Harvesting optimization with stochastic differential equations models: is the optimal enemy of the good?

Nuno M. Brites[a] and Carlos A. Braumann[b]

[a]ISEG – School of Economics and Management, Universidade de Lisboa; REM - Research in Economics and Mathematics, CEMAPRE, Portugal.

[b]Departamento de Matemática, Escola de Ciências e Tecnologia, Universidade de Évora & Centro de Investigação em Matemática e Aplicações, Instituto de Investigação e Formação Avançada, Universidade de Évora, Portugal.

ARTICLE HISTORY

Compiled November 10, 2021

ABSTRACT

We can describe the size evolution of a harvested population in a randomly varying environment using stochastic differential equations. Previously, we have compared the profit performance of four harvesting policies: i) optimal variable effort policy, based on variable effort; ii) optimal penalized variable effort policies, penalized versions based on including an artificial running energy cost on the effort; iii) stepwise policies, staircase versions where the harvesting effort is determined at the beginning of each year (or of each biennium) and kept constant throughout that year (or biennium); iv) constant harvesting effort sustainable policy, based on constant effort. They have different properties, so it is also worth looking at combinations of such policies and studying the single and cross-effects of the amount of penalization, the absence or presence and type of steps, and the restraints on minimum and maximum allowed efforts. Using data based on a real harvested population and considering a logistic growth model, we perform such a comparison study of pure and mixed policies in terms of profit, applicability, and other relevant properties. We end up answering the question: is the optimal enemy of the good?

KEYWORDS

CONTACT Nuno M. Brites. Email: nbrites@iseg.ulisboa.pt

Optimal control; profit optimization; stochastic differential equations; logistic growth; penalized policy; stepwise effort; mixed policies.

1. Introduction

Let $X(t)$ be the size, at time t, of a harvested population under the influence of environmental random fluctuations. The population growth dynamics can be described by the stochastic differential equation (SDE)

$$dX(t) = f(X(t))X(t)dt - qE(t)X(t)dt + \sigma X(t)dW(t), \quad X(0) = x, \tag{1}$$

where $f(X)$ is the *per capita* natural growth rate, $q > 0$ is the catchability coefficient, $E(t) \geq 0$ is the harvesting effort, $H(t) = qE(t)X(t)$ represents the yield from harvesting, $\sigma > 0$ measures the strength of environmental fluctuations, $W(t)$ is a standard Wiener process and $X(0) = x > 0$ is the population size at time 0, which we assume to be known. A particular case of SDE (1), the one most commonly used in the literature and that we will consider here for illustrative purposes, is the logistic model

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t),\ X(0) = x.$$

The parameter $r > 0$ represents the intrinsic growth rate and $K > 0$ is the environment's carrying capacity. In previous work, we discussed the use of a variable effort optimal policy versus a constant effort optimal sustainable policy, considering the logistic model (see [4–6]) and the Gompertz model (see [2,3,6]), to derive harvesting policies based on profit optimization. We have shown that the optimal policy with variable effort, based on the stochastic optimal control theory, has several shortcomings, namely: i) the effort depends on the randomly varying population size, implying the estimation of the population size at each time instant, which is a costly, time consuming and inaccurate task; ii) this policy is inapplicable from the practical point of view due to its frequently and intensely varying effort, and also has frequent periods of no harvesting or harvesting at the maximum possible rate; (iii) this policy poses social problems during the periods of low or no harvesting.

Social problems occur when there is a sizeable or total reduction in harvesting activity, resulting in a partial or total reduction in the number of working hours or the number of vessels, with accompanying effective unemployment of workers. These social problems, besides the personal costs, are often accompanied by extra financial costs like subsidies (unemployment compensations, subsidies to compensate employers' losses), which are usually costs not imputed to the fishing activity but rather to the public sector, possibly covered by taxation revenues. Whether these costs are quantifiable and possibly internalized (and included in the profit function) or not, they should be taken into consideration when designing fishing policies for specific stocks. That is the reason we call attention to them, although they could not in this paper be explicitly quantified and included in the profit function for the fish stock *Hippoglossus hippoglossus* due to data unavailability.

Contrary to the optimal policy, the optimal sustainable policy, based on constant effort and obtained through the theory of stochastic differential equations, has strong advantages: i) leads to sustainable and very easily applicable harvesting policies; ii) the population is driven to a stationary regimen when $t \to +\infty$; iii) does not require knowledge of population size; (iv) poses no social problems. The only disadvantage of this policy is the reduction in profit, which we show to be slight for the models and data considered.

One way to eliminate the social problems posed by the optimal variable effort policy is to incorporate in the model a term that represents an artificial running energy cost designed to reduce the abrupt changes in effort. This was done in [3] considering the Gompertz growth model and by taking several cases with different penalization magnitudes. Unfortunately, the major problem of applicability is maintained since the effort frequently varies across time, although not so intensely. In addition, it is still necessary to keep estimating the population size at each time instant, which is a strong disadvantage.

Also, one can find, for the logistic model ([2]) and the Gompertz model ([4]), a sub-optimal policy, named stepwise policy, where the harvesting effort under the optimal variable effort policy is determined at the beginning of each year period (or at the beginning of a larger period, for instance, two years) and kept constant during that period. The authors showed that this policy is not optimal and still poses some social problems. Still, it has the advantage of being applicable since the changes in the effort are less frequent and compatible with the fishing activity. Furthermore, although it is necessary to keep estimating the fish stock size, it is not necessary to do it so often. Replacing the optimal variable effort policy with a stepwise policy has the advantage of applicability but, at best, considerably reduces the already small profit advantage of the optimal variable effort policy over the optimal constant effort policy. In some cases, the optimal sustainable policy even outperforms this stepwise policy in terms of profit.

This paper is organized as follows. In Section 2 we present these four types of harvesting policies: i) the optimal variable effort policy; ii) the optimal penalized variable effort policies (penalized versions of i)); iii) the stepwise policies (stared sub-optimal versions of i)); iv) the constant harvesting effort sustainable policy. Using data based on a real harvested population and the stochastic logistic growth model, Section 3 refers to the comparisons among these policies and their combinations, together with different combinations of the minimum and maximum efforts allowed. Single and cross effects of the different intervening parameters in these combinations will be studied in terms of profit and in terms of advantages and shortcomings. Finally, some concluding remarks are given in Section 4.

2. Harvesting policies

2.1. Optimal policy with variable effort

To obtain an optimal policy with variable effort based on profit optimization, we follow the stochastic optimal control problem (SOCP) formulated in [1,5,6]. The profit per unit time is defined as $\Pi(t) = R(t) - C(t)$, where $R(t) = (p_1 - p_2 H(t))H(t)$ are the sales revenues per unit time ($p_1 > 0,\ p_2 \geq 0$) and $C(t) = (c_1 + c_2 E(t))E(t)$ represent the fishing costs per unit time ($c_1 > 0,\ c_2 > 0$). So, $\Pi(t) = (p_1 q X(t) - c_1)E(t) - (p_2 q^2 X^2(t) + c_2)E^2(t)$. The SOCP consists in maximizing the present value, i.e. the expected accumulated discounted profit over a finite time interval $[0,T]$:

$$V^* := J^*(x,0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} J(x,0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right],$$

subject to the population dynamics given by the SDE (1), to the control restrictions $0 \leq E_{min} \leq E(t) \leq E_{max} < \infty$ and to a terminal condition $J(X(T),T) = 0$. Note that we use the short notation $\mathbb{E}[\ldots|X(t) = y] = \mathbb{E}_{t,y}[\ldots]$ and that

$$J(y,t) := \mathbb{E}_{t,y}\left[\int_t^T e^{-\delta(\tau-t)}\Pi(\tau)d\tau\right]$$

is, at time t, the expected discounted future profits when the population size at that time is y. The parameter $\delta > 0$ refers to a discount rate accounting for interest rate and cost of opportunity losses and other social rates. In addition, we assume that optimization starts at time $t = 0$ and harvesting continues up to the time horizon T.

The above SOCP can be solved by stochastic dynamic programming theory through Bellman's principle of optimality (see, for instance, [10]). In terms of optimization theory, the problem resorts to finding the effort (i.e., the control) that maximizes the present value $V := J(x,0)$, subject to the growth dynamics given by Eq. (1) and to the constraints on effort and the terminal condition given above. The control value that leads to the maximum V^* will be called the optimal variable effort and is denoted by $E^*(t)$. The Hamilton-Jacobi-Bellman (HJB) equation associated with the SOCP is

$$\begin{aligned} -\frac{\partial J^*(X(t),t)}{\partial t} &= \Big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E^*(t)\Big)E^*(t) - \delta J^*(X(t),t) \\ &\quad + \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE^*(t)\Big)X(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t), \end{aligned}$$

and the optimal variable effort is

$$E^*(t) \;=\; \begin{cases} E_{min}, & \text{if} \quad E^*_{free}(t) < E_{min} \\ E^*_{free}(t), & \text{if} \quad E_{min} \le E^*_{free}(t) \le E_{max} \\ E_{max}, & \text{if} \quad E^*_{free}(t) > E_{max}, \end{cases}$$

where

$$E^*_{free}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) qX(t) - c_1}{2\left(p_2 q^2 X(t)^2 + c_2\right)}$$

is the unconstrained effort (see [5]). The HJB equation is a parabolic PDE and an explicit solution is not available. Hence, to solve it numerically we apply a Crank-Nicolson discretization scheme as in [1–6].

2.2. Optimal variable effort penalized policies

In Section 1 we have mentioned that the optimal variable effort frequently varies across time, having periods of zero/low and maximum/high values. This behaviour, typical in optimal control problems, is not compatible with the logistic of fisheries. In addition, periods of zero or low effort pose social burdens, as explained in Section 1. One way to eliminate this problem is to incorporate in the model a term that represents a running energy cost based on the effort (see, for instance, [12]). This extra cost term is not a real cost, just an artificial way of penalizing the profit values when, at each time instant, the effort takes abrupt changes from a reference effort value, say E_{ref}. One can choose, for instance, E_{ref} as the optimal effort value of the constant effort policy (see Subsection 2.4). In so doing, the resulting optimal penalized policy will not give us the optimal real profit. Still, it will behave better than the optimal variable effort non-penalized policy with milder effort changes.

To implement this approach, the profit per unit time to be optimized is not the real profit (the one presented at the beginning of Section 2.1) but rather an artificial profit $\Pi_\varepsilon(t) := R(t) - C(t) - P_\varepsilon(t)$, with the artificial penalty cost $P_\varepsilon(t) = \varepsilon(E(t) - E_{ref})^2$, where $\varepsilon \geq 0$ is a tuning parameter (representing the penalization magnitude). Thus, we now solve (numerically) the maximization problem

$$\max_{\substack{E(\tau) \\ 0 \leq \tau \leq T}} \mathbb{E}_x \left[\int_0^T e^{-\delta \tau} \left(\Pi(\tau) - \varepsilon (E(\tau) - E_{ref})^2 \right) d\tau \right],$$

where we maximize the artificial expected accumulated discounted profit with an artificial running energy cost, still subject to the population dynamics (1) and to the restrictions on the effort and the terminal condition. Let $E^*_\varepsilon(t)$ be the maximizing effort, which will be called optimal penalized effort. Note, however, that the real expected accumulated discounted profit when we adopt the optimal penalized effort $E^*_\varepsilon(t)$ should use the real costs and so its expression is

$$V^*_\varepsilon \ := \ \mathbb{E}_x \left[\int_0^T e^{-\delta \tau} \left((p_1 q X(t) - c_1) E^*_\varepsilon(t) - (p_2 q^2 X^2(t) + c_2) E^{*^2}_\varepsilon(t) \right) d\tau \right].$$

Considering an artificial energy cost will not eliminate all the major shortcomings of the optimal variable effort policy. The introduction of an energy cost will reduce or even eliminate the social costs arising from the optimal variable effort policy's null or low effort periods. However, it is still necessary to keep estimating the population size at each time instant. In addition, the major problem of the logistic of fisheries will be kept since the effort still varies frequently across time, although not so intensely. Formally, these problems will remain unchanged whatever ε we choose, except for high ε values. The only difference between different choices of ε is not the high frequency of effort changes but the magnitude of such changes. If a low value for ε is chosen, the resulting policy will be similar to the optimal variable effort policy, with almost the same social costs and intense variability in effort between null/low and high values. On the contrary, if a high value for ε is chosen, the resulting policy will still have frequent changes according to population size changes, but the changes will be small in magnitude and the effort will stay close to a constant effort, so that social costs will be eliminated. However, the operability fishing problems remain unchanged. Since the variable effort has values close to the constant effort policy, the profit will be practically indistinguishable from the optimal sustainable constant effort policy profit.

2.3. Sub-optimal policies with variable effort: stepwise policies

To obtain the optimal variable effort policy presented in Subsection 2.1 we need to compute the optimal effort in each of the points of the discretization scheme (as in [1–6]). Since we are dealing with a SOCP without any regularizing penalization, one expects to have frequent and very abrupt changes on the effort, resulting in an inapplicable policy from the point of view of the fishing activity. One way to mitigate this behaviour is to consider sub-optimal policies based on stepwise effort. In a stepwise effort policy, the harvesting effort is determined at the beginning of a time sub-interval with duration p (for instance, 1 or 2 years), and it is kept constant at that value during the whole time sub-interval. So, in this stepwise effort policy, for time t in the period $[lp,(l+1)p[$, we keep the effort $E^*_{step}(t) = E^*(lp)$ constant and equal to the effort of the optimal policy at the beginning of the period. For convenience, we use p as a multiple of the time step Δt used in the numerical computations and Monte Carlo simulations.

Notice that this policy is obviously not optimal. Since it is a stepwise modification of the optimal variable effort policy, it is not even optimal among the stepwise policies. However, it is, as it should, non-anticipative, i.e., it does not use future values of the fish population size, which are unknown at the time of the decision. This policy is applicable since changes in effort, although still abrupt, will occur much less frequently (at most once a year or once every two years).

2.4. Optimal sustainable policy with constant effort

To apply a constant effort policy, one considers a particular case of Eq. (1) with $E(t) \equiv E$, i.e.,

$$dX(t) = f(X(t))X(t)dt - qEX(t)dt + \sigma X(t)dW(t),\ X(0) = x.$$

For the logistic growth model $f(X) = rX(1 - X/K)$, in [7,9] one can find conditions to avoid population extinction, to have a unique solution and to grant a stationary density for the population size, such that we have a stochastic equilibrium (in the sense that the probability distribution of population size converges, in distribution, to a probability distribution with that stationary density). Namely, it is sufficient to have $0 \leq E < \frac{r}{q}\left(1 - \frac{\sigma^2}{2r}\right)$. Such conditions for more general models can be seen in [8].

The state space of X is $(0, +\infty)$ and, if the above conditions on E hold, the boundaries $X = 0$ and $X = +\infty$ are non-attractive. The non-attractiveness of $X = 0$ ensures that there is a zero probability of mathematical extinction. The non-attractiveness of $X = +\infty$ ensures non-explosion and the existence and uniqueness of the solution for all $t > 0$. Thus, it may happen that the transient distribution of $X(t)$ stabilizes and converges, as $t \to +\infty$, to a stationary density. This is indeed the case when the above conditions on E are met. Denoting by X_∞ the random variable with such stationary density, a good approximation of the expected size of the population $\mathbb{E}[X_t]$, for large t, is the expected value of X_∞.

In [4] one can find, for the logistic growth model, the expected value of X_∞ as $\mathbb{E}[X_\infty] = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)$. The sustainable profit per unit time is similar as in the case of the optimal variable effort policy and is defined as $\Pi_\infty := (p_1 q X_\infty - c_1)E - (p_2 q^2 X_\infty^2 + c_2)E^2$. The aim is to determined the optimal sustainable effort E^{**}, i.e. the value of the constant effort E that maximizes the expected sustainable profit per unit time $\mathbb{E}[\Pi_\infty] := (p_1 q \mathbb{E}[X_\infty] - c_1)E - (p_2 q^2 \mathbb{E}[X_\infty^2] + c_2)E^2$. The optimal expected sustainable profit per unit time is then given by

$$\begin{aligned}\mathbb{E}[\Pi_\infty^{**}] = &\left(p_1 q K\left(1 - \frac{qE^{**}}{r} - \frac{\sigma^2}{2r}\right) - c_1\right)E^{**} \\ &- \left(p_2 q^2 K^2\left(1 - \frac{qE^{**}}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE^{**}}{r}\right) + c_2\right)E^{**2}.\end{aligned}$$

3. Comparison of policies

Since the sustainable policy maximizes the expected profit per unit time at the stochastic equilibrium while the others maximize the present value (expected accumulated discounted profit over a finite time horizon $[0, T]$), we need to use a common ground for comparison purposes. For that purpose, we shall always use the present values, for which we need the profits per unit time for the several policies used:

$$\begin{aligned}\Pi^*(t) &= (p_1 q X(t) - c_1)E^*(t) - (p_2 q^2 X^2(t) + c_2){E^*}^2(t),\\ \Pi^{**}(t) &= (p_1 q X(t) - c_1)E^{**} - (p_2 q^2 X^2(t) + c_2){E^{**}}^2,\\ \Pi_i^+(t) &= (p_1 q X(t) - c_1)E_i^+(t) - (p_2 q^2 X^2(t) + c_2){E_i^+}^2(t),\ i = 1, \ldots, 54,\end{aligned}$$

where $\Pi^*(t)$ is the profit per unit time under the optimal variable effort $E^*(t)$ policy, Π^{**} is the profit per unit time under the optimal sustainable policy with constant effort E^{**}, and, for each simulated scenario i, $\Pi_i^+(t)$ is the profit per unit time for the effort $E_i^+(t)$ used in that scenario. Of course, for non-mixed policy scenarios i, namely optimal variable effort policy scenarios, optimal variable effort penalized policy scenarios and stepwise policy scenarios, $E_i^+(t)$ will coincide with the corresponding $E^*(t)$, $E_\varepsilon^*(t)$ and $E_{step}^*(t)$, respectively.

The quantities to be compared, in terms of the expected accumulated discounted profits (present values), for each policy, are:

$$V^* = \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi^*(\tau)d\tau\right], \quad V^{**} = \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi^{**}(\tau)d\tau\right] \quad \text{and}$$

$$V_i^+ = \mathbb{E}_x\left[\int_0^T e^{-\delta\tau}\Pi_i^+(\tau)d\tau\right],\ i = 1, \ldots, 54,$$

respectively for the optimal variable effort policy, the optimal sustainable constant effort policy and the policies $E_i^+(t)$ of the scenarios $i = 1, \ldots, 54$.

To compute the above profit values, we resort to Monte Carlo simulations of the population, based on an Euler scheme and a 1000 sample paths, and obtain the corresponding efforts and profits. We have assumed logistic growth and used realistic biological and economic parameters from the Pacific halibut (*Hippoglossus hippoglossus*) that can be found in [11]. Other parameters were taken from [1]. The full list of parameters is shown in Table 1. For the application of the Crank-Nicolson discretiza-

Table 1. (Adapted from [1]). Values used in the simulations for the base scenario S1 (i=1). For other scenarios, the same parameter values are kept, except for ε, E_{min} and E_{max}, which values are indicated in Table 2. [a]SFU represents the Standardized Fishing Unit. The definition can be found in [11]. y stands for year.

Item	Description	Value	Unit[a]
r	Intrinsic growth rate	0.71	y^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	$\mathrm{SFU}^{-1}\mathrm{y}^{-1}$
σ	Strength of environmental fluctuations	0.2	$\mathrm{y}^{-1/2}$
x	Initial population size	$0.5K$	kg
p_1	Linear price parameter	1.59	$\$\mathrm{kg}^{-1}$
p_2	Quadratic price parameter	$5 \cdot 10^{-9}$	$\$\mathrm{y} \cdot \mathrm{kg}^{-2}$
c_1	Linear cost parameter	$96 \cdot 10^{-6}$	$\$\mathrm{SFU}^{-1}\mathrm{y}^{-1}$
c_2	Quadratic cost parameter	10^{-7}	$\$\mathrm{SFU}^{-2}\mathrm{y}^{-1}$
T	Time horizon	50	y
δ	Discount factor	0.05	y^{-1}
ε	Penalization factor	0	–
E_{min}	Minimum fishing effort	0	SFU
E_{max}	Maximum fishing effort	$0.7r/q$	SFU

tion scheme, applied to solve the HJB equation obtained in Sections 2.1 – 2.3, the time and space grid was designed with $n = 150$ intervals for time (with a time step of $\Delta t = 4$ months) and with $m = 75$ intervals for the state space (with space step $\Delta x = 2.15 \cdot 10^6$ kg and $X_{max} = 2K$).

Figure 1 shows, for scenario $S1$, what will happen when applying the optimal variable effort harvesting policy (left side) and the optimal constant effort sustainable policy (right side), in terms of the evolution, during 50 years, of the population size (top), optimal effort (middle), and profit per unit time (bottom). The black thin lines show one path chosen randomly from the 1000 simulated sample paths, and the thick gray lines show the mean of the 1000 simulated sample paths, which estimates the expected value. Dashed lines show the exact values (only available for the constant effort policy). Looking at what the harvester typically experiences (thin lines in Figure 1), one can see that the two policies behave quite differently. In fact, while the constant effort policy is easily applicable since we apply the same effort E^{**} irrespective of the population size path and of the environmental conditions (middle right, where, of course, we cannot distinguish between the solid, the thin and the dashed lines),

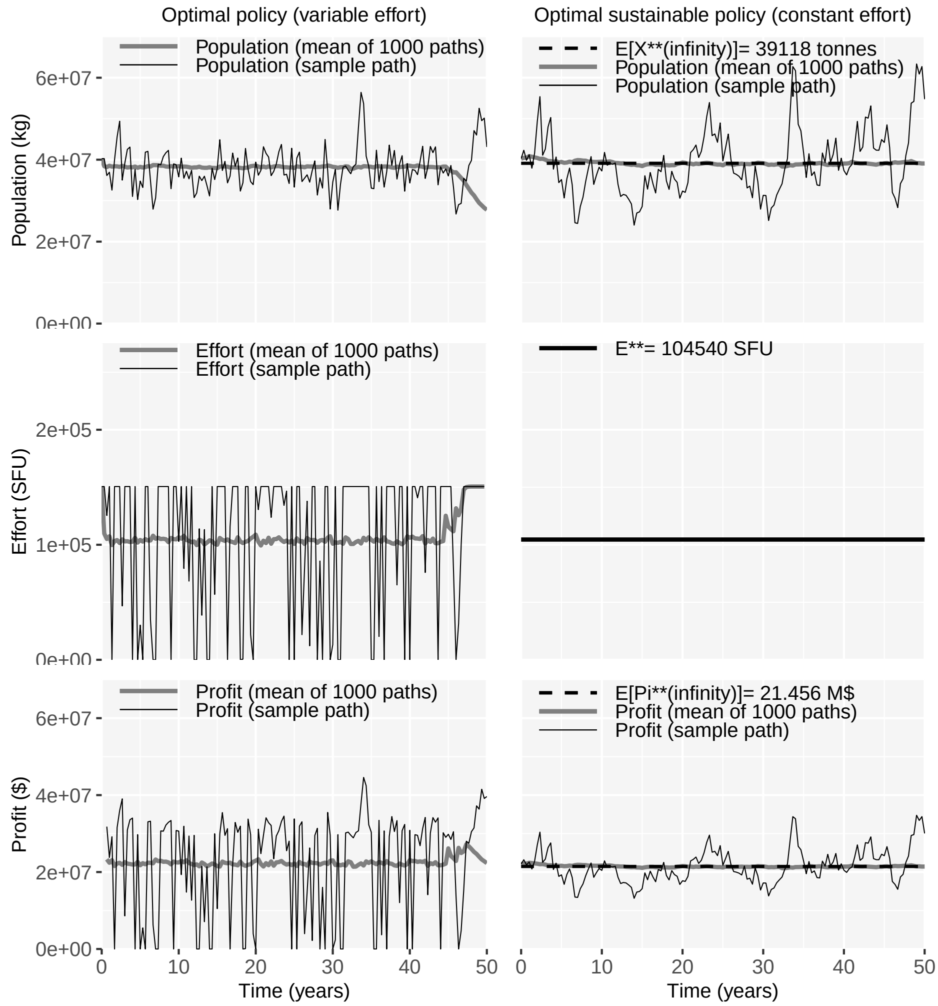

Figure 1. Mean (thick lines) and randomly chosen sample path (thin lines) for the population (first row), the effort (second row) and the profit per unit time (third row) for the optimal variable effort policy (left side) and the optimal sustainable policy (right side). Image adapted from [5].

the optimal policy is inapplicable since the effort $E^*(t)$ changes quite frequently and abruptly (thin line on the middle left). We see that its values depend on time and the fish population size (which is influenced by the random fluctuations of environmental conditions), requiring constant evaluation of the fish stock. Furthermore, it exhibits periods of no or low harvesting, posing social burdens and possible extra costs of unemployment compensation (not considered in our cost structure), and periods of harvesting at the maximum effort E_{max}, which may also involve extra costs (e.g., investment in backup equipment or hiring of extra employees not trained in fishing).

Table 2 shows alternative scenarios $S1$ (Base scenario) to $S54$ corresponding to different combinations of the parameters: E_{min} (0, $0.2r/q$ $0.4r/q$), E_{max} ($0.7r/q$ and

$0.6r/q$), step parameter (no steps, 1-year steps and 2-year steps), penalization level ε (no penalization $\varepsilon = 0$, weak penalization $\varepsilon = 0.0001$ and strong penalization $\varepsilon = 0.01$). For each scenario Si ($i = 1, \dots, 54$), we have computed the corresponding profit present value V_i^+, the relative profit difference Δ_i^* between V_i^+ and $V^* = V_1^+$ and the relative profit difference Δ_i^{**} between V_i^+ and V^{**} (computed for the parameter values of Table 1). Thus, one can easily compare the harvesting policies in terms of profit.

Table 2.: Numerical comparisons for the alternative scenarios $S1$ (Base scenario) to $S54$. The profit value V_i^+ denotes the value of V^+ for scenario $i = 1, \dots, 54$. The relative profit difference between V_i^+ and $V^* = V_1^+$ is given by Δ_i^*, and the relative profit difference between V_i^+ and V^{**} is given by Δ_i^{**}. Profit values are in million dollars.

Scenario i	E_{min}	E_{max}	Step	Penalization	V_i^+	Δ_i^*	Δ_i^{**}
1 (Base)	0	$0.7r/q$	No	$\varepsilon = 0$	413,586	0,00%	4,33%
2				$\varepsilon = 0.0001$	413,384	-0,05%	4,28%
3				$\varepsilon = 0.01$	398,884	-3,55%	0,62%
4			Yes (1-year)	$\varepsilon = 0$	406,716	-1,66%	2,60%
5				$\varepsilon = 0.0001$	410,650	-0,71%	3,59%
6				$\varepsilon = 0.01$	398,645	-3,61%	0,56%
7			Yes (2-years)	$\varepsilon = 0$	390,499	-5,58%	-1,49%
8				$\varepsilon = 0.0001$	403,158	-2,52%	1,70%
9				$\varepsilon = 0.01$	398,336	-3,69%	0,48%
10		$0.6r/q$	No	$\varepsilon = 0$	410,055	-0,85%	3,44%
11				$\varepsilon = 0.0001$	409,887	-0,89%	3,40%
12				$\varepsilon = 0.01$	398,884	-3,55%	0,62%
13			Yes (1-year)	$\varepsilon = 0$	405,444	-1,97%	2,28%
14				$\varepsilon = 0.0001$	407,788	-1,40%	2,87%
15				$\varepsilon = 0.01$	398,645	-3,61%	0,56%
16			Yes (2-years)	$\varepsilon = 0$	393,718	-4,80%	-0,68%

Continued on next page

17				$\varepsilon = 0.0001$	402,248	-2,74%	1,47%
18				$\varepsilon = 0.01$	398,336	-3,69%	0,48%
19	$0.2r/q$	$0.7r/q$	No	$\varepsilon = 0$	413,413	-0,04%	4,29%
20				$\varepsilon = 0.0001$	413,152	-0,10%	4,22%
21				$\varepsilon = 0.01$	398,884	-3,55%	0,62%
22			Yes (1-year)	$\varepsilon = 0$	409,296	-1,04%	3,25%
23				$\varepsilon = 0.0001$	410,407	-0,77%	3,53%
24				$\varepsilon = 0.01$	398,645	-3,61%	0,56%
25			Yes (2-years)	$\varepsilon = 0$	401,794	-2,85%	1,35%
26				$\varepsilon = 0.0001$	404,795	-2,13%	2,11%
27				$\varepsilon = 0.01$	398,336	-3,69%	0,48%
28		$0.6r/q$	No	$\varepsilon = 0$	409,854	-0,90%	3,39%
29				$\varepsilon = 0.0001$	409,681	-0,94%	3,34%
30				$\varepsilon = 0.01$	398,884	-3,55%	0,62 %
31			Yes (1-year)	$\varepsilon = 0$	407,289	-1,52%	2,74%
32				$\varepsilon = 0.0001$	407,675	-1,43%	2,84%
33				$\varepsilon = 0.01$	398,645	-3,61%	0,56%
34			Yes (2-years)	$\varepsilon = 0$	402,297	-2,73%	1,48%
35				$\varepsilon = 0.0001$	403,632	-2,41%	1,82%
36				$\varepsilon = 0.01$	398,336	-3,69%	0,48%
37	$0.4r/q$	$0.7r/q$	No	$\varepsilon = 0$	411,164	-0,59%	3,72%
38				$\varepsilon = 0.0001$	411,098	-0,60%	3,70%
39				$\varepsilon = 0.01$	398,884	-3,55%	0,62%
40			Yes (1-year)	$\varepsilon = 0$	408,654	-1,19%	3,09%
41				$\varepsilon = 0.0001$	408,902	-1,13%	3,15%
42				$\varepsilon = 0.01$	398,645	-3,61%	0,56%
43			Yes (2-years)	$\varepsilon = 0$	404,930	-2,09%	2,15%
44				$\varepsilon = 0.0001$	405,533	-1,95%	2,30%
45				$\varepsilon = 0.01$	398,336	-3,69%	0,48%
46		$0.6r/q$	No	$\varepsilon = 0$	408,090	-1,33%	2,94%

Continued on next page

47		$\varepsilon = 0.0001$	408,047	-1,34%	2,93%
48		$\varepsilon = 0.01$	398,884	-3,55%	0,62%
49	Yes (1-year)	$\varepsilon = 0$	406,622	-1,68%	2,57%
50		$\varepsilon = 0.0001$	406,607	-1,69%	2,57%
51		$\varepsilon = 0.01$	398,645	-3,61%	0,56%
52	Yes (2-years)	$\varepsilon = 0$	404,455	-2,21%	2,03%
53		$\varepsilon = 0.0001$	404,467	-2,20%	2,03%
54		$\varepsilon = 0.01$	398,336	-3,69%	0,48%

We begin by looking at the individual effect of each type of modification on the base scenario (scenario $S1$), which corresponds to $E_{min} = 0$, $E_{max} = 0.7r/q$ and the use of the optimal variable effort policy (without steps and without penalization).

Several comments can be done from Table 2, as follows. Clearly, reducing E_{max} reduces the profit and keeps the inapplicability of the policy and the need to estimate the population size. A reduction on E_{max} might be used, for instance, whenever outer information exists recommending a decrease in harvesting effort due to high fuel price or when there is a suspicion for the population being depleted.

On the other hand, depending on the value, increasing E_{min} reduces or eliminates social problems but does not affect the inapplicability of the policy or the need to estimate the population size. When compared with the $E_{min} = 0$ case, there is a small profit reduction (0.59%) when $E_{min} = 0.4r/q$ and a negligible reduction (0.04%) when $E_{min} = 0.2r/q$.

We now refer to the effect of penalization w.r.t. the optimal policy and the optimal sustainable policy. We use $E_{ref} = E^{**}$ as a reference, but the value of E_{ref} has a small influence (we have considered other E_{ref} close to E^{**} but the profit differences were minimal). The application of a penalized policy keeps the frequent oscillations in the effort, implying the inapplicability of the harvesting policy. Also, the need to estimate population size all the time is kept. Values much smaller than 0.0001 (previous paper) show almost negligible differences w.r.t. the optimal policy. The value 0.0001 attenuates the range of variability of the effort but not very much. The profit is also very similar to the corresponding policy with $\varepsilon = 0$, but it helps reduce social problems.

Considering $\varepsilon = 0.01$ attenuates the variability range, making an effort close to a constant, thus avoiding social problems. However, the profit is close to the profit of the constant effort policy.

Regarding the effect of stepwise policies w.r.t. the optimal policy and the optimal sustainable policy, they eliminate the inapplicability problem and the need to estimate population size is reduced to once a year (or once every two years). Social problems, when they exist, are not attenuated. The profit is reduced, particularly for 2-year steps, when it sometimes becomes lower than the profit of the constant effort policy.

Figures 2 – 4 show one sample path for the effort (the thin lines) obtained by the application of each policy according to Table 2 and represent the effort that would typically be applied if that policy was chosen. The randomly chosen sample path of $W(t)$ (underlying cumulative environmental noise effect) is the same for all scenarios, but the corresponding effort varies according to the scenario chosen. The thick lines correspond to the average of over 1000 simulated paths and are a close estimate of the expected value of the effort.

We now look at cross-effects, i.e., the joint influence of different types of parameter modifications on the base scenario:

- Cross-effects of E_{max} and E_{min}: has no qualitative interaction w.r.t. profit, the changes being almost additive.
- Cross-effects of E_{max} and penalization: for $\varepsilon = 0.01$, since the effort changes in a small neighbourhood of a constant, the value of E_{max} has no effect. For $\varepsilon = 0.0001$, reducing E_{max} reduces the profit almost in the same way as for the $\varepsilon = 0$ case.
- Cross-effects of E_{max} and stepwise: increasing E_{max} increases the profit for no-step policies (as we have seen already) or 1-year step policies, but it has almost no effect or has an opposite effect for 2-year step policies.
- Cross-effects of E_{min} and penalization: for $\varepsilon = 0.01$, since the effort changes in a small neighbourhood of a constant, the value of E_{min} has no effect. For $\varepsilon = 0.0001$, increasing E_{min} reduces the profit almost in the same way as for

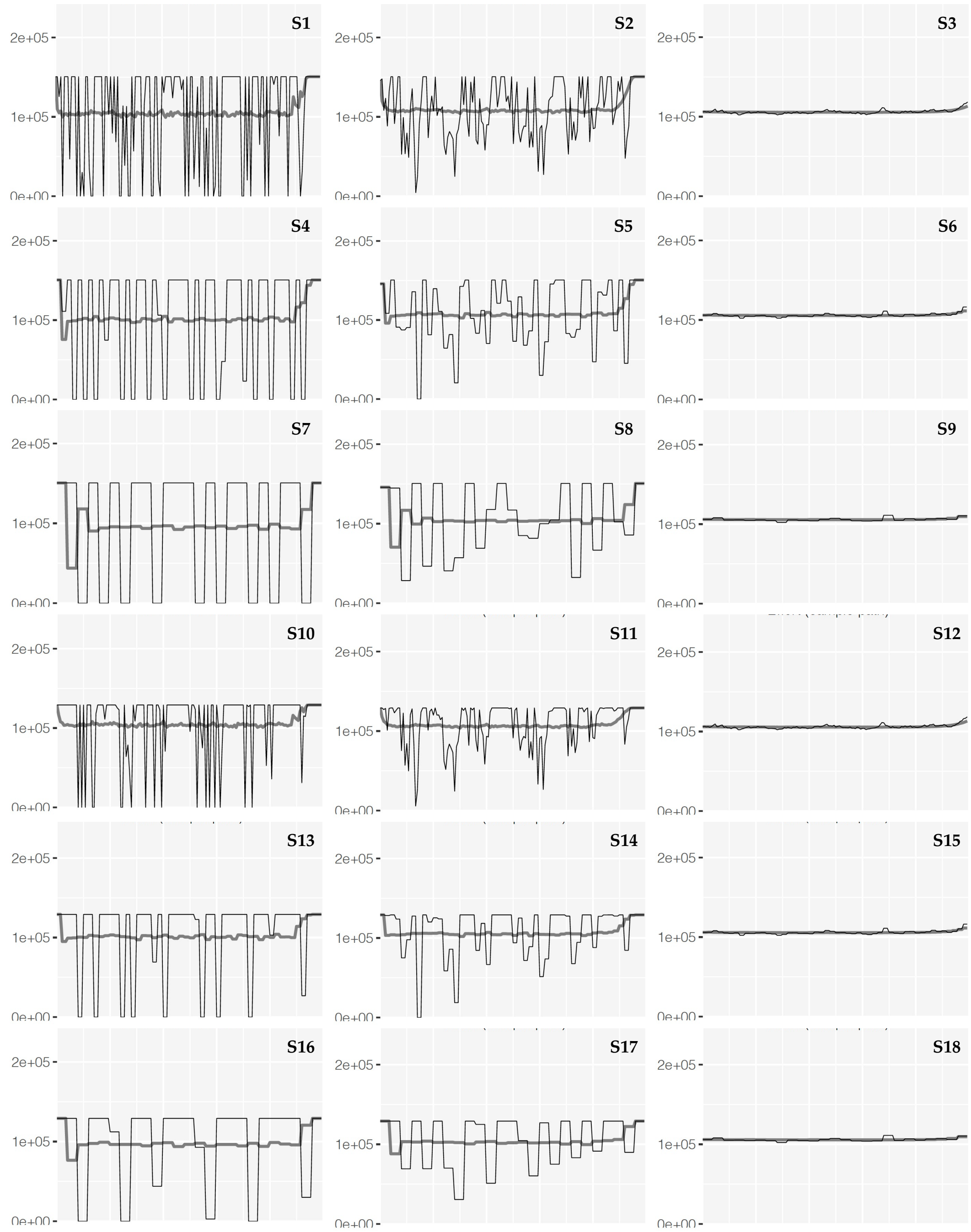

Figure 2. Mean (thick line) and randomly chosen sample path (thin line) for the effort (in SFU units) for each of the scenarios 1 to 18 in Table 2 with $E_{min} = 0$. Each row represents 3 consecutive scenarios differing in penalization level ε (no penalization $\varepsilon = 0$ on the left, $\varepsilon = 0.0001$ in the center and $\varepsilon = 0.01$ on the left), with different lines corresponding to different combinations of the E_{max} and the step parameters. In each figure, the horizontal axis corresponds to time varying from 0 to 50 years.

the $\varepsilon = 0$ case. We have seen that increasing E_{min} has little effect on the profit and reduces or eliminates (depending on the value of E_{min}) the social problems. In contrast, a penalization of $\varepsilon = 0.0001$ has a similar or better result on the

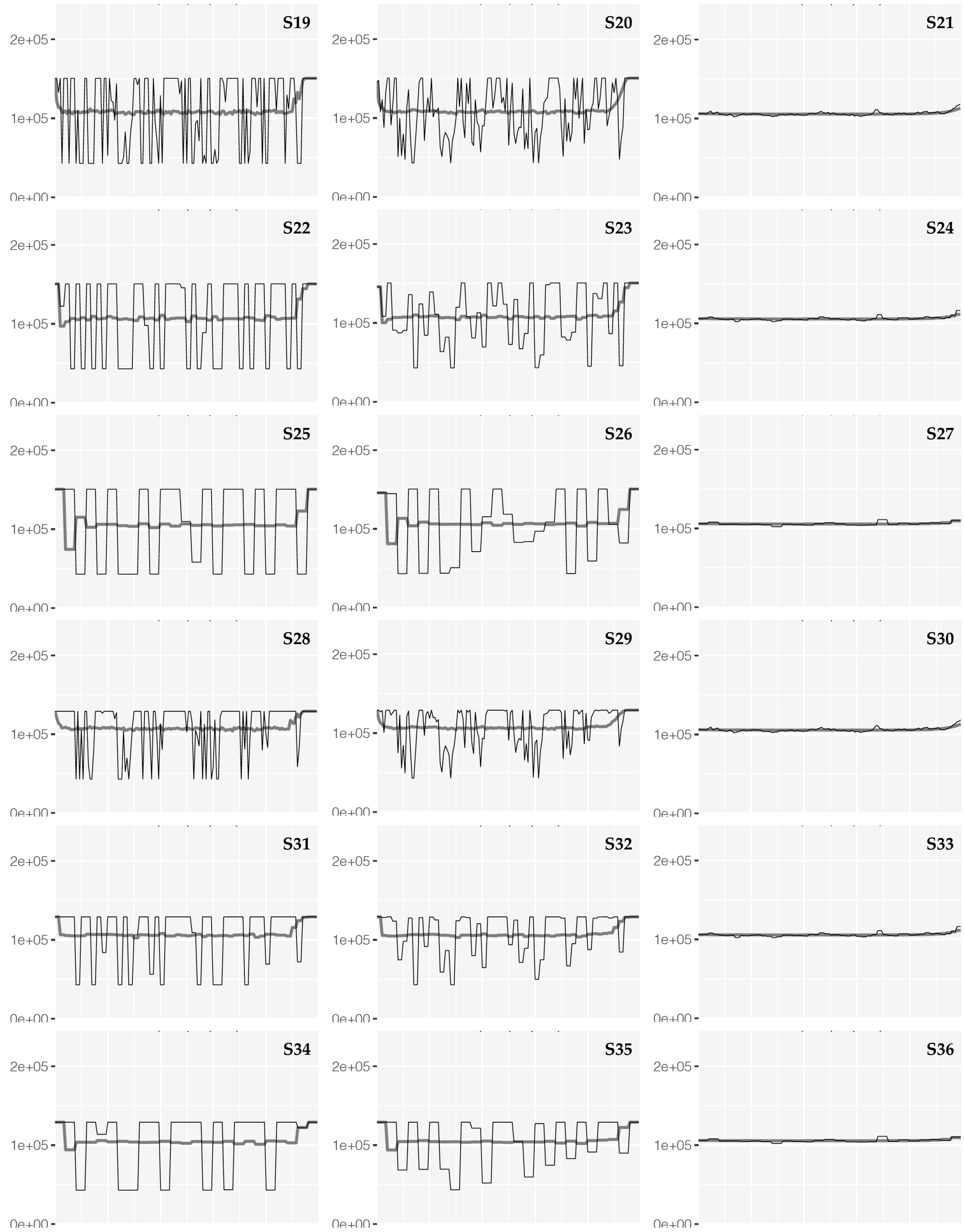

Figure 3. Mean (thick line) and randomly chosen sample path (thin line) for the effort (in SFU units) for each of the scenarios 1 to 18 in Table 2 with $E_{min} = 0.2r/q$. Each row represents 3 consecutive scenarios differing in penalization level ε (no penalization $\varepsilon = 0$ on the left, $\varepsilon = 0.0001$ in the center and $\varepsilon = 0.01$ on the left), with different lines corresponding to different combinations of the E_{max} and the step parameters. In each figure, the horizontal axis corresponds to time varying from 0 to 50 years.

profit reduction but may not attenuate the social problems. A value of $\varepsilon = 0.01$ solves the social problems but has a strong profit reduction. Non-applicability is common to both actions.

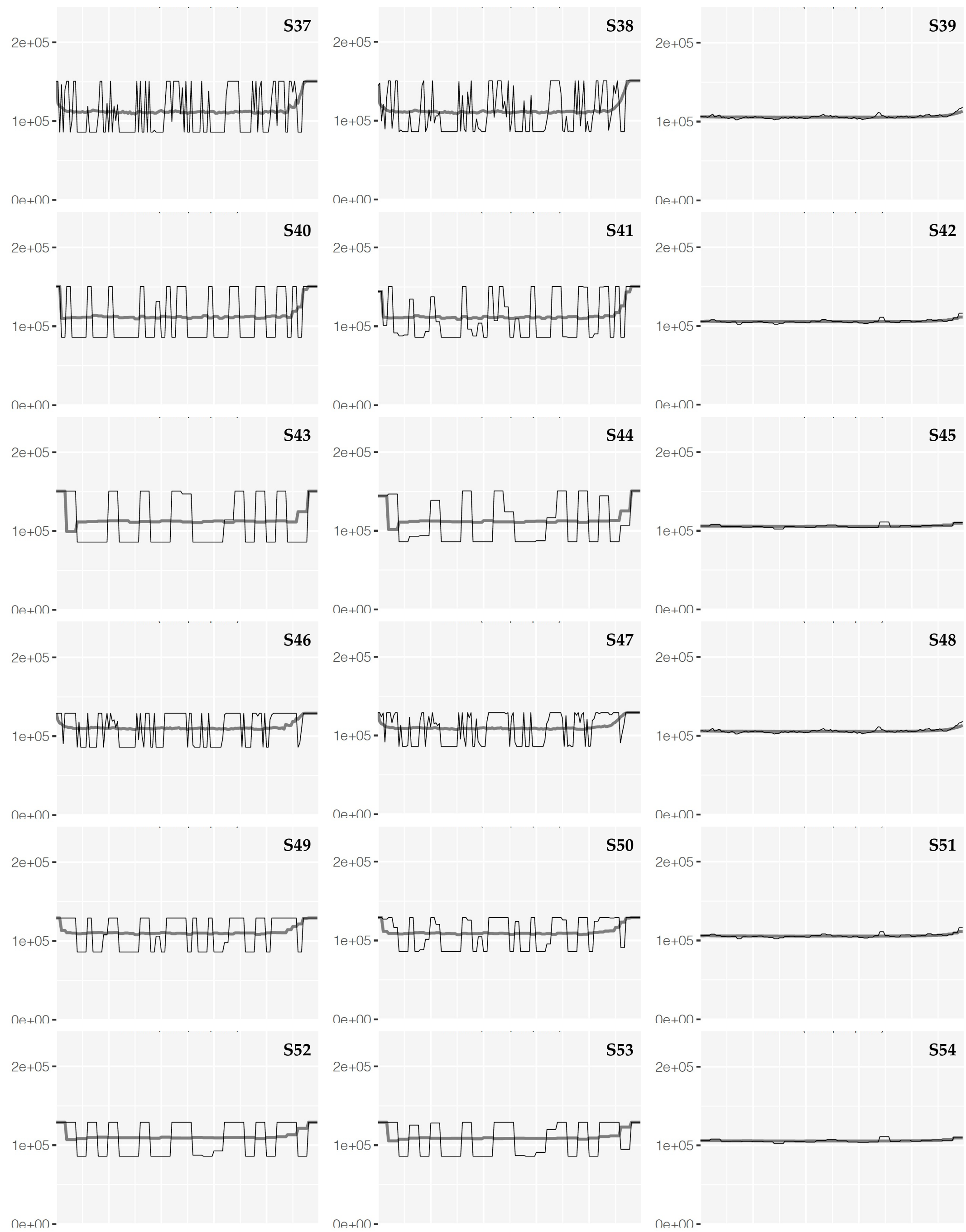

Figure 4. Mean (thick line) and randomly chosen sample path (thin line) for the effort (in SFU units) for each of the scenarios 1 to 18 in Table 2 with $E_{min} = 0.4r/q$. Each row represents 3 consecutive scenarios differing in penalization level ε (no penalization $\varepsilon = 0$ on the left, $\varepsilon = 0.0001$ in the center and $\varepsilon = 0.01$ on the left), with different lines corresponding to different combinations of the E_{max} and the step parameters. In each figure, the horizontal axis corresponds to time varying from 0 to 50 years.

- Cross-effects of E_{min} and stepwise: increasing E_{min} produces a small reduction in the profit for no-step policies (as we have seen already) or 1-year step policies, but it improves the profit for 2-year steps.

- Cross-effects of penalization and stepwise: the joint use of steps and penalization keeps the benefits of steps in terms of applicability and reduction of the need to estimate the population size. When using steps, a large penalization ($\varepsilon = 0.01$) considerably reduces social problems (when they exist) but makes a lower profit. When using steps, a small penalization ($\varepsilon = 0.0001$) improves the profit, strongly for $E_{min} = 0$ and slightly for $E_{min} = 0.2r/q$, but has a negligible effect (sometimes favourable, sometimes unfavourable) for $E = 0.4r/q$. So, for small E_{min}, it somehow smoothens the effect of the step discretization.

Keeping in mind all of the above, we now suggest policies – the best choices – that are a compromise between the profit and the applicability and social issues. A policy with no social problems and no applicability problems that provides a reasonable profit reduction w.r.t. the base optimal policy seems to be the 1-year stepwise penalized policy with $\varepsilon = 0.0001$, $E_{max} = 0.7r/q$ and with an $E_{min} = 0.2r/q$ or $E_{min} = 0.4r/q$ (depending on what is the threshold for relevant social problems to occur). When compared with the base scenario, the profit differences are, respectively, -0.77% and -1.13%. Comparing with the base scenario sustainable policy, the profit gain is, respectively, 3.53% and 3.15%. Of course, both scenarios still require estimating the population size once a year, which cost is not considered in our cost structure. If one has to estimate so often for other reasons, there is no need to consider such costs for optimization purposes. If one does not need to estimate so often for other reasons and those costs are sizeable, one may deduct the extra estimation discounted costs from the overall profit of this policy and compare with the profit of the constant effort policy (which does not require population size estimation). In this case, it might be preferable to use the constant effort policy, which has the advantage of straightforward implementation.

4. Conclusions

This work presents numerical profit comparisons among harvesting policies based on constant, variable (with and without penalization), and stepwise effort for populations living in a randomly varying environment. To obtain the profit values, we have performed 1000 Monte Carlo simulations using a Crank-Nicolson discretization scheme in time and space of the HJB equation and an Euler scheme for the population paths and using realistic biological and economic parameters from the Pacific halibut (*Hippoglossus hippoglossus*) that can be found in [11].

The optimal policy with variable effort has frequent strong changes in effort, including periods of null effort, posing serious applicability problems, producing social burdens and out-of-model costs (such as unemployment compensations), and leading to great instability in the profit earned by the harvester. On the contrary, the optimal sustainable policy based on constant effort does not have these shortcomings, is very easy to implement, and drives the population to a stochastic equilibrium. It also avoids the need for frequent estimation of population size, a difficult and costly process (with costs not considered in the cost structure).

Since the optimal policy is not applicable, we have presented sub-optimal policies, named stepwise policies, based on variable effort but with periods of constant effort. These policies are not optimal but have the advantage of being applicable since the changes in the effort are not so frequent and can be compatible with fishing activity. Furthermore, although we still need to estimate the fish stock size, we do not need to do it so often. The stepwise policies share with the optimal variable effort policy the disadvantage of having periods of null or low fishing and fishing at the highest rate, with the corresponding social implications and out-of-model costs.

One way to eliminate the social problems posed by the optimal variable effort policy is to incorporate in the model a term that represents an artificial running energy cost designed to tame the abrupt changes in effort. This was done by considering several cases with different penalization magnitudes. Unfortunately, the major problem of applicability is maintained since the effort frequently varies across time, although not so intensely. Also, it is still necessary to keep estimating the population size at each instant, which is a strong disadvantage.

We have also compared, for realistic data and the for the stochastic logistic model, these several policies, as well as changes in E_{min} and E_{max} and looked at 54 scenarios resulting from different combinations of these modifications.

Since the optimal policy is not applicable and has social problems, we suggest the best choice policies that are a compromise between the profit and the applicability and social issues. These policies, which presents no social nor applicability problems, incorporate a 1-year stepwise effort, a light penalization level $\varepsilon = 0.0001$, have $E_{max} = 0.7r/q$ and have $E_{min} = 0.2r/q$ or $E_{min} = 0.4r/q$ (depending on the threshold for relevant social burdens). They still require the estimation of the population size once a year. If this involves extra costs (not considered in the profit structure), one should compare their adjusted profits with the profit of the optimal sustainable constant effort policy to decide on the final policy to apply. We can now answer the question: "Is the optimal enemy of the good?" In this case, yes!

Acknowledgements

The authors would like to thank the anonymous reviewers for their helpful comments and suggestions which helped to improve the paper. Nuno M. Brites was partially supported by the Project CEMAPRE/REM - UIDB/05069/2020 - financed by FCT/MCTES through national funds. Carlos A. Braumann belongs to the research centre Centro de Investigação em Matemática e Aplicações, Universidade de Évora, supported by FCT (Fundação para a Ciência e a Tecnologia, Portugal), project UID/04674/2020.

References

[1] Brites NM, Braumann CA. Stochastic differential equations harvesting policies: Allee effects, logistic-like growth and profit optimization. Appl Stochastic Models Bus Ind. 2020;36:825–835.

[2] Brites NM, Braumann CA. Harvesting in a random varying environment: optimal, stepwise and sustainable policies for the Gompertz model. Statistics Opt. Inform. Comput. 2019;7:533–544.

[3] Brites NM, Braumann CA. Fisheries management in randomly varying environments: Comparison of constant, variable and penalized efforts policies for the Gompertz model. Fisheries Research. 2019;216:196–203.

[4] Brites NM, Braumann CA. Harvesting policies with stepwise effort and logistic growth in

a random environment. In: Ventorino E, Aguiar M.A.F., Stollenwek N, Braumann CA, Kooi B, Pugliese A, editors. Dynamical Systems in Biology and Natural Sciences: 2019; Berlin: SEMA SIMAI Springer Series; 21. p. 95–110.

[5] Brites NM, Braumann CA. Fisheries management in random environments: Comparison of harvesting policies for the logistic model. Fisheries Research. 2017;195:238–246.

[6] Brites NM. Stochastic differential equation harvesting models: sustainable policies and profit optimization [PhD thesis]: Universidade de Évora; 2017.

[7] Braumann CA. Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance. New York: John Wiley & Sons, Inc.; 2019.

[8] Braumann CA. Variable effort harvesting models in random environments: generalization to density-dependent noise intensities. Mathematical Biosciences. 2002;177 & 178:229–245.

[9] Braumann CA. Stochastic differential equation models of fisheries in an uncertain world: extinction probabilities, optimal fishing effort, and parameter estimation. In: Capasso V, Grosso E, Paveri-Fontana SL, editors. Mathematics in Biology and Medicine: 1985; Berlin: Springer; p. 201–206.

[10] Bellman R. Dynamic Programming. New Jersey: Princeton University Press; 1957.

[11] Hanson FB, Ryan D. Optimal harvesting with both population and price dynamics. Math. Biosci. 1998;148(2):129–146.

[12] Iolov A, Ditlevsen S, Longtin A. Stochastic optimal control of single neuron spike trains. J. Neural Eng. NIH. 2014;11(4):1–22.