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Equality and Near-Equality in a Nonstandard World

Abstract. In the context of nonstandard analysis, the somewhat vague equality relation of *near-equality* allows us to relate objects that are indistinguishable but not necessarily equal. This relation appears to enable us to better understand certain paradoxes, such as the paradox of Theseus's ship, by identifying identity at a time with identity over a short period of time. With this view in mind, I propose and discuss two mathematical models for this paradox.

Keywords: equality; nonstandard analysis; paradoxes of identity

1. Introduction

A superficial look at equality may create the impression that it is a rather simple matter. By contrast, a more attentive analysis reveals that, in some cases, equality is more involved and subtle than expected. For example, the so-called *paradoxes of identity* [cf., e.g., Deutsch, 2008] have provoked countless philosophical discussions, interpretations and developments which have retained their interest throughout the history of philosophy.

One may be led to believe that the situation is dramatically different in mathematics. Indeed, many dictionaries of mathematical terms either don't bother to define equality (even though that term is widely used throughout) or contain definitions that are far too vague to be useful. For example [Dic, 2003, p. 75] defines being equal as "being the same in some sense determined by context" and Boursin [1983] claims that

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(Reflexivity)	$\forall x(x = x)$
(Symmetry)	$\forall x \forall y(x = y \to y = x)$
(Transitivity)	$\forall x \forall y \forall z(x = y \wedge y = z \to x = z)$

Figure 1. Axioms of equality

> A igualdade é uma relação muito simples, relação de equivalência e de ordem simultaneamente. Escreve-se a igualdade $x = y$ se qualquer propriedade verificada para x se verifica para y.[1]

The latter, not only claims that equality is a facile matter but presupposes the law known as *Leibniz law*, which states that no two distinct things can have exactly the same properties. Nevertheless, this principle is not universally agreed upon, being widely discussed in the literature (see for example [Bender, 2019; Black, 1952]), and may even be false in the quantum domain [Cortes, 1976; French, 2019].

In the following I will consider *identity* to be the binary relation that any object has with itself and only with itself, and that thus fails to hold between distinct objects. As for *equality*, it is generally (but not universally) understood as a reflexive, symmetric and transitive binary relation between terms of a formal language (see Figure 1).

The fact that, from a mathematical perspective, equality is not a trivial matter is made clear by the many different forms that it takes in mathematics. In this paper I argue that equality can be used to understand some paradoxes of identity by considering the notion of near-equality in the context of nonstandard analysis. As argued in Section 3, this notion seems to be especially useful for understanding how equality behaves over time. Indeed, near-equality can be used to identify identity at a time with identity over a "short" period of time. I will look into the perspective that everything is constantly changing, as advocated by Heraclitus, and to the paradox of Theseus's ship. The "short" period of time mentioned above may be outside of the scale of human perception, even infinitesimal, thus spawning infinitesimal differences over infinitesimal periods of time, making clear the relevance of using nonstandard analysis. To this end, the necessary concepts from nonstandard analysis are in

[1] A direct translation would be: "Equality is a very simple relation which is simultaneously an equivalence relation and an order relation. One writes $x = y$ if any property which is verified for x is also verified for y."

fact quite basic and common to almost any presentation of nonstandard analysis. In Section 2, a simple and "economical" version of nonstandard analysis that suffices for the purpose of this paper is presented.

2. Nonstandardness and equality

The advent of nonstandard analysis by Abraham Robinson [1961; 1966] allowed for a first formal consistent treatment of infinitesimals. In spite of some criticism (notably by George Berkeley [2005]), infinitesimals were crucial for the intuitive development of mathematical knowledge by authors such as Archimedes of Syracuse, Simon Stevin, Pierre de Fermat, Gottfried Leibniz, Leonard Euler and even Augustin-Louis Cauchy, to name but a few [Bair et al., 2018; Bair et al., 2020; Katz and Sherry, 2013].[2] The key factor for the arguments in this paper is the fact that the existence of infinitesimals allows us to define different orders of magnitude and a notion of near-equality. For that purpose, a very "economical" version of Nonstandard Analysis due to Edward Nelson [1987, Chapter 4], dubbed ENA^- in [Dinis and van den Berg, 2019, Chapter 1], for Elementary Nonstandard Analysis, is sufficient. Using the notation of the latter reference, I will describe that system below. Nevertheless, almost any other theory that incorporates infinitesimals can be used instead so that the arguments below still make sense [cf., e.g., di Nasso, 1999; Fletcher et al., 2017]. In spite of that, there are important philosophical differences in the different approaches to nonstandard analysis. For example, let us consider Robinson's approach versus Nelson's approach towards the real numbers. In Robinson's approach, there are two sets of real numbers, the usual real numbers (denoted R) and the *hyperreal* numbers (denoted $^*\mathsf{R}$), which include the usual reals as well as the *new* nonstandard elements. In Nelson's approach there is only one set of real numbers that has — and always had — nonstandard elements. The nonstandard elements are now "accessible" because we are considering a richer language. The major advantage of the Nelson's approach

[2] The formal *existence* of infinitesimals is not really required by these authors. So, one may wish to think of infinitesimals as "useful fictions", as for example Leibniz himself admitted in a letter to Samuel Masson [Leibniz, 1989, p. 230]. Hendrik Bos claims that "[Leibniz] had to treat the infinitesimals as 'fictions' which need not correspond to actually existing quantities, but which nevertheless can be used in the analysis of problems" [Bos, 1974, pp. 54–55]. A more detailed discussion of this matter can be found in [Bair et al., 2018].

is its familiarity: one can work there almost as if one were in ZFC. Furthermore, the weaker version presented below is characterized by its simplicity, indeed it is aimed to be "[...] readily available to anyone who can add, multiply and reason" [Nelson, 1987, p. vii].

Classical mathematics is usually formalized by the axioms ZFC of Zermelo-Fraenkel Set Theory with the Axiom of Choice, in a language which only contains one undefined non-logical symbol, '$\in$', for set membership [cf., e.g., Kunen, 1980; Potter, 2004]. The theory ENA^- (cf. Figure 2) is ruled by a simple set of axioms after adding to the language a new predicate 'st'. One should read $\mathrm{st}(x)$ as 'x is standard'. Formulas which involve the predicate 'st' are called *external* and formulas which do not involve that predicate, i.e. formulas in the language of classical mathematics, are called *internal*. The first two axioms state that the usual natural numbers are standard[3] and the third axiom postulates the existence of nonstandard natural numbers. The fourth and final axiom is indeed an axiom scheme called *external induction*. It is a form of induction that allows to conclude that some property is true for all standard natural numbers given that it is true for zero and that whenever it is true for some standard n, then it is also true for its successor $n+1$. Since ENA^- is a conservative extension of classical mathematics, the usual form of induction is still valid, albeit only for internal properties. To see why such restriction is required, consider the formula $\Phi(n) :\equiv \mathrm{st}(n)$. If one could apply internal induction to Φ the conclusion would be that every natural number is standard, in contradiction to the third axiom.

The theory ENA^- allows us to define different orders of magnitude in the following way. A real number x is said to be *infinitesimal*[4] if its absolute value is smaller than the inverse of any positive standard natural number; *limited*, if it is, in absolute value, bounded by some standard natural number and *unlimited*, or *infinitely large* if x is not limited. Finally, the number x is said to be *appreciable* if it is limited but not infinitesimal. Two real numbers x, y whose distance is infinitesimal are said to be *infinitely close* or *near equal* and we write $x \simeq y$. As it

[3] Georges Reeb called "naive" the natural numbers which can be obtained from zero by the successive addition of one, and then claimed that not all natural numbers are naive. "Les entiers naïfs ne remplissent pas N" [Diener and Reeb, 1989]. Indeed, the first two axioms imply that the nonstandard natural numbers cannot be accessed, in a naive way, by the successor function and can, at least in that sense, be considered *ideal* elements.

[4] Such a real number, if different from zero, is necessarily nonstandard.

(1)	$\mathrm{st}(0)$
(2)	$\forall n \in \mathsf{N}(\mathrm{st}(n) \to \mathrm{st}(n+1))$
(3)	$\exists \omega(\neg \mathrm{st}(\omega))$
(4)	$(\Phi(0) \wedge \forall^{\mathrm{st}} n(\Phi(n) \to \Phi(n+1))) \to \forall^{\mathrm{st}} n\, \Phi(n)$,
	where Φ is an arbitrary formula (internal or external) and $\forall^{\mathrm{st}} n\, \Phi(n)$ is an abbreviation of $\forall n(\mathrm{st}(n) \to \Phi(n))$.

Figure 2. The axioms of ENA^-

turns out, given a real number x, there is only one standard real number which is infinitely close to x. Such number is called the *shadow* (or *standard part*[5]) of x and is denoted ${}^\circ(x)$. It is not difficult to see that ${}^\circ(-x) = -{}^\circ(x)$ and ${}^\circ(y+x) = {}^\circ(x) + {}^\circ(y)$.

It is also a simple exercise to check that the Leibniz's rules hold [Callot, 1992; Dinis and van den Berg, 2017; Lutz, 1987]. Leibniz's rules are rules for orders of magnitude. For example, the sum and product of infinitesimals are infinitesimal, the sum and product of appreciable numbers are appreciable, the product of an infinitely large number with an appreciable number is infinitely large, etc. These were the intuitive rules, used by Leibniz and others, that it is possible to formalize and prove rigorously with nonstandard analysis. Indeed, nonstandard analysis allows us to treat rigorously concepts such as 'big' and 'small' as well as the fact that 'small' values can somewhat be neglected, unless there are "too many" things to neglect. This art of neglecting is quite common in physics and in some heuristic methods in mathematics, namely in asymptotics.[6] This approach allows one to take into account the important measurements or factors and neglect the not so important ones.

The notion of near-equality can be seen as a flexible notion of identity, imbued with a certain vagueness: two objects are only different when one can tell them apart, i.e. when they are appreciably different. This means that if the difference between two objects is infinitesimal, then

[5] The standard part map can be seen as a formalization of heuristic principles such as Leibniz's *transcendental law of homogeneity* [Katz and Sherry, 2013, Section 5.3] and of Fermat's *adequality* [Bascelli et al., 2014, p. 854].

[6] The interested reader may consult [Dinis and van den Berg, 2019; van den Berg, 1987] for approaches using nonstandard analysis which can be seen as a formalization of Johannes Van der Corput's *ars negligendi* [van der Corput, 1959/1960].

in practice one is not able to tell them apart and therefore one regards, for all practical purposes, the two objects as being the *same* object. Nevertheless, Leibniz's rules ensure that near-equality still satisfies the axioms for equality. Indeed, reflexivity holds because 0 is infinitesimal; symmetry follows from the fact that if an element is infinitesimal then its symmetric is also infinitesimal and transitivity follows from the fact that the sum of two infinitesimals is still infinitesimal.

An important remark should be made here. Since, near-equality is an external relation one cannot use transitivity indefinitely. Indeed, consider the sequence (u_n) defined by $u_n = \frac{n}{\nu}$, where ν is infinitely large. Observe that $\frac{1}{\nu}$ is infinitesimal. So, one obtains the following sequence of near-equalities:

$$0 \simeq u_1, \quad u_1 \simeq u_2, \quad u_2 \simeq u_3, \quad \ldots, \quad u_{n-1} \simeq u_n, \quad \ldots \tag{1}$$

For standard n one may conclude that $u_0 \simeq u_n$ but for nonstandard n the same conclusion is not acceptable. For example, one could take $n = \nu$, thus deriving that $u_n = 1$ which is clearly not near-equal to 0.

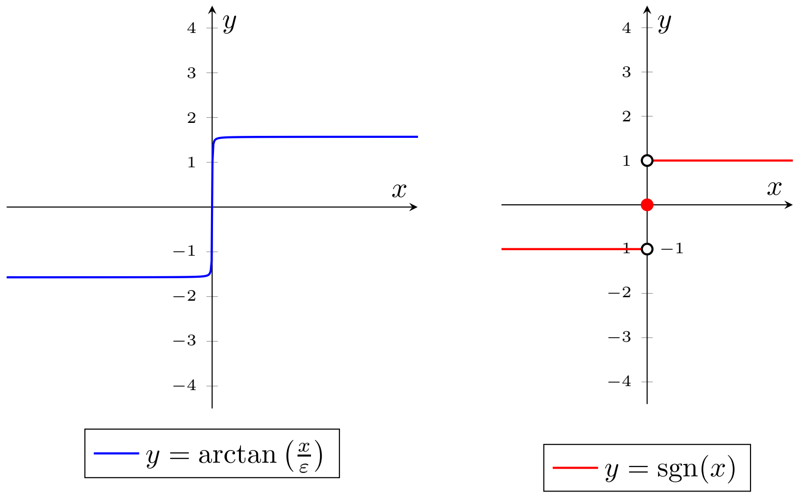
A near-equal version of extensionality is not valid in general as in some cases there is some chaotic behaviour: near-equal inputs produce outputs that are very far apart. Nevertheless, there is a sort of extensionality related with the so-called S-notions. Consider, as an illustrative example, the notion of an S-*continuous* function [Diener and Diener, 1995, Section 1.3], which states that whenever x is near-equal to y, the respective images must also be near-equal. S-continuous functions are functions that appear to be continuous (even if they are not!). The following two examples are instructive. Let ε be a non-zero infinitesimal. Consider the real-valued functions f and g defined respectively by

$$f(x) := \begin{cases} 0 & x \text{ is rational} \\ \varepsilon & \text{otherwise} \end{cases} \qquad\qquad g(x) := \arctan\left(\frac{x}{\varepsilon}\right).$$

The function f is S-continuous in spite of being discontinuous at every point and the function g is continuous but not S-continuous. Around the origin, the function g grows "too fast" and, to the "naked eye" (after rescaling by a factor of $\frac{\pi}{2}$) it seems like the sign function sgn, defined by

$$\operatorname{sgn}(x) := \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases},$$

which is not continuous at the origin.



3. Applications of near-equality

In this section, the notion of near-equality is applied to some concrete case studies. The main goal is to question certain intriguing kinds of reasoning involving equality, namely the proof that $0.999\ldots = 1$, the problem of continuous change and the paradox of Theseus's ship, and to propose nonstandard analysis as a possible framework to better understand these kinds of reasoning.

3.1. From 0.999... to 1

Does $0.999\ldots$ equal 1? A classical proof of the equality of these two representations goes as follows. Let $x = 0.999\ldots$. Then $10x = 9.999\ldots$ which entails $9x = 10x - x = 9.999\ldots - 0.999\ldots = 9$ and therefore x must be equal to 1. Even in the presence of a proof, this is still a puzzling fact. Intuitively, $0.999\ldots$ seems to always be a little shy of 1, nevertheless these are two representations of the same number. The discomfort of not having a unique representation in decimal expansion may be overcome, in a practical way, by banning every decimal expansion ending with an infinite number of 9's. This indeed results in a unique representation in spite of the rather harsh restriction. Other authors take their criticism to another level and propose that indeed $0.999\ldots$ and 1 are not the same number [Bedürftig and Murawski, 2018] (see also [Katz and Katz, 2010a,b]). This is made possible using near-equality and the notion of a shadow. Let us assume, that $0.999\ldots$ is infinitely close but not equal to 1. Then there must exist an $\varepsilon \simeq 0$ separating the two numbers, i.e. such that $0.999\ldots + \varepsilon = 1$. The latter is clearly equivalent to $0.999\ldots = 1 - \varepsilon$. Going back to "standard" mathematics via the notion of shadow one obtains

$$^\circ(0.999\ldots) = {}^\circ(1 - \varepsilon) = {}^\circ(1) - {}^\circ(\varepsilon) = 1 - 0 = 1.$$

This means that standard methods cannot distinguish the difference between $0.999\ldots$ and 1 but a nonstandard framework offers that possibility, mostly due to the fact that one is working with an enriched language. In this way, nonstandard analysis acts as a "microscope" on the real line allowing one to distinguish elements that before were impossible to tell apart [cf., e.g., Kusraev and Kutateladze, 1994, p. 26].

3.2. Heraclitus and continuous change

The pre-Socratic philosopher Heraclitus of Ephesus claimed that all things were constantly flowing (πάντα ῥεῖ) in such a way that one could not bathe oneself in the same river twice as both the person and the river had changed in the meantime. Moreover, he added that "We both step and do not step into the same rivers; we both are and are not" [fragment B49a, Barnes, 2002, p. 49]. The full meaning of this sentence is not entirely clear but it seems to advocate that this constant flux implies some sort of inner contradiction. One can find something similar in everyday life, such as when people say things like "I'm not quite myself today" without necessarily intending a contradiction.[7] Is then reflexivity of equality at stake? At an atomic level an individual is constantly changing; however, one still thinks of oneself at a non-atomic (psychological) level as the same person. Can the idea of constant change be reconciled with Wittgenstein's "meaning as use" [1953, Section 43]? Or with Quine's argument that terms can name the same thing but differ in meaning [1951]? Moreover, as seen in more detail in Subsection 3.3, the

[7] Also related to this conception are the notions of personal identity: being the same man is not the same as being the same person [Locke, 2015, Chapter XXVII] and of relative identity [cf., e.g., Noonan, 2017].

transitivity of equality is not entirely clear and may also be questioned since it seems to be related to vagueness. Of course, one could use the distinction between equality and identity to argue that being the same person is not the same as being the same individual. But one is then forced to accept that there is a certain vagueness in the notion of 'being the same person'. This vagueness is deeply related with paradoxes which arise from the inability of taking into account the different orders of magnitude, interpreted as a whole. So, and more generally, is equality (even in a small degree) a vague relation?

Given two infinitely close instances of time, sometimes one can argue that very small changes have occurred, in which case the changes can be modelled by an S-continuous function. If, on the other hand, the changes are drastic, for example when the so-called *butterfly effect* occurs, such a model is no longer adequate. However, even in the latter case, models using nonstandard analysis seem adequate as they permit to take into account several orders of magnitude. Indeed, one can (formally) create a mathematical model for which infinitesimal differences in time produce effects which are appreciable or infinitely large.

The fact that near-equality is used here is crucial in several ways. Firstly, changes over time can be explained by the fact that appreciable changes are a consequence of an accumulation of infinitesimal changes. Secondly, using the notion of a shadow one can argue that at a human scale there is no perceptible change. As such, it is possible to say that the river, at two very close instances in time, is near-equal but not equal and so, at least in this sense, it is and it is not the same. Finally, near-equality is an external relation. As such, one cannot use induction, nor transitivity an unlimited number of times. This is further explored in the next subsection.

3.3. Two models of the paradox of Theseus's ship

The well-known paradox of Theseus's ship [Clark, 2012, pp. 230–233] can roughly be stated as follows. The ship sailed by Theseus was entirely made of wood. As the years went by, some of the wooden boards began to rot and were replaced by new ones. Eventually, all boards composing the ship were replaced. One can then ask if the restored ship is still the same object as the original ship. Given the difference of scale between the ship and a single wooden board, intuitively one may be led to refer to the initial and final ships as being distinct, while if, say, only one small repair is made as still being the same ship. Presented in this way, it resembles the so-called sorites paradox. Indeed, we are in the presence of vagueness of the sorites kind because if the ships are distinct when all the parts are replaced, then at what point does the new ship arises?

In the spirit of [Dinis and van den Berg, 2019, Chapter 10], using nonstandard analysis, I will consider two simple thought experiments and give the corresponding mathematical formulations, hopefully providing some insight into the paradox.

The first model relies essentially on near-equality, mimicking (1). For that matter, let us assume that there is an initial ship S_0 which is composed of a nonstandard natural number n of boards of wood (or atoms, for that matter) and that a difference of just one component is not sufficient for one to claim that she is in the presence of a new ship.

Consider the sequence of ships $S_0, S_1, \ldots, S_n$, where S_i, $i \in \{0, \ldots, n\}$ denotes the ship after i components have been replaced. So, S_0 denotes the original ship, and S_n the resulting ship after all the components have been replaced. This reasoning can be represented as follows

$$\begin{cases} S_0 \simeq S_0 \\ S_0 \simeq S_1 \\ S_1 \simeq S_2 \\ \quad\vdots \\ S_{n-1} \simeq S_n \\ S_0 \not\simeq S_n \end{cases} \tag{2}$$

Is there a paradox here? The fact that near-equality is an external relation ensures that transitivity can only be used a standard number of times. At the same time, one can choose to (re)interpret the symbol $\simeq$ as meaning ‘near-equal but not equal’, thus accounting for the fact that, say the ships S_0 and S_1 are different but imperceptibly so. This sends us back to Heraclitus’s view that these are and are not the same ship. Note that if equality is used in (2) instead of near-equality, then the reasoning is indeed paradoxical. The relation is no longer an external relation and it is possible to use transitivity nonstandardly many times and derive both $S_0 = S_n$ and $S_0 \neq S_n$. By distinguishing between internal and external relations, our model allows us to explain the paradox, at the cost of “pretending” that the ship is made of a nonstandard number of wooden boards.

The use of nonstandard analysis is nevertheless in line with the intuition that if "few" (standardly many) changes are made then we are in the presence of the same ship whereas if "many" (nonstandardly many) changes occur then the ship is no longer the same. The fact that large changes come as the result of the accumulation of small changes is then a realization of the well known fact that an infinitely large sum of infinitesimals may be appreciable or even infinitely large.

It is possible to slightly modify the model so that if the number of changes is infinitesimal, then we are talking about the same ship whereas if the set of changes is appreciable (i.e. not infinitesimal and not infinitely large) or even infinitely large the ships are indeed different.

Finally, let us now consider a second approach based on (external) induction. Again, assume that the ship is composed of nonstandardly many boards of wood. Let $E(x, y)$ represent the relation 'x is imperceptibly different from but not equal to y' and let s_n represent the n-th ship in Theseus's ship paradox, i.e. the ship after n boards have been replaced. By the assumption that replacing only one board of wood does not produce a different ship, one has $E(s_0, s_1)$ and, for all standard n, if $E(s_0, s_n)$ then $E(s_0, s_{n+1})$. External induction allows one to conclude that if only a standard number of boards have been replaced, then the ship is indeed different, but imperceptibly so. Hence, in order to obtain perceptible differences one must replace a nonstandard number of boards. The argument can then be represented as follows:

$$\begin{cases} (E(s_0, s_1) \wedge \forall^{\mathrm{st}} n \, (E(s_0, s_n) \rightarrow E(s_0, s_{n+1}))) \rightarrow \forall^{\mathrm{st}} n \, E(s_0, s_n) \\ \exists \omega \, (\neg E(s_0, s_\omega)) \end{cases}$$

Again, the nonstandard framework is crucial here. If one were to use (usual) induction instead of external induction, the conclusion would be that, no matter how many boards are replaced, one could not perceive a difference in the ship. Moreover, the reasoning would contradict the second line of the argument, claiming the existence of a number of changes which indeed allows us to perceive a difference in the ship(s).

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