



\mathcal{A} -Variational Principles

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Abstract. Though the \mathcal{A} -quasiconvexity condition has been fully explored since its introduction, no explicit examples of associated variational principles have been considered except in the classical curl-case. Our aim is to propose such a family of problems in the div – curl-situation, and explore the corresponding \mathcal{A} -polyconvexity condition as the main structural assumption to ensure weak lower semicontinuity and existence results.

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1. Introduction

Vector variational problems for an integral functional of the form

$$E(u) = \int_{\Omega} W(x, u(x), \nabla u(x)) \, dx$$

are of paramount importance for hyper-elasticity, where the internal energy density

$$W(x, u, U) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times N} \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \Omega \subset \mathbb{R}^N,$$

has to comply with a bunch of important conditions. In particular, the structural properties of W with respect to its gradient variable U are central to the existence of equilibrium states under typical boundary conditions. Another fundamental motivation to examine this family of variational problems is the study of non-linear systems of PDEs. As a matter of fact, this variational approach is the main method to show existence of weak solutions for such systems beyond the convex case. Check [7] for a discussion on these topics.

In these classic problems, the presence of the gradient variable $U = \nabla u$ stands as a major feature to be understood to the point that it determines the kind of structural assumption to be demanded on the dependence of W with respect to U . This leads to the quasiconvexity condition in the sense of Morrey [24], property that the community is still struggling to understand. This condition is precisely

equivalent to the weak lower semicontinuity of functional $E(u)$ above over usual Sobolev classes of functions.

The main source of quasiconvex functions which are not convex is the class of polyconvex integrands. These can be defined and determined in a very natural way once the collection of weak continuous integrands has been identified. In known existence results for minimizers of functional E under usual boundary conditions in hyper-elasticity, the polyconvexity of W with respect to variable U is always assumed, though it is known that the class of quasiconvex integrands is strictly larger than that of polyconvex densities.

The weak lower semicontinuity property has been treated in a much more general framework in which a constant-rank, linear partial differential operator of the form

$$\mathcal{A}v = \sum_i A_i \frac{\partial v}{\partial x_i} \quad (1.1)$$

is involved. The concept of \mathcal{A} -quasiconvexity is then suitably introduced [10], and shown to be necessary and sufficient [16] for the weak lower semicontinuity of a functional of the form

$$I(v) = \int_{\Omega} W(x, v(x)) \, dx, \quad (1.2)$$

under the differential constraint $\mathcal{A}v = 0$, possibly in addition to suitable boundary restrictions. The importance of such an extension cannot be underestimated as it expands in an unbelievable way the analytical framework. For the particular case $\mathcal{A} = \text{curl}$, we fall back to the classical gradient case.

This new theory is by now very well understood covering the most fundamental developments: Young measures and \mathcal{A} -free measures, relaxation, homogenization, regularity, dynamics, etc, (see [1, 3, 8, 12–15, 17, 18, 20, 21, 23, 28], among others); and yet explicit variational problems under more general differential constraints of the kind $\mathcal{A}v = 0$ have not been systematically pursued, probably due to a lack of such examples of a certain relevance in Analysis or in applications. In the same vein, the natural and straightforward concept of \mathcal{A} -polyconvexity, as far as we can tell, has not been explicitly examined (except recently in [18], and in a different form in [4]), again possibly because of lack of examples where such concept could go beyond plain convexity, and use in a fundamental way to show existence of solutions for such variational problems.

Our goal in this contribution is two-fold. On the one hand, we would like to focus on a particular example of a differential operator of the above kind distinct from the gradient situation (but not far), provide and motivate some examples of variational problems like (1.2) of a certain interest, and resort to \mathcal{A} -polyconvexity as the main hypothesis allowing existence of minimizers. On the other hand, given that the representation form of polyconvexity in terms of compositions of convex and weak continuous functions involves some ambiguity, we have pushed some standard ideas and explicit calculations to gain more insight into this issue.

1.1. \mathcal{A} -Polyconvexity

Suppose a certain non-linear, vector function $A(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is given. This function will play the role of encoding all (or some) independent weak continuous functions for a certain constant-rank, differential operator of the kind given above. Yet for the computations to be performed, it is not important where the vector function A comes from. It is simply assumed to be given. For this reason, we will not make any reference to the operator \mathcal{A} here.

Definition 1. A function $\psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is declared A -polyconvex if there is a function $\Psi(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, convex in the usual sense, such that $\psi(x) = \Psi(x, A(x))$.

It is well-known that the representation of $\psi(x)$ in terms of $\Psi(x, y)$ is non-unique. There is however a canonical representative which is the largest such convex function Ψ . This is in fact standard (check for instance [11]). Consider the function

$$\psi_A(x, y) := \psi(x) \mathbb{1}_{\text{graph}(A)}(x, y) = \begin{cases} \psi(x), & y = A(x) \\ +\infty, & \text{otherwise} \end{cases},$$

for a function $\psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$. More in general, if $\phi(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, then

$$\phi_A(x, y) := \phi(x, y) \mathbb{1}_{\text{graph}(A)}(x, y) = \begin{cases} \phi(x, y), & y = A(x) \\ +\infty, & \text{otherwise} \end{cases}.$$

Proposition 2. A function $\psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is A -polyconvex if and only if

$$\psi(x) \equiv C\psi_A(x, A(x)),$$

where C stands for convexification in the usual sense. Moreover, if $\psi(x) = \phi(x, A(x))$ with ϕ convex, then

$$\phi(x, y) \leq C\psi_A(x, y).$$

The conclusion in this proposition lets us define a canonical representative for A -polyconvex functions.

Definition 3. Let $\psi(x) = \phi(x, A(x))$ be A -polyconvex with $\phi(x, y)$, convex. The function $C\phi_A(x, y)$ is called its canonical representative as a A -polyconvex function.

Though the definition of the convex hull $C\phi_A(x, y)$ is pretty clear, it may be worthwhile to go through explicit calculations to check its explicit form in some distinguished examples like the ones below. Even in very simple situations, calculations turn out to be pretty demanding but they may help in gain insight into the nature of A -polyconvexity. The following are the situations examined here:

1. First scalar case. Take $A(x) = x^3$, and find the canonical representative of $\psi(x) = x^2$.
2. Second scalar case. Take again $A(x) = x^3$, and find the canonical representative for $\psi(x) = x^4 + x^3$ as a A -polyconvex function.
3. Separately convex case. Take $A(x, y) = xy$, and check if $\psi(x, y) = x^2 + y^2$ is its own canonical representative.

4. The true polyconvex case (2×2 -situation). In this case

$$A(\xi) = \det \xi, \quad \psi(\xi) = |\xi|^2, \quad \xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}.$$

Check if ψ is its own canonical representative.

1.2. \mathcal{A} -Polyconvexity

Suppose a constant-rank operator \mathcal{A} as in (1.1) is given.

Definition 4. A continuous, non-linear function $a(v) : \mathbb{R}^m \rightarrow \mathbb{R}$ is weak continuous under the differential constraint $\mathcal{A}v = 0$ if, for a suitable exponent $p \geq 1$,

$$v_j \rightharpoonup v \text{ in } L^p(\Omega, \mathbb{R}^m), \quad \mathcal{A}v_j = \mathcal{A}v = 0,$$

implies

$$a(v_j) \rightharpoonup a(v) \text{ in } \mathcal{D}'(\Omega) \quad (\text{in the sense of distributions}).$$

Assume the components of the vector function $A(v) : \mathbb{R}^m \rightarrow \mathbb{R}^d$ are independent, weakly continuous functions under $\mathcal{A}v = 0$ according to Definition 4. Note how this time the vector function A is precisely coming from a (partial) list of weak continuous functions for the operator \mathcal{A} as indicated in the above definition. It is known that every component of A is a certain polynomial [25].

Definition 5. Let the constant-rank operator \mathcal{A} as in (1.1) be given. An integrand $W(v)$ is said to be \mathcal{A} -polyconvex if it can be written in the form $W(v) = \Psi(A(v))$ where $A(v) : \mathbb{R}^m \rightarrow \mathbb{R}^d$ is a collection of weak continuous functions for \mathcal{A} according to Definition 4, and $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a convex function in the usual sense.

Consider the variational problem

$$\text{Minimize in } v(x) : \quad I(v) = \int_{\Omega} W(x, v(x)) \, dx,$$

under suitable boundary conditions for feasible $v \in L^p(\Omega, \mathbb{R}^m)$ or associated potentials, where $\Omega \subset \mathbb{R}^N$ is a bounded open set with Lipschitz boundary and $W : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a Carathéodory function, in addition to complying with the previous differential constraint. We take for granted that boundary conditions imposed in the problem represent a set of competing fields that is weakly closed so that weak limits preserve such boundary conditions. We will denote by

$$\mathcal{L} \subset \mathcal{L}_{\mathcal{A}}^p = \{v \in L^p(\Omega, \mathbb{R}^m) : \mathcal{A}v = 0\},$$

the set of such competing fields.

The following existence result is but a natural generalization of important existence results in hyperelasticity [2].

Theorem 6. *Suppose, in addition to assumptions just indicated, that the integrand $W(x, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ complies with:*

1. *It satisfies the growth condition*

$$W(x, F) \geq a|F|^p + b,$$

for some $p \geq 2$, $a > 0$, $b \in \mathbb{R}$; and

2. It is \mathcal{A} -polyconvex in the sense of Definition 5 for a.e. $x \in \Omega$, for a family of weak continuous functions $A : \mathbb{R}^m \rightarrow \mathbb{R}^d$ such that

$$|A(F)| \leq C|F|^r, \quad r < p, \quad C > 0.$$

Then the above variational problem admits minimizers $v \in \mathcal{L}$.

The main example corresponds to $\mathcal{A} = \text{curl}$. This is the case for the classic Calculus of Variations and it is very well studied. Another one is the div case when complementary principles are utilized [6]. This particular case has been recently retaken [9]. We would like to focus and motivate the div – curl case. Though variational principles under the div – curl-constraint have been considered sporadically [19], they have not been pursued systematically as far as we can tell.

To be specific, we will stick to the framework in which $\Omega \subset \mathbb{R}^N$, and competing pairs of fields $(v, \nabla u)$ belong to suitable Sobolev spaces under the differential constraint $\text{div } v = 0$ in Ω . The family of integrands we would like to consider are of the form

$$W(v, \nabla u) = \Psi(A(v, \nabla u)), \quad A(v, \lambda) = (v, \lambda, v \cdot \lambda), \quad v, \lambda \in \mathbb{R}^N,$$

for

$$\Psi(v, \lambda, t) = \frac{a|v|^p + b|\lambda|^q}{t^r}, \quad (1.3)$$

if $t > 0$, and $\Psi = +\infty$, if $t \leq 0$, that is to say

$$W(v, \nabla u) = \frac{a|v|^p + b|\nabla u|^q}{(v \cdot \nabla u)^r}, \quad v \cdot \nabla u > 0. \quad (1.4)$$

Densities $a(x)$ and $b(x)$, and exponents p, q, r may vary in suitable ranges, and small coercive perturbation terms are typically added to it to ensure coercivity. More specifically, the following lemma is elementary.

Lemma 7. *If $a, b > 0$ and $\min\{p, q\} \geq r + 1$, the functionals given by (1.4) are div–curl-polyconvex.*

The proof of this lemma is elementary. Simply note that functions in (1.3) are convex under the conditions just given.

In addition to proving Theorem 6, we will justify the form of the integrands in (1.4), show an existence theorem for minimizers (that will be a corollary of Theorem 6), and treat an easy argument to better understand the role of small perturbations added to ensure coerciveness. Needless to say, the div–curl Lemma [26, 30] is central to state some non-standard results in this context. It yields the weak continuity of the inner product $v \cdot \nabla u$ precisely under the differential constraint $\text{div } v = 0$.

2. Examples for \mathcal{A} -Polyconvexity

We cover in this section the calculations involved for the explicit examples listed in Sect. 1.1.

Example 8. In this first scalar case, we take $\psi, A : \mathbb{R} \rightarrow \mathbb{R}$ with $A(x) = x^3$ and $\psi(x) = \phi(x, x^3) = x^2$.

$$\phi_A(x, y) := \phi(x, y)\mathbb{1}_{\text{graph}(x^3)}(x, y) = \begin{cases} x^2, & y = x^3 \\ +\infty, & \text{otherwise} \end{cases},$$

$$C\phi_A(x, y) = \inf_{\alpha_i, x_i, y_i} \left\{ \sum_{i=1}^3 \alpha_i \phi_A(x_i, y_i) : \alpha_i \geq 0, \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i (x_i, y_i) = (x, y) \right\}.$$

We can treat separately the case where only one of the α_i is not zero, because it is trivial to conclude that we get x^2 for points (x, y) with $y = x^3$ and $+\infty$ otherwise. Now, if at least two of the α_i are non-zero, the above equality can be written as

$$\inf_{\alpha_i, x_i} \left\{ \sum_{i=1}^3 \alpha_i x_i^2 : \alpha_i \geq 0, \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i (x_i, x_i^3) = (x, y) \right\}.$$

As a preliminary step, we study the case where exactly one of the α_i is zero. Without loss of generality, we take $\alpha_3 = 0$ and so, for each (x, y) , we have to minimize

$$\alpha x_1^2 + (1 - \alpha)x_2^2$$

subject to

$$\begin{cases} \alpha x_1 + (1 - \alpha)x_2 = x \\ \alpha x_1^3 + (1 - \alpha)x_2^3 = y \\ 0 < \alpha < 1 \end{cases}.$$

From $\alpha x_1 + (1 - \alpha)x_2 = x$ we get $\alpha = \frac{x_2 - x}{x_2 - x_1}$, for $x_2 \neq x_1$ (otherwise we fall again in a trivial case). From here and supposing e.g. $x_1 < x < x_2$, one is lead to minimize

$$-x_1x_2 + x(x_1 + x_2)$$

subject to

$$y + x_1x_2(x_1 + x_2) - x(x_1^2 + x_1x_2 + x_2^2) = 0.$$

The obtained optimality conditions are

$$\begin{cases} -x_2 + x + \lambda(x_2(x_1 + x_2) + x_1x_2 - x(2x_1 + x_2)) = 0 \\ -x_1 + x + \lambda(x_1(x_1 + x_2) + x_1x_2 - x(x_1 + 2x_2)) = 0 \\ y + x_1x_2(x_1 + x_2) - x(x_1^2 + x_1x_2 + x_2^2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (x_2 - x)[-1 + \lambda(2x_1 + x_2)] = 0 \\ (x_1 - x)[-1 + \lambda(x_1 + 2x_2)] = 0, \\ y + x_1x_2(x_1 + x_2) - x(x_1^2 + x_1x_2 + x_2^2) = 0 \end{cases}$$

which have no solutions in its domain. The conclusion is that we have that x^2 is the minimum and it is attained for the points where $y = x^3$ (with $x_1 = x_2 = x$) and that the minimum cannot be attained when $y \neq x^3$.

To prove this, take the case where $y > x^3$ (the other being analogue). It is easy to observe that the infimum cannot be smaller than x^2 , because by hypotheses $\alpha(x_1, x_1^3) + (1 - \alpha)(x_2, x_2^3) = (x, y)$, so in particular $\alpha x_1 + (1 - \alpha)x_2 = x$ and as x^2 is convex, then $\alpha x_1^2 + (1 - \alpha)x_2^2 \geq x^2$. We will prove that the infimum is x^2 and it is obtained when $x_1 \nearrow x$ (and $x_2 \rightarrow +\infty$). To do this, suppose that (x, y) is fixed (but arbitrary), with $y > x^3$. We take three sequences of real numbers, α_n, x_1^n, x_2^n such that $x_1^n < x < x_2^n$, $\alpha_n \in (0, 1)$ and

$$\alpha_n(x_1^n, (x_1^n)^3) + (1 - \alpha_n)(x_2^n, (x_2^n)^3) = (x, y), \quad \forall n \in \mathbb{N}. \tag{2.1}$$

If we take $x_1^n \nearrow x$ then we have $\alpha_n = \frac{x_2^n - x}{x_2^n - x_1^n} \nearrow 1$. From the other equation in (2.1), we have

$$x_2^n = (1 - \alpha_n)^{-\frac{1}{3}} \sqrt[3]{y - \alpha_n(x_1^n)^3}.$$

One must take some care while computing this limit, avoiding possible indeterminations. But as $y - \alpha_n(x_1^n)^3 \rightarrow y - x^3 > 0$ in a monotone way, then

$$\exists p \in \mathbb{N}, \exists c_1, c_2 \in \mathbb{R}^+ : \forall n \geq p \Rightarrow c_2 \geq \sqrt[3]{y - \alpha_n(x_1^n)^3} \geq c_1 > 0.$$

From here, we have

$$x_2^n \rightarrow +\infty$$

and

$$\alpha_n(x_1^n)^2 + (1 - \alpha_n)(x_2^n)^2 = \alpha_n(x_1^n)^2 + (1 - \alpha_n)^{\frac{1}{3}}(y - \alpha_n(x_1^n)^3)^{\frac{2}{3}} \rightarrow x^2.$$

To finish the computations, one has to consider the case with 3 points. But given the solution obtained in the previous step, and again as the function x^2 is convex, the best possible value to obtain is indeed x^2 , and this value cannot be lowered by taking convex combinations of 2 or more points. From here, the conclusion is that in this case, one has

$$C\phi_A(x, y) = x^2,$$

that is, ψ is its own canonical representation as a A -polyconvex function.

We consider again a scalar function.

Example 9. In this second scalar case, we take $\psi, A : \mathbb{R} \rightarrow \mathbb{R}$ with $A(x) = x^3$ and $\psi(x) = \phi(x, x^3) = x^4 + x^3$.

$$\phi_A(x, y) := \phi(x, y) \mathbb{1}_{\text{graph}(x^3)}(x, y) = \begin{cases} x^4 + x^3, & y = x^3 \\ +\infty, & \text{otherwise} \end{cases},$$

$$C\phi_A(x, y) = \inf_{\alpha_i, x_i, y_i} \left\{ \sum_{i=1}^3 \alpha_i \phi_A(x_i, y_i) : \alpha_i \geq 0, \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i(x_i, y_i) = (x, y) \right\}$$

$$= \inf_{\alpha_i, x_i} \left\{ \sum_{i=1}^3 \alpha_i (x_i^4 + x_i^3) : \alpha_i \geq 0, \sum_{i=1}^3 \alpha_i = 1, \sum_{i=1}^3 \alpha_i (x_i, x_i^3) = (x, y) \right\},$$

in which we supposed that at least two of the α_i are non-zero, to avoid trivialities. We will start again by dealing with the situation where one of the α_i is zero. For each (x, y) , we have to minimize

$$\alpha(x_1^4 + x_1^3) + (1 - \alpha)(x_2^4 + x_2^3)$$

subject to

$$\begin{cases} \alpha x_1 + (1 - \alpha)x_2 = x \\ \alpha x_1^3 + (1 - \alpha)x_2^3 = y \\ 0 \leq \alpha \leq 1 \end{cases} .$$

For $x_1 \neq x_2$ we have $\alpha = \frac{x_2 - x}{x_2 - x_1}$, (notice that, as before, if $x_2 = x_1 = x$ then $y = x^3$ and so the minimum is $x^4 + x^3$). From here and supposing without loss of generality, that $x_1 < x < x_2$, one is lead to minimize

$$-x_1 x_2 (x_1^2 + x_1 x_2 + x_2^2 + x_1 + x_2) + x(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 + x_1^2 + x_1 x_2 + x_2^2)$$

subject to

$$y + x_1 x_2 (x_1 + x_2) - x(x_1^2 + x_1 x_2 + x_2^2) = 0.$$

The optimality conditions are

$$\begin{cases} (x - x_2)(3x_1^2 + 2x_1 x_2 + x_2^2 + 2x_1 + x_2) - \lambda(x - x_2)(2x_1 + x_2) = 0 \\ (x - x_1)(x_1^2 + 2x_1 x_2 + 3x_2^2 + x_1 + 2x_2) - \lambda(x - x_1)(x_1 + 2x_2) = 0 \\ y + x_1 x_2 (x_1 + x_2) - x(x_1^2 + x_1 x_2 + x_2^2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (1 - \lambda)(2x_1 + x_2) + (x_1 + x_2)^2 + 2x_1^2 = 0 \\ (1 - \lambda)(x_1 + 2x_2) + (x_1 + x_2)^2 + 2x_2^2 = 0 \\ y + x_1 x_2 (x_1 + x_2) - x(x_1^2 + x_1 x_2 + x_2^2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_i^2 + \frac{(1-\lambda)}{2} x_i - \frac{(1-\lambda)^2}{8} = 0, \quad i = 1, 2 \\ \dots \\ \dots \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{1}{4}(-1 - \sqrt{3})\gamma \\ x_2 = \frac{1}{4}(-1 + \sqrt{3})\gamma \\ \gamma^3 - 6\gamma^2 x + 16y = 0 \end{cases} ,$$

with $\gamma = (1 - \lambda)$ in the last system. As

$$\gamma^3 - 6\gamma^2 x + 16y = 0 \tag{2.2}$$

is a cubic equation in γ , it always has at least one real solution. Substituting they/them into the expressions of x_1, x_2 and then in the function that we are minimizing will lead us to the minimum. Unfortunately it is not possible to have an closed expression for the solution in this case, but at least we can still hope to compute $C\phi_A(x, y)$, for each (x, y) . Nevertheless, some interesting conclusions could be made: for fixed γ , equation (2.2) defines the line that passes through the points (x_1, x_1^3) and (x_2, x_2^3) . This means that the minimum for any (x, y) that belongs to this segment will always be attained at the extreme points (x_1, x_1^3) and (x_2, x_2^3) and, furthermore, $C\phi_A(x, y)$ will be a linear function along this segments which, in particular, this implies that we cannot have $C\phi_A(x, y) = x^4 + y$, if the minimum is attained in the case where one of the α_i is zero.

In order to obtain $C\phi_A(x, y)$, we must analyze the situation when $\alpha_1\alpha_2\alpha_3 \neq 0$. We followed here a different technique: instead of solving the corresponding system of optimality conditions, we will consider two sub-cases, each of which will be reduce to the situation of the previous case. The idea of the reduction came to our minds after a couple of hundreds of numerical computations, where we always obtained optimal solutions with only two distinct points instead of three.

First, we consider the sub-case when of the points $((x_3, x_3^3), (x_4, x_4^3)$ or $(x_5, x_5^3))$ of the candidate to be a minimizer coincides with one of the two points (x_1, x_1^3) , (x_2, x_2^3) of the minimizer obtained from solving (2.2) (see Fig. 1). This case is easy to rule out: suppose that the coinciding point is (x_1, x_1^3) . It is easy to see that

$$\begin{aligned} & \alpha_3(x_3^4 + x_3^3) + \alpha_4(x_4^4 + x_4^3) + \alpha_5(x_5^4 + x_5^3) \\ &= \alpha_3(x_3^4 + x_3^3) + (\alpha_4 + \alpha_5) \left[\frac{\alpha_4}{\alpha_4 + \alpha_5}(x_4^4 + x_4^3) + \frac{\alpha_5}{\alpha_4 + \alpha_5}(x_5^4 + x_5^3) \right] \\ &\leq \alpha_3(x_3^4 + x_3^3) + (\alpha_4 + \alpha_5) [\beta(x_1^4 + x_1^3) + (1 - \beta)(x_2^4 + x_2^3),] \end{aligned}$$

with

$$\frac{\alpha_4}{\alpha_4 + \alpha_5}(x_4, x_4^3) + \frac{\alpha_5}{\alpha_4 + \alpha_5}(x_5, x_5^3) = \beta(x_1, x_1^3) + (1 - \beta)(x_2, x_2^3),$$

for some $\beta \in (0, 1)$, because for any point the lies within the segment $\overline{(x_1, x_1^3), (x_2, x_2^3)}$, the minimum is attained exactly as a linear combination of the images of (x_1, x_1^3) and (x_2, x_2^3) .

In the second and last sub-case, none of the points $(x_3, x_3^3), (x_4, x_4^3), (x_5, x_5^3)$ coincides with any point of the two points $(x_1, x_1^3), (x_2, x_2^3)$, but at least of the three segments $\overline{(x_3, x_3^3), (x_4, x_4^3)}, \overline{(x_4, x_4^3), (x_5, x_5^3)}$ or $\overline{(x_5, x_5^3), (x_3, x_3^3)}$ has to intersect the segment $\overline{(x_1, x_1^3), (x_2, x_2^3)}$ (see Fig. 2). This case is also possible to rule out. Suppose without loss of generality, that $\overline{(x_4, x_4^3), (x_5, x_5^3)}$ intersect the segment $\overline{(x_1, x_1^3), (x_2, x_2^3)}$. Then

$$\begin{aligned} & \alpha_3(x_3^4 + x_3^3) + \alpha_4(x_4^4 + x_4^3) + \alpha_5(x_5^4 + x_5^3) \\ &= \alpha_3(x_3^4 + x_3^3) + (\alpha_5 + \alpha_4) \left[\frac{\alpha_5}{\alpha_5 + \alpha_4}(x_5^4 + x_5^3) + \frac{\alpha_4}{\alpha_5 + \alpha_4}(x_4^4 + x_4^3) \right] \\ &\leq 0(x_3^4 + x_3^3) + (\alpha_5 + \alpha_4)(\beta(x_1^4 + x_1^3) + (1 - \beta)(x_2^4 + x_2^3)) \end{aligned}$$

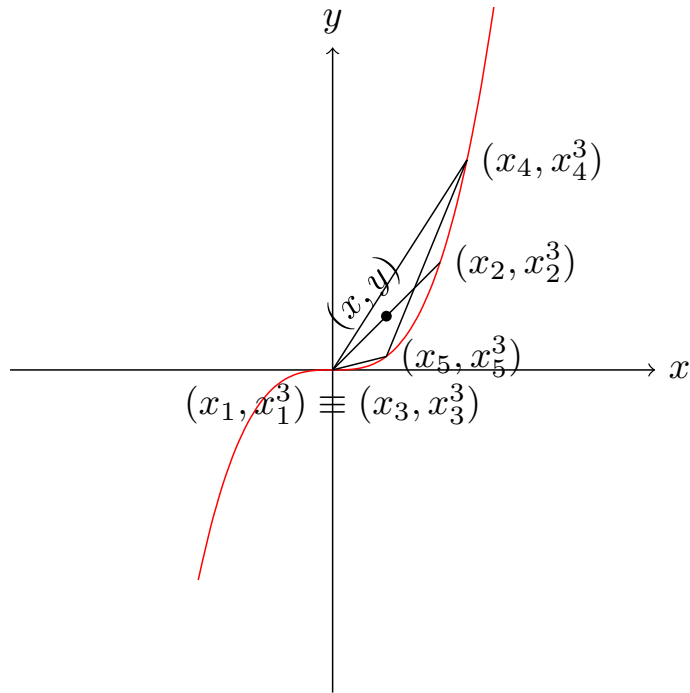


FIGURE 1 Sub-case 1

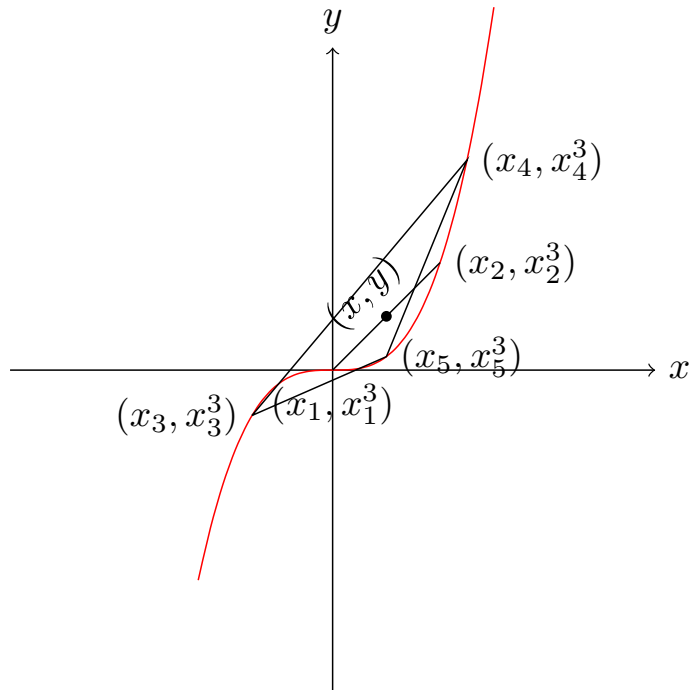


FIGURE 2 Sub-case 2

for some $\beta \in (0, 1)$, otherwise the minimum at the common point of segments $(x_5, x_5^3), (x_3, x_3^3)$ and $(x_1, x_1^3), (x_2, x_2^3)$, obtained with convex combinations of two support points in the graph of x^3 , would be smaller than the minimum at (x, y) , which it is not possible.

Consequently the minimum is attained in the case where one of the α_i is zero and so we cannot have $C\phi_A(x, y) = x^4 + y$ and so ψ is not its own canonical representative. Nevertheless, we can compute $C\phi_A(x, y)$ at each (x, y) by the algorithm described before.

Next we proceed with the separately convex case. Although it is not associated with a constant rank operator, it is a well-known important example which was introduced by Tartar [31] as a model to study rank-one convexity.

Example 10. For $\psi, A : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $A(x, y) = xy$ and $\psi(x, y) = \phi(x, y, xy) = x^2 + y^2$, we have

$$\begin{aligned} \phi_A(x, y, z) &:= \phi(x, y, z)\mathbb{1}_{\text{graph}(xy)}(x, y, z) = \begin{cases} x^2 + y^2, & z = xy \\ +\infty, & \text{otherwise} \end{cases}, \\ C\phi_A(x, y, z) &= \inf_{\alpha_i, x_i, y_i, z_i} \left\{ \sum_{i=1}^4 \alpha_i \phi_A(x_i, y_i, z_i) : \alpha_i \geq 0, \sum_{i=1}^4 \alpha_i = 1, \sum_{i=1}^4 \alpha_i (x_i, y_i, z_i) = (x, y, z) \right\} \\ &= \inf_{\alpha_i, x_i, y_i} \left\{ \sum_{i=1}^4 \alpha_i (x_i^2 + y_i^2) : \alpha_i \geq 0, \sum_{i=1}^4 \alpha_i = 1, \sum_{i=1}^4 \alpha_i (x_i, y_i, x_i y_i) = (x, y, z) \right\}. \end{aligned}$$

Again, we supposed that at least two of the α_i are non-zero, to avoid trivialities, in the last equality. Once again, in order to simplify the minimization problem, we will start again by dealing with the situation where two of the α_i are zero. We take $\alpha_3 = \alpha_4 = 0$, without loss of generality. For each (x, y, z) , we are left to minimize

$$\alpha(x_1^2 + y_1^2) + (1 - \alpha)(x_2^2 + y_2^2)$$

subject to

$$\begin{cases} \alpha x_1 + (1 - \alpha)x_2 = x \\ \alpha y_1 + (1 - \alpha)y_2 = y \\ \alpha x_1 y_1 + (1 - \alpha)x_2 y_2 = z \\ \alpha \in (0, 1) \end{cases}.$$

The optimality conditions are

$$\left\{ \begin{array}{l} 2\alpha x_1 + \lambda_1 \alpha + \lambda_3 \alpha y_1 = 0 \\ 2\alpha y_1 + \lambda_2 \alpha + \lambda_3 \alpha x_1 = 0 \\ 2(1 - \alpha)x_2 + \lambda_1(1 - \alpha) + \lambda_3(1 - \alpha)y_2 = 0 \\ 2(1 - \alpha)y_2 + \lambda_2(1 - \alpha) + \lambda_3(1 - \alpha)x_2 = 0 \\ x_1^2 + y_1^2 - x_2^2 - y_2^2 + \lambda_1(x_1 - x_2) + \lambda_2(y_1 - y_2) + \lambda_3(x_1 y_1 - x_2 y_2) = 0 \\ \alpha x_1 + (1 - \alpha)x_2 = x \\ \alpha y_1 + (1 - \alpha)y_2 = y \\ \alpha x_1 y_1 + (1 - \alpha)x_2 y_2 = z \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} \alpha = \frac{(y_2 - y)^2}{(y_2 - y)^2 + xy - z} \\ \lambda_1 = -2x - 2y \\ \lambda_2 = -2x - 2y \\ \lambda_3 = 2 \\ x_1 = \frac{-xy_2 + z}{-y_2 + y} \\ x_2 = x - y_2 + y \\ y_1 = -\frac{-y^2 + y_2 y - xy + z}{-y_2 + y} \\ y_2 \in \mathbb{R} \setminus \{y\} \end{array} \right. \vee \left\{ \begin{array}{l} \alpha = \frac{(y_2 - y)^2}{(y_2 - y)^2 - xy + z} \\ \lambda_1 = -2x + 2y \\ \lambda_2 = 2x - 2y \\ \lambda_3 = -2 \\ x_1 = \frac{-xy_2 + z}{-y_2 + y} \\ x_2 = x + y_2 - y \\ y_1 = \frac{y^2 - y_2 y - xy + z}{-y_2 + y} \\ y_2 \in \mathbb{R} \setminus \{y\} \end{array} \right.$$

The first solution is always well defined for $z \leq xy$ and the second for $z \geq xy$ (we recall that, in particular, we can take y in the interior of the segment whose extreme points are y_1 and y_2 ; otherwise we interchange the papers of y_2 with x_2). Computing the minimum, one gets

$$C\phi_A(x, y, z) = \max\{(x + y)^2 - 2z, (x - y)^2 + 2z\}$$

$$= \begin{cases} (x + y)^2 - 2z, & z \leq xy \\ (x - y)^2 + 2z, & z \geq xy \end{cases}.$$

Notice that one need not take more than two of the $\alpha_i \neq 0$ in the minimization problem, because the function that we have obtained is already convex and lies below $\phi_A(x, y, z)$. As in the previous case, $\psi(x, y)$ is not its own canonical representative.

Proposition 11. *The canonical representative $C\phi_A(x, y, z)$ of $\psi(x, y) = x^2 + y^2$ as a separately convex function of the two variables x, y , is*

$$C\phi_A(x, y, z) = \begin{cases} (x + y)^2 - 2z, & z \leq xy \\ (x - y)^2 + 2z, & z \geq xy \end{cases}.$$

We now proceed to the typical example of polyconvexity in $\mathbb{R}^{2 \times 2}$ case, see e.g. [11].

Example 12. In this fully vector example, we take

$$\psi, A : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}, \quad \xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \mapsto (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}),$$

with $A(\xi) = \det \xi$ and

$$\psi(\xi) = \phi(\xi, \det \xi) = |\xi|^2 = \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2.$$

We have

$$\begin{aligned} \phi_A(\xi, t) &:= \phi(\xi, t) \mathbb{1}_{\text{graph}(\det \xi)}(\xi, t) = \begin{cases} |\xi|^2, & t = \det \xi \\ +\infty, & \text{otherwise} \end{cases}, \\ C\phi_A(\xi, t) &= \inf_{\alpha_i, \xi_i, t_i} \left\{ \sum_{i=1}^6 \alpha_i \phi_A(\xi_i, t_i) : \alpha_i \geq 0, \sum_{i=1}^6 \alpha_i = 1, \sum_{i=1}^6 \alpha_i (\xi_i, t_i) = (\xi, t) \right\} \\ &= \inf_{\alpha_i, \xi_i} \left\{ \sum_{i=1}^6 \alpha_i |\xi_i|^2 : \alpha_i \geq 0, \sum_{i=1}^6 \alpha_i = 1, \sum_{i=1}^6 \alpha_i (\xi_i, \det \xi_i) = (\xi, t) \right\}, \end{aligned}$$

(in the last equality, we supposed as usual that at least two of the α_i are non-zero) Once again, in order to simplify the minimization problem, we will start again by dealing with the simplest situation, which in here will be the case where four of the α_i are zero, which we take $\alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = 0$, without loss of generality. For each $(\xi, t) \in \mathbb{R}^5$, we have to minimize

$$\alpha |\xi_1|^2 + (1 - \alpha) |\xi_2|^2$$

subject to

$$\begin{cases} \alpha \xi_1 + (1 - \alpha) \xi_2 = \xi \\ \alpha \det \xi_1 + (1 - \alpha) \det \xi_2 = t \\ \alpha \in (0, 1) \end{cases}.$$

The optimality conditions are

$$\left\{ \begin{array}{l} 2\alpha\xi_1 + \alpha\lambda + \lambda_5\alpha \operatorname{adj} \xi_1 = (0, 0, 0, 0) \\ 2(1 - \alpha)\xi_2 + (1 - \alpha)\lambda + \lambda_5(1 - \alpha) \operatorname{adj} \xi_2 = (0, 0, 0, 0) \\ |\xi_1|^2 - |\xi_2|^2 + \langle \lambda, \xi_1 - \xi_2 \rangle + \lambda_5(\det \xi_1 - \det \xi_2) = 0 \quad , \\ \alpha\xi_1 + (1 - \alpha)\xi_2 = \xi \\ \alpha \det \xi_1 + (1 - \alpha) \det \xi_2 = t \end{array} \right.$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $\langle a, b \rangle$ stands for the inner product of the vectors $a, b \in \mathbb{R}^4$ and $\operatorname{adj} \xi_i$ is the matrix (taken as a vector in \mathbb{R}^4) of all the 1×1 minors of matrix ξ_i . The solutions of the above system are (where, in order to drop some of the indexes we will take, when necessary, $\xi = (\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}) = (x, y, z, w)$ and $\xi_i = (\xi_{i,11}, \xi_{i,12}, \xi_{i,21}, \xi_{i,22}) = (x_i, y_i, z_i, w_i)$).

$$\left\{ \begin{array}{l} \alpha = \frac{(x_2-x)^2+(z_2-z)^2}{(x_2-x)^2+(z_2-z)^2-\det \xi+t} \\ \lambda_1 = 2w - 2x \\ \lambda_2 = -2y - 2z \\ \lambda_3 = -2y - 2z \\ \lambda_4 = -2w + 2x \\ \lambda_5 = -2 \\ x_1 = \frac{-x_2yz+x_2^2x+x_2xw-2x_2x^2-x_2t+xz_2^2-2xz_2z+xyz+xz^2-x^2w+x^3+xt}{(x_2-x)^2+(z_2-z)^2} \\ x_2 \in \mathbb{R} \setminus \{x\} \\ y_1 = -\frac{-z_2t+zt-yz_2^2-zxw-x_2^2y-x^2y+2x_2xy+z_2yz+z_2xw}{(x_2-x)^2+(z_2-z)^2} \\ y_2 = y + z - z_2 \\ z_1 = \frac{z_2^2z-z_2yz-2z_2z^2+z_2xw-z_2t+yz^2+z^3+zx_2^2-2zx_2x-zxw+zx^2+zt}{(x_2-x)^2+(z_2-z)^2} \\ z_2 \in \mathbb{R} \setminus \{z\} \\ w_1 = \frac{-x_2t+xt+wx_2^2-x_2yz-x_2xw+xyz+wx_2^2+wz^2-2wz_2z}{(x_2-x)^2+(z_2-z)^2} \\ w_2 = w - x + x_2 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \alpha = \frac{(x_2-x)^2+(z_2-z)^2}{(x_2-x)^2+(z_2-z)^2+\det \xi-t} \\ \lambda_1 = -2w - 2x \\ \lambda_2 = -2y + 2z \\ \lambda_3 = 2y - 2z \\ \lambda_4 = -2w - 2x \\ \lambda_5 = 2 \\ x_1 = -\frac{-x_2yz+x_2xw-x_2t-x_2^2x+2x_2x^2+xyz-x^2w+xt-xz_2^2+2xz_2z-xz^2-x^3}{(x_2-x)^2+(z_2-z)^2} \\ x_2 \in \mathbb{R} \setminus \{x\} \\ y_1 = -\frac{-zxw-yz_2^2-z_2t+zt+z_2yz+z_2xw+2x_2xy-x_2^2y-x^2y}{(x_2-x)^2+(z_2-z)^2} \\ y_2 = y - z + z_2 \\ z_1 = -\frac{-z_2yz+z_2xw-z_2t-z_2^2z+2z_2z^2+yz^2-zxw+zt-z^3-zx_2^2+2zx_2x-zx^2}{(x_2-x)^2+(z_2-z)^2} \\ z_2 \in \mathbb{R} \setminus \{z\} \\ w_1 = \frac{-x_2yz-x_2xw+xyz-x_2t+xt+wx_2^2-2wz_2z+wz_2^2+wz^2}{(x_2-x)^2+(z_2-z)^2} \\ w_2 = x + w - x_2 \end{array} \right.$$

The first solution is well defined for $t \geq \det \xi$ and the second for $t \leq \det \xi$ (notice that $\xi_1, \xi_2 \neq \xi$ so, interchanging variables for which we solve the system of optimality conditions, one can suppose without loss of generality that $(x_2-x)^2+(z_2-z)^2 > 0$). Computing the minimum, one gets

$$\begin{aligned} C\phi_A(\xi, t) &= \max\{|\xi|^2 - 2 \det \xi + 2t, |\xi|^2 + 2 \det \xi - 2t\} \\ &= \begin{cases} |\xi|^2 - 2 \det \xi + 2t, & t \geq \det \xi \\ |\xi|^2 + 2 \det \xi - 2t, & t \leq \det \xi \end{cases} \end{aligned}$$

Notice that, as in the separately convex case, one need not to take more than two of the $\alpha_i \neq 0$ in the minimization problem, because the function that we have obtained is already convex and lies below $\phi_A(\xi, t)$. And so, the answer to the question of whether ψ is its own canonical representative is again negative.

Proposition 13. *The canonical representative $C\phi_A(\xi, t)$ of $\psi(\xi) = |\xi|^2$ as a polyconvex function is*

$$C\phi_A(\xi, t) = \begin{cases} |\xi|^2 - 2 \det \xi + 2t, & t \geq \det \xi \\ |\xi|^2 + 2 \det \xi - 2t, & t \leq \det \xi \end{cases}.$$

3. Existence Theorems

We start by proving our general existence result Theorem 6. The proof is quite standard (see e.g. [29]), and does not require any special ingredient. We restate it here. It refers to the variational problem

$$\text{Minimize in } v(x) \in \mathcal{L} : \quad I(v) = \int_{\Omega} W(x, v(x)) \, dx,$$

where

$$\mathcal{L} \subset \mathcal{L}_{\mathcal{A}}^p = \{v \in L^p(\Omega, \mathbb{R}^m) : \mathcal{A}v = 0\},$$

is a weakly closed subset.

Theorem 14. *Suppose that the integrand $W(x, \cdot) : \mathbb{R}^m \rightarrow \mathbb{R}$ is a Carathéodory integrand and complies with:*

1. *It satisfies the growth condition*

$$W(x, F) \geq a|F|^p + b,$$

for some $p \geq 2$, $a > 0$, $b \in \mathbb{R}$; and

2. *It is \mathcal{A} -polyconvex in the sense of Definition 5 for a.e. $x \in \Omega$, for a family of weak continuous functions $A : \mathbb{R}^m \rightarrow \mathbb{R}^d$ such that*

$$|A(F)| \leq C|F|^r, \quad r < p, \quad C > 0.$$

Then the above variational problem admits minimizers $v \in \mathcal{L}$.

Proof. Suppose that (v_j) is a minimizing sequence. Then, by coercivity of W , there exists $c \in \mathbb{R}^+$ such that

$$|v_j|_{L^p(\Omega, \mathbb{R}^m)} \leq c$$

and so, for an appropriate subsequence ($p \geq 2$), one has

$$v_j \rightharpoonup u \text{ in } L^p(\Omega, \mathbb{R}^m) \tag{3.1}$$

We first prove that I is strongly lower semicontinuous. Suppose that instead of weak convergence in $L^p(\Omega)$ we have strong convergence, that is

$$v_j \rightarrow v \text{ in } L^p(\Omega, \mathbb{R}^m)$$

Then, for an appropriated subsequence, it is true that

$$v_j \rightarrow v \text{ a.e. in } \Omega$$

By hypothesis, $W(x, F) \geq 0$ (eventually by adding some constant) so, applying Fatou's Lemma,

$$I(v) \leq \liminf_{j \rightarrow \infty} I(v_j).$$

This holds for all subsequences, so it follows for the sequence itself. We now prove the (sequentially) weak lower semicontinuity of I . From (3.1), and as we are supposing that $W(x, \cdot)$ is \mathcal{A} -polyconvex in the sense of Definition 5, and due to the upper bound assumed on A , we have

$$A_i(v_j) \rightharpoonup A_i(v) \text{ in } L^q(\Omega, \mathbb{R}), i = 1, \dots, d,$$

for some $q > 1$, and so

$$(v_i, A_1(v_i), \dots, A_d(v_i)) \rightharpoonup (v, A_1(v), \dots, A_d(v))$$

in $L^p(\Omega, \mathbb{R}^m) \times L^q(\Omega, \mathbb{R}^d)$. From Mazur Lemma, we can find a sequence of convex combinations

$$(u_i, U_i^1, \dots, U_i^d) = \sum_{i=j}^{\tau(j)} \alpha_{j,i}(v_i, A_1(v_i), \dots, A_d(v_i)),$$

with $0 \leq \alpha_{j,i} \leq 1$ and $\sum_{i=j}^{\tau(j)} \alpha_{j,i} = 1$ such that

$$(u_i, U_i^1, \dots, U_i^d) \rightarrow (v, A_1(v), \dots, A_d(v)) \tag{3.2}$$

in $L^p(\Omega, \mathbb{R}^m) \times L^q(\Omega, \mathbb{R}^d)$. As $W(x, \cdot)$ is \mathcal{A} -polyconvex, there exists a convex $\Psi(x, \cdot)$ such that $W(x, X) = \Psi(x, A(X))$, and so

$$I(u_i) = \int_{\Omega} \Psi \left(x, \sum_{i=j}^{\tau(j)} \alpha_{j,i}(v_i(x), A_1(v_i(x)), \dots, A_d(v_i(x))) \right) dx \leq \sum_{i=j}^{\tau(j)} \alpha_{j,i} I(v_i). \tag{3.3}$$

Put

$$m := \liminf_{i \rightarrow \infty} I(v_i).$$

Taking a subsequence, we can assume that $I(v_i)$ converges to m , and taking \liminf in both sides of (3.3), we have

$$\liminf_{i \rightarrow \infty} I(u_i) \leq m.$$

On the other hand, by (3.2) and because I is strongly lower semicontinuous,

$$I(v) \leq \liminf_{i \rightarrow \infty} I(u_i) \leq m = \liminf_{i \rightarrow \infty} I(v_i).$$

In particular the boundary conditions are ensured by the weak continuity of the trace and consequently v is a minimizer. □

As indicated in the Introduction, Theorem 6 is a natural extension of important results in classical settings. Our motivation was to look for new interesting explicit examples in non-standard situations. We have been inspired by some variational approaches to deal with the classic Calderón’s problem [5] in inverse 3D-conductivity [22, 27], in which we presume to aim at $v = \gamma \nabla u$ for a certain unknown conductivity coefficient $\gamma(x)$. We intend to recover such unknown coefficient γ by minimizing the functional corresponding to the integrand

$$W(v, \nabla u) = \Psi(A(v, \nabla u)), \quad A(v, \lambda) = (v, \nabla u, v \cdot \lambda), \quad v, \lambda \in \mathbb{R}^3,$$

with

$$\Psi(v, \lambda, t) = \frac{1}{2} \frac{|v|^2 + |\lambda|^2}{t}, \quad t > 0. \tag{3.4}$$

We take $\Psi = +\infty$ whenever $t \leq 0$. We plan to examine this approach more specifically in the future.

An existence theorem, as a corollary of Theorem 6, can then be proved based on *div-curl*-polyconvexity. It can be very easily adapted to cover more general situations for integrands of the form (1.4).

Corollary 15. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, and outer normal vector n to $\partial\Omega$. Consider the class*

$$\mathcal{L}^+ := \left\{ v \in L^2(\Omega, \mathbb{R}^3), \quad u \in W^{1,2}(\Omega, \mathbb{R}), \quad \operatorname{div} v = 0, \quad v \cdot \nabla u > 0 \text{ in } \Omega, \right. \\ \left. (v \cdot n, u) = (v_0, u_0) \text{ on } \partial\Omega \right\}$$

and for each, $\varepsilon > 0$, the functional

$$I_\varepsilon(v, u) = \int_\Omega W_\varepsilon(v, \nabla u) \, dx,$$

where

$$W_\varepsilon(v, \nabla u) = \frac{1}{2} \frac{|v|^2 + |\nabla u|^2}{v \cdot \nabla u} + \varepsilon(|v|^2 + |\nabla u|^2 + |v \cdot \nabla u|^2). \tag{3.5}$$

Then, for each $\varepsilon > 0$, the minimization problem

$$\min_{(v,u) \in \mathcal{L}^+} I_\varepsilon(v, u)$$

has at least one solution.

Proof. Consider an arbitrary, but fixed, $\varepsilon > 0$. W_ε is \mathcal{A} -polyconvex and satisfies the required growth condition. If $(v_j, \nabla u_j, v_j \cdot \nabla u_j)$ is a minimizing sequence, then, possible for a subsequence, coercivity entails

$$v_j \rightharpoonup v \text{ in } L^2(\Omega, \mathbb{R}^3) \tag{3.6}$$

$$\nabla u_j \rightharpoonup \nabla u \text{ in } L^2(\Omega, \mathbb{R}^3) \tag{3.7}$$

$$v_j \cdot \nabla u_j \rightharpoonup X \text{ in } L^2(\Omega, \mathbb{R}) \tag{3.8}$$

From (3.6) and (3.7), remembering that $\operatorname{div} v_j = \operatorname{div} v = 0$, the *div-curl* Lemma [26, 30] gives us

$$v_j \cdot \nabla u_j \rightharpoonup v \cdot \nabla u$$

in the sense of distributions; in fact, in $L^2(\Omega, \mathbb{R})$ because of (3.8). Though elementary to check, it is relevant to stress that the function in (3.4) is convex in the usual sense, and so we are entitled to apply Theorem 6 (or rather its proof) with

$$w = (v, \nabla u), \quad A(w) = A(v, \nabla u) = v \cdot \nabla u, \quad p = 2,$$

□

to conclude that $(v, u) \in \mathcal{L}^+$ is a minimizer.

There is nothing to prevent us from proving a similar existence theorem for a family of integrands like in (1.3) based on Lemma 7.

It is finally interesting to stress that the effect of the small perturbation added in (3.5) to ensure coercivity is quite innocent in the following sense.

Proposition 16. *Let $\{(v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ be a sequence where, for each $\varepsilon > 0$, $(v_\varepsilon, u_\varepsilon)$ is the minimizer of I_ε , that is,*

$$I_\varepsilon(v_\varepsilon, u_\varepsilon) = \min_{u, v \in \mathcal{L}^+} I_\varepsilon(v, u).$$

If we put $I_0(v, u) = I_\varepsilon(v, u)|_{\varepsilon=0}$, then $\{(v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ is minimizing for I_0 .

Proof. We have, by one side,

$$I_\varepsilon(v_\varepsilon, u_\varepsilon) \leq I_\varepsilon(v, u) = I_0(v, u) + \varepsilon|(v, \nabla u, v \cdot \nabla u)|^2, \quad \forall (v, u) \in \mathcal{L}^+, \quad \forall \varepsilon > 0$$

and, on the other side,

$$I_\varepsilon(v_\varepsilon, u_\varepsilon) = I_0(v_\varepsilon, u_\varepsilon) + \varepsilon|(v_\varepsilon, \nabla u_\varepsilon, v_\varepsilon \cdot \nabla u_\varepsilon)|^2 \geq I_0(v_\varepsilon, u_\varepsilon).$$

In particular,

$$I_0(v_\varepsilon, u_\varepsilon) \leq I_0(v, u) + \varepsilon|(v, \nabla u, v \cdot \nabla u)|^2, \quad \forall (v, u) \in \mathcal{L}^+, \quad \forall \varepsilon > 0.$$

Taking limits in both sides of this last inequality leads us to

$$\liminf_{\varepsilon \rightarrow 0} I_0(v_\varepsilon, u_\varepsilon) \leq I_0(v, u), \quad \forall (v, u) \in \mathcal{L}^+,$$

from where it follows that $\{(v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0}$ is a minimizing sequence for I_0 due to the arbitrariness of $(v, u) \in \mathcal{L}^+$. \square

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