+\Delta(\phi_n+\frac{2\pi m}{N}))}\alpha_2 + O\left(a_2^2\right),$$

that we develop in powers of δ, doing this sectionally and collecting the results lapwise we have

$$\begin{aligned} g\left(\phi\right) &= -\sum_{m=0}^{N-1} \frac{\sin\left(\phi_n + \frac{2\pi m}{N}\right)}{(A+\Delta(\phi_n+\frac{2\pi m}{N}))^k}\alpha_2 + O\left(a_2^2\right) \\ &= -\sum_{m=0}^{N-1} \frac{\sin\left(\phi_n + \frac{2\pi m}{N}\right)}{A}\left(1 - \frac{\Delta(\phi_n+\frac{2\pi m}{N})}{A} + \dots\right)\alpha_2 + O\left(a_2^2\right)\end{aligned}$$

since

$$\sum_{m=0}^{N-1} \sin\left(\phi_n + \frac{2\pi m}{N}\right) = 0$$

(as we have seen in the proof of Theorem 6.2.2) and remembering that Δ is either

$-\delta$, 0 or δ depending on the value of its argument, we have

$$\begin{aligned} g(\phi) &= \sum_{m=0}^{N-1} \frac{\sin\left(\phi_n + \frac{2\pi m}{N}\right)}{A} \left(\frac{\Delta(\phi_n + \frac{2\pi m}{N})}{A} - \dots \right) \alpha_2 + O\left(a_2^2\right) \\ &= O\left(a_2^2, \alpha_2 \frac{\delta}{A}\right). \end{aligned}$$

Thus, the dynamical system for the synchronization of this type of clocks is given by the solution of the difference equation

$$\phi_{n+1} = \phi_n - 2\pi\varepsilon - \frac{\sin\phi_n}{A}\alpha_1 \mod 2\pi,$$

and the synchronization is dictated by the slow clock. Moreover, this phenomenon aligns the beat of the slow clock with the beats of the fast clock such that when the slow clock bursts, the perturbation upon the fast one that clock is at phase near 0 or near π, in the same way as we proved the same result for the perfect circle case (Theorem 6.2.3).

□

6.4 Simulations

We performed numerical simulations of two dynamical systems. The first system features a perfect circular limit cycle corresponding to equation 6.8, where we set $\omega = 1$, varied $n = 1, 2, 3, 4$, and used $R = 2$, representing a limit cycle with a radius of 2.

$$\begin{aligned} \dot{x}_1 &= x_1 + y_1 - x_1(x_1^2 + y_1^2)/R^2, \\ \dot{y}_1 &= -x_1 + y_1 - y_1(x_1^2 + y_1^2)/R^2, \\ \dot{x}_2 &= x_2 + (n+\varepsilon)y_2 - x_2(x_2^2 + y_2^2)/R^2, \\ \dot{y}_2 &= -(n+\varepsilon)x_2 + y_2 - y_2(x_2^2 + y_2^2)/R^2, \end{aligned} \tag{6.32}$$

The numerical analysis of this system yielded results consistent with those found in our previous work, providing experimental confirmation of our earlier findings. We chose a limit cycle with a radius of 2 to facilitate comparison with the solutions of the van der Pol equation.

The second system we examined was the well-known van der Pol oscillator [116]. Since this system is strongly nonlinear and does not have a perfectly circular limit cycle (except in rare cases), this study provided strong numerical evidence that similar results apply to general oscillators with non-circular limit cycles, interacting via bursts.

The van der Pol equations are:

$$\begin{aligned}\dot{x}_1 &= y_1,\\ \dot{y}_1 &= \mu y_1(1-x_1^2) - x_1,\\ \dot{x}_2 &= y_2,\\ \dot{y}_2 &= \mu y_2(1-x_2^2) - (n^2+\varepsilon)x_2,\end{aligned} \tag{6.33}$$

The parameter μ controls the strength of the nonlinearity in the equation. When μ is near zero, the limit cycle becomes a near-perfect circle with radius 2 [116]. We set $\mu = 0.01$, a relatively large value, meaning the limit cycle is not circular, and the nonlinearity is fairly strong.

We used Wolfram Mathematica 14.0 from Wolfram Research, Inc., Champaign, Illinois, USA, along with its built-in packages for solving differential equations numerically, specifically handling the differential equations between bursts.

We first solved the uncoupled differential equations (i.e., setting $\varepsilon = 0$) over 1000 periods to ensure the system settled into its limit cycle (this is essential for the van der Pol oscillator). We then determined the period numerically with an error margin of 10^{-6}, which was again important in the case of the van der Pol system, since the system 6.32 has always the same period. The time $t = 0$ was set when the solution for the slow oscillator reached $x_1 = 0$ with $y_1 > 0$, after 1000 cycles. From that point, we started the numerical solution of the system with $\varepsilon \neq 0$. The process involved solving the equations between bursts using Mathematica's numerical built in solver, then adjusting the initial conditions at each burst according to equation 6.2, and restarting the process. For near-identical oscillators, we used 1000 cycles of the slow oscillator, and for the $2:1$, $3:1$, and $4:1$ cases, we used 2000 cycles, since in those cases the convergence to the locked state is slower. This algorithm worked equally well for both systems, governed by equations 6.32 and 6.33.

We computed the numerical solutions, periods, and frequencies of both the uncoupled and coupled systems, and compared the results with our predictions. Additionally, we calculated the phase differences over time, using the slow oscillator's phase as a reference in both cases.

The figures corresponding to these simulations can be found in Table 6.1 and Table 6.2, respectively.

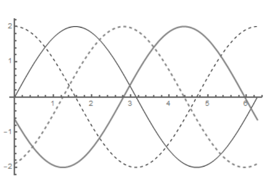
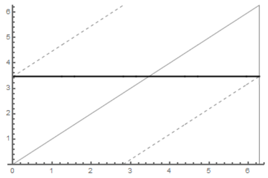
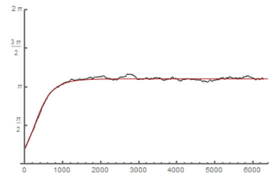
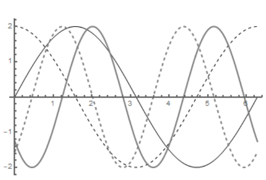
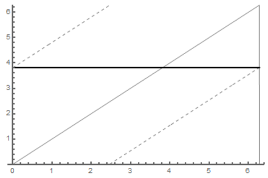
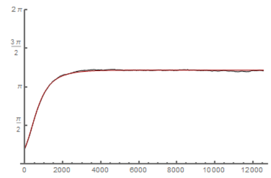
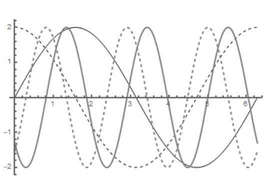
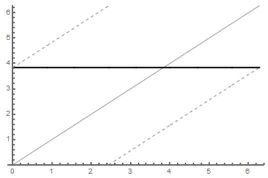
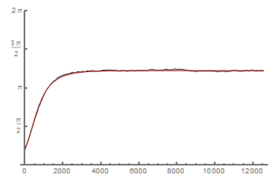
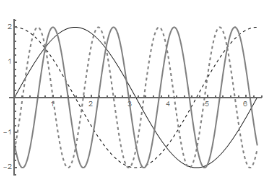
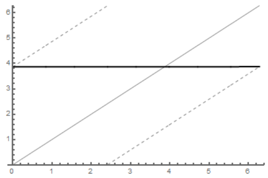
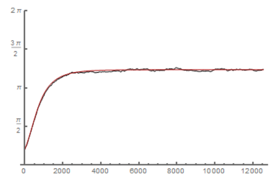
$n:1$	Solutions (phase locking)	Phase difference (phase locking)	Evolution to phase locking
$1:1$			
$2:1$			
$3:1$			
$4:1$			

Figure 6.1: Simulation results for the system in equation 6.32 with circular limit cycles.

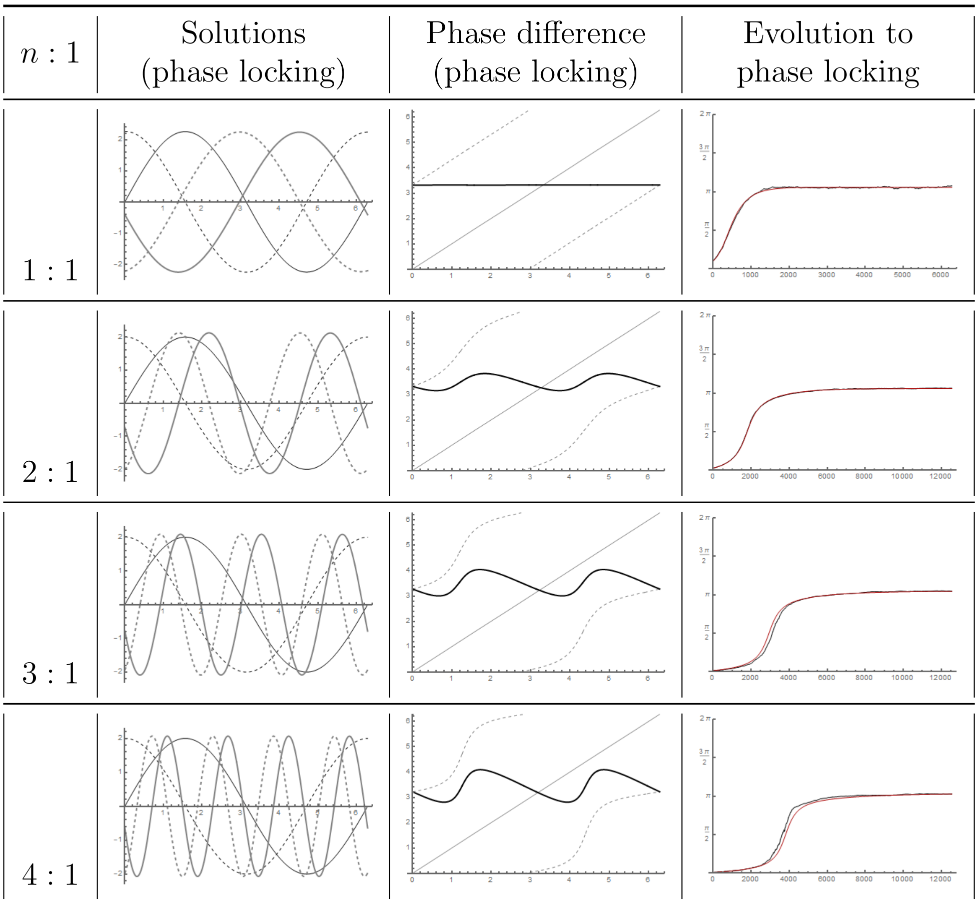


Figure 6.2: Simulation results for the synchronization of van der Pol equations 6.33.

For each system, we analyzed the cases with frequency ratios of $1:1$, $2:1$, $3:1$, and $4:1$. For each case, we present:

1. The actual solutions under phase locking within the interval $[0, 2\pi[$ show that, in all cases, when the slow clock completes one cycle, the other clocks complete n cycles. We normalized the solutions to facilitate a better comparison of $x_1(t)$, $y_1(t)$, $x_2(t)$, and $y_2(t)$.

2. The phase of each clock, ϕ_1, ϕ_2/n, and the phase difference $\phi_1 - \phi_2/n \mod 2\pi$, is measured along the final cycle of the slow clock in the phase-locked state after the completion of the iterative process.

3. The evolution of the phase differences $(\phi_1 - \phi_2) \mod 2\pi$ over the number of cycles is measured at the exact burst/fire points of the slow clock, from the moment the coupling is established until the prescribed number of cycles, i.e., 1000 for symmetric coupling or 2000 for $n : 1$ coupling when $n \neq 1$. These phase differences are taken stroboscopically [77], one per cycle of the reference (slow) oscillator. We plotted 1000 or 2000 phase differences in each graph. The phase difference is always seen in the perspective of the slow clock when its phase is $0 \mod 2\pi$.

The plots for the evolution of the solutions at the final locked state show $x_1(t)$ in solid black, $y_1(t)$ in dashed black, $x_2(t)$ in grey, and $y_2(t)$ in dashed grey. These plots are displayed at the first column of figures 6.1 and 6.2.

In the second column of of figures 6.1 and 6.2 we can observe the phase difference after phase locking is attaained, the plots shows the phase of clock 1, $\Phi_1(t)$, in solid grey, the phase of clock 2, $\Phi_2(t)/n$, in dashed grey, and the difference $(\Phi_1(t) - \Phi_2(t)/n) \, (\mod 2\pi)$ in solid black.

In the last column of figures 6.1 and 6.2 we show the plots of the evolution of phase differences until phase opposition, these plots include a simulation without noise in solid red and the curve with a noise component overlaid in solid black (see more on noise below).

6.4.1 Perfect circle limit cycle

In the case of two coupled oscillators with circular limit cycles, under phase locking, the fast clock completes n cycles for every cycle of the slow clock in an $n : 1$ frequency ratio (first column of figure 6.1).

We observe that the normalized phase difference (in solid black in the second column) between both clocks remains constant throughout the cycle and does not depend on n. This naturally occurs since, in this system, the angular velocity (the derivative of the phase) is constant, and the limit cycle is a perfect circle. The horizontal lines in the second column of figure 6.1 confirm what was expected.

Phase locking is reached at approximately π, with small deviations due to the frequency offset ϵ and the perturbation α (third column of figures 6.1 and 6.2).

The simulations were performed with $\epsilon = 0.001\,\text{rad/s}$ and $\alpha = 0.02$ to enhance the visual effect in the plots. However, in the experimental setup in [94], both ϵ and α were on the order of 10^{-4}. Despite the effect of energy transfer α causing deviations toward phase opposition, this additional energy is insufficient to destabilize the limit cycle, as theoretically shown in [16], confirming that the system remains structurally stable.

The addition of noise (seen in the last column of figure 6.1 in solid black) did not destroy the effects of synchronization, phase locking, and the master-master relationship for the $1:1$ case, or the master-slave relationship for the $n:1$ cases where $n>1$, as observed in our simulations.

For the $1:1$ case, we added a random perturbation in each cycle with a normal probability distribution, with mean 0, and a standard deviation of 2×10^{-4} in the frequency of both oscillators, along with a similar random perturbation in α, with a standard deviation of 4×10^{-3}, which is 20% of the value of α. The level of noise was suggested by the experimental measurements of [94].

For the $n:1$ cases where $n>1$, we added a random perturbation in each cycle with a normal probability distribution, with mean 0, and a standard deviation of 10^{-4} in the frequency of both oscillators, along with a similar random perturbation in α, with a standard deviation of 2×10^{-3}, which is 10% of the value of α.

All the oscillators entered a locked state. The relevant parameters are enumerated below:

1. $1:1$, Initial period of the slow clock: 6.28319, initial period of the fast clock: 6.27691, final period in the locked state for the two oscillators: 6.28005, which is exactly the average of the initial periods. The final state is master-master, as predicted by our results. The mean period of the slow clock in the locked state with noise was 6.2803, and the standard deviation of the periods with noise was 4.6×10^{-4}. The two clocks remained at a phase difference of 3.46 rad at the locked state.

2. $2:1$, Initial period of the slow clock: 6.28319, initial period of the fast clock: 3.14002, final period in the locked state for the two oscillators: 6.28319 and 3.14157, which is the original period of the slow clock and half of this period. The final state is master-slave, as predicted by our results, with the slow clock dictating the final locked state. The mean period with noise was 6.2803, with a standard deviation of 9.3×10^{-4}. The two clocks remained at a phase difference of 3.82 rad at the locked state.

3. $3:1$, Initial period of the slow clock: 6.28319, initial period of the fast clock: 2.0937, final period in the locked state for the two oscillators: 6.28319 and 2.09441, which is the original period of the slow clock and one-third of this period. The final state is master-slave, as predicted by our results, with the slow clock again dictating the final locked state. The mean period with noise was 6.2832, with a standard deviation of 9.9×10^{-4}. The two clocks remained at a phase difference of 3.84 rad at the locked state.

4. $4:1$, Initial period of the slow clock: 6.28319, initial period of the fast clock: 1.5704, final period in the locked state for the two oscillators: 6.2833 and

1.570821, which is extremely close to the original period of the slow clock and one-fourth of this period. The final state is master-slave, as predicted by our results, with the slow clock once again dictating the final locked state. The mean period with noise was 6.283, with a standard deviation of 1×10^{-3}. The two clocks remained at a phase difference of 3.87 rad at the locked state.

6.4.2 Van der Pol

Similar simulations were performed for the system of coupled van der Pol oscillators, revealing similar behaviour, which is very promising since these dynamical systems have limit cycles that are not circular. The simulations show that, in the presence of fire/burst synchronization, there is strong evidence that the slow oscillator also determines the locked dynamics.

Although the phase difference (column 2 of figure 6.2) under phase locking is not constant as in the circular case, it is periodic and returns to the same value after each cycle of the slow clock. This is naturally explained by the fact that the pace of the representative point along the limit cycle is not uniform for the case $n : 1$ when $n > 1$, as the limit cycle is not circular and the phase velocity differs for these coupled oscillators. In the symmetric $1 : 1$ case, however, the phase difference is strictly constant after phase locking is attained for the entire period of the slow oscillator.

We observe that the evolution towards phase opposition exhibits a curve with slower convergence toward phase locking at a phase difference of approximately π, as n increases. Nevertheless, even for van der Pol oscillators under mutual perturbation, phase locking still occurs.

The simulations were performed using the same values as in subsection 6.4.1, with $\mu = 0.01$ in the van der Pol equations 6.33.

Once again, the addition of noise (shown in the last column of figure 6.1 in solid black) did not destroy the effects of synchronization, phase locking, or the master-master relationship for the $1 : 1$ case, nor the master-slave relationship for the $n : 1$ cases where $n > 1$, as observed in our simulations.

All the oscillators reached a locked state. The parameters are listed below, and the van der Pol equations and equations 6.32 yield strikingly similar results:

1. $1 : 1$, Initial period of the slow clock: 6.28322, initial period of the fast clock: 6.28009, final period in the locked state for the two oscillators: 6.28171, which is very close to the average of the initial periods. The final state is master-master. The mean period of the slow clock in the locked state with noise was 6.2815,

and the standard deviation of the periods with noise was 3.5×10^{-4}. The two clocks maintained a phase difference of 3.32 rad at the locked state.

2. 2 : 1, Initial period of the slow clock: 6.28322, initial period of the fast clock: 3.14121, final period in the locked state for the two oscillators: 6.28322 and 3.14161, which corresponds to the original period of the slow clock and half of this period. The final state is master-slave, with the slow clock dictating the locked state. The mean period with noise was 6.28318, with a standard deviation of 1.1×10^{-3}. The two clocks maintained a phase difference of 3.32 rad at the locked state.

3. 3 : 1, Initial period of the slow clock: 6.28322, initial period of the fast clock: 2.09428, final period in the locked state for the two oscillators: 6.28326 and 2.09442, which is very close to the original period of the slow clock and one-third of this period. The final state is master-slave with the slow clock dictating the locked state. The mean period with noise was 6.28332, with a standard deviation of 1.0×10^{-3}. The two clocks maintained a phase difference of 3.27 rad at the locked state.

4. 4 : 1, Initial period of the slow clock: 6.28322, initial period of the fast clock: 1.5708, final period in the locked state for the two oscillators: 6.2832 and 1.57081, which corresponds to the original period of the slow clock and one-fourth of this period. The final state is master-slave with the slow clock again dictating the locked state. The mean period with noise was 6.283, with a standard deviation of 1.1×10^{-3}. The two clocks maintained a phase difference of 3.22 rad at the locked state.

6.5 Discussion

When one oscillator operates at a frequency ω and another oscillator operates at a frequency near a non-trivial multiple of the first one, $n\omega$, where n is an integer greater than 1, the synchronization between them tends to exhibit a consistent pattern of a master-slave relationship.

This phenomenon can be geometrically explained by the interaction dynamics: the multiple bursts of the faster clock tend to cancel each other out when impacting the slower clock, while the single burst of the slower clock on the faster one has a cumulative and pronounced effect. We hypothesize that this synchronization pattern resulting from perturbative impacts is prevalent in various physical systems. We also conjecture that oscillators with limit cycles not necessarily circular, like the van der Pol oscillator, exhibit similar behaviour.

It would be enlightening to gather experimental evidence corroborating our mathematical findings. We are planning experiments in chemistry with oscillating reactions and in electronics with electronic oscillators, to gather experimental evidence supporting both our theoretical findings and simulations.

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