Universidade de Évora - Instituto de Investigação e Formação Avançada

Programa de Doutoramento em Matemática

Área de especialização | Matemática e Aplicações

Tese de Doutoramento

Lower and upper solutions method on higher order boundary value problems including differential equations and coupled systems

Infeliz Carvalho Coxe

Orientador(es) | Feliz Manuel Minhós

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Lower and upper solutions method on higher order boundary value problems including diferential equations and coupled sistems

Infeliz Carvalho Coxe

Contents

To my parents Freitas Coxe and Maria Carvalho (in memory). With them, I started the adventure of learning, through which I also learnt to be a happy person. Their teachings taught me that in our lives we only succeed fighting completely for what we want.

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Método de sub e sobre-soluções para problemas de ordem superior com valores na fronteira incluindo equações diferenciais e sistemas acoplados

Resumo

A escassez, na literatura, de problemas de valor fronteira envolvendo sistemas de equações não lineares acopladas, com todas as não linearidades completas, quer em domínios limitados ou ilimitados, levou a elaboração do presente trabalho. As principais técnicas fazem uso do método de sub e sobre soluções, para obter condições suficientes para a existência de solução em problemas que envolvem equações diferenciais de ordem superior e vários tipos de condições de fronteira, definidas na semi – recta real, bem como sistemas de equações diferencias não lineares e não autónomos, com condições de fronteira funcionais. Foi ainda estudado a localização das soluções dos sistemas em espaços de Banach, seguindo vários argumentos e abordagens tais como o teorema de ponto fixo de Schauder, as funções de Green ou as suas estimativas, truncaturas e perturbações convenientes, a condição de Nagumo apresentada em váias versões, entre outros. Em todos os casos foram considerados diversas aplicações quer a problemas teóricos quer a fenómenos da vida real.

Palavras Chaves: Sistema de equações diferencias, Domínios não compactos, Teoria de ponto fixo, Sub e sobre-soluções.

Lower and upper solutions method on higher order boundary value problems including differential equations and coupled systems

Abstract

The scarcity, in the literature, of boundary value problems involving systems of coupled nonlinear equations, with complete nonlinearities, whether in bounded or unbounded, led to the elaboration of the present work. The main techniques make use of the method of lower and upper solutions, to obtain sufficient conditions for the existence of solutions for problems involving differential equations of higher order and various types of boundary conditions, eventually defined in the real semi-line, as well as systems of non-linear and non-autonomous differential equations, with functional boundary conditions.

The location of solutions in Banach spaces was also studied, following various arguments and approaches such as Schauder's fixed-point theorem, Green's functions, or his convenient estimates, truncations and perturbations, Nagumo's condition presented in several versions, among others. In all cases, several applications were considered, both to theoretical problems and to real-life phenomena.

Key words: Systems of differential equations, non compact domains, fixed point theory, lower and upper solutions.

Introduction

This doctoral dissertation approaches interesting facts in the area of Ordinary Differential Equations, having as title "Lower and upper solutions method on higher order boundaryvalue problems including differential equations and coupled systems", in which a sequence of applications illustrating the use of Ordinary Differential Equations (ODE) is presented, to describe mathematically some natural forces and study them.

A system of differential equations is a set of equations, in which each one relates itself to the values of the function and its derivatives. When there are interactions and dependency between variables in the system of differential equations, it is said to be a coupled system. The phenomena, laws and systems that operate the universe are not independent nor isolated. The interaction is a characteristic of everything that surrounds the universe.

The natural phenomena are generally non-linear and are shaped with non-linear differential equations systems of superior sequence. Moreover, these systems are also used to study and explain several important problems of science and Engineering, which can not be analyzed with non-linear systems.

Non-linear differential equation systems, have had a growing interest in last years, especially, due to its applications in diverse areas, as dynamic of applications, mechanic, optimum control, and harvest.

The present work on non-linear differential equations systems of superior sequences with problems of borderline value, in which the considered systems are linked.

The main percussive study of differential equations, according to the history of mathematics, were Gottfried Leibniz (1646 -1716), Isaac Newton (1643 – 1727), Bernoulli brothers, that is, James Bernoulli (1654 – 1705), and John Bernoulli (1667 -1748). In 1675, Leibniz was the first to study differential equations in considering and solve the trivial equation

$$\int ydy = \frac{1}{2}y^2$$

providing tools as a signal of integral and tangents inverse problem. It was Leibniz who discovered the technique to separate variables, studying the solution of the equation $f(x)dx = g(y)dy$, developing the technique to solve the homogeneous equation, $dy = f\left(\frac{x}{y}\right)dy$ as well as innumerable contributions and applications. Parallel to Leibniz, Newton had a great impact as a contributor in differential equations theory. One of the most notable Newton's significative contributions in this area, were in study of fluxes and their applications. In study of fluxes he established that $f(x, y) = 0$, with x and y functions of t

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0.$$

At the same time, the Bernoulli brothers created a number of methods (used up today), to solve and motivate the development of differential equations.

The determination of solutions for non-linear differential equations systems of higher-order, with boundary values or initial value, is not easy to generalize, and in most cases, it is impossible to solve.

In literature, the methods are generally illustrated by numerical methods for a determination of approximated solutions. Even using numerical methods, nothing guarantees that all equations or systems of non-linear equations present unchanging solutions. Thus, in this work, we present sufficient conditions for existence of solutions, and in some cases its location , in differential equations with several fully equations, with boundary values on finite or infinite intervals, with generalized impulses, multipoint conditions, with Phi-laplacians, among others. By our arguments it will be able to confirm the existence of several solutions, especially in bounded or unbounded, homoclinics and heteroclinics, and several applications for theoretical problems as well as for real-life phenomena.

The uniqueness or multiplicity of solutions and their structures depend not on the involved linearity and on the type of boundary conditions. Thus, the study of the solvability of non-linear differential equations systems of higher order considers these two arguments: regularity and properties of the nonlinearities and several types of boundary conditions.

In this dissertation we pretend to add to the literature some new methods and techniques, presenting results with complete nonlinearities defined on bounded unbounded intervals, based on compact operators defined in Banach spaces. In more detail, our arguments apply a system of integral equations,with Green's functions as the kernel component, and some auxiliaries compact operators, in which it is applied the fixed point theory. Lower and upper coupled solutions, provide a tool of location to establish not only the equivalence between the auxiliary problems and the initial ones, but also give some qualitative properties of solution. Conditions of the Nagumo type, unilateral or bilateral ones, play an important role in our arguments, to control higher order derivatives growth.

In particular, the third order differential equations model many phenomena in Physics, Engineering, and Physiology, among others. For example, we made reference to fine flux film of viscous fluid on a surface, the deflection of a clamped beam with a constant transverse section or variable, the wave solutions of Korteweg - de Vries' equations, and non-linear suspensions of vehicles.

Differential equations of the fourth order, may translate the flexion of an elastic beam, and in this sense, we consider them as beam equations. They have received a growing interest from several fields of Science and of Engineering. Equations defined on noncompact intervals, require more delicate techniques to obtain sufficient conditions for the existence of a solution on half-line boundary value problems. As examples, we refer the extension of solutions on correspondent bounded intervals, via truncation and perturbation techniques, together with the fixed point theory in some Banach spaces, and the method of lower and upper solutions.

The generalization of some cases to non-linear differential equations a general higher order n, is presented: Applying an adequate auxiliary integral problem, truncated and with bounded perturbations, we obtain existence and location results.

The dissertation is organized in five chapters, each one dealing with a particular boundary value problem and with example and applications to real-life phenomena. More precisely:

Chapter 1 considers a third order coupled systems where the nonlinearities include a second derivative dependence, via cone theory. In this case the standard cone theory in the literature can not be applied because the second derivative of the Green's functions, associated to the linear form, changes sign. Our method applies an integral system defined by the Green's functions as the kernel component, and some auxiliary compact operators, in which an adequate truncature plays a key role. Coupled lower and upper solutions provide a localization tool which complements the existence result.

In Chapter 2, the previous third order coupled system is generalized such that the nonlinearities include all the derivatives of both variables. The technique used applies an integral system defined with the Green's functions, and some auxiliary compact integral operators with an adequate truncature. Coupled lower and upper solutions are useful not only to give a location tool and the equivalence between auxiliary and initial problems, but also to obtain some qualitative properties of the solution. Last section contains an example to show the applicability of the main theorem and the utility of the localization tool to have some qualitative properties on the solution.

The Chapter 3 approaches solvability of generalized fourth order coupled systems with two-point boundary conditions. We underline that the nonlinearities can depend on all derivatives of both unknown functions. This dependence is due to an adequate auxiliary integral problem with a new type of truncature based on lower and upper solutions and some bounded perturbations. A Nagumo-type condition allows *a priori* estimations on the third derivatives. The main theorem is an existence and localization result, gives some qualitative data on the system solutions such as, sign, variation, growth, bounds, and convexity or concavity. This fourth order equations can be applies to beam theory, and the boundary conditions define the beam type, that its behavior on the endpoints. That is why these equations are often called beam equations. Four the first time, as far as we know, these fourth-order coupled system were used to model a coupled clamped beam set.

Chapter 4 contains a generalization of previous techniques to coupled higher order systems that, to the best of our knowledge, is new in the literature, and opens the possibility of new types of models. The main existence tool remains lower and upper solutions, but, in this case, applied to a homotopic problem together with Leray-Schauder topological degree theory. Moreover, this chapter contains two applications for higher order coupled systems´: the first one, for $n = 2$, to a family of Lorentz-Lagrangian systems, and the second one, for $n = 3$, to some stationary coupled system of Korteweg-de Vries equations with damping and forced terms.

Finally, Chapter 5 deals with the solvability of n^{th} order coupled systems with full nonlinearities defined on an unbounded interval, with functional boundary conditions. We combine all the above features, taking advantage of all of them and allowing their application to a wider range of real-life problems and phenomena. To deal with the loss of compacity, an adequate operator is defined in a weighted Banach space, with weighted norms, for which it can be proved, an uniform bound, equicontinuity and the equiconvergence at infinity. Sufficient conditions are given to have fixed points, via Schauder's fixed point theorem, solutions of the auxiliary problem. Applying lower and upper solutions method is proved that these fixed points are solutions of the initial problem, too. **Moreover, despite the localization part, we stress that these solutions may be unbounded.** Last section contains an application for $n = 4$: the study of the bending of infinite beams with different types of foundations. We point out that, the functional boundary conditions allow us to consider new types of models, where, for example, global data on the beam could be considered, which is new on the literature.

Chapter 1

On third order coupled systems with full nonlinearities

This work gives sufficient conditions for the existence of solution, positive or not, of the nonlinear third order coupled system composed by the differential equations

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t), v''(t)) \\ -v'''(t) = h(t, u(t), u'(t), u''(t)), \end{cases} \tag{1.1}$$

where $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are L^1-Carathéodory functions, and the two-point boundary conditions

$$\begin{cases} u(0) = u'(0) = u'(1) = 0 \\ v(0) = v'(0) = v'(1) = 0. \end{cases} \tag{1.2}$$

Moreover, as it is applied lower and upper solutions technique, the localization part of the result allows us to have some qualitative data about solutions sign, growth or variation, as suggested in [75].

Higher order nonlinear systems of differential equations have had an increasing interest in last years, mostly due to their applications in several fields such as populations dynamics, mechanics, optimal control, harvesting,..., as it can be seen in [7, 24, 42, 43, 46, 47, 50, 56, 59, 63, 65, 67] and the references therein.

In particular, third order equations model many phenomenons in physics, engineering and physiology, among others. As examples, we mention the flow of a thin film of viscous fluid over a solid surface (see[11, 92]), the deflection of a curved beam having a constant or varying cross section, the solitary waves solution of the Korteweg–de Vries equation ([66]), the thyroid-pituitary interaction ([25]) or vehicles nonlinear suspensions ([51]).

The methods used in the literature for third order coupled systems can not deal with the second derivatives of the unknown functions. See, for example, [82] where the author proves the existence of at least three positive solutions for the boundary-value problem

$$\begin{cases} u'''(t) + a(t)\ f(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ v'''(t) + b(t)\ h(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u'(1) = \beta u'(\eta) \\ v(0) = v'(0) = 0, \quad v'(1) = \beta v'(\eta), \end{cases}$$

where $f, h : [0,1] \times [0,\infty)^2 \to [0,\infty)$ are continuous and $0 < \eta < 1,\ 1 < \beta < 1/\eta$, $a(t), b(t) \in C([0,1],[0,\infty))$ and are not identically zero on $[\eta/\beta, \eta]$, applying the Leggett-Williams fixed point theorem. And [52], where it is studied the third order differential equations

$$u_i'''(t) + f_i(t, u_1(t), ..., u_n(t), u_1'(t), ..., u_n'(t)) = 0,\ 0 < t < 1,\ i = 1, ..., n,$$

where $f_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions, with multi-point integral boundary conditions, via the Guo-Krasnosels'kii fixed point theorem in a cone.

Motivated by the above papers and by those applications which include a second derivative dependence, we consider problem (1.1), (1.2). Remark that standard cone theory can not be applied to our problem because the second derivative of the Green's functions, associated to the linear form of (1.1), changes sign.

Our arguments apply an integral system defined with the Green's functions as the kernel component, and some auxiliary compact operators, in which an adequate truncature plays a key role. Coupled lower and upper solutions provide a localization tool to establish not only the equivalence between auxiliary and initial problems, but also to give some qualitative properties of the solution. Moreover, a Nagumo-type condition allows an *a priori* control on second derivatives, as in [40].

This chapter is organized as it follows: Section 2 contains the expression of the Green's functions, coupled lower and upper solutions definitions and *a priori* estimations for the second derivatives. The main theorem, an existence and localization result, is in Section 3. In last section is presented an example to show the applicability of the main result.

1.1 Definitions and auxiliary results

Let $E = C^2[0,\ 1]$ be the Banach space equipped with the norm $\|\cdot\|_{C^2}$, defined by

$$\|w\|_{C^2} := \max\{\|w\|, \|w'\|, \|w''\|\},$$

where

$$\|y\| := \max_{t\in[0,\ 1]} |y(t)|$$

and $E^2 = \left(C^2[0,\ 1]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \max\{\|u\|_{C^2}, \|v\|_{C^2}\}.$$

For the reader's convenience we present the definition of L^1-Carathéodory function:

Definition 1.1 *A function $g : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ is a L^1-Carathéodory function, if it satisfies the following properties:*

1. *$g(t,\cdot,\cdot,\cdot)$ is continuous in $\mathbb{R}^3$ for a.e. $t \in [0,1]$.*
2. *$g(\cdot, x, y, z)$ is measurable in $[0,1]$ for all $(x,y,z) \in \mathbb{R}^3$.*
3. *For every $L > 0$ there exists $\psi_L \in L^1[0,1]$ such that, for a.e. $t \in [0,1]$ and all $(x,y,z) \in \mathbb{R}^3$ with $\|(x,y,z)\| \leq L$,*

$$|g(t,x,y,z)| \leq \psi_L(t)$$

Lemma 1.2 *The pair of functions $(u(t), v(t)) \in \left(C^2[0,\ 1],\ \mathbb{R}\right)^2$ is a solution of problem (1.1)-(1.2) if and only if $(u(t), v(t))$ it is a solution of the following system of integral equations*

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s,\ v(s),\ v'(s), v''(s))ds \\ \\ v(t) = \int_0^1 G(t,s) h(s,\ u(s),\ u'(s), u''(s))ds, \end{cases} \tag{1.3}$$

where $G(t,s)$ is the Green's function associated to problem (1.1)-(1.2), defined by

$$G(t,s) = \begin{cases} -\frac{t^2 s}{2} - \frac{s^2}{2} + ts & ,\ 0 \leq s \leq t \leq 1 \\ \\ -\frac{t^2 s}{2} + \frac{t^2}{2} & ,\ t \leq s \leq 1 \end{cases} \tag{1.4}$$

The proof follows standard arguments and it is omitted.

Definition 1.3 *The pair of functions* $(\alpha_1, \alpha_2) \in (C^3[0,1])^2$ *is called coupled lower solutions of (1.1)-(1.2) if*

$$\begin{cases} -\alpha_1'''(t) \leq f(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)) \\ -\alpha_2'''(t) \leq h(t, \alpha_2(t), \alpha_2'(t), \alpha_2''(t)) \end{cases}$$

with

$$\alpha_1(0) \leq 0, \alpha_1'(0) \leq 0, \alpha_1'(1) \leq 0 \tag{1.5}$$

and

$$\alpha_2(0) \leq 0, \alpha_2'(0) \leq 0, \alpha_2'(1) \leq 0.$$

The pair $(\beta_1, \beta_2) \in (C^3[0,1])^2$ *is said to be coupled upper solutions of (1.1)-(1.2) if they verify the reversed inequalities.*

To control the growth of the second derivatives we need Nagumo-type conditions:

Definition 1.4 *The* L^1*-Carathéodory functions* $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *satisfy Nagumo-type conditions if there are positive continuos functions* ϕ_1, ϕ_2 *such that*

$$|f(t, v_0, v_1, v_2)| \leq \phi_1(v_2) \tag{1.6}$$

and

$$|h(t, u_0, u_1, u_2)| \leq \phi_2(u_2) \tag{1.7}$$

with

$$\int_0^{+\infty} \frac{s}{\phi_1(s)} ds = +\infty \quad \text{and} \quad \int_0^{+\infty} \frac{s}{\phi_2(s)} ds = +\infty. \tag{1.8}$$

Next lemma gives *a priori* estimations for $u''(t)$ and $v''(t)$:

Lemma 1.5 *Let* $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *be* L^1*-Carathéodory functions satisfying (1.6), (1.7) and (1.8), in* $[0,1] \times \mathbb{R}^3$*. Then there exist* $R_1, R_2 > 0$ *(not depending on* (u, v) *) such that for every solution of (1.1) verifying*

$$\begin{aligned} \alpha_1^{(i)}(t) &\leq u^{(i)}(t) \leq \beta_1^{(i)}(t) \\ \alpha_2^{(i)}(t) &\leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \text{ for } i = 0, 1, \text{ and } t \in [0,1], \end{aligned}$$

we have

$$||u''|| < R_1 \quad \text{and} \quad ||v''|| < R_2. \tag{1.9}$$

Proof. Let (u, v) be a solution of (1.1), such that

$$\alpha_1(t) \leq u(t) \leq \beta_1(t),\ \alpha_1'(t) \leq u'(t) \leq \beta_1'(t),\ \text{for } t \in [0,1]\,, \tag{1.10}$$

and

$$\alpha_2(t) \leq v(t) \leq \beta_2(t),\ \alpha_2'(t) \leq v'(t) \leq \beta_2'(t),\ \text{for } t \in [0,1]\,.$$

Define $r > 0$ such that

$$r := \max\left\{\beta_1'(0) - \alpha_1'(1), \beta_1'(1) - \alpha_1'(0), \beta_2'(0) - \alpha_2'(1), \beta_2'(1) - \alpha_2'(0)\right\} \tag{1.11}$$

and take $R_1, R_2 > 0$ such that

$$\int_r^{R_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t) \tag{1.12}$$

and

$$\int_r^{R_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \beta_2'(t) - \min_{t\in[0,1]} \alpha_2'(t)\,.$$

Let us prove the *a priori* estimation for $u''(t)$, as for $v''(t)$ the technique is identically.

If, by contradiction, $|u''(t)| > r, \forall t \in [0,1]$, in the case $u''(t) > r$, for $t \in [0,1]$, by (1.10) and (1.11), we have the contradiction

$$\beta_1'(1) - \alpha_1'(0) \geq u'(1) - u'(0) = \int_0^1 u''(t)dt > \int_0^1 r\ dt \geq \beta_1'(1) - \alpha_1'(0).$$

In the case where $u''(t) < -r$, for $t \in [0,1]$, we achieve a similar contradiction. Therefore there exists $t \in [0,1]$ such that $|u''(t)| < r$.

If $|u''(t)| < r$, $\forall t \in [0,1]$, the proof would be finished assuming $R_1 > r$.

Consider that there is $t_0 \in [0,1[$ such that $|u''(t_0)| > r$. If $u''(t_0) > r$, there is $t^* \in [0,1]$, with $t^* < t_0$, $u''(t^*) = r$ and $u''(t) > r, \forall t \in]t^*, t_0]$.

By a change of variable,

$$\begin{aligned}
\int_{u''(t^*)}^{u''(t_0)} \frac{s}{\phi_1(s)} ds &= \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} u'''(s)\, ds \\
&= \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} f(s, v(s), v'(s), v''(s))\, ds \\
&\leq \int_{t^*}^{t_0} u''(s)\, ds = u'(t_0) - u'(t^*) \\
&\leq \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t) < \int_{r}^{R_1} \frac{s}{\phi_1(s)} ds.
\end{aligned}$$

As t_0 is taken arbitrarily on the values where $u''(t_0) > r$, then $u''(t) < R_1, \forall t \in [0,1]$.

If we assume $u''(t_0) < -r$, the method is analogous. Therefore, $||u''|| < R_1$.

Applying a similar technique and (2.8) it can be shown that $\|v''\| < R_2$, for some $R_2 > 0$. ■

The existence tool will be the Schauder's fixed point theorem:

Theorem 1.6 *([101]) Let Y be a nonempty, closed, bounded and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

1.2 Existence and localization theorem

The main theorem will provide the existence and the localization of a solution for problem (1.1)-(1.2).

Theorem 1.7 *Let $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ be L^1-Carathéodory functions satisfying the Nagumo type conditions (1.6), (1.7) and (1.8).*
If there are coupled lower and upper solutions of (1.1)-(1.2), (α_1, α_2) and (β_1, β_2), respectively, such that

$$(\alpha_1', \alpha_2') \leq (\beta_1', \beta_2'),$$

that is,

$$\alpha_1'(t) \leq \beta_1'(t) \text{ and } \alpha_2'(t) \leq \beta_2'(t), \ \forall t \in [0,1],$$

then there is at least a pair $(u(t), v(t)) \in \left(C^3[0,\ 1],\ \mathbb{R}\right)^2$ *solution of (1.1)-(1.2) and, moreover, for* $i = 0, 1$,

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t)$$

and

$$\alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \ \forall t \in [0,1].$$

Remark 1.8 *If* $\alpha_1'(t) \leq u'(t) \leq \beta_1'(t)$ *for* $t \in [0,1]$, *then by integration in* $[0,t]$, *and, by (1.5) and (1.2),*

$$\alpha_1(t) \leq \alpha_1(t) - \alpha_1(0) \leq u\,(t) \leq \beta_1\,(t) - \beta_1\,(0) \leq \beta_1(t), \text{ for } t \in [0,1].$$

Analogously if $\alpha_2'(t) \leq v'\,(t) \leq \beta_2'(t)$, $\forall t \in [0,1]$, *then*

$$\alpha_2(t) \leq v(t) \leq \beta_2(t) \text{ for } t \in [0,1].$$

Proof. Define the operators $T_1 : E^2 \rightarrow E$, $T_2 : E^2 \rightarrow E$ and

$$T\,(u,v) = (T_1\,(u,v)\,, T_2\,(u,v)) \tag{1.13}$$

with

$$\begin{aligned}
(T_1\,(u,v))\,(t) &= \int_0^1 G\,(t,s)\, f\,(s, v\,(s)\,, v'\,(s)\,, v''\,(s))\,ds \\
(T_2\,(u,v))\,(t) &= \int_0^1 G\,(t,s)\, h\,(s, u\,(s)\,, u'\,(s)\,, u''\,(s))\,ds.
\end{aligned}$$

By Lemma 1.2, the fixed points of T are solutions of (1.1)-(1.2). In the following we prove that T has a fixed point.

Consider the auxiliary operators $T^* : E^2 \rightarrow E^2$, $T^*\,(u,v) = (T_1^*\,(u,v)\,, T_2^*\,(u,v))$ where $T_1^* : E^2 \rightarrow E$ is given by

$$T_1^*\,(u\,(t)\,, v(t)) = \int_0^1 G\,(t,s)\, F\,(s, u\,(s)\,, v(s))\,ds$$

with $F(t,u(t),v(t)) := F$ defined as

$$F=\begin{cases} f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } u'(t)>\beta_1'(t), \\ \hfill v'(t)>\beta_2'(t) \\ f\left(t,u(t),u'(t),u''(t)\right)-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } \alpha_1'(t)\leq u'(t)\leq \beta_1'(t), \\ \hfill v'(t)>\beta_2'(t) \\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}+\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } u'(t)<\alpha_1'(t), \\ \hfill v'(t)>\beta_2'(t) \\ f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} \text{ if } u'(t)>\beta_1'(t), \\ \hfill \alpha_2'(t)\leq v'(t)\leq \beta_2'(t) \\ f\left(t,v(t),v'(t),v''(t)\right) \text{ if } \alpha_1'(t)\leq u'(t)\leq \beta_1'(t),\ \alpha_2'(t)\leq v'(t)\leq \beta_2'(t) \\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} \text{ if } u'(t)<\alpha_1'(t), \alpha_2'(t)\leq v'(t)\leq \beta_2'(t) \\ f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}+\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } u'(t)>\beta_1'(t), \\ \hfill v'(t)<\alpha_2'(t) \\ f\left(t,u(t),u'(t),u''(t)\right)-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } \alpha_1'(t)\leq u'(t)\leq \beta_1'(t), v'(t)<\alpha_2'(t) \\ f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } u'(t)<\alpha_1'(t), \\ \hfill v'(t)<\alpha_2'(t), \end{cases}$$

and $T_2^* : E^2 \to E$ by

$$T_2^*(u(t), v(t)) = \int_0^1 G(t,s)\, H(s, u(s), v(s))\, ds$$

with $H(t,u(t),v(t)) := H$ given by

$$H = \begin{cases} h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } u'(t) > \beta_1'(t), \\ \hfill v'(t) > \beta_2'(t) \\ h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ v'(t) > \beta_2'(t) \\ h\left(t,\beta_2(t),\beta_2'(t),\beta_2''(t)\right) + \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|} \text{ if } u'(t) < \alpha_1'(t), \\ \hfill v'(t) > \beta_2'(t) \\ h\left(t,v(t),v'(t),v''(t)\right) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} \text{ if } u'(t) > \beta_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,u(t),u'(t),u''(t)\right) \text{ if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,v(t),v'(t),v''(t)\right) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} \text{ if } u'(t) < \alpha_1'(t),\ \alpha_2'(t) \le v'(t) \le \beta_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) + \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } u'(t) > \beta_1'(t), \\ \hfill v'(t) < \alpha_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } \alpha_1'(t) \le u'(t) \le \beta_1'(t),\ v'(t) < \alpha_2'(t) \\ h\left(t,\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|} \text{ if } u'(t) < \alpha_1'(t), \\ \hfill v'(t) < \alpha_2'(t). \end{cases}$$

As f and h are L^1-Carathéodory functions, therefore F and H are L^1-Carathéodory functions, too. Define the compact subset of E^2

$$K = \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} \le L\right\},$$

with $L > 0$ given by

$$L > \max\left\{R_1, R_2, \left\|\alpha_i^{(j)}\right\|, \left\|\beta_i^{(j)}\right\|, i=1,2,\ j=0,1,2\right\}, \tag{1.14}$$

where R_1, R_2 are defined in (1.9). Therefore, by Definition 1.1, for $(u,v) \in K$, there are positive functions $\psi_{1L}, \psi_{2L} : [0,1] \to (0,+\infty)$ such that $\psi_{1L}, \psi_{2L} \in L^1[0,1]$ and, for $(u,v) \in K$,

$$|F(t,u(t),v(t))| \le \psi_{1L}(t), \text{ for a.e. } t \in [0,1], \tag{1.15}$$

and

$$|H(t,u(t),v(t))| \le \psi_{2L}(t), \text{ for a.e. } t \in [0,1]. \tag{1.16}$$

The Green's function $G(t,s)$ is continuous in $[0,1]\times[0,1]$ and, by Remark 4.5, functions $F(t,u(t),v(t))$ and $H(t,u(t),v(t))$ are bounded. Then $T_1^*(u,v)$

and $T_2^*(u,v)$ are well defined and continuous in E^2, and so, the operator T^* is well defined and continuous in E^2.

Step 1: T_1^* *and* T_2^* *are completely continuous in* $(C^2[0,1])^2$.

The operator T_1^* is continuous in $(C^2[0,1])^2$ as $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ are continuous and f is a L^1-Carathéodory function. Moreover, $\frac{\partial^2 G(t,s)}{\partial t^2}$ is bounded and therefore

$$\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2} F(s,u(s),v(s))\,ds \text{ is continuous.}$$

In the same way, T_2^* is continuous in $(C^2[0,1])^2$.

Claim 1.1. T_1^* *and* T_2^* are uniformly bounded in $(C^2[0,1])^2$.

Define

$$M(s) := \max\left\{\max_{0\le t\le 1}|G(t,s)|, \max_{0\le t\le 1}\left|\frac{\partial G}{\partial t}(t,s)\right|, \sup_{0\le t\le 1}\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right|\right\}$$

.

Then, by Lemma (3.5) and (1.15),

$$|(T_1^*(u(t),v(t))| \le \int_0^1 |G(t,s)|\,|F(s,u(s),v(s))|\,ds \le \int_0^1 M(s)\ \psi_{1L}(s)\,ds < k_0.$$

Analogously, it can be proved that

$$|\,(T_1^*(u(t),v(t)))'\,| < k_1 and |\,(T_1^*(u(t),v(t)))''\,| < k_2,$$

for some $k_0, k_1, k_2 > 0$.

As, for $T_2^*(u,v)$, by (2.8),

$$|(T_2^*(u(t),v(t))| \le \int_0^1 |G(t,s)|\,|H(s,u(s),v(s))|\,ds \le \int_0^1 M(s)\ \psi_{2L}(s)\,ds < \eta_0,$$

for $\eta_0 > 0$. By similar arguments, we have

$$|\,(T_2^*(u(t),v(t)))'\,| < \eta_1 \text{ and } |\,(T_2^*(u(t),v(t)))''\,| < \eta_2,$$

for some $\eta_1, \eta_2 > 0$.

Therefore T^* is uniformly bounded in $\left(C^2[0,1]\right)^2$.

Claim 1.2. *T_1^* and T_2^* are equicontinuous in* $\left(C^2[0,1]\right)^2$.

For the first operator T_1^*, consider t_1, $t_2 \in [0,1]$ and, without loss of generality, suppose $t_1 \le t_2$. So, by (1.15),

$$\left|T_1^*(u,v)(t_1) - T_1^*(u,v)(t_2)\right| \le \int_0^1 \left|G(t_1,s) - G(t_2,s)\right| \ \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2.$$

By similar arguments,

$$\left|(T_1^*(u,v))'(t_1) - (T_1^*(u,v))'(t_2)\right| \le \int_0^1 \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| \ \psi_{1L}(s)\,ds \to 0$$

as $t_1 \to t_2$, and

$$\begin{aligned}\left|(T_1^*(u,v))''(t_1) - (T_1^*(u,v))''(t_2)\right| &\le \int_0^1 \left|\frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2}\right| \ \psi_{1L}(s)\,ds \\ &\le \int_{t_1}^{t_2} \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2.\end{aligned}$$

The proof that T_2^* is equicontinuous in $\left(C^2[0,1]\right)^2$ follows as above.

By the Arzèla-Ascoli theorem, the operator $T^*(u,v)$ is completely continuous.

Step 3: *$T^* : E^2 \to E^2$ has a fixed point.*

In order to apply Theorem 1.6 for operator $T^*(u,v)$ it remains to prove that $T^*D \subset D$, for some closed, bounded and convex $D \subset E^2$.

Consider $D \subset E^2$ given by $D := \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} \le \rho\right\}$, with $\rho > 0$ such that

$$\rho > \max\{L, k_i, \eta_i, i = 0,1,2\},$$

where R_1, R_2 are given by (1.9), L by (1.14), and k_i, η_i, $i = 0,1,2$, are as in Claim 1.1.

By Claim 1.1, $\left\|(T_1^*(u,v)^{(i)}\right\| \leq k_i$, $i = 0,1,2$, and $\left\|(T_2^*(u,v)^{(i)}\right\| \leq \eta_i$, $i = 0,1,2$. Therefore $\|(T_1^*(u,v)\|_E < \rho$ and $\|(T_2^*(u,v)\|_E < \rho$, that is,

$$\|T^*(u,v)\|_{E^2} < \rho.$$

So, $T^*D \subset D$, and, by Theorem 1.6, T^* has a fixed point $(u,v) \in D \subset E^2$.

Step 4: *This fixed point (u,v) of T^* is also a fixed point of T, given by (1.13).*

As (u,v) is a fixed point of $T^*(u,v)$ it means that (u,v) is a fixed point of $T_1^*(u,v)$ and of $T_2^*(u,v)$.

By standard arguments it can be shown that

$$-u'''(t) = F(t, u(t), v(t))$$

and

$$-v'''(t) = H(t, u(t), v(t)).$$

So, to prove this step it will be enough to show that

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t) \text{ and } \alpha_2'(t) \leq v'(t) \leq \beta_2'(t),\ \forall t \in [0,1]. \tag{1.17}$$

For the first inequality suppose, by contradiction, that there is $t \in [0,1]$ such that $\alpha_1'(t) > u'(t)$. Define

$$\max_{0\leq t<1} (\alpha_1'(t) - u'(t)) := \alpha_1'(t_0) - u'(t_0) > 0.$$

By (1.2) and (1.5), $t_0 \neq 0$ because $\alpha'(0) - u'(0) = \alpha'(0) \leq 0$. Analogously $t_0 \neq 1$. So $t_0 \in]0,1[$ and

$$\alpha_1''(t_0) = u_1''(t_0),\ \alpha_1'''(t_0) - u'''(t_0) \leq 0.$$

As $(\alpha_1'(t) - u'(t)) \in C[0,1]$, there is $I \subset [0,1]$ such that $t_0 \in I$ and

$$\begin{aligned} \alpha_1'(t) - u'(t) &> 0, \\ \alpha_1'''(t) - u'''(t) &\leq 0,\ \forall t \in I. \end{aligned}$$

For all possible values of $v'(t_0)$, we obtain the following contradictions by the truncature F and Definition 3.3:

If $v'(t_0) < \alpha_2'(t_0)$, and as $v'(t) - \alpha_2'(t) \in C[0,1]$, then there is $J_0 \subset [0,1]$ such that $t_0 \in J_0$ and $v'(t) - \alpha_2'(t) < 0$, $\forall t \in J_0$.

As $t_0 \in I \cap J_0$ then $I \cap J_0 \neq \varnothing$ and

$$\begin{aligned}
0 &\geq \int\limits_{I\cap J_0} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int\limits_{I\cap J_0} \Bigg(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|} \\
&\quad - \frac{v'(t) - \alpha_2'(t)}{1 + |u'(t) - \alpha_2'(t)|}\Bigg) dt \\
&> \int\limits_{I\cap J_0} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \geq 0.
\end{aligned}$$

If $\alpha_2'(t_0) \leq v'(t_0) \leq \beta_2'(t_0)$, and as $v', \alpha_2', \beta_2' \in C[0,1]$ then there exists $J_1 \subset [0,1]$ such that $t_0 \in J_1$ and $\alpha_2'(t) \leq v'(t) \leq \beta_2'(t)$, $\forall t \in J_1$.

As $I \cap J_1 \neq \varnothing$ then

$$\begin{aligned}
0 &\geq \int\limits_{I\cap J_1} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int\limits_{I\cap J_1} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|}\right) dt \\
&> \int\limits_{I\cap J_1} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \geq 0.
\end{aligned}$$

If $v'(t_0) > \beta_2'(t_0)$, and as $v'(t) - \beta_2'(t) \in C[0,1]$ then there is $J_2 \subset [0,1]$ such that $t_0 \in J_2$ and $v'(t) - \beta_2'(t) > 0$, $\forall t \in J_2$.

As $t_0 \in I \cap J_2$ then $I \cap J_2 \neq \varnothing$ and

$$\begin{aligned}
0 &\geq \int_{I\cap J_2} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int_{I\cap J_2} \left(\alpha_1'''(t) - f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right) - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|} \right. \\
&\quad \left. + \frac{v'(t) - \beta_2'(t)}{1 + |v'(t) - \beta_2'(t)|}\right) dt \\
&> \int_{I\cap J_2} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t)\right)\right) dt \geq 0.
\end{aligned}$$

Therefore, $\alpha_1'(t) \leq u'(t), \forall t \in [0,1]$. By similar arguments it can be proved that $u'(t) \leq \beta_1'(t), \forall t \in [0,1]$, and so,

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t), \ \forall t \in [0,1]. \tag{1.18}$$

Applying the same technique with the truncature H, it can be achieved that

$$\alpha_2'(t) \leq v'(t) \leq \beta_2'(t), \ \forall t \in [0,1]. \tag{1.19}$$

So, the fixed point (u, v) of T^* is also a fixed point of T, given by (1.13), and by Lemma 3.2, $(u(t), v(t))$ is a solution of problem (1.1)-(1.2). ■

1.3 Example

Consider the system of nonlinear and nonautonomous differential equations

$$\left\{\begin{array}{rl} -u'''(t) = & v^3(t) + e^{v'(t)} - 6\sqrt[3]{\left(v''(t)\right)^2} \\ \\ -v'''(t) = & \frac{t}{4} - \arctan\left(u(t)\right) + \left(u'(t)\right)^3 + 2\left(u''(t)\right)^2 \end{array}\right. \tag{1.20}$$

with the boundary conditions (1.2).

In fact (1.20) is a particular case of (1.1) with

$$f(t, x, y, z) = x^3 + e^y - 6\sqrt[3]{z^2}$$

and

$$h\left(t,x,y,z\right)=\frac{t}{4}-\arctan x+y^{3}+2z^{2},$$

where f and h given above are L^1-Carathéodory functions.

By easy computations, it can be seen that functions

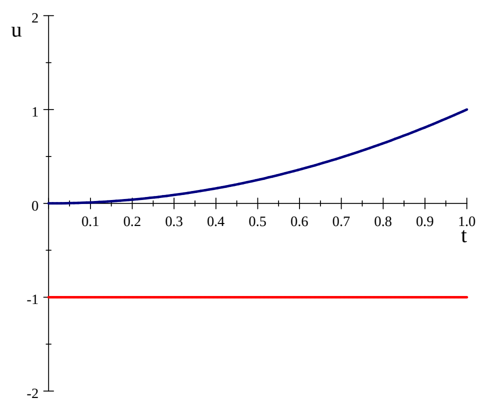
$$\begin{aligned}\alpha_1(t) &= -1, \quad \beta_1(t)=t^2\\ \alpha_2(t) &= -t^2, \quad \beta_2(t)=1\end{aligned}$$

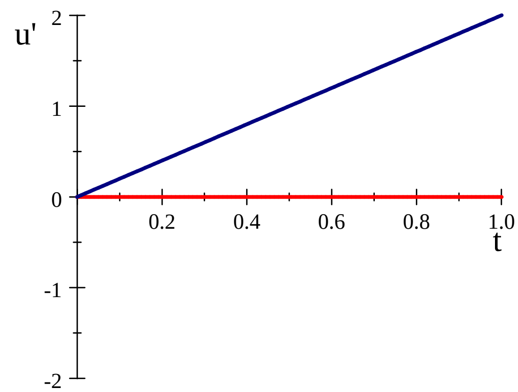
are coupled lower and upper solutions of (1.20), (1.1).

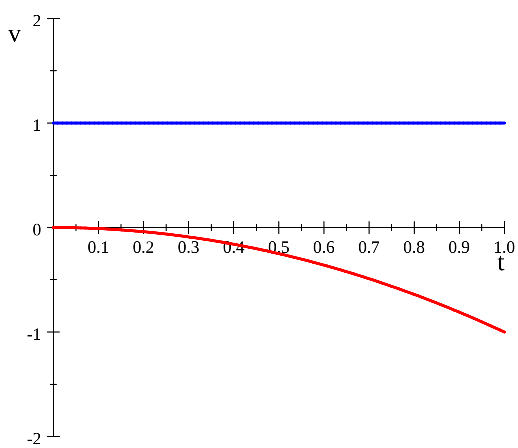
By Theorem 4.4 there is a solution (u,v) of (1.20), (1.2) such that

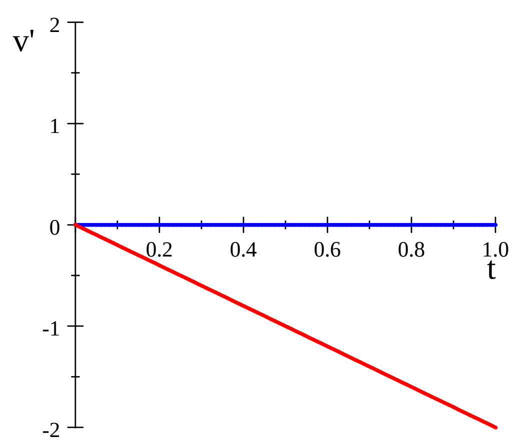
$$\begin{aligned}-1 &\leq u(t)\leq t^2, \quad -t^2\leq v(t)\leq 1\\ 0 &\leq u'\left(t\right)\leq 2t, \quad -2t\leq v'\left(t\right)\leq 0, \text{ for } t\in[0,1].\end{aligned}$$

From the localization part, $u\left(t\right)$ is a nondecreasing function and $v\left(t\right)$ is a nonincreasing one, as it can be seen in next figures.









Chapter 2

Solvability of generalized third order coupled systems with two-point boundary conditions

In this chapter, it is proved the existence of solutions for the nonlinear third-order coupled system with fully differential equations

$$\begin{cases} -u'''(t) = f(t, u(t), u'(t), u''(t), v(t), v'(t), v''(t)) \\ -v'''(t) = h(t, u(t), u'(t), u''(t), v(t), v'(t), v''(t)), \end{cases} \tag{2.1}$$

where $f, h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are L^1-Carathéodory functions, together with the two-point boundary conditions

$$\begin{cases} u(0) = u'(0) = u'(1) = 0 \\ v(0) = v'(0) = v'(1) = 0. \end{cases} \tag{2.2}$$

Remark that the nonlinearities can depend on all derivatives of both unknown functions, which, to the best of our knowledge, is new in the literature. This dependence is allowed by an adequate auxiliary integral problem with a truncature by lower and upper solutions with bounded perturbations. The main theorem is an existence and localization result, giving some extra qualitative data on the system solution, such as, sign, variation, growth, bounds,..., as it is suggested in [75].

Higher order nonlinear differential equations and systems have been developed in last years, mainly by their applications in many fields such as populations dynamics, mechanics, optimal control, ..., as it can be seen in

[5, 8, 24, 42, 46, 47, 50, 56, 63, 65, 68] and the references therein. Third order equations, in particular, can model a large number of phenomena in engineering, physics, physiology, and chemistry, among others. For instance, we refer the flow of a thin film of viscous fluid over a solid surface (see [11, 92]), solitary waves of the Korteweg–de Vries equation ([66]), the thyroid-pituitary homeostatic interaction ([25]), or vehicles suspensions ([51]).

The usual methods for nonlinear third order coupled systems can not consider the second derivatives of the unknown functions, whenever the second derivative of the associated Green's function change sign. This is the case of [82], where the author proves the existence of three positive solutions for the system

$$\begin{cases} u'''(t) + a(t)\ f(t, u(t), v(t)) = 0, & 0 < t < 1, \\ v'''(t) + b(t)\ h(t, u(t), v(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = 0, \quad u'(1) = \beta u'(\eta) \\ v(0) = v'(0) = 0, \quad v'(1) = \beta v'(\eta), \end{cases}$$

with $f, h : [0,1] \times [0,\infty)^2 \to [0,\infty)$ continuous functions, $0 < \eta < 1$, $1 < \beta < 1/\eta$, $a(t), b(t) \in C([0,1],[0,\infty))$ are not identically zero on $[\eta/\beta, \eta]$, applying the Leggett-Williams fixed point theorem. And of [52], where it is considered n third order differential equations of the type

$$u_i'''(t) + f_i(t, u_1(t), ..., u_n(t), u_1'(t), ..., u_n'(t)) = 0,\ 0 < t < 1,\ i = 1, ..., n,$$

where $f_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions, and multi-point integral boundary conditions, via the Guo-Krasnosels'kii fixed point theorem in cones.

These papers and the above applications with dependence on second derivatives, motivated our problem (1.1), (1.2), where we apply an integral system defined with the Green's functions, and some auxiliary compact integral operators with an adequate truncature. Coupled lower and upper solutions are a location tool to give not only the equivalence between auxiliary and initial problems, but also to obtain some qualitative properties of the solution. Moreover, a Nagumo-type condition allows *a priori* estimations on second derivatives, as suggested in [40].

The chapter is organized in this way: Section 2 contains the explicit Green's functions, the definitions of coupled lower and upper solutions and a lemma with *a priori* bounds for the second derivatives. The main result, an existence and localization theorem, is in Section 3. Last section contains an example to show the applicability of the main theorem and the utility of the localization tool.

2.1 Preliminary results

Consider the Banach space $E = C^2[0,\,1]$ equipped with the norm $\|\cdot\|_{C^2}$, given by

$$\|w\|_{C^2} := \max\left\{\|w\|, \|w'\|, \|w''\|\right\},$$

with

$$\|y\| := \max_{t\in[0,\,1]} |y(t)|,$$

and the product space $E^2 = \left(C^2[0,\,1]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \|u+v\|_{C^2}.$$

Next lemma gives the integral system related to (1.1)-(1.2):

Lemma 2.1 *The continuous functions $(u(t),v(t)) \in \left(C^2[0,\,1],\,\mathbb{R}\right)^2$ are a solution of problem (2.1)-(2.2) if and only if $(u(t),v(t))$ is a solution of the system of integral equations*

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s, u(s),\, u'(s), u''(s),\, v(s),\, v'(s), v''(s))ds \\ \\ v(t) = \int_0^1 G(t,s) h(s,\, u(s),\, u'(s), u''(s), v(s),\, v'(s), v''(s))ds, \end{cases} \tag{2.3}$$

where $G(t,s)$ is the Green's function associated to problem (2.1)-(2.2), defined by

$$G\left(t,s\right) = \begin{cases} -\frac{t^2 s}{2} - \frac{s^2}{2} + ts \quad , \; 0 \le s \le t \le 1 \\ \\ -\frac{t^2 s}{2} + \frac{t^2}{2} \qquad\quad , \; t \le s \le 1 \end{cases} \tag{2.4}$$

The proof applies standard calculus and it is omitted.

Definition 2.2 *The functions $(\alpha_1, \alpha_2) \in \left(C^3\left[0,1\right]\right)^2$ are coupled lower solutions of (2.1)-(2.2) if*

$$\begin{cases} -\alpha_1'''\left(t\right) \le f\left(t, \alpha_1\left(t\right), \alpha_1'\left(t\right), \alpha_1''\left(t\right), \alpha_2\left(t\right), \alpha_2'\left(t\right), \alpha_2''\left(t\right)\right) \\ -\alpha_2'''\left(t\right) \le h\left(t, \alpha_1\left(t\right), \alpha_1'\left(t\right), \alpha_1''\left(t\right), \alpha_2\left(t\right), \alpha_2'\left(t\right), \alpha_2''\left(t\right)\right) \end{cases}$$

with

$$\alpha_1\left(0\right) \le 0, \alpha_1'\left(0\right) \le 0, \alpha_1'\left(1\right) \le 0 \tag{2.5}$$

and

$$\alpha_2\left(0\right) \le 0, \alpha_2'\left(0\right) \le 0, \alpha_2'\left(1\right) \le 0. \tag{2.6}$$

Functions $(\beta_1, \beta_2) \in \left(C^3\left[0,1\right]\right)^2$ are coupled upper solutions of (2.1)-(2.2) if they verify the reversed inequalities.

An order relation between pairs will be used forward with the following sense

$$(x,y) \leq (z,w) \iff x \leq z \wedge y \leq w, \ \forall x,y,z,w \in \mathbb{R}.$$

An *a priori* estimation on the second derivatives of u and v is given by Nagumo-type conditions:

Definition 2.3 *The continuous functions $f,h : [0,1] \times \mathbb{R}^6 \to \mathbb{R}$ satisfy Nagumo-type conditions in the set*

$$S = \left\{ \begin{array}{c} (t,x_0,x_1,x_2,y_0,y_1,y_2) \in [0,1] \times \mathbb{R}^6 : \\ \alpha_1^{(i)}(t) \leq x_i \leq \beta_1^{(i)}(t), \ \alpha_2^{(i)}(t) \leq y_i \leq \beta_2^{(i)}(t), \ \textit{for } i=0,1 \end{array} \right\},$$

if there are positive and continuos functions $\phi_i : [0,+\infty[\to]0,+\infty[$, $i = 0,1$, such that

$$|f(t,u_0,u_1,u_2,v_0,v_1,v_2)| \leq \phi_1(|u_2|) \ \textit{ and } \ |h(t,u_0,u_1,u_2,v_0,v_1,v_2)| \leq \phi_2(|v_2|) \tag{2.7}$$

with

$$\int_0^{+\infty} \frac{s}{\phi_1(s)} ds = +\infty \ \textit{ and } \int_0^{+\infty} \frac{s}{\phi_2(s)} ds = +\infty \tag{2.8}$$

Lemma 2.4 *Let $f,h : [0,1] \times \mathbb{R}^6 \to \mathbb{R}$ be continuous functions satisfying (2.7) and (2.8), in S. Then there exist $R_1, R_2 > 0$ (not depending on (u,v)) such that for every solution (u,v) of (2.1)-(2.2), in S, we have*

$$\|u''\| < R_1 \ \textit{ and } \ \|v''\| < R_2. \tag{2.9}$$

Proof. Let (u,v) be a solution of (2.1), such that

$$\alpha_1(t) \leq u(t) \leq \beta_1(t), \ \alpha_1'(t) \leq u'(t) \leq \beta_1'(t), \tag{2.10}$$

and

$$\alpha_2(t) \leq v(t) \leq \beta_2(t), \ \alpha_2'(t) \leq v'(t) \leq \beta_2'(t), \ \text{for } t \in [0,1].$$

For each $r > 0$, take $R_1, R_2 > r$ such that

$$\int_r^{R_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t). \tag{2.11}$$

and

$$\int_{r}^{R_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \beta_2'(t) - \min_{t\in[0,1]} \alpha_2'(t). \qquad (2.12)$$

Assume that $|u''(t)| \geq r, \forall t \in [0,1]$. In the case where $u''(t) \geq r$, for $t \in [0,1]$, by (2.2), we have the contradiction

$$r > 0 = \int_0^1 (u''(t)) \; dt \geq \int_0^1 r \; dt = r.$$

In the case where $u''(t) < -r$, for $t \in [0,1]$, we achieve a similar contradiction. Therefore there exists $t \in [0,1]$ such that $|u''(t)| < r$.

If $|u''(t)| < r$, $\forall t \in [0,1]$, the proof would be finished, assuming $R_1 > r$.

Consider now that there is $t_0 \in [0,1[$ such that $|u''(t_0)| > r$. If $u''(t_0) > r$, by the mean value theorem and (2.2), there is $\bar{t} \in]0,1[$ such that $u''(\bar{t}) = 0$.

Therefore there is $t^* \in [0,1]$, with $t^* < t_0$, $u''(t^*) = r$ and $u''(t) > r, \forall t \in]t^*, t_0]$.

By an adequate change of variable (2.7) and (2.11),

$$\int_{u''(t^*)}^{u''(t_0)} \frac{s}{\phi_1(s)} ds = \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} (u'''(s)) \, ds$$

$$\leq \int_{t^*}^{t_0} \frac{u''(s)}{\phi_1(u''(s))} \left| -f(s, u(s), u'(s), u''(s), v(s), v'(s), v''(s)) \right| ds$$

$$\leq \int_{t^*}^{t_0} (u''(s)) \, ds = u'(t_0) - u'(t^*) \leq \beta_1'(t_0) - \alpha_1'(t^*)$$

$$\leq \max_{t\in[0,1]} \beta_1'(t) - \min_{t\in[0,1]} \alpha_1'(t) < \int_r^{R_1} \frac{s}{\phi_1(s)} ds.$$

As t_0 is taken arbitrarily on the values where $u''(t_0) > r$, then $u''(t) < R_1, \forall t \in [0,1]$.

If $u''(t_0) < -r$, the method is analogous to obtain $-u''(t_0) < R_1$ and therefore, $||u''|| < R_1$.

Applying similar arguments it can be proved that $||v''|| < R_2$. ■

Remark 2.5 *The a priori estimation given by (2.9) does not depend on the boundary conditions.*

2.2 Existence of solution

The main theorem will provide the existence and the localization of a solution for problem (2.1)-(2.2).

Theorem 2.6 *Assume that $f, h : [0,1] \times \mathbb{R}^6 \to \mathbb{R}$ are continuous functions satisfying the Nagumo type conditions (2.7) and (2.8).*
If there are coupled lower and upper solutions of (2.1)-(2.2), (α_1, α_2) and (β_1, β_2), respectively, such that

$$(\alpha_1', \alpha_2') \leq (\beta_1', \beta_2')$$

and

$$\begin{aligned} f(t,\alpha_1(t),x_1,x_2,\alpha_2(t),y_1,y_2) &\leq f(t,x_0,x_1,x_2,y_0,y_1,y_2) \quad (2.13)\\ &\leq f(t,\beta_1(t),x_1,x_2,\beta_2(t),y_1,y_2), \\ h(t,\alpha_1(t),x_1,x_2,\alpha_2(t),y_1,y_2) &\leq h(t,x_0,x_1,x_2,y_0,y_1,y_2) \quad (2.14)\\ &\leq h(t,\beta_1(t),x_1,x_2,\beta_2(t),y_1,y_2), \end{aligned}$$

for $\alpha_1(t) \leq x_0 \leq \beta_1(t)$, $\alpha_2(t) \leq y_0 \leq \beta_2(t)$ and $(t,x_1,x_2,y_1,y_2) \in [0,1] \times \mathbb{R}^4$ fixed, then there is $(u(t),v(t)) \in \left(C^3[0,\,1],\,\mathbb{R}\right)^2$ solution of (2.1)-(2.2) such that, for $i = 0,1$,

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t)$$

and

$$\alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t),\ \forall t \in [0,1].$$

Remark 2.7 *If $\alpha_1'(t) \leq u'(t) \leq \beta_1'(t)$ for $t \in [0,1]$, then the relation $\alpha_1(t) \leq u(t) \leq \beta_1(t)$, for $t \in [0,1]$, can be easily obtained by integration in $[0,t]$, by (2.5) and (2.2).*
Analogously if $\alpha_2'(t) \leq v'(t) \leq \beta_2'(t)$, $\forall t \in [0,1]$, we achieve $\alpha_2(t) \leq v(t) \leq \beta_2(t)$ for $t \in [0,1]$.

Proof. Define the operators $T_1 : E^2 \to E$, $T_2 : E^2 \to E$ and $T : E^2 \to E^2$ given by

$$T(u,v) = (T_1(u,v), T_2(u,v)) \quad (2.15)$$

with

$$(T_1(u,v))(t) = \int_0^1 G(t,s)\, f(s, u(s), u'(s), u''(s), v(s), v'(s), v''(s))\, ds$$

and

$$(T_2(u,v))(t) = \int_0^1 G(t,s)\, h(s, u(s), u'(s), u''(s), v(s), v'(s), v''(s))\, ds.$$

The fixed points of T are, from Lemma 2.1, solutions of (2.1)-(2.2). Forward we shall prove that T has a fixed point.

Consider the auxiliary operator $T^* : E^2 \to E^2$, $T^*(u,v) = (T_1^*(u,v), T_2^*(u,v))$ where $T_1^* : E^2 \to E$ is given by

$$T_1^*(u(t), v(t)) = \int_0^1 G(t,s)\, F(s, u(s), v(s))ds$$

with $F(t, u(t), v(t))$ defined as

- $f(t, \beta_1(t), \beta_1'(t), \beta_1''(t), \beta_2(t), \beta_2'(t), \beta_2''(t)) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|} - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$,
  if $u'(t) > \beta_1'(t)$, $v'(t) > \beta_2'(t)$
- $f(t, u(t), u'(t), u''(t), v(t), v'(t), v''(t)) - \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$,
  if $\alpha_1'(t) \le u'(t) \le \beta_1'(t)$, $v'(t) > \beta_2'(t)$
- $f(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} + \frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$,
  if $u'(t) < \alpha_1'(t)$, $v'(t) > \beta_2'(t)$
- $f(t, \beta_1(t), \beta_1'(t), \beta_1''(t), \beta_2(t), \beta_2'(t), \beta_2''(t)) - \frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}$, if $u'(t) > \beta_1'(t)$,
  $\alpha_2'(t) \le v'(t) \le \beta_2'(t)$
- $f(t, u(t), u'(t), u''(t), v(t), v'(t), v''(t))$, if $\alpha_1'(t) \le u'(t) \le \beta_1'(t)$,
  $\alpha_2'(t) \le v'(t) \le \beta_2'(t)$
- $f(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}$, if $u'(t) < \alpha_1'(t)$,
  $\alpha_2'(t) \le v'(t) \le \beta_2'(t)$

- $f\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t),\beta_2(t),\beta_2'(t),\beta_2''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}+\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$, if $u'(t)>\beta_1'(t),\ v'(t)<\alpha_2'(t)$
- $f\left(t,u(t),u'(t),u''(t),v(t),v'(t),v''(t)\right)-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$, if $\alpha_1'(t)\le u'(t)\le\beta_1'(t), v'(t)<\alpha_2'(t)$
- $f\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t),\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$, if $u'(t)<\alpha_1'(t),\ v'(t)<\alpha_2'(t)$,

and $T_2^*:E^2\to E$ as

$$T_2^*(u(t),v(t))=\int_0^1 G(t,s)\,H(s,u(s),v(s))ds$$

with $H(t,u(t),v(t))$ given by

- $h\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t),\beta_2(t),\beta_2'(t),\beta_2''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$, if $u'(t)>\beta_1'(t),\ v'(t)>\beta_2'(t)$
- $h\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t),\beta_2(t),\beta_2'(t),\beta_2''(t)\right)-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$, if $\alpha_1'(t)\le u'(t)\le\beta_1'(t),\ v'(t)>\beta_2'(t)$
- $h\left(t,\beta_1(t),\beta_1'(t),\beta_1''(t),\beta_2(t),\beta_2'(t),\beta_2''(t)\right)+\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}-\frac{v'(t)-\beta_2'(t)}{1+|v'(t)-\beta_2'(t)|}$, if $u'(t)<\alpha_1'(t),\ v'(t)>\beta_2'(t)$
- $h\left(t,u(t),u'(t),u''(t),v(t),v'(t),v''(t)\right)-\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}$, if $u'(t)>\beta_1'(t)$, $\alpha_2'(t)\le v'(t)\le\beta_2'(t)$
- $h\left(t,u(t),u'(t),u''(t),v(t),v'(t),v''(t)\right)$, if $\alpha_1'(t)\le u'(t)\le\beta_1'(t)$, $\alpha_2'(t)\le v'(t)\le\beta_2'(t)$
- $h\left(t,u(t),u'(t),u''(t),v(t),v'(t),v''(t)\right)-\frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|}$, if $u'(t)<\alpha_1'(t)$, $\alpha_2'(t)\le v'(t)\le\beta_2'(t)$
- $h\left(t,\alpha_1(t),\alpha_1'(t),\alpha_1''(t),\alpha_2(t),\alpha_2'(t),\alpha_2''(t)\right)+\frac{u'(t)-\beta_1'(t)}{1+|u'(t)-\beta_1'(t)|}-\frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$, if $u'(t)>\beta_1'(t),\ v'(t)<\alpha_2'(t)$

- $h\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right) - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$,
  if $\alpha_1'(t) \leq u'(t) \leq \beta_1'(t),\ v'(t) < \alpha_2'(t)$

- $h\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right) - \frac{u'(t)-\alpha_1'(t)}{1+|u'(t)-\alpha_1'(t)|} - \frac{v'(t)-\alpha_2'(t)}{1+|v'(t)-\alpha_2'(t)|}$,
  if $u'(t) < \alpha_1'(t),\ v'(t) < \alpha_2'(t)$.

Consider the subset of E^2

$$K = \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} \leq L\right\},$$

with $L > 0$ defined as

$$L > \max\left\{R_1, R_2, \left\|\alpha_i^{(j)}\right\|, \left\|\beta_i^{(j)}\right\|, i = 1,2,\ j = 0,1,2\right\}, \tag{2.16}$$

where R_1 and R_2 are given by (2.9).

By the continuity of f and h and Lemma 2.4, there exist positive functions $\psi_{1L}, \psi_{2L} : [0,1] \to (0,+\infty)$ such that $\psi_{1L}, \psi_{2L} \in L^1[0,1]$ and, for $(u,v) \in K$,

$$|F(t,u(t),v(t))| \leq \psi_{1L}(t), \text{ for a.e. } t \in [0,1], \tag{2.17}$$

and

$$|H(t,u(t),v(t))| \leq \psi_{2L}(t), \text{ for a.e. } t \in [0,1]. \tag{2.18}$$

The Green function $G(t,s)$ is continuous in $[0,1]\times[0,1]$ and, by Remark 2.7, the truncature functions $F(t,u(t),v(t))$ and $H(t,u(t),v(t))$ are bounded. Then $T_1^*(u,v)$, $T_2^*(u,v)$ and T^* are well defined and continuous in E^2.

Step 1: T^* is *completely continuous in* E^2.

The step will be proved if T_1^* and T_2^* are completely continuous in E^2.

As $G(t,s)$ and $\frac{\partial G(t,s)}{\partial t}$ are continuous and f is a L^1-Carathéodory function, then the operator T_1^* is continuous in E^2

Moreover, $\frac{\partial^2 G(t,s)}{\partial t^2}$ is bounded and therefore

$$\int_0^1 \frac{\partial^2 G(t,s)}{\partial t^2} F(s,u(s),v(s))\,ds \text{ is continuous.}$$

By similar arguments, T_2^* is continuous in E^2.

Claim 2.1. T^* is uniformly bounded in E^2.

To prove that T_1^* and T_2^* are uniformly bounded in E^2, we define

$$M(s) := \max\left\{ \max_{0\leq t\leq 1} |G(t,s)|, \max_{0\leq t\leq 1}\left|\frac{\partial G}{\partial t}(t,s)\right|, \sup_{0\leq t\leq 1}\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right| \right\}.$$

.

By Lemma 2.4 , (2.7) and (2.8),

$$\begin{aligned} |(T_1^*(u(t),v(t))| &\leq \int_0^1 |G(t,s)|\,|F(s,u(s),v(s))|\,ds \qquad (2.19) \\ &\leq \int_0^1 M(s)\ \psi_{1L}(s)\,ds < k_0. \end{aligned}$$

and, analogously,

$$|(T_1^*(u(t),v(t)))'| < k_1 \text{ and } |(T_1^*(u(t),v(t)))''| < k_2, \qquad (2.20)$$

for some $k_0, k_1, k_2 > 0$.

For $T_2^*(u,v)$, by (2.7) and (2.18), it can be shown that

$$\begin{aligned} |(T_2^*(u(t),v(t))| &\leq \int_0^1 |G(t,s)|\,|H(s,u(s),v(s))|\,ds \qquad (2.21) \\ &\leq \int_0^1 M(s)\ \psi_{2L}(s)\,ds < \eta_0, \end{aligned}$$

and

$$|(T_2^*(u(t),v(t)))'| < \eta_1 \text{ and } |(T_2^*(u(t),v(t)))''| < \eta_2, \qquad (2.22)$$

for some $\eta_0, \eta_1, \eta_2 > 0$. So,

$$\begin{aligned} \|T^*(u,v)\|_{E^2} &= \|(T_1^*(u,v), T_2^*(u,v))\|_{E^2} \\ &= \max\left\{ \begin{array}{c} \|T_1^*(u,v) + T_2^*(u,v)\|, \left\|(T_1^*(u,v))' + (T_2^*(u,v))'\right\|, \\ \left\|(T_1^*(u,v))'' + (T_2^*(u,v))''\right\| \end{array} \right\}. \end{aligned}$$

For the first norm, by (2.19) and (2.21),

$$\begin{aligned} \|T_1^*(u,v) + T_2^*(u,v)\| &= \max_{t\in[0,1]} |(T_1^*(u,v)) + T_2^*(u,v)| \\ &\leq \max_{t\in[0,1]} |T_1^*(u,v)| + \max_{t\in[0,1]} |T_2^*(u,v)| \leq k_0 + \eta_0, \end{aligned}$$

that is, $\|T_1^*(u,v) + T_2^*(u,v)\| \leq k_0 + \eta_0$.

With similar arguments it can be proved, by (2.20) and (2.22), that

$$\left\|(T_1^*(u,v))' + (T_2^*(u,v))'\right\| \leq k_1+\eta_1 \text{ and } \left\|(T_1^*(u,v))'' + (T_2^*(u,v))''\right\| \leq k_2+\eta_2.$$

Therefore T^* is uniformly bounded in E^2.

Claim 2.2. T^* is equicontinuous in E^2.

Let us prove that T_1^* and T_2^* are equicontinuous in E^2.

For T_1^*, consider t_1, $t_2 \in [0,1]$ and, without loss of generality, assume that $t_1 \leq t_2$. By (2.17),

$$\begin{aligned} |T_1^*(u,v)(t_1) - T_1^*(u,v)(t_2)| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \; F(s,u(s),v(s))\,ds \\ &\leq \int_0^1 |G(t_1,s) - G(t_2,s)| \; \psi_{1L}(s)\,ds \to 0, \end{aligned}$$

as $t_1 \to t_2$,

$$\left|(T_1^*(u,v))'(t_1) - (T_1^*(u,v))'(t_2)\right| \leq \int_0^1 \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| \psi_{1L}(s)\,ds \to 0,$$

as $t_1 \to t_2$, and

$$\begin{aligned} \left|(T_1^*(u,v))''(t_1) - (T_1^*(u,v))''(t_2)\right| &\leq \int_0^1 \left|\frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2}\right| \; \psi_{1L}(s)\,ds \\ &\leq \int_{t_1}^{t_2} \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2. \end{aligned}$$

The proof that T_2^* is equicontinuous in E^2 follows as above.

So, T^* is equicontinuous in E^2 and, by the Arzèla-Ascoli theorem, $T^*(u,v)$ is completely continuous in E^2.

Step 3: *$T^* : E^2 \to E^2$ has a fixed point.*

By Theorem 1.6, it remains to show that, for some closed, bounded and convex $D \subset E^2$, $T^*D \subset D$.

Assume that $D \subset E^2$ is given by $D := \{(u,v) \in E^2 : \|(u,v)\|_{E^2} \leq \rho\}$, with $\rho > 0$ such that

$$\rho > \max\{L, k_i + \eta_i, i = 0, 1, 2\},$$

where L is given by (2.16), and k_i, η_i, $i = 0, 1, 2$, by Claim 1.1.

For $(u,v) \in D$ we have that $\left\|(T_1^*(u,v)^{(i)}\right\| \leq k_i$, for $i = 0, 1, 2$, $\left\|(T_2^*(u,v)^{(i)}\right\| \leq \eta_i$, for $i = 0, 1, 2$, and

$$\|T^*(u,v)\|_{E^2} < \rho.$$

Therefore, $T^*D \subset D$, and, by Theorem 1.6, T^* has a fixed point $(u,v) \in D \subset E^2$.

Step 4: (u,v) *is also a fixed point of* T, *given by (2.15).*

As (u,v) is a fixed point of $T^*(u,v)$ then (u,v) is a fixed point of $T_1^*(u,v)$ and $T_2^*(u,v)$.

By standard arguments we have

$$-u'''(t) = F(t, u(t), v(t))$$

and

$$-v'''(t) = H(t, u(t), v(t)).$$

So, to prove this step it will be enough to show the equivalence between the operators $T^*(u,v)$ and $T.(u,v)$, that is if

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t) \text{ and } \alpha_2'(t) \leq v'(t) \leq \beta_2'(t),\ \forall t \in [0,1].$$

For first inequality assume, by contradiction, that there is $t \in [0,1]$ where $\alpha_1'(t) > u'(t)$, and define

$$\max_{0 \leq t < 1} (\alpha_1'(t) - u'(t),) := \alpha_1'(t_0) - u'(t_0) > 0.$$

By the boundary conditions (2.2) and (2.5), $t_0 \neq 0$, as $\alpha'(0) - u'(0) = \alpha'(0) \leq 0$, and $t_0 \neq 1$. Therefore $t_0 \in]0,1[$ and $\alpha_1'''(t_0) - u'''(t_0) \leq 0$.

As $(\alpha_1'(t) - u'(t)) \in C[0,1]$, there is $J \subset [0,1]$ such that $t_0 \in J$ and

$$\begin{aligned} \alpha_1'(t) - u'(t) &> 0, \\ \alpha_1'''(t) - u'''(t) &\leq 0,\ \forall t \in J. \end{aligned}$$

For all possible cases of the value of $v'(t_0)$, we obtain contradictions by the truncature $F(t, u(t), v(t))$ and Definition 2.2:

If $v'(t_0) < \alpha_2'(t_0)$, and as $v'(t) - \alpha_2'(t) \in C[0,1]$, then there exists $I_0 \subset [0,1]$ such that $t_0 \in I_0$ and $v'(t) - \alpha_2'(t) < 0$, $\forall t \in I_0$.

As $t_0 \in J \cap I_0$ then $J \cap I_0 \neq \varnothing$ and

$$\begin{aligned}
0 &\geq \int\limits_{J\cap I_0} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int\limits_{J\cap I_0} \Big(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right) \\
&\quad -\frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|} - \frac{v'(t) - \alpha_2'(t)}{1 + |u'(t) - \alpha_2'(t)|}\Big) dt \\
&> \int\limits_{J\cap I_0} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right)\right) dt \geq 0.
\end{aligned}$$

If $\alpha_2'(t_0) \leq v'(t_0) \leq \beta_2'(t_0)$, and as $v', \alpha_2', \beta_2' \in C[0,1]$ then there is $I_1 \subset [0,1]$ such that $t_0 \in I_1$ and $\alpha_2'(t) \leq v'(t) \leq \beta_2'(t)$, $\forall t \in I_1$.

As $J \cap I_1 \neq \varnothing$ then

$$\begin{aligned}
0 &\geq \int\limits_{J\cap I_1} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int\limits_{J\cap I_1} \Big(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right) \\
&\quad -\frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|}\Big) dt \\
&> \int\limits_{J\cap I_1} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right)\right) dt \geq 0.
\end{aligned}$$

If $v'(t_0) > \beta_2'(t_0)$, and as $v'(t) - \beta_2'(t) \in C[0,1]$ then there exists $I_2 \subset [0,1]$ such that $t_0 \in I_2$ and $v'(t) - \beta_2'(t) > 0$, $\forall t \in I_2$.

As $t_0 \in J \cap I_2$ then $J \cap I_2 \neq \varnothing$ and

$$\begin{aligned}
0 &\geq \int\limits_{J\cap I_2} \left(\alpha_1'''(t) - u'''(t)\right) dt \\
&= \int\limits_{J\cap I_2} \Big(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right) \\
&\quad - \frac{u'(t) - \alpha_1'(t)}{1 + |u'(t) - \alpha_1'(t)|} + \frac{v'(t) - \beta_2'(t)}{1 + |v'(t) - \beta_2'(t)|}\Big) dt \\
&> \int\limits_{J\cap I_2} \left(\alpha_1'''(t) + f\left(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t)\right)\right) dt \geq 0.
\end{aligned}$$

Therefore, $\alpha_1'(t) \leq u'(t), \forall t \in [0,1]$. By a similar technique it can be shown that $u'(t) \leq \beta_1'(t), \forall t \in [0,1]$, and so,

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t), \ \forall t \in [0,1]. \tag{2.23}$$

Applying the same method with the truncature $H(t, u(t), v(t))$, we can obtain that

$$\alpha_2'(t) \leq v'(t) \leq \beta_2'(t), \ \forall t \in [0,1]. \tag{2.24}$$

So, the fixed point (u, v) of T^* is a fixed point of T, given by (2.15), and, from Lemma 2.1, $(u(t), v(t))$ is a solution of problem (2.1)-(2.2). ■

2.3 Example

Consider the third order nonlinear and nonautonomous system

$$\begin{cases}
-u'''(t) = & t^3 + (u(t))^2 \ u'(t) - \sqrt[5]{u''(t) + 1} + (v(t))^3 + e^{2v'(t)} \\
& -4\sin\left(v''(t) + \frac{\pi}{4}\right) \\
-v'''(t) = & t \ u(t) \ v(t) - (u'(t))^3 v'(t) - \sqrt[3]{u''(t)} - \sqrt[5]{v''(t)}
\end{cases} \tag{2.25}$$

with boundary conditions (2.2).

Clearly (2.25) is a particular case of (2.1) with

$$f(t, x_0, x_1, x_2, y_0, y_1, y_2) = t^3 + x_0^2 \ x_1 - \sqrt[5]{x_2(t) + 1} + y_0^3 + e^{2y_1} - 4\sin\left(y_2 + \frac{\pi}{4}\right)$$

and

$$h(t, x_0, x_1, x_2, y_0, y_1, y_2) = tx_0y_0 - (x_1)^3 y_1 - \sqrt[3]{x_2} - \sqrt[5]{y_2}.$$

These functions f and h verify the assumptions of Theorem 2.6, and, by simple computations, we see that

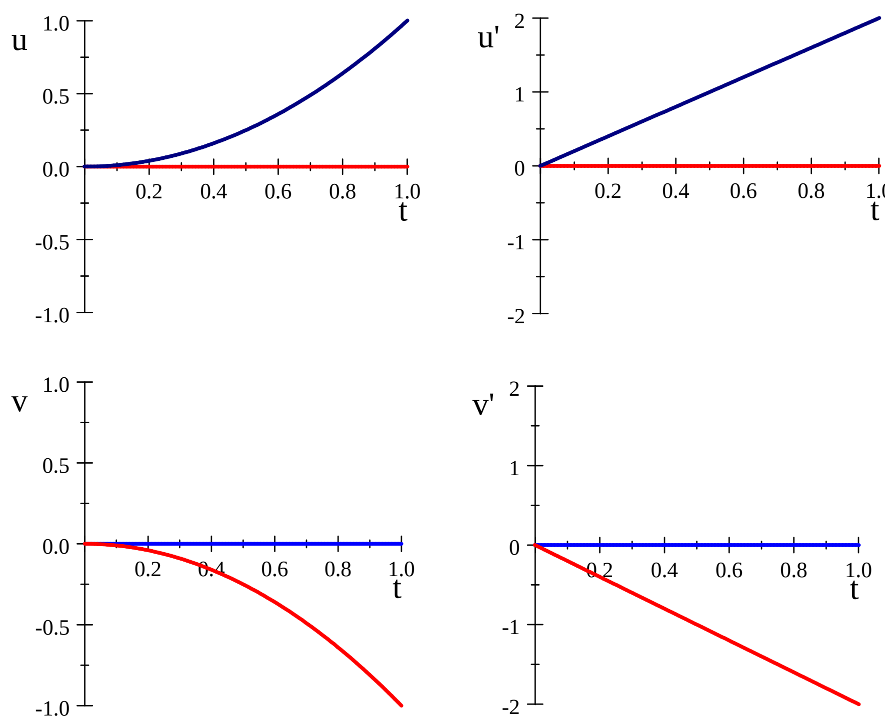
$$\begin{aligned} \alpha_1(t) &= 0, \quad \beta_1(t) = t^2 \\ \alpha_2(t) &= -t^2, \quad \beta_2(t) = 0 \end{aligned}$$

are coupled lower and upper solutions of (2.25), (2.2).

By Theorem 2.6,there is a solution (u, v) of (2.25), (2.2) such that

$$\begin{aligned} 0 &\leq u(t) \leq t^2, \quad -t^2 \leq v(t) \leq 0 \\ 0 &\leq u'(t) \leq 2t, \quad -2t \leq v'(t) \leq 0, \text{ for } t \in [0,1]. \end{aligned}$$

From the localization part, $u(t)$ is a non negative and nondecreasing function and $v(t)$ is a nonpositive and nonincreasing function, as it is illustrated by the following figures.



Chapter 3

Systems of coupled clamped beams equations with full nonlinear terms: existence and location results

In this chapter we consider the fourth order coupled system

$$\left\{\begin{array}{l} u^{(4)}(t)=f\left(t,u(t),u'(t),u''(t),u'''(t),v(t),v'(t),v''(t),v'''(t)\right)\\ v^{(4)}(t)=h\left(t,u(t),u'(t),u''(t),u'''(t),v(t),v'(t),v''(t),v'''(t)\right)\end{array}\right. \tag{3.1}$$

with $f,h:[0,1]\times\mathbb{R}^8\to\mathbb{R}$ some L^1- Carathéodory functions and the boundary conditions

$$\left\{\begin{array}{l} u(0)=A_{0,}\ u'(0)=A_1,\ u''(0)=\ u''(1)=A_2\\ v(0)=B_0,\ v'(0)=B_1,\ v''(0)=\ v''(1)=B_2,\end{array}\right. \tag{3.2}$$

with $A_i,B_i\in\mathbb{R}$, for $i=0,1,2$.

Fourth order differential equations have been studied by many authors with different types of boundary conditions, as it can be seen in [9, 15, 18, 28, 56, 63, 69, 83, 84, 98, 102, 104] and the references therein. The applications can be found in several fields, such as the deflection of beams resting on elastic foundations ([49]), to describe the motion of the road bed of suspension bridges ([21]), nonlocal elasticity theory, more precisely the study of nonlinear vibrations of an Euler-Bernoulli nanobeam ([90]), study of the flow of certain fluids over stretching or shrinking sheets ([36]), among others.

Boundary value problems composed by systems of fourth order differential equations are more scarce (see, for instance, [5, 46, 68, 105]). In [68] the authors consider the existence of multiple positive solutions for coupled singular system of second and fourth order ordinary differential equations

$$\begin{cases} u^{(4)} = f(t, v), \quad (t, v) \in (0, 1) \times \mathbb{R}^+, \\ -v'' = g(t, u), \quad (t, u) \in (0, 1) \times \mathbb{R}^+, \\ u(0) = u(1) = u''(0) = u''(1) = 0 \\ v(0) = v(1) = 0, \end{cases}$$

where $f \in C[(0,1) \times \mathbb{R}^+, \mathbb{R}^+]$ and $g \in C[(0,1) \times \mathbb{R}^+, \mathbb{R}^+]$, applying a fixed point theorem of cones expansion and compression.

In [105] it is studied the fourth-order nonlinear singular semipositone system

$$\begin{cases} x^{(4)}(t) = f(t, x(t), y(t), x''(t), y''(t)), \\ y^{(4)}(t) = g(t, x(t), y(t), x''(t), y''(t)), \qquad t \in (0, 1), \\ x(0) = x(1) = x''(0) = x''(1) = 0 \\ y(0) = y(1) = y''(0) = y''(1) = 0, \end{cases}$$

with $f, g \in C((0, 1) \times [0, \infty) \times [0, \infty) \times (-\infty, 0] \times (-\infty, 0], \mathbb{R})$, by approximating the fourth-order singular semipositone system to a second-order singular system and using a fixed point index theorem on cones, to guarantee the existence of positive solutions of the problems.

Motivated by the above papers we consider in this chapter the problem (3.1), (3.2), where the nonlinearities can depend on all derivatives of both unknown functions, which, to the best of our knowledge, is new in the literature. This dependence is due to an adequate auxiliary integral problem with a truncature by lower and upper solutions with some bounded perturbations. Moreover, we underline that the classical cone theory can not be applied to our problem as the second derivative of the Green's functions change sign.

Our method applies an integral system defined with the Green's functions, and some auxiliary compact integral operators with an adequate truncature. Coupled lower and upper solutions are a location tool to give not only the equivalence between auxiliary and initial problems, but also to obtain some qualitative properties of the solution. A Nagumo-type condition allows *a priori* estimations on the third derivatives, as suggested in [40].

The main theorem is an existence and localization result, gives some qualitative data on the system solutions such as, sign, variation, growth, bounds, convexity/concavity..., as it is suggested in [75].

The chapter is organized in this way: Section 2 contains the explicit Green's functions, the definitions of coupled lower and upper solutions and a lemma with *a priori* bounds for the second derivatives. The main result, an existence and localization theorem, is in Section 3. Last section contains an example to show the applicability of the main theorem and the utility of the localization tool.

3.1 Preliminary results

Let $E = C^3[0,\,1]$ be the Banach space equipped with the norm $\|\cdot\|_{C^3}$, defined by

$$\|w\|_{C^3} := \max\left\{\|w\|, \|w'\|, \|w''\|, \|w'''\|\right\},$$

where

$$\|y\| := \max_{t\in[0,\,1]} |y(t)|$$

and $E^2 = \left(C^3[0,\,1]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \max\left\{\|u\|_{C^3}, \|v\|_{C^3}\right\}.$$

For the reader's convenience we present the definition of L^1-Carathéodory function:

Definition 3.1 *A function $g : [0,1]\times\mathbb{R}^8 \to \mathbb{R}$ is a L^1-Carathéodory function, if it satisfies the following properties:*

1. *$g(t,\cdot,\cdot)$ is continuous in $\mathbb{R}^8$ for a.e. $t \in [0,1]$;*
2. *$g(\cdot,\overline{x},\overline{y})$, with $\overline{x} = (x_0,x_1,x_2,x_3)$ and $\overline{y} = (y_0,y_1,y_2,y_3)$, is measurable in $[0,1]$ for all $(\overline{x},\overline{y}) \in \mathbb{R}^8$;*
3. *for every $L > 0$ there exists $\psi_L \in L^1[0,1]$ such that, for a.e. $t \in [0,1]$ and all $(\overline{x},\overline{y}) \in \mathbb{R}^8$ with $\|(\overline{x},\overline{y})\|_{E^2} \le L$,*

$$|g(t,\overline{x},\overline{y})| \le \psi_L(t).$$

Lemma 3.2 *The functions* $(u(t), v(t)) \in \left(C^3[0,\,1],\,\mathbb{R}\right)^2$ *are solution of problem (3.1), (3.2) if and only if* $(u(t), v(t))$ *is a solution of the following system of integral equations*

$$\begin{cases} u(t) = \int_0^1 G(t,s) f(s, u(s), u'(s), u''(s), u'''(s), v(s), v'(s), v''(s), v'''(s))ds \\ \\ v(t) = \int_0^1 G(t,s) h(s, u(s), u'(s), u''(s), u'''(s), v(s), v'(s), v''(s), v'''(s))ds, \end{cases} \tag{3.3}$$

where $G(t,s)$ *is the Green's function associated to problem (3.1), (3.2), defined by*

$$G(t,s) = \begin{cases} -\frac{s^3}{6} + \frac{s^2 t}{2} - \frac{st^2}{2} + \frac{st^3}{6}, & 0 \le s \le t \\ \\ \frac{st^3}{6} - \frac{t^3}{6}, & t \le s \le 1. \end{cases} \tag{3.4}$$

The proof follows standard arguments and it is omitted.

Definition 3.3 *The pair of functions* $(\alpha_1, \alpha_2) \in \left(C^4[0,1]\right)^2$ *is called coupled lower solutions of (3.1), (3.2) if*

$$\begin{cases} \alpha_1^{(4)}(t) \ge f(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_1'''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t), \alpha_2'''(t)) \\ \alpha_2^{(4)}(t) \ge h(t, \alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_1'''(t), \alpha_2(t), \alpha_2'(t), \alpha_2''(t), \alpha_2'''(t)) \end{cases}$$

with

$$\alpha_1(0) \le A_0, \alpha_1'(0) \le A_1, \alpha_1''(0) \le A_2, \alpha_1''(1) \le A_2 \tag{3.5}$$

and

$$\alpha_2(0) \le B_0, \alpha_2'(0) \le B_1, \alpha_2''(0) \le B_2, \alpha_2''(1) \le B_2.$$

The pair $(\beta_1, \beta_2) \in \left(C^4[0,1]\right)^2$ *is said to be coupled upper solutions of (3.1), (3.2) if they verify the reversed inequalities.*

Throughout the chapter we apply an order relation between pairs defined as

$$(x,y) \le (z,w) \iff x \le z \wedge y \le w, \ \forall x,y,z,w \in \mathbb{R}.$$

To control the growth of the second derivatives we need Nagumo-type conditions:

Definition 3.4 *The L^1-Carathéodory functions $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ satisfy Nagumo-type conditions if there exist positive continuos functions $\phi_i : [0,+\infty[\to]0,+\infty[$, $i = 0,1$, such that*

$$\begin{aligned} |f(t,u_0,u_1,u_2,u_3,v_0,v_1,v_2,v_3)| &\leq \phi_1(|u_3|) \\ |h(t,u_0,u_1,u_2,u_3,v_0,v_1,v_2,v_3)| &\leq \phi_2(|v_3|), \end{aligned} \tag{3.6}$$

for

$$\alpha_1^{(i)}(t) \leq u_i(t) \leq \beta_1^{(i)}(t)$$
$$\alpha_2^{(i)}(t) \leq v_i(t) \leq \beta_2^{(i)}(t), \text{ for } i = 0,1,2 \text{ and } t \in [0,1],$$

with

$$\int_0^{+\infty} \frac{s}{\phi_1(s)} ds = +\infty \text{ and } \int_0^{+\infty} \frac{s}{\phi_2(s)} ds = +\infty \ . \tag{3.7}$$

Next lemma gives a priori estimations for $u'''(t) + v'''(t)$:

Lemma 3.5 *Let $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ be L^1-Carathéodory functions satisfying (3.6), (3.7), in $[0,1] \times \mathbb{R}^8$. Then there exist $R_1, R_2 > 0$ (not depending on (u,v)) such that for every solution verifying*

$$\begin{gathered} \alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t) \\ \alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \text{ for } i = 0,1,2, \text{ and } t \in [0,1], \end{gathered} \tag{3.8}$$

we have

$$||u'''|| < R_1 \text{ and } ||v'''|| < R_2. \tag{3.9}$$

Proof. Let (u,v) be a solution of (3.1), verifying (3.8).

For each $r > 0$, take $R_1, R_2 > r$ such that

$$\int_r^{R_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \beta_1''(t) - \min_{t\in[0,1]} \alpha_1''(t), \tag{3.10}$$

and

$$\int_r^{R_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \beta_2''(t) - \min_{t\in[0,1]} \alpha_2''(t).$$

If $|u'''(t)| > r, \forall t \in [0,1]$, and in the case where $u'''(t) > r$ for $t \in [0,1]$, by (3.8) and (3.2), we have the following contradiction,

$$r > 0 = \int_0^1 (u'''(t))\ dt > \int_0^1 r\ dt = r.$$

In the case where $u'''(t) < -r$, for $t \in [0,1]$, we achieve a similar contradiction. Therefore there exists $t \in [0,1]$ such that $|u'''(t)| < r$.

If $|u'''(t)| < r, \forall t \in [0,1]$, the proof would be finished with $||u'''|| < r < R_1$.

Consider that there is $t_0 \in [0,1[$ such that $|u'''(t_0)| > r$. If $u'''(t_0) > r$, by the mean value theorem and the boundary conditions, there is $t^* \in [0,1]$ where $u'''(t^*) = r$.

If $t^* < t_0$, then $u'''(t) > r, \forall t \in]t^*, t_0]$. By a convenient change of variable and (3.10), defining, for short,

$$\overline{u}(t) := (u(t), u'(t), u''(t), u'''(t)) \text{ and } \overline{v}(t) := (v(t), v'(t), v''(t), v'''(t)), \tag{3.11}$$

we have

$$\begin{aligned}
\int_{u'''(t^*)}^{u'''(t_0)} \frac{s}{\phi_1(s)} ds &= \int_{t^*}^{t_0} \frac{u'''(s)}{\phi_1(u''(s))} \left(u^{(4)}(s)\right) ds \\
&= \int_{t^*}^{t_0} \frac{u'''(s)}{\phi_1(u'''(s))} \left|f\left(s, \overline{u}(s), \overline{v}(s)\right)\right| ds \\
&\leq \int_{t^*}^{t_0} u'''(s)\, ds = u''(t_0) - u''(t^*) \\
&\leq \max_{t\in[0,1]} \left(\beta_1''(t)\right) - \min_{t\in[0,1]} \left(\alpha_1''(t)\right) < \int_r^{R_1} \frac{s}{\phi_1(s)} ds.
\end{aligned}$$

If $t^* > t_0$, then $u'''(t) > r, \forall t \in [t_0, t^*[$, and the calculus follows as in the previous case. As t_0 is taken arbitrarily on the values where $u'''(t) > r$, then $u'''(t) < R_1, \forall t \in [0,1]$.

If we assume $u'''(t_0) < -r$, the method is analogous. Therefore, $||u'''|| < R_1$.

Applying similar arguments it can be proved that $||v'''|| < R_2$. ∎

3.2 Main result

The main theorem will provide the existence and the localization of a solution for problem (3.1), (3.2).

Theorem 3.6 *Let $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ be L^1-Carathéodory functions satisfying the Nagumo type conditions (3.6) and (3.7).*
If there are coupled lower and upper solutions of (3.1), (3.2), (α_1, α_2) and (β_1, β_2), respectively, such that

$$(\alpha_1'', \alpha_2'') \leq (\beta_1'', \beta_2''), \tag{3.12}$$

then there is at least a pair $(u(t), v(t)) \in (C^4[0,\ 1], \mathbb{R})^2$ solution of (3.1), (3.2) and, moreover, for $i = 0, 1, 2$,

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t)$$

and

$$\alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t),\ \forall t \in [0,1].$$

Remark 3.7 *If $\alpha_1''(t) \leq u''(t) \leq \beta_1''(t)$ for $t \in [0,1]$, then by integration in $[0,t]$, and, by (3.5) and (3.2),*

$$\alpha_1'(t) \leq u'(t) \leq \beta_1'(t) \text{ and } \alpha_1(t) \leq u(t) \leq \beta_1(t), \text{ for } t \in [0,1].$$

Analogously if $\alpha_2''(t) \leq v''(t) \leq \beta_2''(t)$, $\forall t \in [0,1]$, then

$$\alpha_2'(t) \leq v'(t) \leq \beta_2'(t) \text{ and } \alpha_2(t) \leq v(t) \leq \beta_2(t) \text{ for } t \in [0,1].$$

Proof. Define the operators $T_1 : (C^3[0,\ 1])^2 \to C^3[0,\ 1]$, $T_2 : (C^3[0,\ 1])^2 \to C^3[0,\ 1]$ and

$$T(u,v) = (T_1(u,v), T_2(u,v)) \tag{3.13}$$

with

$$\begin{aligned}(T_1(u,v))(t) &= \int_0^1 G(t,s)\, f(s, \overline{u}(s), \overline{v}(s))ds \\ (T_2(u,v))(t) &= \int_0^1 G(t,s)\, h(s, \overline{u}(s), \overline{v}(s))ds,\end{aligned}$$

where $\overline{u}(t)$ and $\overline{v}(t)$ are given by (3.11).

By Lemma 3.2, the fixed points of T are solutions of (3.1), (3.2). In the following we prove that T has a fixed point.

Consider the auxiliary operators $T^* : (C^3[0,\,1])^2 \to (C^3[0,\,1])^2$, $T^*(u,v) = (T_1^*(u,v), T_2^*(u,v))$,

$$T_1^*(u(t), v(t)) = \int_0^1 G(t,s)\, F(s, u(s), v(s))\, ds$$

with $F(t, u(t), v(t)) := F(t)$ defined as

$$F(t) = \begin{cases} f\left(t, \overline{\beta}_1(t), \overline{\beta}_2(t)\right) - \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} - \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } u''(t) > \beta_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ f(t, \overline{u}(t), \overline{v}(t)) - \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ f(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)) - \frac{u''(t)-\alpha_1''(t)}{1+|u''(t)-\alpha_1''(t)|} + \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } u''(t) < \alpha_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ f\left(t, \overline{\beta}_1(t), \overline{\beta}_2(t)\right) - \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} \text{ if } u''(t) > \beta_1''(t), \\ \qquad \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ f(t, \overline{u}(t), \overline{v}(t)) \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t),\ \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ f(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)) - \frac{u''(t)-\alpha_1''(t)}{1+|u(t)-\alpha_1''(t)|} \text{ if } u''(t) < \alpha_1''(t), \\ \qquad \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ f\left(t, \overline{\beta}_1(t), \overline{\beta}_2(t)\right) - \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} + \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } u''(t) > \beta_1''(t), \\ \qquad v''(t) < \alpha_2''(t) \\ f(t, \overline{u}(t), \overline{v}(t)) - \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t), \\ \qquad v''(t) < \alpha_2''(t) \\ f(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)) - \frac{u''(t)-\alpha_1''(t)}{1+|u''(t)-\alpha_1''(t)|} - \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } u''(t) < \alpha_1''(t), \\ \qquad v''(t) < \alpha_2''(t), \end{cases}$$

where

$$\begin{aligned} \overline{\alpha}_1(t) &:= (\alpha_1(t), \alpha_1'(t), \alpha_1''(t), \alpha_1'''(t)), \\ \overline{\alpha}_2(t) &:= (\alpha_2(t), \alpha_2'(t), \alpha_2''(t), \alpha_2'''(t)), \\ \overline{\beta}_1(t) &:= (\beta_1(t), \beta_1'(t), \beta_1''(t), \beta_1'''(t)), \\ \overline{\beta}_2(t) &:= (\beta_2(t), \beta_2'(t), \beta_2''(t), \beta_2'''(t)). \end{aligned} \tag{3.14}$$

Similarly, we define $T_2^* : E^3 \to E$ by

$$T_2^*(u\,(t)\,, v\,(t)) = \int_0^1 G\,(t,s)\,H(t,u(t),v(t)ds$$

with $H(t,u(t),v(t)) := H(t)$ given by

$$H(t) = \begin{cases} h\left(t,\overline{\beta}_1(t),\overline{\beta}_2(t)\right) - \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} - \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } u''(t) > \beta_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ h\left(t,\overline{\beta}_1(t),\overline{\beta}_2(t)\right) - \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ h\left(t,\overline{\beta}_1(t),\overline{\beta}_2(t)\right) + \frac{u''(t)-\alpha_1''(t)}{1+|u''(t)-\alpha_1''(t)|} - \frac{v''(t)-\beta_2''(t)}{1+|v''(t)-\beta_2''(t)|} \text{ if } u''(t) < \alpha_1''(t), \\ \qquad v''(t) > \beta_2''(t) \\ h\,(t,\overline{u}(t),\overline{v}(t)) - \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} \text{ if } u''(t) > \beta_1''(t),\ \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ h\,(t,\overline{u}(t),\overline{v}(t)) \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t),\ \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ h\,(t,\overline{u}(t),\overline{v}(t)) - \frac{u''(t)-\alpha_1''(t)}{1+|u''(t)-\alpha_1''(t)|} \text{ if } u''(t) < \alpha_1''(t),\ \alpha_2''(t) \le v''(t) \le \beta_2''(t) \\ h\,(t,\overline{\alpha}_1,\overline{\alpha}_2) + \frac{u''(t)-\beta_1''(t)}{1+|u''(t)-\beta_1''(t)|} - \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } u''(t) > \beta_1''(t), \\ \qquad v''(t) < \alpha_2''(t) \\ h\,(t,\overline{\alpha}_1,\overline{\alpha}_2) - \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } \alpha_1''(t) \le u''(t) \le \beta_1''(t), \\ \qquad v''(t) < \alpha_2''(t) \\ h\,(t,\overline{\alpha}_1,\overline{\alpha}_2) - \frac{u''(t)-\alpha_1''(t)}{1+|u''(t)-\alpha_1''(t)|} - \frac{v''(t)-\alpha_2''(t)}{1+|v''(t)-\alpha_2''(t)|} \text{ if } u''(t) < \alpha_1''(t), \\ \qquad v''(t) < \alpha_2''(t). \end{cases}$$

As f and h are L^1-Carathéodory functions, therefore F and H are L^1-Carathéodory functions, too. Define the compact subset of $\left(C^3[0,\,1]\right)^2$

$$K = \left\{(u,v) \in \left(C^3[0,\,1]\right)^2 : \|(u,v)\|_{E^2} \le L\right\},$$

with $L > 0$ given by

$$L > \max\left\{R, \left\|\alpha_i^{(j)}\right\|, \left\|\beta_i^{(j)}\right\|, i = 1,2,\ j = 0,1,2,3\right\}, \tag{3.15}$$

where R is defined in (3.9). Therefore, by Definition 3.1, for $(u,v)\in K$, there are positive functions $\psi_{1L},\psi_{2L}:[0,1]\to(0,+\infty)$ such that $\psi_{1L},\psi_{2L}\in L^1[0,1]$ and, for $(u,v)\in K$,

$$|F(t,u(t),v(t))|\le\psi_{1L}(t),\ \text{for a.e. } t\in[0,1]\,, \tag{3.16}$$

and

$$|H(t,u(t),v(t))|\le\psi_{2L}(t),\ \text{for a.e. } t\in[0,1]\,. \tag{3.17}$$

The Green's function $G(t,s)$ is continuous in $[0,1]\times[0,1]$ and, by Remark 4.5, functions $F(t,u(t),v(t))$ and $H(t,u(t),v(t))$ are bounded. Then $T_1^*(u,v)$ and $T_2^*(u,v)$ are well defined and continuous in $\left(C^3[0,\,1]\right)^2$, and so, the operator T^* is well defined and continuous in $\left(C^3[0,\,1]\right)^2$.

Step 1: *T^* is completely continuous in $\left(C^3[0,1]\right)^2$.*

The operator T_1^* is continuous in $\left(C^3[0,1]\right)^2$ as $G(t,s)$. $\frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial^2 G(t,s)}{\partial t^2}$ are continuous and f is a L^1-Carathéodory function. Moreover, $\frac{\partial^3 G(t,s)}{\partial t^3}$ is bounded and therefore

$$\int_0^1\frac{\partial^3 G(t,s)}{\partial t^3}F(s,u(s),v(s))\,ds \text{ is continuous.}$$

In the same way, T_2^* is continuous in $\left(C^3[0,1]\right)^2$.

Claim 1.1. T^* is uniformly bounded in $\left(C^3[0,1]\right)^2$.

Define

$$M(s):=\max\left\{\max_{0\le t\le1}|G(t,s)|,\max_{0\le t\le1}\left|\frac{\partial G}{\partial t}(t,s)\right|,\max_{0\le t\le1}\left|\frac{\partial^2 G}{\partial t^2}(t,s)\right|,\sup_{0\le t\le1}\left|\frac{\partial^3 G}{\partial t^3}(t,s)\right|\right\}$$

Then, by Lemma 3.5 and (3.16),

$$|(T_1^*(u(t),v(t))|\le\int_0^1|G(t,s)|\,|F(s,u(s),v(s))|\,ds\le\int_0^1 M(s)\ \psi_{1L}(s)\,ds<k_0.$$

Analogously, it can be proved that

$$|\left(T_1^*(u(t),v(t))\right)'|<k_1,\ |\left(T_1^*(u(t),v(t))\right)''|<k_2 \text{ and } |\left(T_1^*(u(t),v(t))\right)'''|<k_3,$$

for some $k_0, k_1, k_2, k_3 > 0$.

As, for $T_2^*(u,v)$, by (3.17),

$$|(T_2^*(u(t),v(t))| \leq \int_0^1 |G(t,s)|\,|H(s,u(s),v(s))|\,ds \leq \int_0^1 M(s)\ \psi_{2L}(s)\,ds < \eta_0,$$

for $\eta_0 > 0$. By similar arguments, we have

$$|\,(T_2^*(u(t),v(t)))'\,| < \eta_1\ ,\ |\,(T_2^*(u(t),v(t)))''\,| < \eta_2 \text{ and } |\,(T_1^*(u(t),v(t)))'''\,| < \eta_3$$

for some $\eta_1, \eta_2, \eta_3 > 0$.

Moreover,

$$\begin{aligned}\|T^*(u,v)\|_{E^2} &= \|(T_1^*(u,v), T_2^*(u,v))\|_{E^2} \\ &= \max\left\{\left\|(T_1^*(u,v))^{(i)}\right\|, \left\|(T_2^*(u,v))^{(i)}\right\|, i=0,1,2,3\right\} \\ &\leq \max\{k_i, \eta_i,\ i=0,1,2,3\}.\end{aligned}$$

Therefore T^* is uniformly bounded in $(C^3[0,1])^2$.

Claim 1.2. T^* is equicontinuous in $(C^3[0,1])^2$.

Consider $t_1, t_2 \in [0,1]$ and, without loss of generality, assume that $t_1 \leq t_2$. By (3.16)

$$|T_1^*(u,v)(t_1) - T_1^*(u,v)(t_2)| \leq \int_0^1 |G(t_1,s) - G(t_2,s)|\ \psi_{1L}(s)\,ds \to 0, \text{ as } t_1 \to t_2.$$

By similar arguments,

$$\begin{aligned}\left|(T_1^*(u,v))'(t_1) - (T_1^*(u,v))'(t_2)\right| &\leq \int_0^1 \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right|\ \psi_{1L}(s)\,ds \to 0, \\ \left|(T_1^*(u,v))''(t_1) - (T_1^*(u,v))''(t_2)\right| &\leq \int_0^1 \left|\frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2}\right|\ \psi_{1L}(s)\,ds \to 0,\end{aligned}$$

as $t_1 \to t_2$, and

$$\left|(T_1^*(u,v))'''(t_1) - (T_1^*(u,v))'''(t_2)\right| \leq \int_0^1 \left|\frac{\partial^3 G(t_1,s)}{\partial t^3} - \frac{\partial^3 G(t_2,s)}{\partial t^3}\right| \psi_{1L}(s)\,ds$$

$$\leq \int_{t_1}^{t_2} \psi_{1L}(s)\,ds \to 0 \text{ as } t_1 \to t_2.$$

By the same technique it can be proved that T_2^* is equicontinuous in $\left(C^3[0,1]\right)^2$.

Therefore, T^* is equicontinuous in $\left(C^3[0,1]\right)^2$ and, by the Arzèla-Ascoli theorem, $T^*(u,v)$ is completely continuous in $\left(C^3[0,1]\right)^2$.

Step 3: $T^* : \left(C^3[0,1]\right)^2 \to \left(C^3[0,1]\right)^2$ *has a fixed point.*

In order to apply the well known Schauder's fixed point theorem for operator $T^*(u,v)$, it remains to prove that $T^*D \subset D$, for some closed, bounded and convex $D \subset \left(C^3[0,1]\right)^2$.

Consider

$$D := \left\{(u,v) \in \left(C^3[0,1]\right)^2 : \|(u,v)\|_{E^2} \leq \rho\right\},$$

with $\rho > 0$ such that

$$\rho > \max\{M, k_i, \eta_i,\ i = 0,1,2,3\},$$

where L is given by (3.15) and k_i, η_i, $i = 0,1,2,3$, are defined in Claim 1.1.

By Claim 1.1, $\left\|(T_1^*(u,v)^{(i)} + (T_2^*(u,v)^{(i)}\right\| \leq k_i + \eta_i$, $i = 0,1,2,3$ and, therefore,

$$\|T^*(u,v)\|_{E^2} < \rho.$$

So, $T^*D \subset D$, and, by Schauder's fixed point theorem, T^* has a fixed point $(u,v) \in D \subset \left(C^3[0,1]\right)^2$.

Step 4: If (u,v) *is a fixed point of* T^* *then* (u,v) *is a fixed point of* T, *given by (3.13).*

Let (u,v) be a fixed point of $T^*(u,v)$. Therefore (u,v) is a fixed point of the operators $T_1^*(u,v)$ and $T_2^*(u,v)$.

By standard calculus it can be proved that $u^{(4)}(t) = F(t,u(t),v(t))$ and $v^{(4)}(t) = H(t,u(t),v(t))$.

To finish the proof it is enough to see that

$$\alpha_1^{(i)}(t) \le u^{(i)}(t) \le \beta_1^{(i)}(t) \text{ and } \alpha_2^{(i)}(t) \le v^{(i)}(t) \le \beta_2^{(i)}(t), \ \forall t \in [0,1], \ i = 0,1,2. \tag{3.18}$$

Suppose, by contradiction, that the first inequality does not hold for $i = 2$. Then there is $t \in [0,1]$ such that $\alpha_1''(t) > u''(t)$. Define

$$\max_{0 \le t < 1} \left(\alpha_1''(t) - u''(t), \right) := \alpha_1''(t_0) - u''(t_0) > 0.$$

By (3.2) and (3.5), $t_0 \neq 0$ because $\alpha''(0) - u''(0) \le 0$. Analogously $t_0 \neq 1$. So $t_0 \in]0,1[$ and

$$\alpha_1^{(4)}(t_0) - u^{(4)}(t_0) \le 0.$$

As $(\alpha_1''(t) - u''(t)) \in C[0,1]$, there is $I \subset [0,1]$ such that $t_0 \in I$ and

$$\begin{aligned} \alpha_1''(t) - u''(t) &> 0, \\ \alpha^{(4)}(t) - u^{(4)}(t) &\le 0, \ \forall t \in I. \end{aligned}$$

For all possible values of $v''(t_0)$, we obtain the following contradictions by the truncature F and Definition 3.3:

If $v''(t_0) < \alpha_2''(t_0)$, and as $v''(t) - \alpha_2''(t) \in C[0,1]$, then there is $J_0 \subset [0,1]$ such that $t_0 \in J_0$ and $v''(t) - \alpha_2''(t) < 0$, $\forall t \in J_0$.

As $t_0 \in I \cap J_0$ then $I \cap J_0 \neq \varnothing$ and

$$\begin{aligned} 0 &\ge \int_{I \cap J_0} \left(\alpha_1^{(4)}(t) - u^{(4)}(t) \right) dt \\ &= \int_{I \cap J_0} \left(\alpha_1^{(4)}(t) + f(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)) - \frac{u''(t) - \alpha_1''(t)}{1 + |u''(t) - \alpha_1''(t)|} - \frac{v''(t) - \alpha_2''(t)}{1 + |v''(t) - \alpha_2''(t)|} \right) dt \\ &> \int_{I \cap J_0} \left(\alpha_1^{(4)}(t) + f(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)) \right) dt \ge 0, \end{aligned}$$

with $\overline{\alpha}_1(t)$ and $\overline{\alpha}_2(t)$ given by (3.14).

If $\alpha_2''(t_0) \le v''(t_0) \le \beta_2''(t_0)$, and as $v'', \alpha_2'', \beta_2'' \in C[0,1]$ then there exists $J_1 \subset [0,1]$ such that $t_0 \in J_1$ and $\alpha_2''(t) \le v''(t) \le \beta_2''(t)$, $\forall t \in J_1$.

As $I \cap J_1 \neq \varnothing$ then

$$\begin{aligned} 0 &\geq \int\limits_{I\cap J_1} \left(\alpha_1^{(4)}(t) - u^{(4)}(t)\right) dt \\ &= \int\limits_{I\cap J_1} \left(\alpha_1^{(4)}(t) + f\left(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)\right) - \frac{u''(t) - \alpha_1''(t)}{1 + |u''(t) - \alpha_1''(t)|}\right) dt \\ &> \int\limits_{I\cap J_1} \left(\alpha_1^{(4)}(t) + f\left(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)\right)\right) dt \geq 0. \end{aligned}$$

If $v''(t_0) > \beta_2''(t_0)$, and as $v''(t) - \beta_2''(t) \in C[0,1]$ then there is $J_2 \subset [0,1]$ such that $t_0 \in J_2$ and $v''(t) - \beta_2''(t) > 0$, $\forall t \in J_2$.

As $t_0 \in I \cap J_2$ then $I \cap J_2 \neq \varnothing$ and

$$\begin{aligned} 0 &\geq \int\limits_{I\cap J_2} \left(\alpha_1^{(4)}(t) - u^{(4)}(t)\right) dt \\ &= \int\limits_{I\cap J_2} \left(f\left(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)\right) - \frac{u''(t) - \alpha_1''(t)}{1 + |u''(t) - \alpha_1''(t)|}\right. \\ &\quad \left. + \frac{v''(t) - \beta_2''(t)}{1 + |v''(t) - \beta_2''(t)|}\right) dt \\ &> \int\limits_{I\cap J_2} \left(\alpha_1^{(4)}(t) + f\left(t, \overline{\alpha}_1(t), \overline{\alpha}_2(t)\right)\right) dt \geq 0. \end{aligned}$$

Therefore, $\alpha_1''(t) \leq u''(t), \forall t \in [0,1]$. By similar arguments it can be proved that $u''(t) \leq \beta_1''(t), \forall t \in [0,1]$, and so,

$$\alpha_1''(t) \leq u''(t) \leq \beta_1''(t),\ \forall t \in [0,1]. \tag{3.19}$$

Applying a similar technique with the truncature H, we can achieve that

$$\alpha_2''(t) \leq v''(t) \leq \beta_2''(t),\ \forall t \in [0,1]. \tag{3.20}$$

The other inequalities of (3.18) are obtained by integration in $[0,t]$, with $t \in [0,1]$, and by Remark 3.7.

So, the fixed point (u,v) of T^* is also a fixed point of T, given by (3.13).

By Lemma 3.2, $(u(t), v(t))$ is a solution of problem (3.1), (3.2). ■

Corollary 3.8 *Assume the hypothesis of Theorem 3.6 hold.*

(i) *If, for* $i = 0, 1, 2,$

$$0 \leq \alpha_1^{(i)}(t) \leq \beta_1^{(i)}(t) \text{ and } \alpha_2^{(i)}(t) \leq \beta_2^{(i)}(t) \leq 0, \ \forall t \in [0, 1],$$

then there is at least a pair $(u(t), v(t)) \in (C^4[0,\ 1],\ \mathbb{R})^2$ *solution of (3.1), (3.2) such that*

$$0 \leq \alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t) \text{ and } \alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t) \leq 0, \ \forall t \in [0, 1].$$

(ii) *If, for* $i = 0, 1, 2,$

$$\alpha_1^{(i)}(t) \leq \beta_1^{(i)}(t) \leq 0, \text{ and } 0 \leq \alpha_2^{(i)}(t) \leq \beta_2^{(i)}(t), \ \forall t \in [0, 1],$$

then there is at least a pair $(u(t), v(t)) \in (C^4[0,\ 1],\ \mathbb{R})^2$ *solution of (3.1), (3.2) such that*

$$\alpha_1^{(i)}(t) \leq u^{(i)}(t) \leq \beta_1^{(i)}(t) \leq 0, \text{ and } 0 \leq \alpha_2^{(i)}(t) \leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \ \forall t \in [0, 1].$$

3.3 Example

Consider the third order nonlinear and nonautonomous system

$$\begin{cases} u^{(4)}(t) = & u(t)\, v(t) + (u'(t))^3\, v'(t) + \sqrt[3]{u''(t) + 1} + v''(t) + u'''(t) + \sqrt[3]{v'''(t)} \\ v^{(4)}(t) = & (u(t))^2\, v(t) + u'(t)\,(v'(t))^2 + e^{u''(t)} \sin(v''(t)) + 4\sqrt[5]{u'''(t)} + v'''(t) \end{cases} \tag{3.21}$$

with the boundary conditions

$$\begin{array}{r} u(0) = 1\ u'(0) = u''(0) = \ u''(1) = 0, \\ v(0) = -1,\ v'(0) = v''(0) = v''(1) = 0. \end{array} \tag{3.22}$$

The null functions are not a pair of solutions of system (3.21), (3.22), which is a particular case of (3.1), (3.2) with

$$\begin{array}{rcl} f(t, \overline{x}, \overline{y}) & = & x_0 y_0 + (x_1)^3\, y_1 + \sqrt[3]{x_2 + 1} + y_2 + x_3 + \sqrt[3]{y_3}, \\ h(t, \overline{x}, \overline{y}) & = & (x_0)^2\, y_0 + x_1\,(y_1)^2 + e^{x_2} \sin(y_2) + 4\sqrt[5]{x_3} + y_3, \end{array}$$

where $\overline{x} = (x_0, x_1, x_2, x_3)$ and $\overline{y} = (y_0, y_1, y_2, y_3)$, and $A_0 = 1$, $A_1 = A_2 = 0$, $B_0 = -1$, $B_1 = B_2 = 0$.

These L^1-Carathéodory functions f and h verify the Nagumo conditions in the set

$$E = \left\{ (\overline{x}, \overline{y}) \in \mathbb{R}^8 : \begin{array}{c} 1 \leq x_0 \leq 2,\ 0 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 6, \\ -2 \leq y_0 \leq -1,\ \ -3 \leq y_1 \leq 0,\ \ -6 \leq y_2 \leq 0 \end{array} \right\},$$

with

$$\phi\left(|u_3 + v_3|\right) := K + |x_3| + \sqrt[3]{|y_3|} + 4\sqrt[5]{|x_3|} + |y_3|,$$

for some $K > 0$.

As the functions

$$\begin{aligned} \alpha_1(t) &= 1,\ \ \beta_1(t) = t^3 + 1 \\ \alpha_2(t) &= -t^3 - 1,\ \ \beta_2(t) = -1 \end{aligned}$$

are coupled lower and upper solutions of (3.21), (3.22), then, by Theorem 3.6 there is a solution (u, v) of (3.21), (3.22), such that

$$\begin{aligned} 1 &\leq u(t) \leq t^3 + 1, \quad -t^3 - 1 \leq v(t) \leq -1, \\ 0 &\leq u'(t) \leq 3t^2,\ \ -3t^2 \leq v'(t) \leq 0, \\ 0 &\leq u''(t) \leq 6t,\ \ -6t \leq v''(t) \leq 0,\ \text{for } t \in [0, 1]. \end{aligned}$$

From the localization part, $u(t)$ is a strictly positive, nondecreasing and convex function, and $v(t)$ is a strictly negative, nonincreasing and concave function.

3.4 Coupled clamped beams

Assuming that a beam has a uniform cross-section and no axial load is applied, the equation for a moderately large deflection $u(x)$ of Bernoulli-Euler-v. Karman beam is expressed (see [49]) as

$$EIu^{(4)}(x) - \frac{3}{2}EA\left(u'(x)\right)^2 u''(x) = q(x),$$

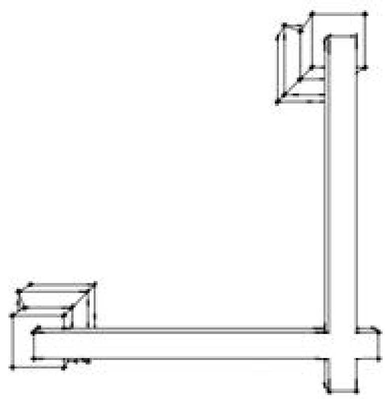
where E is the Young's modulus, I the mass moment of inertia, A the cross-sectional area, and $q(x)$ the distributed net load.

If we consider that the nonlinear Bernoulli-Euler-v. Karman beam is resting on a linear elastic foundation, the net loading $q(x)$ consists of the shear force $u'''(x)$ downward and the spring force $ku(x)$ upward, in which k is the spring constant, that is

$$q(x) = u'''(x) - ku(x).$$

For simplicity, we neglect the weight of the beam.

Based on this model, we analyze the bending of two coupled beams of length L, as in next figure,



Coupled beams, clamped on left
endpoint and free on the other endpoint

modelled by the following system

$$\begin{cases} u^{(4)}(x) = \frac{3}{2}\frac{A_1}{I_1}\left[(u'(x))^2 u''(x) + (v'(x))^2 v''(x)\right] \\ \qquad -\frac{1}{E_1 I_1}(k_1 u(x) + k_2 v(x)) + u'''(x) + v'''(x) \\ \\ v^{(4)}(x) = \frac{3}{2}\frac{A_3}{I_2}\left[(u'(x))^2 u''(x) + (v'(x))^2 v''(x)\right] \\ \qquad -\frac{1}{E_2 I_2}(k_1 u(x) + k_2 v(x)) + u'''(x) + v'''(x) \end{cases} \tag{3.23}$$

where E_1, I_1 and A_1, are, respectively, the Young's modulus, the mass moment of inertia, the cross-sectional area related to the beam u, and E_2, I_2

and A_2 the equivalent data related to beam v. The terms $k_1 u(x)$ and $k_2 v(x)$, $(k_1 \geq 0, k_2 \geq 0)$, are the spring forces related to the elastic foundations of u and v, respectively.

The beams are clamped of the left endpoint and free on the right endpoint, so, the corresponding boundary conditions are

$$\begin{cases} u(0) = 0, \ u'(0) = 0, \ u''(0) = 0, \ u''(L) = 0 \\ v(0) = 0, \ v'(0) = 0, \ v''(0) = 0, \ v''(L) = 0. \end{cases} \tag{3.24}$$

As we are dealing with the interval $[0, L]$, the Green's function depends on L, and Lemma 3.2 is replaced by the following result:

Lemma 3.9 *The functions $(u(t), v(t)) \in (C^4[0, L], \mathbb{R})^2$ are solution of problem (3.23)-(3.24) if and only if $(u(t), v(t))$ is a solution of the following system of integral equations*

$$\begin{cases} u(x) = \int_0^L \Big[G_L(x,s) \Big(\frac{3}{2}\frac{A_1}{I_1} \left[(u'(s))^2 u''(s) + (v'(s))^2 v''(s) \right] \\ \qquad -\frac{1}{E_1 I_1} (k_1 u(s) + k_2 v(s)) + u'''(s) + v'''(s) \Big) \Big] ds \\ \\ v(x) = \int_0^L \Big[G_L(x,s) \Big(\frac{3}{2}\frac{A_3}{I_2} \left[(u'(x))^2 u''(x) + (v'(x))^2 v''(x) \right] \\ \qquad -\frac{1}{E_2 I_2} (k_1 u(x) + k_2 v(x)) + u'''(x) + v'''(x) \Big) \Big] ds, \end{cases} \tag{3.25}$$

where $G_L(x,s)$ is the Green's function associated to problem (3.23)-(3.24), defined by

$$G_L(x,s) = \begin{cases} \frac{s\,x^3}{6L}(-s+L) - \frac{s^3}{6} + \frac{s^2 x}{2} - \frac{s x^2}{2} + \frac{x^3}{6}, & 0 \leq s \leq x \\ \\ \frac{x^3}{6L}(s-L), & x \leq s \leq L. \end{cases} \tag{3.26}$$

This problem (3.23), (3.24) is a particular case of (3.1), (3.2) with

$$\begin{aligned} f(x, \overline{u}, \overline{v}) &= \frac{3}{2}\frac{A_1}{I_1} \left[(u_1)^2 u_2 + (v_1)^2 v_2 \right] - \frac{1}{E_1 I_1} (k_1 u_0 + k_2 v_0) + u_3 + v_3, \\ h(x, \overline{u}, \overline{v}) &= \frac{3}{2}\frac{A_2}{I_2} \left[(u_1)^2 u_2 + (v_1)^2 v_2 \right] + \frac{1}{E_2 I_2} (k_1 u_0 + k_2 v_0) + u_3 + v_3, \end{aligned} \tag{3.27}$$

with $\overline{u} = (u_0, u_1, u_2, u_3)$ and $\overline{v} = (v_0, v_1, v_2, v_3)$.

The functions

$$\begin{aligned} \alpha_1(x) &= 0, \quad \alpha_2(x) = -x^3 \\ \beta_1(x) &= x^3, \quad \beta_2(x) = 0 \end{aligned}$$

are coupled lower and upper solutions of (3.23)-(3.24), respectively, for

$$k_1, k_2 \leq \min\{6E_1I_1,\ 6E_2I_2\},$$

verifying (3.12).

The functions f and h given by (3.27) are L^1-Carathéodory with

$$\begin{aligned} |f(x,\overline{u},\overline{v})| &\leq \psi_{1M}(t) \equiv 162L^3\frac{A_1}{I_1} + \frac{L^3}{E_1I_1}(k_1+k_2)+6, \\ |h(x,\overline{u},\overline{v})| &\leq \psi_{2M}(t) \equiv 162L^3\frac{A_2}{I_2} + \frac{L^3}{E_2I_2}(k_1+k_2)+6, \end{aligned}$$

for

$$\|(\overline{u},\overline{v})\|_{E^2} \leq M := \max\left\{L^3, 3L^2, 6L, 6\right\},$$

and verify the Nagumo condition with

$$\phi\left(|u_3+v_3|\right) := 324L^3\frac{A_1}{I_1} + \frac{2L^3}{E_1I_1}(k_1+k_2) + |u_3+v_3|,$$

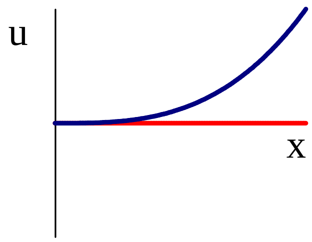
for

$$\begin{aligned} 0 &\leq u_0 \leq L^3,\ 0 \leq u_1 \leq 3L^2,\ 0 \leq u_2 \leq 6L, \\ -L^3 &\leq v_0 \leq 0,\ -3L^2 \leq v_1 \leq 0,\ -6L \leq v_2 \leq 0. \end{aligned}$$

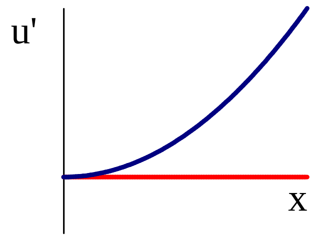
Then, by Corollary 3.8, there is a solution $(u(x), v(x))$ of (3.23)-(3.24), such that

$$\begin{aligned} 0 &\leq u(x) \leq x^3, \quad -x^3 \leq v(x) \leq 0, \\ 0 &\leq u'(x) \leq 3x^2, \quad -3x^2 \leq v'(x) \leq 0, \\ 0 &\leq u''(x) \leq 6x, \quad -6x \leq v''(x) \leq 0, \text{ for } x \in [0,L]. \end{aligned}$$

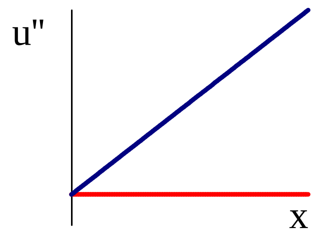
From the localization part, the displacement $u(x)$ is nonnegative, with a nonnegative slope and a convex bending moment stiffness. On the other hand, the displacement $v(x)$ is nonpositive, with a nonpositive slope and a concave bending moment stiffness, as it is illustrated by the following figures:



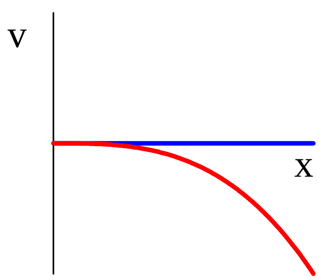
The displacement of the beam u is non-negative



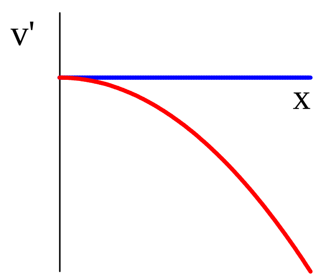
The slope of beam u is non-negative



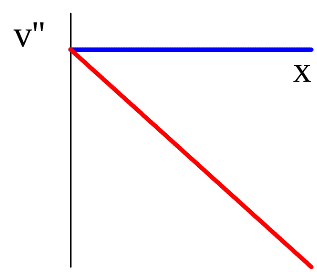
The bending moment stiffnessof the beam u is non-negative



The displacement of the beam v is non-positive



The slope of beam v is non-positive



The bending moment stiffnessof the beam v is non-positive

Chapter 4

Solvability for nth order coupled systems with full nonlinearities

In this chapter we consider the nth order coupled system composed of the fully coupled differential equations

$$\begin{cases} u^{(n)}(t) = f\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right) \\ v^{(n)}(t) = g\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right) \end{cases} \tag{4.1}$$

with $f, h : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ some continuous functions and the boundary conditions

$$\begin{cases} u^{(i)}(0) = A_i, \ u^{(n-2)}(1) = B, \\ v^{(i)}(0) = C_i, \ v^{(n-2)}(1) = D, \ i = 0, 1, ..., n-2, \end{cases} \tag{4.2}$$

for $A_i, B, C_i, D \in \mathbb{R}$.

Coupled systems of nonlinear boundary value problems of second and higher order with ordinary differential equations have received, in these last years, a great deal of attention in the literature, by means of different methods and several types of arguments. For recent trends in this field, we recommend interested readers to [3, 45, 68, 77, 85, 93, 96, 99, 100, 102], and the references therein.

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)) \\ -v'''(t) = h(t, u(t), u'(t)) \\ u(0) = u'(0) = 0, u'(1) = \alpha u'(\eta) \\ v(0) = v'(0) = 0, v'(1) = \alpha v'(\eta), \end{cases}$$

with non-negative continuous functions $f,\ h \in C\left([0,\,1] \times [0,\,+\infty)^2,\ [0,\,+\infty)\right)$ verifying adequate superlinear and sublinear conditions near 0 and $+\infty$, $0 < \eta < 1$ and the parameter α such that $1 < \alpha < \frac{1}{\eta}$. Applying the Guo–Krasnosel'skiĭ theorem on expansion-compression cones, and defining an adequate cone, to overcome the dependence on the first derivatives, it is proved the existence of a positive and increasing solution of the system.

In [86], it is studied the existence of solutions for a system of bending elastic beam equations

$$\begin{cases} u''''(t) = f(t,\, u(t),\, v(t),\, u'(t),\, v'(t)),\ t \in (0,1)\,, \\ v''''(t) = g(t,\, u(t),\, v(t),\, u'(t),\, v'(t)),\ t \in (0,1)\,, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

via the fixed point index theory, assuming sufficient conditions, some of them of the lipschitzian type.

In [79, 78], the authors present the system

$$\begin{cases} u^{(4)}\,(t) = f\,(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)) \\ v^{(4)}\,(t) = h\,(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)) \end{cases}$$

with $f, h : [0,1] \times \mathbb{R}^8 \to \mathbb{R}$ some L^1- Carathéodory functions, together with the boundary conditions

$$\begin{cases} u\,(0) = u'\,(0) = u''\,(0) = u''\,(1) = 0 \\ v\,(0) = v'\,(0) = v''\,(0) = v''\,(1) = 0, \end{cases}$$

and prove its solvability applying Green's functions, with integral operators theory and Schauder's fixed point.

In [94], it is considered the nth-order nonlinear boundary value problem

$$\begin{cases} u^{(n)}\,(t) + f\,(t, u(t), v(t)) = 0,\ 0 < t < 1, \\ v^{(n)}\,(t) + g\,(t, u(t), v(t)) = 0,\ 0 < t < 1, \\ u^{(i)}\,(0) = u\,(1) = 0,\ i = 0, 1, ..., n-2, \\ v^{(i)}\,(0) = v\,(1) = 0,\ i = 0, 1, ..., n-2, \end{cases}$$

where $n \geq 2$ and $f, g \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+,\, \mathbb{R}^+)$, $(\mathbb{R}^+ := [0, \infty))$. Based on *a priori* estimates, achieved by Jensen's integral inequality, fixed point index theory and assumptions on the nonlinearities, formulated in terms of spectral radii of associated linear integral operators, it is proved the existence of, at least, one positive solution.

This type of coupled systems cover some classical systems of differential equations, as, for instance, Lorenz-Lagrangian systems, [12, 91], and Korteweg-de Vries (KdV) coupled equations, [31, 35, 34, 44, 54, 73, 74, 97], and have a huge variety of applications, such as, in solitary waves theory, [23, 32, 58], the study of the bending of elastic beams, [5, 56, 72, 87], among others.Motivated by the above works, we present a technique for coupled higher order systems that, to the best of our knowledge, is new in the literature, and opens the possibility of new types of models. Our method applies a new Nagumo-type condition for coupled equations, with adequate growth conditions on the nonlinearities, to obtain not only the existence of a solution but also some data about the location of the unknown functions and their derivatives, given by lower and upper solutions method. The existence tool will be given by a homotopic problem and Leray-Schauder topological degree theory. Moreover, this section contains two applications for higher order coupled systems. The first one, for n even, $n = 2$, to a family of Lorentz-Lagrangian systems, and the second one, for $n = 3$, to some stationary coupled system of Korteweg-de Vries equations with damping and forced terms.

The chapter is organized in this way: Section 2 contains the functional framework, definitions, and some *a priori* estimations given by Nagumo-type conditions. The main result in Section 3 is based on some growth assumptions on the nonlinearities. Last two sections contain the applications: Section 4, deals with some Lorentz-Lagrangian systems, and Section 5 with a coupled system of KdV equations.

4.1 Definitions and preliminaries

Let $E := C^{n-1}[0, 1]$ be the Banach space equipped with the norm $\|\cdot\|_{C^{n-1}}$, defined by

$$\|w\|_{C^{n-1}} := \max\left\{\|w\|, ..., \|w^{(n-1)}\|\right\},$$

where

$$\|y\| := \max_{t\in[0,1]} |y(t)|$$

and $E^2 := \left(C^{n-1}[0, 1]\right)^2$ with the norm

$$\|(u, v)\|_{E^2} = \max\{\|u\|_{C^{n-1}}, \|v\|_{C^{n-1}}\}.$$

Throughout the chapter we apply the following relation:

$$(x_0, ..., x_{n-1}, y_0, ..., y_{n-1}) \le (z_0, ..., z_{n-1}, w_0, ..., w_{n-1})$$
$$\iff x_i \le z_i \wedge y_i \le w_i,\ \forall x_i, y_i, z_i, w_i \in \mathbb{R}, i = 0, 1, ..., n-1.$$

For some functions $\gamma_j^i, \delta_j^i \in C[0,\ 1]$, for $j = 1, 2$, and $i = 0, 1, ..., n-2$, such that

$$\gamma_j^i(t) \le \delta_j^i(t), \forall t \in [0, 1],$$

define the set

$$S := S_{\gamma_j^i,\ \delta_j^i} = \left\{ \begin{array}{c} (t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) \in [0,1] \times \mathbb{R}^{2n} : \gamma_1^i(t) \le u_i \le \delta_1^i(t), \\ \gamma_2^i(t) \le v_i \le \delta_2^i(t),\ i = 0, 1, ..., n-2 \end{array} \right\}. \tag{4.3}$$

Throughout the chapter we apply an the following order relation between pairs defined as

$$(x_0, ..., x_{n-1}, y_0, ..., y_{n-1}) \le (z_0, ..., z_{n-1}, w_0, ..., w_{n-1})$$
$$\iff x_i \le z_i \wedge y_i \le w_i,\ \forall x_i, y_i, z_i, w_i \in \mathbb{R}, i = 0, 1, ..., n-1.$$

To control the growth of the $(n-1)$ derivatives we need some Nagumo-type conditions:

Definition 4.1 *The continuous functions* $f, g : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ *satisfy Nagumo-type conditions relative to the set* S, *if there are positive continuos functions* $\phi_i : [0, +\infty[\to]0, +\infty[, i = 1, 2$, *such that*

$$|f\,(t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1})| \le \phi_1\,(|u_{n-1}|) \tag{4.4}$$

and

$$|g\,(t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1})| \le \phi_2\,(|v_{n-1}|) \tag{4.5}$$

for $(t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) \in S$, *with*

$$\int_0^{+\infty} \frac{s}{\phi_i\,(s)} ds = +\infty,\ i = 1, 2. \tag{4.6}$$

Lemma 4.2 *Let $f, g : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous functions satisfying a Nagumo type condition relative to the set S.*
Then there are $N_1, N_2 > 0$ such that, for every solution (u,v) of (4.1)-(4.2) with $\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right) \in S$,

$$\left\| u^{(n-1)} \right\| < N_1 \text{ and } \left\| v^{(n-1)} \right\| < N_2 \, . \tag{4.7}$$

Proof. Let (u,v) be a solution of (4.1) such that

$$\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right) \in S.$$

For $r > B - A_{n-2}$, consider $N_1, N_2 > r$ such that

$$\int_r^{N_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \delta_1^{n-2}(t) - \min_{t\in[0,1]} \gamma_1^{n-2}(t), \tag{4.8}$$

and

$$\int_r^{N_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \delta_2^{n-2}(t) - \min_{t\in[0,1]} \gamma_2^{n-2}(t). \tag{4.9}$$

If $\left| u^{(n-1)} \right| \le r, \forall t \in [0,1]$, then this part of the proof is finished, as $\left\| u^{(n-1)} \right\| \le r < N_1$.

On the other hand if $\left| u^{(n-1)}(t) \right| \ge r, \forall t \in [0,1]$, we obtain the following contradiction for the case where $u^{(n-1)}(t) > r$,

$$r > B - A_{n-2} = \int_0^1 u^{(n-1)}(t)\, dt \ge \int_0^1 r dt = r.$$

If $u^{(n-1)}(t) < -r$ the contradiction is analogous.

By (4.2) and the Mean Value Theorem, there is $t_0 \in]0,1[$ such that $u^{(n-1)}(t_0) > r$, $t_2 \in]0,1[$, $t_2 < t_0$, with $u^{(n-1)}(t_2) = r$ and $u^{(n-1)}(t) > r$, $\forall t \in]t_2, t_0]$.

Then, by (4.1), (4.5) and (4.9),

$$\begin{aligned}
\int_{u^{(n-1)}(t_2)}^{u^{(n-1)}(t_0)} \frac{s}{\phi_1(s)}ds &= \int_{t_2}^{t_0} \frac{u^{(n-1)}(s)}{\phi_1\left(u^{(n-1)}(s)\right)} u^{(n)}(s)\,ds \\
&\le \int_{t_2}^{t_0} \frac{u^{(n-1)}(s)}{\phi_1\left(u^{(n-1)}(s)\right)} \left| f\begin{pmatrix} s, u(s), ..., u^{(n-1)}(s), \\ v(s), ..., v^{(n-1)}(s) \end{pmatrix} \right| ds \\
&\le \int_{t_2}^{t_0} u^{(n-1)}(s)\,ds = u^{(n-2)}(t_0) - u^{(n-2)}(t_2) \\
&\le \max_{t\in[0,1]} \delta_1^{n-2}(t) - \min_{t\in[0,1]} \gamma_1^{n-2}(t) < \int_r^{N_1} \frac{s}{\phi_1(s)}ds.
\end{aligned}$$

By the arbitrariness of t_0 related to the values where $u^{(n-1)}(t_0) > r$, we have

$$u^{(n-1)}(t) < N_1, \forall t \in [0,1].$$

For $t_2 > t_0$ with $u^{(n-1)}(t_2) = r$ and $u^{(n-1)}(t) > r, \forall t \in [t_0, t_2[$, the arguments are similar.

In the case where $u^{(n-1)}(t) < -r$ the technique is analogous and, therefore, $\left\|u^{(n-1)}\right\| \le N_1$.

Applying the same method as above, it can be proved, by (4.5) and (4.9), that $\left\|v^{(n-1)}\right\| \le N_2$. ■

Lower and upper functions will be defined as a pair, as follows:

Definition 4.3 *A pair of functions $(\alpha_1, \alpha_2) \in E^2$ is a coupled lower solution of (4.1), (4.2) if*

$$\begin{aligned}
\alpha_1^{(n)}(t) &\ge f\left(t, \alpha_1(t), ..., \alpha_1^{(n-1)}(t), \alpha_2(t), ..., \alpha_2^{(n-2)}(t), v_{n-1}\right), \\
\textit{for } t &\in [0,1] \textit{ and } v_{n-1} \in \mathbb{R}, \qquad (4.10)\\
\alpha_2^{(n)}(t) &\ge g\left(t, \alpha_1(t), ..., \alpha_1^{(n-2)}(t), u_{n-1}, \alpha_2(t), ..., \alpha_2^{(n-1)}(t), \right), \\
\textit{for } t &\in [0,1] \textit{ and } u_{n-1}, \in \mathbb{R},
\end{aligned}$$

with

$$\begin{matrix} \alpha_1^{(i)}(0) \le A_i, \ \alpha_1^{(n-2)}(1) \le B, \\ \alpha_2^{(i)}(0) \le C_i, \ \alpha_2^{(n-2)}(1) \le D. \end{matrix} \qquad (4.11)$$

The pair $(\beta_1, \beta_2) \in E^2$ is said to be a coupled upper solution of (4.1), (4.2) if they verify the reversed inequalities.

4.2 Main result

The main theorem is an existence and localization result, meaning that, it provides not only the existence, but also some data about the localization of the unknown functions and their derivatives:

Theorem 4.4 *Let $f, g : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous functions. Suppose that there are coupled lower and upper solutions of (4.1), (4.2), (α_1, α_2) and (β_1, β_2), respectively, such that*

$$\left(\alpha_1^{(n-2)}(t), \alpha_2^{(n-2)}(t)\right) \leq \left(\beta_1^{(n-2)}(t), \beta_2^{(n-2)}(t)\right), \forall t \in [0,1].$$

Assume that f and g verify the Nagumo conditions relative to the set $S_{\alpha^{(i)},\beta^{(i)}}$, $i = 0, 1, ...n-2$, and the growth conditions

$$\begin{aligned} &f\left(t, \alpha_1(t), ..., \alpha_1^{(n-3)}(t), u_{n-2}, u_{n-1}, \alpha_2(t), ..., \alpha_2^{(n-2)}(t), v_{n-1}\right) \\ &\geq f(t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) \\ &\geq f\left(t, \beta_1(t), ..., \beta_1^{(n-3)}(t), u_{n-2}, u_{n-1}, \beta_2(t), ..., \beta_2^{(n-2)}(t), v_{n-1}\right) \end{aligned} \tag{4.12}$$

for $\alpha_1^{(i)}(t) \leq u_i \leq \beta_1^{(i)}(t)$, $i = 0, 1, ..., n-3$, $\alpha_2^{(j)}(t) \leq v_j \leq \beta_2^{(j)}(t)$, $j = 0, 1, ..., n-2$, and $(t, u_{n-2}, u_{n-1}, v_{n-1}) \in [0,1] \times \mathbb{R}^3$, and

$$\begin{aligned} &g\left(t, \alpha_1(t), ..., \alpha_1^{(n-2)}(t), u_{n-1}, \alpha_2(t), ..., \alpha_2^{(n-3)}(t), v_{n-2}, v_{n-1}\right) \\ &\geq g(t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) \\ &\geq g\left(t, \beta_1(t), ..., \beta_1^{(n-2)}(t), u_{n-1}, \beta_2(t), ..., \beta_2^{(n-3)}(t), v_{n-2}, v_{n-1}\right), \end{aligned} \tag{4.13}$$

for $\alpha_1^{(j)}(t) \leq u_j \leq \beta_1^{(j)}(t)$, $j = 0, 1, ..., n-2$, $\alpha_2^{(i)}(t) \leq v_i \leq \beta_2^{(i)}(t)$, $i = 0, 1, ..., n-3$, and $(t, u_{n-1}, v_{n-2}, v_{n-1}) \in [0,1] \times \mathbb{R}^3$.

Then problem (4.1), (4.2) has, at least, a solution $(u, v) \in E^2$, such that,

$$\begin{aligned} \alpha_1^{(i)}(t) &\leq u^{(i)}(t) \leq \beta_1^{(i)}(t), \ \forall t \in [0,1], \\ \alpha_2^{(i)}(t) &\leq v^{(i)}(t) \leq \beta_2^{(i)}(t), \text{for } i = 0, 1, ..., n-2, \end{aligned}$$

and

$$\left\|u^{(n-1)}\right\| < N_1, \ \left\|v^{(n-1)}\right\| < N_2,$$

where $N_1$and N_2 are given by (4.7).

Remark 4.5 *If* $\alpha_1^{(n-2)}(t) \leq \beta_1^{(n-2)}(t)$ *for* $t \in [0,1]$, *then by integration in* $[0,t]$, *(4.2) and (4.11),*

$$\alpha_1^{(i)}(t) \leq \beta_1^{(i)}(t), \ \text{for } i = 0,1,...,n-3, \ \text{and } t \in [0,1].$$

Analogously from $\alpha_2^{(n-2)}(t) \leq \beta_2^{(n-2)}(t), \forall t \in [0,1]$, *by integration in* $[0,t]$, *then*

$$\alpha_2^{(i)}(t) \leq \beta_2^{(i)}(t), \ \text{for } i = 0,1,...,n-3, \ \text{and } t \in [0,1].$$

Proof. Define the continuous functions, for $i = 0,1,...,n-2$, $j = 1,2$,

$$\delta_{j,i}(t, w_i) = \begin{cases} \beta_j^{(i)} & \text{if } w_i > \beta_j^{(i)} \\ w_i & \text{if } \alpha_j^{(i)} \leq w_i \leq \beta_j^{(i)}, \\ \alpha_j^{(i)} & \text{if } w_i < \alpha_j^{(i)}. \end{cases}$$

For $\lambda \in [0,1]$, consider the homotopic problem composed by the equations

$$\begin{cases} u^{(n)}(t) = \lambda f \begin{pmatrix} t, \delta_{1,0}(t, u(t)), ..., \delta_{1,n-2}\left(t, u^{(n-2)}(t)\right), u^{(n-1)}(t), \\ \delta_{2,0}(t, v(t)), ..., \delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), \delta_{N_2}\left(t, v^{(n-1)}(t)\right) \end{pmatrix} \\ \qquad + u^{(n-2)}(t) - \lambda \delta_{1,n-2}\left(t, u^{(n-2)}(t)\right) \\ v^{(n)}(t) = \lambda g \begin{pmatrix} t, \delta_{1,0}(t, u(t)), ..., \delta_{1,n-2}\left(t, u^{(n-2)}(t)\right), \delta_{N_1}\left(t, u^{(n-1)}(t)\right), \\ \delta_{2,0}(t, v(t)), ..., \delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), v^{(n-1)}(t) \end{pmatrix} \\ \qquad + v^{(n-2)}(t) - \lambda \delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), \end{cases} \tag{4.14}$$

for $t \in [0,1]$, together with boundary conditions together with boundary conditions

$$\begin{cases} u^{(i)}(0) = \lambda A_i, \ u^{(n-2)}(1) = \lambda B, \ i = 0,1,...,n-2, \\ v^{(i)}(0) = \lambda C_i, \ v^{(n-2)}(1) = \lambda D. \end{cases} \tag{4.15}$$

Take $r_1, r_2 > 0$ such that, for $u^{(n-1)}(t), v^{(n-1)}(t), \in \mathbb{R}$,

$$-r_j < \alpha_j^{(n-2)}(t) \leq \beta_j^{(n-2)}(t) \leq r_j, \ j = 1,2, \ \text{for } t \in [0,1], \tag{4.16}$$

$$|A_{n-2}| < r_1, \ \ |B| < r_1, \ \ |C_{n-2}| < r_2, \ \ |D| < r_2, \tag{4.17}$$

$$f\left(t, \alpha_1(t), ..., \alpha_1^{(n-2)}(t), 0, \alpha_2(t), ..., \alpha_2^{(n-2)}(t), v^{(n-1)}(t)\right) - r_1 - \alpha_1^{(n-2)}(t) < 0, \tag{4.18}$$

$$f\left(t, \beta_1(t), ..., \beta_1^{(n-2)}(t), 0, \beta_2(t), ..., \beta_2^{(n-2)}(t), v^{(n-1)}(t)\right) + r_1 - \beta_1^{(n-2)}(t) > 0, \tag{4.19}$$

$$g\left(t, \alpha_1(t), ..., \alpha_1^{(n-2)}(t), u^{(n-1)}(t), \alpha_2(t), ..., \alpha_2^{(n-2)}(t), 0\right) - r_2 - \alpha_2^{(n-2)}(t) < 0, \tag{4.20}$$

$$g\left(t, \beta_1(t), ..., \beta_1^{(n-2)}(t), u^{(n-1)}(t), \beta_2(t), ..., \beta_2^{(n-2)}(t), 0\right) + r_2 - \beta_2^{(n-2)}(t) > 0. \tag{4.21}$$

For clearness the proof will follow several steps :

Step 1: *Every solution (u, v) of (4.14), (4.15) verifies,*

$$\begin{aligned} \left|u^{(n-2)}(t)\right| &< r_1, \ \left|v^{(n-2)}(t)\right| < r_2, \\ \left|u^{(i)}(t)\right| &< r_1 + \sum_{k=i}^{n-3} |A_k| := r_1^i, \\ \left|v^{(i)}(t)\right| &< r_2 + \sum_{k=i}^{n-3} |C_k| := r_2^i, \end{aligned}$$

for $i = 0, 1, ..., n-3$, independently of $\lambda \in [0,1]$.

Suppose, by contradiction, that the first inequality is not verified for $i = n-2$. Then there is a solution $u(t)$ of (4.14), (4.2) and $t \in [0,1]$ such that $\left|u^{(n-2)}(t)\right| \geq r_1$, that is,

$$u^{(n-2)}(t) \geq r_1 \text{ or } u^{(n-2)}(t) \leq -r_1.$$

In the first case define

$$\max_{t\in[0,1]} u^{(n-2)}(t) := u^{(n-2)}(t_0) \geq r_1.$$

As, by *(4.17)*, $t_0 \neq 0$ and $t_0 \neq 1$, then $t_0 \in]0,1[$, $u^{(n-1)}(t_0) = 0$ and $u^{(n)}(t_0) \leq 0$.

Therefore, for $\lambda \in \left]0,1\right]$, it is obtained, by (4.14), (4.16) and (4.19), the following contradiction,

$$\begin{aligned} 0 &\geq u^{(n)}(t_0) \\ &= \lambda f\left(\begin{array}{c} t_0, \delta_{1,0}(t_0, u(t_0)), ..., \delta_{1,n-2}\left(t_0, u^{(n-2)}(t_0)\right), 0, \\ \delta_{2,0}(t_0, v(t_0)), ..., \delta_{2,n-2}\left(t_0, v^{(n-2)}(t_0)\right), v^{(n-1)}(t_0) \end{array}\right) \\ &\quad + u^{(n-2)}(t_0) - \lambda \beta_1^{(n-2)}(t_0) \\ &\geq \lambda \left[\begin{array}{c} f(t_0, \beta_1(t_0), ..., \beta_1^{(n-2)}(t_0), 0, \beta_2(t_0), ..., \beta_2^{(n-2)}(t_0), v^{(n-1)}(t_0)) \\ + r_1 - \beta_1^{(n-2)}(t_0) \end{array}\right] \\ &> 0 \end{aligned}$$

For $\lambda = 0$ the contradiction is given by

$$0 \geq u^{(n)}(t_0) = u^{(n-2)}(t_0) \geq r_1 > 0.$$

Following similar arguments, it can be proved that

$$u^{(n-2)}(t) \geq -r_1, \forall t \in [0,1],$$

and, therefore,

$$\left|u^{(n-2)}(t)\right| \leq r_1, \forall \in [0,1].$$

As

$$\int_0^t u^{(n-2)}(s)\, ds = u^{(n-3)}(t) - \lambda A_{n-3}$$

and

$$-r_1 \leq \int_0^t u^{(n-2)}(s)\, ds \leq r_1,$$

therefore

$$\left|u^{(n-3)}(t)\right| < r_1 + |A_{n-3}|, \forall t \in [0,1].$$

By iteration of this type of arguments we have

$$\left|u^{(i)}(t)\right| < r_1 + \sum_{k=i}^{n-3} |A_k|, \forall t \in [0,1],$$

for $i = 0, 1, ..., n-3$, independently of $\lambda \in [0,1]$.

By the technique above it can be obtained that $\left|v^{(n-2)}(t)\right| < r_2$, and

$$\left|v^{(i)}(t)\right| < r_2 + \sum_{k=i}^{n-3} |C_k|, \forall t \in [0,1],$$

for $i = 0, 1, ..., n-3$, independently of $\lambda \in [0,1]$.

Step 2: *Every solution* (u,v) *of (4.14), (4.15) satisfies* $\left\|u^{(n-1)}\right\| < N_1$, *and* $\left\|v^{(n-1)}\right\| < N_2$, *independently of* $\lambda \in [0,1]$.

For $\lambda \in [0,1]$ define the functions

$$\begin{array}{c} F_\lambda (t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) := \\ \lambda f \left(\begin{array}{c} t, \delta_{1,0}(t, u_0), ..., \delta_{1,n-2}(t, u_{n-2}), u_{n-1}, \delta_{2,0}(t, v_0), \\ ..., \delta_{2,n-2}(t, v_{n-2}), v_{n-1} \end{array} \right) \\ + u_{n-2} - \lambda \delta_{1,n-2}(t, u_{n-2}) \end{array} \tag{4.22}$$

and

$$\begin{array}{c} G_\lambda (t, u_0, ..., u_{n-1}, v_0, ..., v_{n-1}) := \\ \lambda g \left(\begin{array}{c} t, \delta_{1,0}(t, u_0), ..., \delta_{1,n-2}(t, u_{n-2}), u_{n-1}, \delta_{2,0}(t, v_0), \\ ..., \delta_{2,n-2}(t, v_{n-2}), v_{n-1} \end{array} \right) \\ + v_{n-2} - \lambda \delta_{2,n-2}(t, v_{n-2}). \end{array} \tag{4.23}$$

The functions F_λ and G_λ verify the Nagumo conditions (2.7), (4.5) and (4.8), as

$$\begin{array}{rcl} |F_\lambda| & \leq & \left| f \left(\begin{array}{c} t, \delta_{1,0}(t, u_0), ..., \delta_{1,n-2}(t, u_{n-2}), u_{n-1}, \\ \delta_{2,0}(t, v_0), ..., \delta_{2,n-2}(t, v_{n-2}), v_{n-1} \end{array} \right) \right| \\ & & + |u_{n-2}| + |\delta_{1,n-2}(t, u_{n-2})| \\ & \leq & \phi_1(|u_{n-1}|) + 2r_1, \end{array}$$

$$\begin{array}{rcl} |G_\lambda| & \leq & \left| g \left(\begin{array}{c} t, \delta_{1,0}(t, u_0), ..., \delta_{1,n-2}(t, u_{n-2}), u_{n-1}, \\ \delta_{2,0}(t, v_0), ..., \delta_{2,n-2}(t, v_{n-2}), v_{n-1} \end{array} \right) \right| \\ & & + |v_{n-2}| + |\delta_{2,n-2}(t, v_{n-2})| \\ & \leq & \phi_2(|v_{n-1}|) + 2r_2, \end{array}$$

and

$$\int_0^{+\infty} \frac{s}{\phi_1(s) + 2r_1} ds = +\infty, \int_0^{+\infty} \frac{s}{\phi_2(s) + 2r_2} ds = +\infty.$$

By Step 1, and applying Lemma 4.2, with, for $j = 1, 2$,

$$\begin{aligned} \gamma_j^{n-2}(t) &\equiv -r_j,\ \delta_j^{n-2}(t) \equiv r_j, \\ \gamma_j^{i}(t) &\equiv -r_j^i,\ \delta_j^{i}(t) \equiv r_j^i, \end{aligned}$$

for $i = 0, 1, ..., n-3$, there are $N_1, N_2 > 0$ such that

$$\left\| u^{(n-1)} \right\| < N_1 \text{ and } \left\| v^{(n-1)} \right\| < N_2.$$

Step 3: *Problem (4.14), (4.15) has, at least, a solution for* $\lambda = 1$.

Define the operators

$$\mathcal{L} : (C^n([0,1]))^2 \subseteq E^2 \to (C([0,1]))^2 \times \mathbb{R}^{2n}$$

given by

$$\mathcal{L}(u,v) = \begin{pmatrix} u^{(n)}(t), v^{(n)}(t), u(0), ..., u^{(n-2)}(0), u^{(n-2)}(1), \\ v(0), ..., v^{(n-2)}(0), v^{(n-2)}(1) \end{pmatrix},$$

and $\mathcal{N}_\lambda : (C^{n-1}([0,1]))^2 \to (C([0,1]))^2 \times \mathbb{R}^{2n}$, given by

$$\mathcal{N}_\lambda(u,v) = \begin{pmatrix} F_\lambda\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right), \\ G_\lambda\left(t, u(t), ..., u^{(n-1)}(t), v(t), ..., v^{(n-1)}(t)\right), \\ \lambda A_1, ..., \lambda A_{n-2}, \lambda B, \lambda C_1, ..., \lambda C_{n-2}, \lambda D \end{pmatrix},$$

where F_λ and G_λ are defined in (4.22) and (4.23), respectively.

As $\mathcal{L}^{-1}$ is compact then it can be defined the completely continuous operator $\mathcal{T}_\lambda : \left((C^{n-1}([0,1]))^2, \mathbb{R}\right) \to \left((C^{n-1}([0,1]))^2, \mathbb{R}\right)$ given by

$$\mathcal{T}_\lambda(u,v) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u,v).$$

Consider

$$\rho = \max\left\{N_1, N_2, r_j^i, \text{ for } j = 1, 2, i = 0, 1, ..., n-3,\right\},$$

where r_j^i, N_1, N_2, are given in Steps 1 and 2, respectively, and define the set

$$\Omega = \left\{ (u,v) \in E^2 : \|(u,v)\|_{E^2} < \rho + 1 \right\}.$$

Therefore the degree $d(I - \mathcal{T}_\lambda, \Omega, (0,0))$ is well defined for every $\lambda \in [0,1]$, and by the invariance under homotopy,

$$d(I - \mathcal{T}_0, \Omega, (0,0)) = d(I - \mathcal{T}_1, \Omega, (0,0)).$$

The equation $\mathcal{T}_0(u,v) = (u,v)$ is equivalent to the homogeneous problem

$$\begin{cases} u^{(n)}(t) - u^{(n-2)}(t) = 0 \\ v^{(n)}(t) - v^{(n-2)}(t) = 0 \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, \\ v^{(i)}(0) = v^{(n-2)}(1) = 0, \ i = 0, 1, ..., n-2, \end{cases}$$

which admits only the trivial solution.

Then, by degree theory, $d(I - \mathcal{T}_0, \Omega(0,0)) = \pm 1$, and so the equation

$$(u,v) = \mathcal{T}_1(u,v)$$

has at least one solution. That is, by Step 1, the problem composed by the equation (4.14), and the boundary conditions (4.15) has at least a solution $(u_1(t), v_1(t))$ in Ω.

Step 4: *This solution* $(u_1(t), v_1(t))$ *is a solution of (4.1), (4.2).*

To prove this assertion it will be enough, by Steps 1 and 2, to show that

$$\alpha_1^{(i)}(t) \le u_1^{(i)}(t) \le \beta_1^{(i)}(t),$$

and

$$\alpha_2^{(i)}(t) \le v_1^{(i)}(t) \le \beta_2^{(i)}(t), \forall t \in [0,1], \ i = 0, 1, ..., n-2.$$

Suppose, by contradiction, that there exists $t \in [0,1]$ such that

$$u_1^{(n-2)}(t) > \beta_1^{(n-2)}(t),$$

and define

$$\max_{t \in [0,1]} \left[u_1^{(n-2)}(t) - \beta_1^{(n-2)}(t) \right] := u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) > 0. \tag{4.24}$$

As, by (4.2) and Definition 4.3, $t_1 \neq 0$ and $t_1 \neq 1$, then $t_1 \in \left]0,1\right[$, $u_1^{(n-1)}(t_1) = \beta_1^{(n-1)}(t_1)$ and

$$u_1^{(n)}(t_1) \leq \beta_1^{(n)}(t_1). \tag{4.25}$$

So, by (4.12), (4.24), Definition 4.3, Steps 1 and 2, we obtained the following contradiction

$$\begin{aligned} 0 \geq \; & u_1^{(n)}(t_1) - \beta_1^{(n)}(t_1) = \\ & f\left(\begin{array}{c} t_1, \delta_{1,0}(t_1, u_1(t_1)), ..., \delta_{1,n-2}\left(t_1, u_1^{(n-2)}(t_1)\right), u_1^{(n-1)}(t_1), \\ \delta_{2,0}(t_1, v_1(t_1)), ..., \delta_{2,n-2}\left(t_1, v_1^{(n-2)}(t_1)\right), v_1^{(n-1)}(t_1) \end{array} \right) \\ & + u_1^{(n-2)}(t_1) - \delta_{1,n-2}\left(t_1, u_1^{(n-2)}(t_1)\right) - \beta_1^{(n)}(t_1) \\ \geq \; & f(t_1, \beta_1(t_1), ..., \beta_1^{(n-1)}(t_1), \beta_2(t_1), ..., \beta_2^{(n-2)}(t_1), v^{(n-1)}(t)) \\ & + u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) - \beta_1^{(n)}(t_1) \\ \geq \; & u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) > 0. \end{aligned}$$

Therefore,

$$u_1^{(n-2)}(t) \leq \beta_1^{(n-2)}(t), \; \forall t \in [0,1].$$

Applying the same argument, it can be justified that $\alpha_1^{(n-2)}(t) \leq u_1^{(n-2)}(t)$, for $t \in [0,1]$.

Integrating in $[0,t]$ the inequalities

$$\alpha_1^{(n-2)}(t) \leq u_1^{(n-2)}(t) \leq \beta_1^{(n-2)}(t),$$

we have, for the first one,

$$\begin{aligned} \alpha_1^{(n-3)}(t) - A_{n-3} \; & \leq \; \alpha_1^{(n-3)}(t) - \alpha_1^{(n-3)}(0) = \int_0^t \alpha_1^{(n-2)}(s)\, ds \leq \int_0^t u_1^{(n-2)}(s)\, ds \\ & = \; u_1^{(n-3)}(t) - u_1^{(n-3)}(0) = u_1^{(n-3)}(t) - A_{n-3}, \forall t \in [0,1], \end{aligned}$$

and therefore

$$\alpha_1^{(n-3)}(t) \leq u_1^{(n-3)}(t), \forall t \in [0,1].$$

By similar technique, we get

$$\alpha_1^{(i)}(t) \le u_1^{(i)}(t), \forall t \in [0,1], i = 0, 1, ..., n-2.$$

In an analogous way, we can prove that

$$u_1^{(i)}(t) \le \beta_1^{(i)}(t), \text{for } i = 0, 1, ..., n-2,$$

and

$$\alpha_2^{(i)}(t) \le v_1^{(i)}(t) \le \beta_2^{(i)}(t), \forall t \in [0,1], i = 0, 1, ..., n-2.$$

■

4.3 Lorentz- Lagrangian system model

This model was presented by Voigt in 1887, and adopted later by Lorentz in 1904, and by Poincaré in 1906. Lorentz-Lagrangian systems have many analogies with classical Lagrangian systems $q'' + V(q) = 0$, for which the results of existence of periodic and homoclinic solutions were established through a variety of methods.

In [12], the author presents, as example, a system of the Lorentz-Lagrangian type, modelling the motion of a particle in a rotating potential in a frame, that moves with the potential.

Based on the ideas of [12], we consider the Lorentz- Lagrangian system:

$$\begin{cases} u''(t) + k(v(t) - u(t)) - 2(k-1)^2 \frac{u(t)}{(1+u^2(t))^2} = 0, \\ v''(t) - v(t) - k(v'(t) - u(t)) = 0, \end{cases} \tag{4.26}$$

with $k > 1$ a parameter, together with the boundary conditions

$$\begin{array}{c} u(0) = 0, \ u(1) = 1 \\ v(0) = 0, \ v(1) = 1. \end{array} \tag{4.27}$$

The system above is a particular case of problem (4.1), (4.2), with $n = 2$,

$$f(t, u_0, u_1, v_0, v_1) = -k(v_0 - u_0) + 2(k-1)^2 \frac{u_0}{(1+u_0^2)^2}$$

and

$$g(t, u_0, u_1, v_0, v_1) = v_0 + k(v_1 - u_0).$$

Moreover the functions

$$\begin{aligned}\alpha_1(t) &= -t, \quad \beta_1(t) = t \\ \alpha_2(t) &= -t \quad \beta_2(t) = t\end{aligned}$$

are lower and upper solutions of problem (4.26), (4.27), respectively, for $k > 1$, and the nonlinearities f and g satisfy the growth conditions (4.12) and (4.13).

These functions verify the Nagumo conditions (4.4) and (4.5) with

$$\phi_1(u_1) \equiv 2k + \frac{4}{5}(k-1)^2$$

and

$$\phi_2(v_1) \equiv 1 + 2k(|v_1| + 1),$$

with

$$-t \leq u_0 \leq t, \quad -t \leq v_0 \leq t, \text{ for } t \in [0,1].$$

Therefore, by Theorem 4.4, there is a solution (u, v) of problem (4.26), (4.27) for $k > 1$ such that

$$\begin{aligned}-t &\leq u(t) \leq t \\ -t &\leq v(t) \leq t, \ \forall t \in [0,1].\end{aligned}$$

4.4 Application to a coupled system of two Korteweg-de Vries (KdV) equations

The Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0,$$

models in the unidirectional propagation of water waves with small amplitude lying a channel, ([74]), and was first introduced by Boussinesq and then reformulated by Diederik Korteweg and Gustav de Vries.

In [97], it is studied the coupled KdV equations

$$\begin{cases} u_t - \frac{1}{2}(7-3\alpha)u_{xxx} - u_x u - uv_x - \frac{1}{2}(1-\alpha)vu_x + \frac{1}{2}(1+\alpha)vu_x = 0 \\ v_t + v_{xxx} + u_x v + uv_x + vu_x + \frac{1}{2}(1+\alpha)uv_x - \frac{1}{2}(1-\alpha)\,\mathrm{u}u_x = 0, \end{cases} \tag{4.28}$$

with $\alpha^2 = 5$, which are a new model for describing two-layer fluids with different dispersion relations.

It can be observed in [73] that, for the case of constant boundary and initial conditions, various types of steady and transient solutions were derived.

Based on the above model (4.28), we consider a particular case of a stationary coupled system of the KdV equations with damping and forced terms:

$$\left\{ \begin{array}{l} u'''(t) = m(t) - 0.01\,(u(t) + v(t))\,|u'(t)| - u(t) \\ v'''(t) = n(t) - 0.01\,(u(t) + v(t))\,|v'(t)| - v(t) \end{array} \right. \tag{4.29}$$

with $m, n : \mathbb{R} \to \mathbb{R}$ continuous functions and γ, δ positive parameters, together with the boundary conditions

$$\begin{aligned} u(0) &= u'(0) = u'(1) = 0 \\ v(0) &= v'(0) = v'(1) = 0. \end{aligned} \tag{4.30}$$

The functions

$$\begin{aligned} \alpha_1(t) &= \alpha_2(t) = t^3 - 5t^2, \\ \beta_1(t) &= \beta_2(t) = -t^3 + 5t^2 + t \end{aligned}$$

are lower and upper solutions of problem (4.29), (4.30), if the forcing terms verify

$$m(t) \leq 6,\ n(t) \leq 6,\ \forall t \in [0,1] \tag{4.31}$$

It can be easily seen that (4.29), (4.30) is a particular case of problem (4.1), (4.2), with $n = 3$,

$$f(t, u_0, u_1, u_2, v_0, v_1, v_2) = m(t) - 0.01\,(u_0 + v_0)\,|u_1| - u_0$$

and

$$g(t, u_0, u_1, u_2, v_0, v_1, v_2) = n(t) - 0.01\,(u_0 + v_0)\,|v_1| - v_0.$$

These functions verify trivially the Nagumo conditions (4.4) and (4.5), as they have no dependence on the second derivatives. Moreover they satisfy the growth conditions (4.12) and (4.13), by Theorem 4.4, there is a solution (u, v) of problem (4.29), (4.30), for functions m and n verifying (4.31), and

$$\begin{aligned} t^3 - 5t^2 &\leq u(t) \leq -t^3 + 5t^2 + t, \\ t^3 - 5t^2 &\leq v(t) \leq -t^3 + 5t^2 + t,\ \forall t \in [0,1]. \end{aligned}$$

Chapter 5

Higher order functional discontinuous boundary value problems on the half-line

5.1 Introduction

This chapter is concerned with the study of a fully nonlinear higher order discontinuous equation on the half line

$$u^{(n)}(t) = f(t,\, u(t),\, u'(t), ..., u^{(n-1)}(t)), \quad t \in [0, +\infty[, \tag{5.1}$$

where $f : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ is a L^1- Carathéodory function, with the functional boundary conditions,

$$\begin{cases} L_i\left(u^{(i)}, u\left(0\right), u'\left(0\right), ..., u^{(n-2)}\left(0\right)\right) = 0, \; i = 0, ..., n-2, \\ \qquad\qquad L_{n-1}\left(u, u^{(n-1)}\left(+\infty\right)\right) = 0, \end{cases} \tag{5.2}$$

with $u^{(n-1)}(+\infty) := \lim\limits_{t \to +\infty} u^{(n-1)}(t)$, $L_i : C([0, +\infty[) \times \mathbb{R}^{n-1} \to \mathbb{R}$, $i = 0, 1, ..., n-2$, and $L_{n-1} : C([0, +\infty[) \times \mathbb{R} \to \mathbb{R}$ continuous functions.

These types of higher-order boundary value problems have been considered by many authors, not only with a general higher-order derivative n, but also for particular cases of n. Most of all, are studied for continuous nonlinearities, and in bounded intervals, with classical boundary conditions, such as, [55, 64], for linear problems, [4, 38], for two-point separated and Sturm-Liouville boundary conditions, [10, 27, 40], for multipoint problems, [89], for periodic solutions, among others.

The functional boundary conditions in higher-order problems can include global data on the unknown variable and its derivatives, and, in this way, they generalize the usual boundary assumptions, considering local, nonlocal or integro-differential conditions, with deviating arguments, delays or advances, maxima or minima of some variables. For works dealing with these features see [15, 13, 16, 26, 30, 37, 41, 53, 57, 61] and the references therein.

On unbounded intervals there is a lack of compacity on the operator, that can be overcome by applying some adequate techniques to guarantee the solvability. As examples, we mention the extension by continuity of some adequate bounded intervals by a diagonalization method, the definition of suitable Banach spaces and norms to obtain sufficient conditions for the existence of fixed points, and lower and upper solutions technique. The interested readers can see these methods in, for example, [1, 2, 10, 20, 60, 76, 95] and in their references.

In more detail, we refer [62], where the authors study the problem composed by the nth-order differential equation on the half-line

$$-u^{(n)}(t) = q(t)f(t,\, u(t), ..., u^{(n-1)}(t)), \quad t \in (0, +\infty),$$

where $q : (0, +\infty) \to (0, +\infty)$, $f : [0, +\infty)\times\mathbb{R}^n \to \mathbb{R}$ are continuous, together with the boundary conditions

$$\begin{cases} u^{(i)}(0) = A_i, \quad i = 0, 1, ..., n-3, \\ u^{(n-2)}(0) - au^{(n-1)}(0) = B, \\ u^{(n-1)}(+\infty) = C, \end{cases}$$

with $a > 0$, $A_i, B, C \in \mathbb{R}$, $i = 0, 1, ..., n-3$. Applying lower and upper solutions method and the Schäuder fixed point theorem, the authors prove the existence of a solution, and from topological degree theory, of triple solutions.

In [39], it is considered the problem with the ϕ–Laplacian type differential equation

$$-\left(\phi(u^{(n-1)}(t))\right)' = f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right),$$

defined on the bounded interval $(0, 1)$, where $n \geq 2$, ϕ is an increasing homeomorphism and $f : (0, 1) \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, and the functional boundary conditions

$$\begin{cases} g_i\left(u, u', \ldots, u^{(n-1)}, u^{(i)}(0)\right) = 0, \; i = 0, \ldots, n-2, \\ g_{n-1}\left(u, u', \ldots, u^{(n-1)}, u^{(n-2)}(1)\right) = 0, \end{cases}$$

with $g_i : (C[0,1])^n \times \mathbb{R} \to \mathbb{R}$, $i = 0, \ldots, n-1$, continuous functions. Applying lower and upper solutions method, together with a Nagumo-type condition, it is proved that, for $n \geq 3$, the order between the lower and upper solutions and their derivatives is not relevant. The type of order depends on whether n is even or odd and on the existent relationship between the $(n-2)-nd$ derivatives of the lower and upper solutions. Moreover, the monotonic behavior of the nonlinearities is related to the parity of n.

In our problem we combine for the first time, as far as we know, all these features, taking advantage of all of them and allowing their application to a wider range of real-life problems and phenomena. In short, the method is based on the definition of an auxiliary problem, composed by a truncated and perturbed equation, with initial values and the asymptotic behavior of the higher derivative given by truncated functions, which include the functional data. An adequate operator is defined in a weighted Banach space, and the lack of compactness is overcome by considering weighted norms. Sufficient conditions are given to have fixed points, via Schauder's fixed point theorem. Lower and upper solutions method is used to prove that these fixed points, solutions of the auxiliary problem, are solutions to the initial problem, too. Moreover, despite the localization part, we stress that these solutions may be unbounded.

A possible application for higher-order problems defined on unbounded intervals is, for $n = 4$, the study of the bending of infinite beams with different types of foundations, as it can be seen for example, in [21, 48, 49, 70]. We point out that, the functional boundary conditions as (5.2), allow us to consider new types of models, where, for example, global data on the beam could be considered, which is new on the literature.

The chapter is organized as follows: Section 2 contains the definitions of the weighted Banach space and norms, some *a priori* bounds and other auxiliary results. In section 3 it is presented the main result: an existence and localization theorem for the functional problem. The last section is concerned with a numerical example related to the estimate of the deflection of an infinite beam, subject to global conditions.

5.2 Definitions and auxiliary results

In this work, it will be considered the space

$$X = \left\{ x \in C^{n-1}[0, +\infty[: \lim_{t \to +\infty} \frac{x^{(i)}(t)}{1 + t^{n-1-i}} \text{ exists in } \mathbb{R}, \ i = 0, 1, ..., n-1 \right\}$$

with the norm $\|x\|_X := \max \left\{ \left\| x^{(i)} \right\|_i \right\}, \ i = 0, 1, ..., n-1$, where

$$\left\| \omega^{(i)} \right\|_i = \sup_{0 \leq t < +\infty} \left| \frac{\omega^{(i)}(t)}{1 + t^{n-1-i}} \right|, \ i = 0, 1, ..., n-1.$$

It can be proved that $(X, \|.\|_X)$ is a Banach space.

The following definition establishes the regularity of the nonlinear part:

Definition 5.1 *A function $f : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$ is called a L^1- Carathéodory function if it verifies:*

i) for each $(y_0, ..., y_{n-1}) \in \mathbb{R}^n$, $t \mapsto f(t, y_0, ..., y_{n-1})$ is measurable on $[0, +\infty[$;

ii) for almost every $t \in [0, +\infty[$, $(y_0, ..., y_{n-1}) \mapsto f(t, y_0, ..., y_{n-1})$ is continuous in $\mathbb{R}^n$;

iii) for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1[0, +\infty[$, $j = 0, 1, ..., n-1$, such that whenever $(t, y_0, ..., y_{n-1}) \in [0, +\infty[\times \mathbb{R}^n$ satisfies $|y_i| < \rho\,(1 + t^{n-1-i})$, $i = 0, 1, ..., n-1$, one has

$$|f(t, y_0, ..., y_{n-1})| \leq \varphi_\rho(t), \ \text{a.e. } t \in [0, +\infty[.$$

Solutions of the linear problem associated to (5.1)-(5.2) are defined with kernels given by the Green's function, which can be obtained by standard calculus, as in [62], Lemma 2.1:

Lemma 5.2 *Let $h \in L^1[0, +\infty[$. Then the linear boundary value problem composed by*

$$\begin{cases} u^{(n)}(t) = h\,(t)\,, \ \text{a.e. } t \in [0, +\infty[, \\ u^{(i)}\,(0) = A_{i,} \ i = 0, 1, ..., n-2, \\ u^{(n-1)}\,(+\infty) = B, \end{cases} \tag{5.3}$$

with $A_i, B \in \mathbb{R}$, $i = 0, 1, ..., n-2$, has a unique solution given by

$$u(t) = \sum_{i=0}^{n-2} \frac{A_i}{i!} t^i + \frac{B}{(n-1)!} t^{n-1} + \int_0^{+\infty} G(t, s) h\,(s)\, ds, \tag{5.4}$$

where

$$G(t,s)=\begin{cases}\sum_{j=0}^{n-2}\left(\frac{(-1)^j}{(n-2-j)!(j+1)!}s^{j+1}t^{n-2-j}\right), & 0\leq s\leq t<+\infty \\ \frac{1}{(n-1)!}t^{n-1}, & 0\leq t\leq s<+\infty.\end{cases} \tag{5.5}$$

Remark 5.3 *The Green function given by (5.5) satisfies*

$$\lim_{t\to+\infty}\frac{G_i\left(t,s\right)}{1+t^{n-1-i}}\in\mathbb{R},\ for\ i=0,1,...,n-1,$$

with

$$G_i\left(t,s\right):=\frac{\delta^i G}{\delta t^i}\left(t,s\right)=\sum_{j=0}^{n-2-i}\left(\frac{(-1)^{j+i}}{(n-2-j-i)!\left(j+1\right)!}s^{j+1}t^{n-2-j-i}\right). \tag{5.6}$$

In order to apply a fixed point theorem it is important to have an *a priori* estimation for $u^{(n-1)}(t)$. In the literature this bound is obtained from a Nagumo-type growth.

Let $\gamma,\Gamma\in X$ and define the set

$$E=\left\{\begin{array}{c}(t,y_0,...,y_{n-1})\in[0,+\infty[\times\mathbb{R}^n:\gamma^{(i)}(t)\leq y_i\leq\Gamma^{(i)}(t),\\ i=0,1,...,n-2,\ \gamma^{(n-1)}\left(+\infty\right)\leq y_{n-1}\leq\Gamma^{(n-1)}\left(+\infty\right)\end{array}\right\}. \tag{5.7}$$

Definition 5.4 *A L^1- Carathéodory function $f:E\to\mathbb{R}$ is said to satisfy a Nagumo-type growth condition in E if it verifies*

$$|f(t,y_0,...,y_{n-1})|\leq\psi(t)\Phi(|y_{n-1}|),\forall(t,y_0,...,y_{n-1})\in E, \tag{5.8}$$

for some positive continuous functions $\psi,\Phi,$ and some $\nu>1,$ such that

$$\sup_{0\leq t<+\infty}\psi(t)(1+t)^{\nu}<+\infty,\int_0^{+\infty}\frac{s}{\Phi(s)}ds=+\infty. \tag{5.9}$$

Next lemma provides an *a priori* bound:

Lemma 5.5 *Let $f : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a L^1- Carathéodory function satisfying (5.8) and (5.9) in E. Then for every $r > 0$ there exists $R > 0$ (not depending on u) such that every u solution of (5.1), satisfying*

$$\gamma^{(i)}(t) \leq u^{(i)}(t) \leq \Gamma^{(i)}(t), i = 0, 1, ..., n-2, \tag{5.10}$$
$$\gamma^{(n-1)}(+\infty) \leq u^{(n-1)}(+\infty) \leq \Gamma^{(n-1)}(+\infty),$$

for $t \in [0,+\infty[$, verifies

$$\left\|u^{(n-1)}\right\|_{n-1} < R. \tag{5.11}$$

Proof. Let u be a solution of (5.1) such that (5.10) holds. Consider $r > 0$ such that

$$r > \max\left\{\left|\gamma^{(n-1)}(+\infty)\right|, \left|\Gamma^{(n-1)}(+\infty)\right|\right\}. \tag{5.12}$$

By the previous inequality and (5.7), we cannot have $\left|u^{(n-1)}(t)\right| > r, \forall t \in [0,+\infty[$.

If $\left|u^{(n-1)}(t)\right| \leq r, \forall t \in [0,+\infty)$, taking $R > r/2$ the proof is complete as

$$\left\|u^{(n-1)}\right\|_{n-1} = \sup_{0\leq t<+\infty}\left|\frac{u^{(n-1)}(t)}{2}\right| \leq \frac{r}{2} < R.$$

If there exists $t_0 \in [0,+\infty)$ such that $\left|u^{(n-1)}(t_0)\right| > r$, then, in the case that $u^{(n-1)}(t_0) > r$, by (5.9), we can take $R > r$ such that

$$\int_r^R \frac{s}{\Phi(s)}ds > M \max\left\{\begin{array}{c} M_1 + \sup_{0\leq t<+\infty}\frac{\Gamma^{(n-2)}(t)}{1+t}\frac{\nu}{\nu-1}, \\ \\ M_1 - \inf_{0\leq t<+\infty}\frac{\gamma^{(n-2)}(t)}{1+t}\frac{\nu}{\nu-1} \end{array}\right\}$$

with

$$M := \sup_{0\leq t<+\infty}\psi(t)(1+t)^\nu \text{ and } M_1 := \sup_{0\leq t<+\infty}\frac{\Gamma^{(n-2)}(t)}{(1+t)^\nu} - \inf_{0\leq t<+\infty}\frac{\gamma^{(n-2)}(t)}{(1+t)^\nu}.$$

By (5.12), choose $t_1 \in (0,+\infty)$ such that $t_1 > t_0$ and

$$u^{(n-1)}(t_1) = r,\ u^{(n-1)}(t) > r, \forall t \in [t_0, t_1[.$$

Then

$$\begin{aligned}
\int_{u^{(n-1)}(t_1)}^{u^{(n-1)}(t_0)} \frac{s}{\Phi(s)}ds &= \int_{t_1}^{t_0} \frac{u^{(n-1)}(s)}{\Phi|u^{(n-1)}(s)|}u^{(n)}(s)ds \\
&\leq \int_{t_0}^{t_1} \frac{\left|f(s,u(s),...,u^{(n-1)}(s))\right|}{\Phi(u^{(n-1)}(s))}u^{(n-1)}(s)ds \\
&\leq \int_{t_0}^{t_1} \psi(s)\ u^{(n-1)}(s)\ ds \leq M\int_{t_0}^{t_1} \frac{u^{(n-1)}(s)}{(1+s)^{\nu}}\ ds \\
&= M\int_{t_0}^{t_1} \left(\frac{u^{(n-2)}(s)}{(1+s)^{\nu}}\right)' + \frac{\nu u^{(n-2)}(s)}{(1+s)^{1+\nu}}ds \\
&= M\left(\frac{u^{(n-2)}(t_1)}{(1+t_+)^{\nu}} - \frac{u^{(n-2)}(t_0)}{(1+t_*)^{\nu}} + \int_{t_0}^{t_1} \frac{\nu u^{(n-2)}(s)}{(1+s)^{1+\nu}}ds\right) \\
&\leq M\left(M_1 + \sup_{0\leq t<+\infty} \frac{\Gamma^{(n-2)}(t)}{1+t}\int_0^{+\infty} \frac{\nu}{(1+s)^{\nu}}ds\right) \\
&< \int_r^R \frac{s}{\Phi(s)}ds.
\end{aligned}$$

So $u^{(n-1)}(t_0) < R$ and as t_1, t_0 are arbitrary in $[0,+\infty)$, for the values where $u^{(n-1)}(t) > r$, we have that $u^{(n-1)}(t) < R, \forall t \in [0,+\infty[$.

By the same technique, considering t_- and t_* such that $u^{(n-1)}(t_-) < -r$, $u^{(n-1)}(t_*) = -r$, $u^{(n-1)}(t) < -r, \forall t \in [t_-, t_*]$, it can be proved that $u^{(n-1)}(t) > -R, \forall t \in [0,+\infty[$, and, therefore, $\left\|u^{(n-1)}\right\|_{n-1} < \frac{R}{2} < R, \forall t \in [0,+\infty)$. ■

Next result will play a key role to apply a fixed-point theorem.

Lemma 5.6 *([1]) A set $M \subset X$ is relatively compact if the following conditions hold:*

1. *all functions from M are uniformly bounded;*
2. *all functions from M are equicontinuous on any compact interval of $[0,+\infty[$;*
3. *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that, for $i = 0,1,...,n-1$,*

$$\left|\frac{u^{(i)}(t)}{1+t^{n-1-i}} - \lim_{t\to+\infty}\frac{u^{(i)}(t)}{1+t^{n-1-i}}\right| < \epsilon,\ \textit{for all } t > t_\epsilon,\ x \in M.$$

The functions considered as lower and upper solutions for the initial problem are defined as it follows:

Definition 5.7 *A function $\alpha \in C^n[0,+\infty[\cap X$ is said to be a lower solution of problem (5.1), (5.2) if*

$$\begin{cases} \alpha^{(n)}(t) \geq f(t, \alpha(t), \alpha'(t), ..., \alpha^{(n-1)}(t)), \quad t \in [0,+\infty[, \\ L_i(\alpha^{(i)}, \alpha(0), ..., \alpha^{(n-2)}(0) \geq 0, \;\; i = 0,1,...,n-2, \\ L_{n-1}\left(\alpha, \alpha^{(n-1)}(+\infty)\right) > 0 \end{cases}$$

A function β is an upper solution of problem (5.1), (5.2) if the reversed inequalities hold.

Forward, the boundary functions L_j, for $j = 0,1,...,n-1$, must verify the following assumptions:

(H_1) For $i = 0,1,...,n-2$, $L_i(w, y_0, y_1, ..., y_{n-2})$ is nondecreasing in all the arguments except in the $(i+2)-nd$ variable;

(H_2) $\lim_{t\to\infty} L_{n-1}(w,z) \in \mathbb{R}$ for $\alpha \leq w \leq \beta$ and $\alpha^{(n-1)}(+\infty) \leq z \leq \beta^{(n-1)}(+\infty)$;

(H_3) $L_{n-1}(w,z)$ is nondecreasing on w for z fixed.

5.3 Main Result

This section contains an existence and localization result, that is, not only the existence of at least a solution for problem (5.1), (5.2) is proved, but also it provides some localization data for this solution and its derivatives.

Theorem 5.8 *Let $f : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ be a L^1- Carathéodory function, and α, β lower and upper solutions of (5.1),(5.2) respectively, such that*

$$\alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t), \forall t \in [0,+\infty[. \tag{5.13}$$

$$\alpha^{(i)}(0) \leq \beta^{(i)}(0), \;\; i = 0,1,...,n-3, \tag{5.14}$$

and

$$\alpha^{(n-1)}(+\infty) \leq \beta^{(n-1)}(+\infty). \tag{5.15}$$

Assume that f verifies the Nagumo conditions (5.8) and 5.9) in the set

$$E_* = \left\{ \begin{array}{c} (t, y_0, ..., y_{n-1}) \in [0, +\infty[\times \mathbb{R}^n, \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), i = 0, 1, ..., n-2, \\ \alpha^{(n-1)}(+\infty) \leq y_{n-1} \leq \beta^{(n-1)}(+\infty) \end{array} \right\},$$

and

$$\begin{aligned} f\left(t, \alpha(t), \alpha'(t), ..., \alpha^{(n-3)}(t), y_{n-2}, y_{n-1}\right) &\geq f(t, y_0, ..., y_{n-1}) \\ &\geq f\left(t, \beta(t), \beta'(t), ..., \beta^{(n-3)}(t), y_{n-2}, y_{n-1}\right) \end{aligned} \tag{5.16}$$

and the assumptions (H_1), (H_2) and (H_3) hold.
If there is $\rho > 0$ such that,

$$\max \left\{ \begin{array}{c} \|\alpha\|_X, \|\beta\|_X, R, \\ \max\limits_{i=0,1,...,n-2} \left\{ \sum\limits_{j=i}^{n-2} \frac{M_j}{(j-i)!} + \frac{M_\infty}{(n-1-i)!} + \int\limits_0^{+\infty} M_i(s) \left(\varphi_\rho(s) + \frac{1}{1+s^{2n}} \right) ds \right\}, \\ \frac{M_\infty}{2} + \frac{1}{2} \int\limits_0^{+\infty} \left(\varphi_\rho(s) + \frac{1}{1+s^{2n}} \right) ds \end{array} \right\} < \rho \tag{5.17}$$

where

$$\begin{aligned} M_j &:= \max\left\{ \left|\alpha^{(j)}(0)\right|, \left|\beta^{(j)}(0)\right| \right\}, \\ M_\infty &:= \max\left\{ \left|\alpha^{(n-1)}(+\infty)\right|, \left|\beta^{(n-1)}(+\infty)\right| \right\}, \\ M_i(s) &:= \sup_{0 \leq t < +\infty} \frac{|G_i(t,s)|}{1 + t^{n-1-i}}, \; i = 0, 1, ..., n-1, \end{aligned} \tag{5.18}$$

and $G_i(t,s)$ are given by (5.6), then, for R given by (5.11), there is $u \in X$, solution of problem (5.1),(5.2), such that

$$\begin{aligned} \alpha^{(i)}(t) &\leq u^{(i)}(t) \leq \beta^{(i)}(t), i = 0, 1, ..., n-2, \\ -R &< u^{(n-1)}(t) < R, \text{ for } t \in [0, +\infty[, \\ \alpha^{(n-1)}(+\infty) &\leq u^{(n-1)}(+\infty) \leq \beta^{(n-1)}(+\infty). \end{aligned}$$

Proof. By integration of (5.13) and (5.14), $\alpha^{(j)}(t) \leq \beta^{(j)}(t)$, $j = 0, 1, ..., n-3$, for $t \in [0, +\infty[$. Therefore we can consider the modified and perturbed equation

$$u^{(n)}(t) = f\left(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), ..., \delta_{n-2}(t, u^{(n-2)}(t)), u^{(n-1)}(t)\right)$$

$$+\frac{1}{1+t^{2n}}\,\frac{u^{(n-2)}(t) - \delta_{n-2}(t, u^{(n-2)}(t))}{1 + |u^{(n-2)}(t) - \delta_{n-2}(t, u^{(n-2)}(t))|}, t \in [0, +\infty[, \tag{5.19}$$

where the functions $\delta_j : [0, +\infty[\times\mathbb{R} \to \mathbb{R}, i = 0, 1, ..., n-2$, are given by

$$\delta_i(t, y_i) = \begin{cases} \beta^{(i)}(t) & , \ y_i > \beta^{(i)}(t) \\ y_i & , \ \alpha^{(i)}(t) \le y_i \le \beta^{(i)}(t) \\ \alpha^{(i)}(t) & , \ y_i < \alpha^{(i)}(t) \end{cases}, \ i = 0, 1, 2, ..., n-3, \tag{5.20}$$

together with the truncated boundary conditions and

$$\begin{cases} u^{(i)}(0) = \delta_i\left(0, u^{(i)}(0) + L_i\left(u^{(i)}, u(0), u'(0), ..., u^{(n-1)}(0)\right)\right), \\ \text{for } i = 0, 1, ..., n-2, \\ u^{(n-1)}(+\infty) = \delta_\infty\left(u^{(n-1)}(+\infty) + L_{n-1}\left(u, \delta_\infty\left(u^{(n-1)}(+\infty)\right)\right)\right) \end{cases} \tag{5.21}$$

where

$$\delta_\infty(t, y) = \begin{cases} \beta^{(n-1)}(+\infty) & , \quad y > \beta^{(n-1)}(+\infty) \\ y & , \quad \alpha^{(n-1)}(+\infty) \le y \le \beta^{(n-1)}(+\infty) \\ \alpha^{(n-1)}(+\infty) & , \quad y < \alpha^{(n-1)}(+\infty). \end{cases} \tag{5.22}$$

Let us define the operator $T : X \to X$

$$Tu(t) = \sum_{j=0}^{n-2} \frac{A_j}{j!} t^j + \frac{B}{(n-1)!} t^{n-1} + \int_0^{+\infty} G(t, s) F_u(s)\, ds$$

with

$$A_j := \delta_j\left(0, u^{(j)}(0) + L_j\left(u^{(j)}, u(0), u'(0), ..., u^{(n-2)}(0)\right)\right),$$

$$B := \delta_\infty\left(u^{(n-1)}(+\infty) + L_{n-1}(u, \delta_\infty\left(u^{(n-1)}(+\infty)\right))\right),$$

$$F_u(s) \ : \ = f\left(s, \delta_0(s, u(s))), ..., \delta_{n-2}\left(s, u^{(n-2)}(s)\right), u^{(n-1)}(s)\right) + \frac{1}{1+s^{2n}}\frac{u^{(n-2)}(s) - \delta_{n-2}(s, u^{(n-2)}(s))}{1 + |u^{(n-2)}(s) - \delta_{n-2}(s, u^{(n-2)}(s))|}$$

and $G(t,s)$ given by (5.5).

For clearness, the proof will follow several steps:

STEP 1: *T is compact*

(i) *$T : X \to X$ is well defined.*

Let $u \in X$. As f is a L^1-Carathéodory function by, $Tu \in C^{n-1}([0,+\infty[)$ and by Definition 5.1, for $u \in X$ such that $\|u\|_X < \rho_0$, with

$$\rho_0 > \max\left\{\|\alpha\|_X, \|\beta\|_X, R\right\}, \tag{5.23}$$

there is a positive function $\varphi_{\rho_0} \in L^1[0,+\infty[$, such that

$$\int_0^{+\infty} |F_u(s)|\, ds \leq \int_0^{+\infty} \left(\varphi_{\rho_0}(s) + \frac{1}{1+s^{2n}}\right) ds < +\infty.$$

Therefore, F_u is also a L^1- Carathéodory function.

Moreover, for $i = 0,1,...,n-1$, and $G_i(t,s)$ given by (5.6), we have

$$\lim_{t\to+\infty} \frac{(Tu)^i(t)}{1+t^{n-1-i}} =$$

$$\lim_{t\to+\infty} \frac{1}{1+t^{n-1-i}} \left[\sum_{j=i}^{n-2} A_j \frac{t^{j-i}}{(j-i)!} + B\frac{t^{n-1-i}}{(n-1-i)!} + \int_0^{+\infty} G_i(t,s)\, F_u(s)\, ds\right] < +\infty,$$

that is, $Tu \in X$.

(ii) *T is continuous.*

For any convergent sequence $u_n \to u$ in X, there exists $\rho > 0$ such that $\sup_n \|u_n\|_X < \rho$, and

$$\begin{aligned}
\|Tu_n - Tu\|_X &= \max\left\{\left\|(Tu_n)^{(i)} - (Tu)^{(i)}\right\|_i, i = 0,1,...,n-1\right\} \\
&= \max_{i=0,1,...,n-1}\left\{\sup_{0\leq t<+\infty} \frac{\left|(Tu_n)^{(i)}(t) - (Tu_n)^{(i)}(t)\right|}{1+t^{n-1-i}}\right\} \\
&\leq \max_{i=0,1,...,n-1} \int_0^{+\infty} \frac{|G_i(t,s)|}{1+t^{n-1-i}} |F_{u_n}(s) - F_u(s)|\, ds \\
&\leq \max_{i=0,1,...,n-1} \int_0^{+\infty} M_i(s)\, |F_{u_n}(s) - F_u(s)|\, ds \longrightarrow 0\ , \quad n \to +\infty,
\end{aligned}$$

with $M_i(s)$ given by (5.18).

(iii) *T is compact.*

Let $B \subset X$ be a bounded subset. Therefore there is $r > 0$ such that $\|u\|_X < r, \forall u \in B$.

By (i) and (ii) it is clear that

$$\begin{aligned}\|Tu\|_X &= \max\left\{\left\|(Tu)^{(i)}\right\|_i, i = 0, 1, ..., n-1\right\} \\ &= \max_{i=0,1,\ldots,n-1}\left\{\sup_{0\leq t<+\infty}\frac{\left|(Tu)^{(i)}\right|}{1+t^{n-1-i}}\right\} < +\infty\end{aligned}$$

and so TB is uniformly bounded.

In order to prove that TB is equicontinuous, consider $L > 0$ and $t_1, t_2 \in [0, L]$. Suppose, without loss of generality, that $t_1 < t_2$. Then

$$\left|\frac{(Tu)^{(i)}(t_1)}{1+t_1^{n-1-i}} - \frac{(Tu)^{(i)}(t_2)}{1+t_2^{n-1-i}}\right| \leq$$

$$\left|\begin{array}{c}\frac{1}{1+t_1^{n-1-i}}\left[\sum_{j=i}^{n-2} A_j \frac{t_1^{j-i}}{(j-i)!} + B\frac{t_1^{n-1-i}}{(n-1-i)!}\right] \\ -\frac{1}{1+t_2^{n-1-i}}\left[\sum_{j=i}^{n-2} A_j \frac{t_2^{j-i}}{(j-i)!} + B\frac{t_2^{n-1-i}}{(n-1-i)!}\right]\end{array}\right|$$

$$+\int_0^{+\infty}\left|\frac{G_i(t,s)}{1+t_1^{n-1-i}} - \frac{G_i(t,s)}{1+t_2^{n-1-i}}\right| |F_u(s)|\, ds \to 0, \text{ as } t_1 \to t_2.$$

For $i = n-1$, the function G_{n-1} is not continuous for $s = t$, and

$$\left|\frac{(Tu)^{(n-1)}(t_1)}{2} - \frac{(Tu)^{(n-1)}(t_2)}{2}\right| \leq \int_0^{+\infty}\left|\frac{G_{n-1}(t_1,s) - G_{n-1}(t_2,s)}{2}\right| |F_u(s)|\, ds$$

$$\leq \frac{1}{2}\int_{t_1}^{t_2} |F_u(s)|\, ds \to 0, \text{ as } t_1 \to t_2.$$

Moreover TB is equiconvergent at infinity because, by the Lebesgue's Dominated Convergence Theorem, we obtain, for $i = 0, 1, ..., n-2$,

$$\left| \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} - \lim_{t\to+\infty} \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} \right| =$$

$$\left| \begin{array}{c} \frac{1}{1+t^{n-1-i}} \left(\sum_{j=i}^{n-2} A_j \frac{t^{j-i}}{(j-i)!} + B \frac{t^{n-1-i}}{(n-1-i)!} \right) \\ + \int_0^{+\infty} G_i(t,s) F_u(s)\, ds - \frac{B}{(n-1-i)!} \end{array} \right| \to 0,$$

as $t \to +\infty$, and, for $i = n-1$,

$$\left| \frac{(Tu)^{(n-1)}(t)}{2} - \lim_{t\to+\infty} \frac{(Tu)^{(n-1)}(t)}{2} \right| = \frac{1}{2} \int_t^{+\infty} |F_u(s)|\, ds \to 0$$

as $t \to +\infty$

Therefore by Lemma 5.6, TB is relatively compact, and so, T is compact.

STEP 2. *The problem (5.19), (5.21) has at least a solution.*

By Lemma 5.2, the fixed points of T are solutions of problem (5.19), (5.21). So it is enough to prove that T has a fixed point.

To apply Schauder's Fixed Point Theorem, we consider the non empty, closed, bounded and convex set $D \subset X$, defined by

$$D := \{u \in X : \|u\|_X \leq \rho_1\}$$

with $\rho_1 > 0$ given by

$$\max_{i=0,1,...,n-2} \left\{ \begin{array}{c} \sum_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int_0^{+\infty} M_i(s) \left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right) ds, \\ \frac{|B|}{2} + \frac{1}{2} \int_0^{+\infty} \left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}}\right) ds \end{array} \right\} < \rho_1.$$

Let us prove that $TD \subset D$.

For $i=0,1,...,n-2$, and $u\in D$,

$$\begin{aligned}
\left\|(Tu)^{(i)}\right\|_X &= \max_i \left\|(Tu)^{(i)}\right\|_i = \max_i \left\{ \sup_{0\le t<+\infty} \left| \frac{(Tu)^{(i)}(t)}{1+t^{n-1-i}} \right| \right\} \\
&\le \max_i \left[\sup_{0\le t<+\infty} \left(\begin{array}{c} \sum_{j=i}^{n-2} |A_j| \frac{t^{j-i}}{(1+t^{n-1-i})(j-i)!} + |B| \frac{t^{n-1-i}}{(1+t^{n-1-i})(n-1-i)!} \\ + \int_0^{+\infty} |G_i(t,s)| \, |F_u(s)| \, ds \end{array} \right) \right] \\
&\le \max_i \left\{ \sum_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int_0^{+\infty} M_i(s) \, |F_u(s)| \, ds \right\} \\
&\le \max_i \left\{ \sum_{j=i}^{n-2} \frac{|A_j|}{(j-i)!} + \frac{|B|}{(n-1-i)!} + \int_0^{+\infty} M_i(s) \left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}} \right) ds \right\} \\
&< \rho_1.
\end{aligned}$$

For $i=n-1$,

$$\begin{aligned}
\left\|(Tu)^{(n-1)}\right\|_{n-1} &= \sup_{0\le t<+\infty} \left| \frac{(Tu)^{(n-1)}(t)}{2} \right| = \frac{1}{2} \left| B + \int_0^{+\infty} F_u(s) \, ds \right| \\
&\le \frac{|B|}{2} + \frac{1}{2} \int_0^{+\infty} \left(\varphi_{\rho_1} + \frac{1}{1+s^{2n}} \right) ds < \rho_1.
\end{aligned}$$

Therefore $TD\subset D$, and by Schauder's Fixed Point Theorem, the operator T has a fixed point u_* which is a solution of problem (5.19), (5.21).

STEP 3. *Every solution of problem (5.19), (5.21) verifies*

$$\alpha^{(i)}(t)\le u^{(i)}(t)\le \beta^{(i)}(t), i=0,1,...,n-2, \tag{5.24}$$

$$-R<u^{(n-1)}(t)<R, \text{ for } t\in[0,+\infty[, \tag{5.25}$$

$$\alpha^{(n-1)}(+\infty)\le u^{(n-1)}(+\infty)\le \beta^{(n-1)}(+\infty). \tag{5.26}$$

Let u be a solution of the modified problem (5.19), (5.21) and suppose that, by contradiction, there exists $t\in[0,+\infty)$ such that $\alpha^{(n-2)}(t)>u^{(n-2)}(t)$. Therefore

$$\inf_{0\le t<+\infty}(u^{(n-2)}(t)-\alpha^{(n-2)}(t))<0.$$

By (5.14), this infimum can not be attained neither on 0 as, by (5.20) and (5.21),

$$\begin{aligned}(u^{(n-2)}(0)-\alpha^{(n-2)}(0)) = {} & \delta_{n-2}\left(0, u^{(n-2)}(0)+L_{n-2}\left(u^{(n-2)}, u(0), ..., u^{(n-2)}(0)\right)\right) \\ & -\alpha^{(n-2)}(0) \geq 0,\end{aligned}$$

nor at $+\infty$, as, by (5.22),

$$\begin{gathered}(u^{(n-1)}(+\infty)-\alpha^{(n-1)}(+\infty)) = \\ \delta_{\infty}\left(u^{(n-1)}(\infty)+L_{n-1}\left(u, \delta_{\infty}u^{(n-1)}(+\infty)\right)\right)-\alpha^{(n-1)}(+\infty) \geq 0.\end{gathered}$$

Then there is an interior point $t_* \in (0,+\infty)$ such that

$$\min_{0\leq t<+\infty}(u^{(n-2)}(t)-\alpha^{(n-2)}(t)) := u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*) < 0,$$

with $u^{(n-1)}(t_*) = \alpha^{(n-1)}(t_*)$ and $u^{(n)}(t_*)-\alpha^{(n)}(t_*) \geq 0$. Therefore by (5.16) and Definition 5.7, we get the contradiction

$$\begin{aligned} 0 & \leq u^{(n)}(t_*)-\alpha^{(n)}(t_*) \\ & = f(t_*, \delta_0(t_*, u(t_*)), ..., \delta_{n-2}(t_*, u^{(n-2)}(t_*)), u^{n-1}(t_*)) \\ & \quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\delta_{n-2}(t_*, u^{(n-2)}(t_*))}{1+|u^{n-2}(t_*)-\delta_{n-2}(t_*, u^{n-2}(t_*))|}-\alpha^{(n)}(t_*) \\ & = f(t_*, \delta_0(t_*, u(t_*)), ..., \delta_{n-3}(t_*, u^{(n-3)}(t_*)), \alpha^{(n-2)}(t_*), \alpha^{(n-1)}(t_*)) \\ & \quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)|}-\alpha^{(n)}(t_*) \\ & \leq f\left(t_*, \alpha(t_*), \alpha'(t_*), ..., \alpha^{(n-1)}(t_*)\right) \\ & \quad +\frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)|}-\alpha^{(n)}(t_*) \\ & \leq \frac{1}{1+t_*^{2n}}\frac{u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)}{1+|u^{(n-2)}(t_*)-\alpha^{(n-2)}(t_*)|} < 0. \end{aligned}$$

So $u^{(n-2)}(t) \geq \alpha^{(n-2)}(t), \forall t \in [0,+\infty[$.

Analogously it can be shown that $u^{(n-2)}(t) \leq \beta^{(n-2)}(t), \forall t \in [0,+\infty[$.

By (5.14), integrating on $[0,+\infty[$, we have

$$\begin{aligned}\alpha^{(n-3)}(t) \leq u^{(n-3)}(t) & -\delta_{n-3}\left(0, u^{(n-3)}(0)+L_{n-3}\left(u^{(n-3)}, u(0), ..., u^{(n-2)}(0)\right)\right) \\ & +\alpha^{(n-3)}(0) \leq u^{(n-3)}(t),\end{aligned}$$

and by similar arguments

$$\alpha^{(i)}(t) \leq u^{(i)}(t), \text{ for } t \in [0. +\infty[, \text{ and } i = 0, 1, ..., n-3.$$

With the same technique it can be proved that

$$u^{(i)}(t) \leq \beta^{(i)}(t), \text{ for } t \in [0 + \infty[, \text{ and } i = 0, 1, ..., n-2,$$

and, therefore,

$$\alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t), \forall t \in [0, +\infty[, \ i = 0, 1, ..., n-2.$$

So, this solution of problem (5.19), (5.21) belongs to E_* and condition (5.25) is a direct consequence of Lemma 5.5.

Moreover, (5.26) is trivially verified, by (5.22).

STEP 4. *Let u_* be a solution of problem (5.19), (5.21). Then u_* is a solution of problem (5.1), (5.2).*

According to Step 3, to prove this claim it is enough to show that

$$\alpha^{(i)}(0) \leq u_*^{(i)}(0) + L_i\left(u_*^{(i)}, u_*(0), ..., u_*^{(n-2)}(0)\right) \leq \beta^{(i)}(0), \tag{5.27}$$

for $i = 0, 1, ..., n-2$, and

$$\alpha^{(n-1)}(+\infty) \leq u_*^{(n-1)}(+\infty) + L_{n-1}\left(\delta_\infty\left(u_*^{(n-1)}(+\infty)\right)\right) \leq \beta^{(n-1)}(+\infty).$$

Suppose that the first inequality of (5.27) does not hold for $i = n-2$. That is,

$$\alpha^{(n-2)}(0) > u_*^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), ..., u_*^{(n-2)}(0)\right). \tag{5.28}$$

Therefore, by (5.20) and (5.21) we have

$$u_*^{(n-2)}(0) = \alpha^{(n-2)}(0).$$

By Definition 5.7 and (H_1) the following contradiction with (5.28) holds:

$$\begin{aligned} & u_*^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), ..., u_*^{(n-3)}(0), u_*^{(n-2)}(0)\right) \\ = \ & \alpha^{(n-2)}(0) + L_{n-2}\left(u_*^{(n-2)}, u_*(0), ..., u_*^{(n-3)}(0), \alpha^{(n-2)}(0)\right) \\ \geq \ & \alpha^{(n-2)}(0) + L_{n-2}\left(\alpha^{(n-2)}, \alpha(0), ..., \alpha^{(n-3)}(0), \alpha^{(n-2)}(0)\right) \\ \geq \ & \alpha^{(n-2)}(0) \end{aligned}$$

A similar contradiction can be obtained in the remaining inequalities. So, (5.27) holds.

Assume now that

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) < \alpha^{(n-1)}(+\infty)\right). \qquad (5.29)$$

Therefore, by (5.21),

$$u_*^{(n-1)}(+\infty) = \alpha^{(n-1)}(+\infty)$$

which, by (H_3) and Definition 5.7, leads to a contradiction with (5.29):

$$\begin{aligned} & u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right)\right) \\ = \ & \alpha^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \alpha^{(n-1)}(+\infty)\right) \\ \geq \ & \alpha^{(n-1)}(+\infty) + L_{n-1}\left(\alpha, \alpha^{(n-1)}(+\infty)\right) \\ > \ & \alpha^{(n-1)}(+\infty) \end{aligned}$$

Therefore

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) \geq \alpha^{(n-1)}(+\infty)\right)$$

Applying the same technique it can be proved that

$$u_*^{(n-1)}(+\infty) + L_{n-1}\left(u_*, \delta_\infty\left(u_*^{(n-1)}(+\infty)\right) \leq \beta^{(n-1)}(+\infty)\right).$$

Therefore u_* is a solution of problem (5.1), (5.2). ■

5.4 Infinite nonlinear beam resting on nonuniform elastic foundations

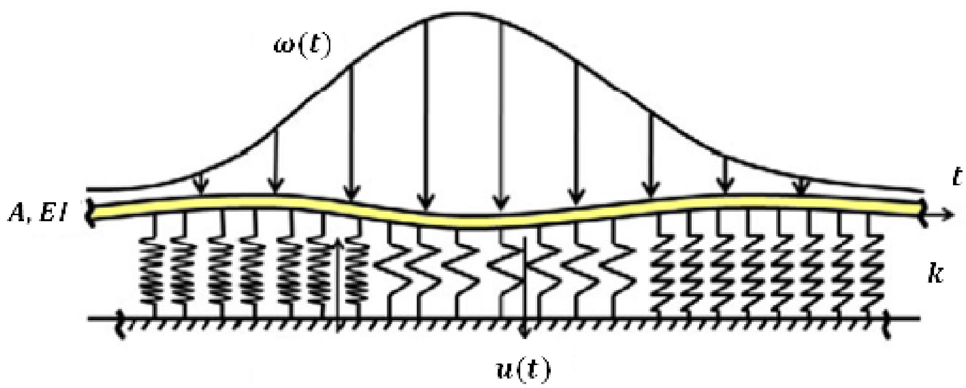
In [49], the author studies the bending of an infinite beam modelled by the nonlinear Bernoulli-Euler-v.Karman differential equation

$$EIu^{(4)}(t) + ku(t) = \frac{3}{2}EA(u'(t))^2 u''(t) + \omega(t), \ t \in \mathbb{R}, \qquad (5.30)$$

together with asymptotic boundary conditions.

This problem analyses moderately deflections of infinite nonlinear beams resting on nonuniform elastic foundations, subject to localized shear forces.

For the readers convenience we recall that E is the Young's modulus, I the mass moment of inertia, $k\ u(t)$ is the spring force upward, with k a spring constant, (neglecting the beam's weight, for simplicity), A the cross-sectional area of the beam and $\omega(t)$ the applied load downward.



Infinite nonlinear beam resting on nonuniform elastic foundations.

Motivated by the above problem, we consider a numerical example composed by the nonlinear fourth order differential equation

$$u^{(4)}(t) = \frac{1}{10}\frac{1}{1+t^4}\left[\frac{|u''(t)|}{1+t}e^{-u'(t)} - k\arctan(u(t)) + 2u'''(t)\right], \qquad (5.31)$$

for $t \in [0,+\infty[$, and $k > 0$.

Assume that $u \in X$, the sum $\sum_{i=1}^{+\infty} \frac{u''(i^2)}{i^4}$, and the integral $\int_0^{+\infty} \frac{|u(t)|}{1+t^6}dt$ are finite. Then, define the functional boundary conditions

$$\begin{cases} u(0) = \min\limits_{t\in[0,+\infty[} u(t) \\ u'(0) = \frac{1}{4}\left\|u'\right\|_1, \\ u''(0) = \frac{1}{7}\sum\limits_{i=1}^{+\infty} \frac{u''(i^2)}{i^4} \\ u'''(+\infty) = \int_0^{+\infty} \frac{|u(t)|}{1+t^6}dt. \end{cases} \qquad (5.32)$$

Indeed, the problem (5.31), (5.32) is a particular case of the initial problem (5.1), (5.2) with $n = 4$,

$$\begin{aligned}
f(t, y_0, y_1, y_2, y_3) &= \frac{1}{10}\frac{1}{1+t^4}\left[\frac{|y_2|}{1+t}e^{-y_1} - k \arctan y_0 + 2y_3\right], \\
L_0\left(w, w_0, w_1, w_2\right) &= \min_{t\in[0,+\infty[} w(t) - w_0, \\
L_1\left(w', w_0, w_1, w_2\right) &= \frac{1}{4}\left\|w'\right\|_1 - w_1, \\
L_2\left(w'', w_0, w_1, w_2\right) &= \frac{1}{7}\sum_{i=1}^{+\infty}\frac{w''(i^2)}{i^4} - w_2, \\
L_3\left(w, w_3\right) &= \int_0^{+\infty}\frac{|w(t)|}{1+t^6}dt - w_3.
\end{aligned}$$

The functions $\alpha, \beta \in X$, defined by $\alpha = \frac{1}{2}$ and $\beta = t^3 + t^2 + t + 1$ are, respectively, lower and upper solution solutions of (5.31), (5.32), for $0 < k \leq 7$, verifying (5.13), (5.14), and (5.15).

The assumptions (H_1), (H_2) and (H_3) are fulfilled and the Nagumo condition is verified in the set

$$E^* = \left\{(t, y_0, y_1, y_2, y_3) \in [0, +\infty[\times\mathbb{R}^4, \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t), i = 0, 1, 2\right\},$$

with

$$\psi(t) = \frac{1}{10}\frac{1}{1+t^4},\ \Phi(|y_3|) = 6 + \frac{7\pi}{2} + 2\left|y_3\right|, \text{ and } \nu = 4.$$

As the Green's function for the homogeneous problem

$$\begin{cases} u^{(4)}(t) = 0 \\ u(0) = 0 \\ u'(0) = 0 \\ u''(0) = 0 \\ u'''(+\infty) = 0 \end{cases}$$

is given by

$$G(t,s) = \begin{cases} \frac{s^3}{6} - \frac{ts^2}{2} + \frac{t^2 s}{2}, 0 \leq s \leq t \\ \\ \frac{t^3}{6}, t \leq s < \infty, \end{cases}$$

then the constants, in (5.17), are

$$\begin{aligned}
\|\alpha\|_X &= \frac{1}{2}, \|\beta\|_X = 6, R = 3, \\
M_0 &= 1, M_1 = 1, M_2 = 2, M_3 = 6, M_\infty = 6, \\
M_0(s) &= \frac{1}{6}, M_1(s) = \frac{1}{2}, M_2(s) = 1, M_3(s) = \frac{1}{2},
\end{aligned}$$

and

$$\varphi_\rho(t) = \frac{1}{10}\frac{1}{1+t^4}\left(\frac{\rho}{1+t} + \frac{7\pi}{2} + 2\rho\right).$$

It can be easily seen, from the corresponding calculus, that condition (5.17) holds for $\rho > 14,52$.

Therefore, by Theorem 5.8, there is a solution u of problem (5.31), (5.32), for $0 < k \leq 7$, such that, for $t \in [0, +\infty[$,

$$\begin{aligned}
\frac{1}{2} &\leq u(t) \leq t^3 + t^2 + t + 1, \\
0 &\leq u'(t) \leq 3t^2 + 2t + 1, \\
0 &\leq u''(t) \leq 6t + 2, \\
-6 &< u'''(t) < 6,
\end{aligned}$$

and

$$0 \leq u'''(+\infty) \leq 6.$$

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