Profit optimization of stochastically fluctuating populations under harvesting: the effects of Allee effects

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ABSTRACT

The main goal of this work is to assess and compare the influence of Allee effects in profit optimization of stochastically fluctuating harvested populations considering several harvesting policies. The policies to compare are based on constant, variable, and stepwise harvesting efforts. For application purposes, the population growth models considered are the logistic model and a logistic-like model with weak Allee effects. In recent work, we have shown that the optimal harvesting policy with variable effort is inapplicable whereas the optimal harvesting policy with constant effort is easily applicable and leads to population sustainability. However, the latter implies profit losses, comparing to the first one. So, we consider a stepwise policy which is applicable but shares some of the problems with the optimal policy based on variable effort. We also show that some of the disadvantages of the optimal policy are eliminated by considering a penalized profit with an artificial running energy cost on the effort. However, the applicability problems remain. Finally, in terms of optimal profit, we study the influence of Allee effects on all policies and check whether Allee effects should or should not be taken into account when designing harvesting policies.

KEYWORDS
Optimal control; profit optimization, stochastic differential equations; Allee effects; logistic growth.

1. Introduction

Let $X(t)$ be the size, at time t, of a harvested population under the influence of environmental random fluctuations. The population growth dynamics can be described by the stochastic differential equation (SDE)

$$dX(t) = f(X(t))X(t)dt - qE(t)X(t)dt + \sigma X(t)dW(t),\ X(0) = x, \tag{1}$$

where $f(X)$ is the *per capita* natural growth rate, $q > 0$ is the catchability coefficient, $E(t) \geq 0$ is the harvesting effort, $H(t) = qE(t)X(t)$ represents the yield from harvesting, $\sigma > 0$ measures the strength of environmental fluctuations, $W(t)$ is a standard

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Wiener process and $X(0) = x > 0$ is the population size at time 0, which we assume to be known. Particular cases of SDE (1), the ones to be here considered, are the logistic model and the logistic-like model with Allee effects given, respectively, by

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t),\ X(0) = x$$

and

$$dX(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\left(\frac{X(t) - A}{K - A}\right)dt - qE(t)X(t)dt + \sigma X(t)dW(t),\ X(0) = x. \tag{2}$$

The parameter $r > 0$ represents the intrinsic growth rate and $K > 0$ is the environment's carrying capacity. The Allee parameter, $A \in (-K, 0)$, stands for the strength of the weak Alee effects. The closer A is to 0, the more intense is the Allee effect. On the contrary, the closer A is to $-K$, the less intense is the Allee effect. Taking $A \to -\infty$ leads to the well-known logistic model. Strong Allee effects occur when $A \in (0, K)$ and they will not be considered here, since population extinction will occur with probability one even in the absence of harvesting.

The term "harvesting" and the models here presented are suitable for populations in the areas of forestry, hunting, agriculture, fishing, and others. Since we will illustrate with applications in fisheries, we often use the term "fishing" in place of "harvesting".

The presence of Allee effects occurs when, for low values of the population size, we observe *per capita* growth rates lower than the high rates one would expect considering the higher availability of resources per individual. Allee effects may be due to several causes, such as the difficulty in finding mating partners or in setting up an effective pack-hunting size or, in the case of prey species, in constructing a strong enough group defence against predators (see, for instance, [1–4]).

Eq. (2) assumes that the natural growth rate follows a logistic-like model inspired by a similar deterministic model (see, for instance, [3]). However, without changing the logistic-like model for the average natural growth rate dynamics, we use a different parametrization of that model in order to allow easier comparisons with the logistic model without Allee effects (as in [2]). In particular, the logistic model and the logistic-like model here considered have in common the same carrying capacity K and the same slope of the natural growth rate at $X = K$.

In previous work we discussed the use of a variable effort optimal policy versus a constant effort optimal policy, considering a logistic-like model with weak Allee effects (see [1]), the logistic model (see [5,6]), and the Gompertz model ([7,8]), in order to derive harvesting policies based on profit optimization. We have shown that the optimal policy with variable effort, obtained using optimal control methods, has several shortcomings, namely: (i) the effort depends on the randomly varying population size, implying the estimation of the population size at each time instant, which is a costly, time consuming and inaccurate task; (ii) this policy is inapplicable from the practical point of view; (iii) this policy poses social problem during the periods of low or no harvesting. In fact, the effort is highly variable and may even have frequent periods of no harvesting or harvesting at the maximum possible rate. On the contrary, the optimal constant effort policy has strong advantages: (i) leads to sustainable policies with the population being driven, when $t \to +\infty$, to a stationary regimen, reaching a stochastic equilibrium with a stationary density (see Section 2.4); (ii) it is easily applicable; (iii) does not require knowledge of population size; (iv) poses no social problems. The only disadvantage of this policy is the reduction in profit, which we show to be slight for the models and data considered.

One way to eliminate the social problems posed by the optimal variable effort policy is to incorporate in the model a term that represents an artificial running energy cost, designed to reduce the abrupt changes in effort. This was done in [7] considering the Gompertz growth model and by taking several cases with different penalization magnitudes. Unfortunately, the major problem of applicability is maintained, since the effort still varies frequently across time, although not so intensely. Also, it is still necessary to keep estimating the population size at each time instant, which is a strong disadvantage.

Also, one can find, for the logistic model ([8]) and the Gompertz model ([5]), a sub-optimal policy, named stepwise policy, where the harvesting effort under the optimal variable effort policy is determined at the beginning of each year period (or at the beginning of a larger period, for instance, two years) and kept constant during that period. The authors showed that this policy is not optimal and still poses some social problems, but has the advantage of being applicable since the changes in the effort are less frequent and compatible with the fishing activity. Furthermore, although there is a need to keep estimating the fish stock size, it is not necessary to do it so often. Replacing the optimal variable effort policy with a stepwise policy has the advantage of applicability but, at best, considerably reduces the already small profit advantage of the optimal variable effort policy over the optimal constant effort policy. In some cases, the optimal sustainable policy even outperforms this stepwise policy in terms of profit.

This paper is organized as follows: in Section 2 we present four harvesting policies: three optimal policies and one sub-optimal policy denoted by the stepwise policy. Section 3 refers to the comparisons between policies using a population under fishing for which we have access to realistic biological and economic data. Finally, some concluding remarks are given in Section 4.

2. Harvesting policies

2.1. Optimal policy with variable effort

To obtain an optimal policy with variable effort based on profit optimization we follow the stochastic optimal control problem (SOCP) formulated in [1] and [6]. The profit per unit time is defined as $\Pi(t) = R(t) - C(t)$, where $R(t) = (p_1 - p_2 H(t))H(t)$ are the sales revenues per unit time ($p_1 > 0,\ p_2 \geq 0$) and $C(t) = (c_1 + c_2 E(t))E(t)$ represent the fishing costs per unit time ($c_1 > 0,\ c_2 > 0$). Hence, $\Pi(t) = (p_1 q X(t) - c_1)E(t) - (p_2 q^2 X^2(t) + c_2)E^2(t)$. The SOCP consists in maximizing the present value, i.e. the expected accumulated discounted profit per unit time over a finite time interval $[0, T]$:

$$V^* := J^*(x,0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} J(x,0) = \max_{\substack{E(\tau) \\ 0\leq\tau\leq T}} \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi(\tau)d\tau\right],$$

subject to the population dynamics given by the SDE (1), to the control restrictions $0 \leq E_{min} \leq E(t) \leq E_{max} < \infty$ and to a terminal condition $J(X(T), T) = 0$. The parameter $\delta > 0$ refers to a discount rate accounting for interest rate and cost of opportunity losses and other social rates. Note that we use the short notation

$\mathbb{E}[\ldots|X(t)=y] = \mathbb{E}_{t,y}[\ldots]$ and that

$$J(y,t) := \mathbb{E}_{t,y}\left[\int_t^T e^{-\delta(\tau-t)}\Pi(\tau)d\tau\right]$$

is, at time t, the expected discounted future profits when the population size at that time is y. In addition, we assume that optimization starts at time $t=0$ and harvesting continues up to the time horizon T.

The above SOCP can be solved by stochastic dynamic programming theory through Bellman's principle of optimality (see, for instance, [9]). In terms of optimization theory, the problem resorts to finding the effort (i.e., the control) that maximizes the present value $V := J(x,0)$, subject to the growth dynamics given by Eq. (1) and to the constraints on effort and the terminal condition given above. The control value that leads to the maximum V^* will be called the optimal variable effort and is denoted by $E^*(t)$, which exists and is unique since $E(t)$ is a Markov control and Π is concave with respect to the control (see, for instance, [10]). The Hamilton-Jacobi-Bellman (HJB) equation associated with the SOCP is

$$\begin{aligned}-\frac{\partial J^*(X(t),t)}{\partial t} &= \Big(p_1 qX(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E^*(t)\Big)E^*(t) - \delta J^*(X(t),t)\\ &+ \frac{\partial J^*(X(t),t)}{\partial X(t)}\Big(f(X(t)) - qE^*(t)\Big)X(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)}\sigma^2 X^2(t),\end{aligned}$$

and the optimal variable effort is

$$E^*(t) = \begin{cases} E_{min}, & \text{if} \quad E^*_{free}(t) < E_{min} \\ E^*_{free}(t), & \text{if} \quad E_{min} \le E^*_{free}(t) \le E_{max} \\ E_{max}, & \text{if} \quad E^*_{free}(t) > E_{max}, \end{cases}$$

where

$$E^*_{free}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) qX(t) - c_1}{2\left(p_2 q^2 X(t)^2 + c_2\right)}$$

is the unconstrained effort (see [1,11]). The HJB equation is a parabolic PDE and an explicit solution is not available. Hence, to solve it numerically we apply a Crank-Nicolson discretization scheme as in [1,5–8,12].

2.2. Optimal variable effort penalized policy

In Section 1 we have mentioned that the optimal variable effort varies frequently across time between periods of zero/low and maximum/high values. This behaviour, typical in optimal control problems, is not compatible with the logistic of fisheries. In addition, periods of zero or low effort poses social burdens, as explained in Section 1. One way to eliminate this problem is to incorporate in the model a term that represents a running energy cost based on the effort (see, for instance, [15]). This extra cost term is not a real cost, just an artificial way of penalizing the profit values when, at each time instant, the effort takes abrupt changes from a reference effort value, say E_{ref}. One can choose, for instance, E_{ref} as the optimal effort value of the constant effort policy. In so doing, the resulting optimal penalized policy will not give us the optimal real profit but will behave better than the optimal variable effort non-penalized policy in terms of having milder effort changes.

To implement this approach, the profit per unit time to be optimized is not the real profit (the one presented at the beginning of Section 2.1) but rather an artificial profit $\Pi_\varepsilon(t) := R(t) - C(t) - P_\varepsilon(t)$, with the artificial penalty cost $P_\varepsilon(t) = \varepsilon(E(t) - E_{ref})^2$, where $\varepsilon \geq 0$ is a tuning parameter (representing the penalization magnitude). Thus, we now solve (numerically) the maximization problem

$$\max_{\substack{E(\tau)\\0\leq\tau\leq T}} \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Big(\Pi(\tau) - \varepsilon(E(\tau) - E_{ref})^2\Big)d\tau\right],$$

where we maximize the artificial expected accumulated discounted profit with an artificial running energy cost, still subject to the population dynamics (1) and to the restrictions on the effort and the terminal condition. Let $E^*_\varepsilon(t)$ be the maximizing effort, which will be called optimal penalized effort. Note, however, that the real expected accumulated discounted profit when we adopt the optimal penalized effort $E^*_\varepsilon(t)$ should use the real costs and so its expression is

$$V^*_\varepsilon \ := \ \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau}\Pi^*_{\varepsilon,real}(\tau)d\tau\right],$$

with $\Pi^*_{\varepsilon,real}(t) := (p_1qX(t) - c_1)E^*_\varepsilon(t) - (p_2q^2X^2(t) + c_2)E^{*^2}_\varepsilon(t)$.

Considering an artificial energy cost will not eliminate all the major shortcomings of the optimal variable effort policy. The introduction of an energy cost will reduce or even eliminate the social costs arising from the null or low effort periods of the optimal variable effort policy, but it is still necessary to keep estimating the population size at each time instant. In addition, the major problem of the logistic of fisheries will persist, since the effort still varies frequently across time, although not so intensely. Formally, these problems will remain unchanged whatever ε we choose, except for high ε values. The only difference between different choices of ε is not in the high frequency of effort changes but in the magnitude of such changes. If a low value for ε is chosen, the resulting policy will be similar to the optimal variable effort policy, with almost the same social costs and intense variability in effort between null/low and high values. On the contrary, if a high value for ε is chosen, the resulting policy will still have frequent changes according to population size changes, but the changes will be small in magnitude and the effort will stay close to a constant effort so that social costs will be eliminated. However, the operability fishing problems remain unchanged and, since the variable effort has values close to the constant effort policy, the profit will be practically indistinguishable from the optimal sustainable constant effort policy profit. In Section 3, we will apply this policy to the logistic model and the logistic model with Allee effects.

2.3. Sub-optimal policies with variable effort: stepwise policies

To obtain the optimal variable effort policy presented in Section 2.1 we need to compute the optimal effort in each of the points of the discretization scheme (as in [1,5–8,12]). Since we are dealing with a SOCP without any regularizing penalization, one expects to have frequent and very abrupt changes on the effort, resulting in an inapplicable policy from the point of view of the fishing activity. One way to mitigate this behaviour is to consider sub-optimal policies based on stepwise effort.

In [5], for the logistic model, and in [8], for the Gompertz model, stepwise policies (without penalization) were introduced and applied to real harvested populations without Allee effects. In a stepwise effort policy, the harvesting effort is determined at the beginning of a time sub-interval with duration p (for instance, 1 or 2 years) and is kept constant at that value during the whole time sub-interval. Therefore, in this stepwise effort policy, for time t in the period $[lp,(l+1)p[$, we keep the effort $E^*_{step}(t) = E^*(lp)$ constant and equal to the effort of the optimal policy at the beginning of the period. For convenience, we use p to be a multiple of the time step Δt used in the numerical computations and Monte Carlo simulations.

Notice that this policy is obviously not optimal. Since it is a stepwise modification of the optimal variable effort policy, it is not even optimal among the stepwise policies. However, it is, as it should, non-anticipative, i.e. it does not use future values of the fish population size, which are unknown at the time of the decision. It will not eliminate abrupt changes in effort but will result in an applicable policy.

2.4. Optimal sustainable policy with constant effort

To apply a constant effort policy, one considers a particular case of Eq. (1) with $E(t) \equiv E$. In [13,14], for the logistic growth model, and in [1] for the logistic-like growth model with Allee effects, one can find conditions to avoid population extinction, to have a unique solution and to grant a stationary density for the population size. Hence, for the logistic model, it is sufficient to have $0 \leq E < \frac{r}{q}\left(1-\frac{\sigma^2}{2r}\right)$ and, for the logistic-like growth model with Allee effects, it is sufficient that $0 \leq E < \frac{r}{q}\left(\frac{A}{A-K}-\frac{\sigma^2}{2r}\right)$. Under such conditions, the distribution of X(t) stabilizes and converges, as $t \to +\infty$, to a distribution with a stationary density. Denoting by X_∞ the random variable with such stationary density, a good approximation of the expected size of the population $\mathbb{E}[X_t]$, for large t, is $\mathbb{E}[X_\infty]$.

Following [1,5–8], the sustainable profit per unit time is defined similarly to the case of the optimal variable effort policy as $\Pi_\infty := (p_1 q X_\infty - c_1)E - (p_2 q^2 X_\infty^2 + c_2)E^2$. The effort that maximizes the expected sustainable profit $\mathbb{E}[\Pi_\infty]$, i.e. the optimal sustainable effort, will be denoted by E^{**} and the corresponding profit by $\Pi^{**}_\infty := (p_1 q X_\infty - c_1)E^{**} - (p_2 q^2 X_\infty^2 + c_2)E^{**2}$.

From [5] and [1], for the logistic model we obtain $\mathbb{E}[X_\infty] = K\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)$ and

$$\begin{aligned}\mathbb{E}[\Pi^{**}_\infty] = &\left(p_1 q K\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)-c_1\right)E^{**}\\ &-\left(p_2 q^2 K^2\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)\left(1-\frac{qE^{**}}{r}\right)+c_2\right)E^{**2}.\end{aligned}$$

and, for the logistic-like model with Allee effects, we obtain $\mathbb{E}[X_\infty] = \frac{I_1(E)}{I_0(E)}$ and

$$\mathbb{E}[\Pi_\infty^{**}] = \left(p_1 q \frac{I_1(E^{**})}{I_0(E^{**})} - c_1\right) E^{**} - \left(p_2 q^2 \frac{I_2(E^{**})}{I_0(E^{**})} + c_2\right) E^{**2},$$

with $I_j(E) = \int\limits_0^{+\infty} z^{\alpha-\beta E+j-1} e^{-\gamma(z-(K+A))^2} dz$, $\alpha = \frac{2rA}{\sigma^2(A-K)} - 1$, $\beta = \frac{2q}{\sigma^2}$, $\gamma = \frac{r}{\sigma^2 K(K-A)}$.

3. Comparison of policies

Policies comparison, in terms of the expected accumulated discounted profit, between the optimal variable effort policy and the optimal sustainable constant effort policy cannot be done directly, since the first one maximizes the accumulated discounted profit over a finite time interval $[0, T]$ and the second one maximizes the profit per unit time as $T \to \infty$. Thus, for comparisons purposes, we will consider:

(A) the optimal profit per unit time under the optimal variable effort $E^*(t)$:

$$\Pi^*(t) = (p_1 q X(t) - c_1) E^*(t) - (p_2 q^2 X^2(t) + c_2) E^{*^2}(t),$$

(B) the optimal profit per unit time under the optimal penalized effort $E_\varepsilon^*(t)$:

$$\Pi_{\varepsilon,real}^*(t) = (p_1 q X(t) - c_1) E_\varepsilon^*(t) - (p_2 q^2 X^2(t) + c_2) E_\varepsilon^{*^2}(t),$$

(C) the optimal profit per unit time under the stepwise variable effort $E_{step}^*(t)$:

$$\Pi_{step}^*(t) = (p_1 q X(t) - c_1) E_{step}^*(t) - (p_2 q^2 X^2(t) + c_2) E_{step}^{*^2}(t),$$

and

(D) the optimal sustainable profit per unit time under the optimal constant effort E^{**}:

$$\Pi^{**}(t) = (p_1 q X(t) - c_1) E^{**} - (p_2 q^2 X^2(t) + c_2) E^{**^2}.$$

The quantities to be compared in terms of the expected accumulated discounted profits (present values), for each policy, are:

$$V^* = \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau} \Pi^*(\tau) d\tau\right], \quad V_\varepsilon^* := \mathbb{E}_{0,x}\left[\int\limits_0^T e^{-\delta\tau} \Pi_{\varepsilon,real}^*(\tau) d\tau\right],$$

$$V_{step}^* = \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau} \Pi_{step}^*(\tau) d\tau\right], \quad V^{**} = \mathbb{E}_{0,x}\left[\int_0^T e^{-\delta\tau} \Pi^{**}(\tau) d\tau\right],$$

which represent the optimal present values under (A), (B), (C) and (D), respectively.

To compute $V^*, V^*_\varepsilon, V^*_{step}$ and V^{**}, we resort to Monte Carlo simulations of the population, based on an Euler scheme and a 1000 sample paths, and obtain the corresponding efforts and profits. We have used realistic biological and economic parameters from the Pacific halibut (*Hippoglossus hippoglossus*) that can be found in [11]. Other parameters were taken from [1]. The full list of parameters is shown in Table 1. For the application of the Crank-Nicolson discretization scheme, applied to solve the HJB equation (see Sections 2.1 – 2.3), the time and space grid was designed with $n = 150$ intervals for time (with a time step of $\Delta t = 4$ months) and with $m = 75$ intervals for the state space (with space step $\Delta x = 2.15 \cdot 10^6$ kg and $X_{max} = 2K$ kg).

Table 1. Values used in the simulations. Adapted from [1].

Item	Description	Value	Unit[a]
	SDE parameters		
r	Intrinsic growth rate	0.71	year^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
A	Allee parameter	$-0.75K, -0.10K$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	SFU^{-1}year^{-1}
σ	Strength of environmental fluctuations	0.2	year$^{-1/2}$
x	Initial population size	$0.5K$	kg
	Bioeconomic parameters		
p_1	Linear price parameter	1.59	\$kg^{-1}
p_2	Quadratic price parameter	$5 \cdot 10^{-9}$	\$year $\cdot$ kg^{-2}
c_1	Linear cost parameter	$96 \cdot 10^{-6}$	\$SFU^{-1}year^{-1}
c_2	Quadratic cost parameter	10^{-7}	\$SFU^{-2}year^{-1}
	Other parameters		
T	Time horizon	50	year
δ	Discount factor	0.05	year^{-1}
E_{min}	Minimum fishing effort	0	SFU
E_{max}	Maximum fishing effort	$0.7r/q$	SFU

[a]SFU represents the Standardized Fishing Unit. The definition can be found in [11].

Table 2. Numerical comparisons among the profits obtained by the application of policies (A) to (D), respective standard deviation (sd) and relative differences (Δ and Δ_1 in percentage), using the logistic model (LM) and two scenarios for the logistic-like model with Allee effects (LMAE). Δ represents the relative difference w.r.t. V^* of the same model and Δ_1 the relative differences w.r.t V^* of the logistic model. Profit values are in million dollars.

		LM			LMAE ($A = -0.75K$)				LMAE ($A = -0.10K$)			
		profit	sd	Δ	profit	sd	Δ	Δ_1	profit	sd	Δ	Δ_1
V^*		413.59	38.32	0.0	296.14	44.66	0.0	-28.4	218.79	44.70	0.0	-47.1
V^*_ε	ε=0.001	407.46	37.99	-1.5	286.87	43.62	-3.1	-30.6	203.07	43.11	-7.2	-50.9
	ε=0.01	398.88	35.84	-3.6	271.75	41.03	-8.2	-34.3	176.88	39.86	-19.2	-57.2
	ε=0.1	396.71	35.06	-4.1	263.63	37.72	-11.0	-36.3	155.27	41.65	-29.0	-62.5
	ε=0.5	396.48	34.97	-4.1	263.27	36.97	-11.1	-36.3	152.08	40.30	-30.5	-63.2
V^*_{step}	1-year	406.72	38.79	-1.7	287.70	44.22	-2.9	-30.4	208.32	44.47	-4.8	-49.6
	2-years	390.50	38.14	-5.6	270.00	42.46	-8.8	-34.7	186.52	44.96	-14.7	-54.9
V^{**}		396.42	34.95	-4.1	261.85	36.19	-11.6	-36.7	83.41	7.40	-61.9	-79.8

In Table 2 we summarize and compare results for the logistic model (LM) and the logistic-like model with Allee effects (LMAE) with Allee parameters $A = -0.75K$ (mild Allee effects) and $A = -0.10K$ (more intense Allee effects). Regarding other values for A (not shown here) we can conclude that when A becomes smaller, the Allee effects have less influence on the policies. When A increases (approaching zero), the Allee effects become more pronounced and imply large differences in terms of profit values. Table 2 shows the values of the profits V^*, V^*_ε, V^*_{step} and V^{**} (where the expectations are obtained approximately by taking the average value over the 1000 Monte Carlo simulations of the observed profits) and their relative differences to the optimal policy (A). Fishermen observe only one trajectory, hence it is interesting to have a measure of the uncertainty of the observed profit. In Table 2 we also show, for each policy, the standard deviation of the observed profits among the 1000 simulated trajectories.

The case $\varepsilon = 0$ corresponds to considering a non-penalized policy with variable effort and the resulting profits are the same as for V^* (first line of Table 2). Increasing ε values from 0.001 to 0.5 implies a decrease of profit between 1.5% and 4.1% when comparing the penalized policy with the non-penalized policy. Taking into account less intense Allee effects ($A = -0.75K$), profit differences also increase with ε values but are higher (from 3.1% to 11.1%). More intense Allee effects ($A = -0.10K$) imply higher profit differences (from 7.2% to 30.5%). The comparison of penalized policies with Allee effects and penalized policies without Allee effects shows very large differences (from 30.6% to 36.3%) in present values, being much higher when Allee effects are more intense (from 50.9% to 63.2%).

For the stepwise policy, we have chosen time sub-intervals with a duration of $p =$ 1 year and $p = 2$ years, i.e., the optimal effort is determined at the beginning of each year/each biennium and kept constant during that period. The application of a stepwise policy instead of the optimal variable effort policy will cause a decrease between 1.7% (1-year step) and 5.6% (2-years step), on average. These differences are more pronounced in the presence of Allee effects (from 2.9% to 8.8% for less intense effects and from 4.8% to 14.7% for a higher intensity).

Table 2 also shows that the optimal policy with constant effort has a profit reduction of 4.1% when compared with the optimal variable effort policy, both without Allee effects. We notice that the policy with constant effort implies identical profit values when compared to the penalized policy (with $\varepsilon \in \{0.01, 0.1, 0.5\}$) and even outperforms the stepwise policy with 2-years steps (-4.1% against -5.6%). However, when Allee effects are taken into account, one observes a decrease of the profit values for the sustainable policy.

Regarding the standard deviation of the profit among the 1000 simulated trajectories, the values obtained for all policies are similar, although slightly higher for the policies with Allee effects.

The black thin lines on Figures 1 to 4 show one path chosen at random from the 1000 simulated sample paths and the thick gray lines show the mean of the 1000 simulated sample paths, which estimates the expected value.

Figure 1 compares the optimal variable policy, showing in the middle line the optimal effort $E^*(t)$ for the three models considered here: (a) logistic model, (b) logistic-like model with Allee effects with $A = -0.75K$, (c) logistic-like model with Allee effects with $A = -0.10K$. The population size and the profit per unit time when applying the optimal effort are shown on the top and the bottom line, respectively. Figures 2 to 4, respectively, for the logistic model, for the logistic model with Allee effects with $A = -0.75K$, and for the logistic model with Allee effects with $A = -0.10K$, compare the optimal efforts obtained by applying the optimal policy with variable effort (first row (a)), the optimal sustainable policy with constant effort (first row (b)), the optimal penalized policy ($\varepsilon = 0.001$) with variable effort (second row (a)), the optimal penalized policy ($\varepsilon = 0.01$) with variable effort (third row (a)), the optimal policy with stepwise effort with 1-year steps (second row (b)) and the optimal policy

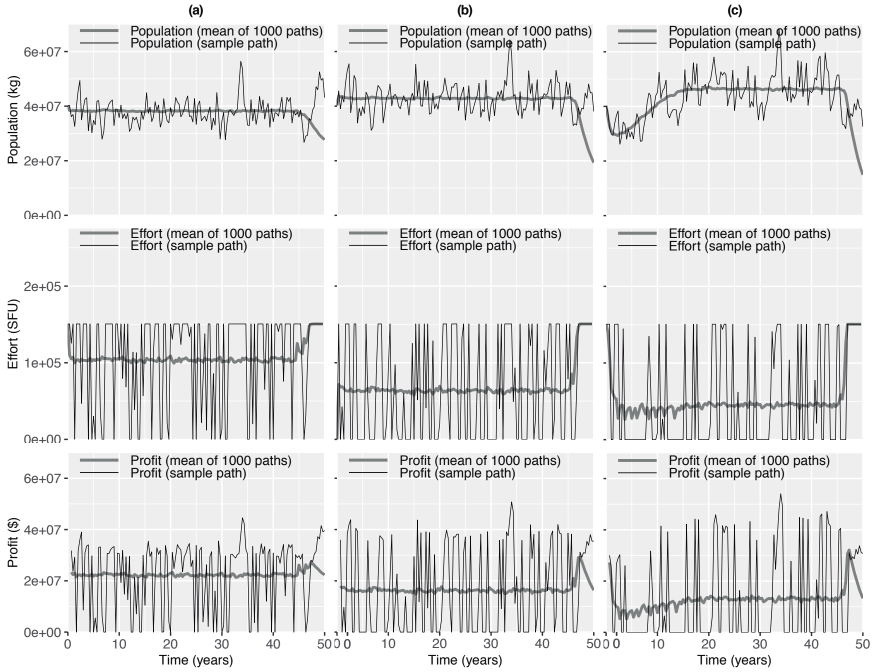

Figure 1. Mean (thick lines) and randomly chosen sample path (thin lines) for the population (first row), the effort (second row) and the profit per unit time (third row) for the optimal variable effort policy when the growth model is: (a) logistic, (b) logistic-like with Allee effects with $A = -0.75K$, (c) logistic-like with Allee effects with $A = -0.10K$.

with stepwise effort with 2-year steps (third row (b)).

The optimal policy (Figure 1 (a)) exhibits frequent and abrupt changes on effort and profit per unit time, contrasting with the corresponding trajectories for the optimal sustainable policy based on constant effort (for the effort dynamics, see first row of Figure 2 (b)). The depicted trajectory is one of the possible outcomes that the harvester may experience. In terms of the population size, the logistic model produces the trajectory with less size oscillations (comparison along the first row of graphics in Figure 1). Note that, for the policies with variable effort, harvesters should adjust the fishing effort at every time instant. Effort adjustments often correspond to changes from periods with zero effort (i.e., the fishery is closed, with possible social problems or the need for compensations not considered in the profit structure) to periods with maximum effort (i.e., fishing with all available equipment and manpower). Clearly, this is not applicable since abrupt and frequent changes on effort are not compatible with the logistic of fisheries and, in addition, it is not feasible to obtain information on population size all the time. On the contrary, the optimal sustainable policy implies the application of a constant effort at every time instant, making this policy very easy to implement by harvesters.

The penalized policies attenuate the social problems since they have obviously effort changes milder than the optimal variable effort policy, as can be seen by comparing the middle line of Figure 1 with the left second rows (for $\varepsilon = 0.001$) and the left third rows (for $\varepsilon = 0.01$, with very mild changes) of Figures 2 to 4. However, the applicability problem due to frequent changes in the effort persists.

Another possible policy, without the shortcomings of the optimal variable policy, is the stepwise policy. This one is suboptimal but has the advantage of being applicable since, although there are still abrupt changes in effort, they occur less frequently (at most once a year or once every two years). The stepwise policies (Figures 2 to 4, middle (1-year step) and bottom (2-years step) images on the right-side (b)) present large variations in terms of the effort per unit time. For the logistic growth model, both stepwise policies (middle and bottom of Figure 2 right side) have similar behaviour as for the logistic model policies without steps (middle-left of Figure 1). However, for the stepwise policies, effort values during a year (or a 2 year) period are very often equal to the maximum effort or to the zero effort; remembering that the effort is computed at the beginning of that period according to the optimal policy, this behaviour may be due to a compensation of the effort having been fixed without any adjustment during the previous period. We can also notice that these policies mimic the optimal variable effort policy in terms of the considerable rise of the average effort near the time horizon. This is not a surprise since both policies are not designed to provide a sustainable behaviour of the population size.

Comparing the effort sample paths from the middle of Figure 1 ((a) and (b)) with the middle and bottom right sides of Figures 3 and 4, one can see long periods with zero effort, corresponding to long periods where the fishery is not working. Closing a fishery is not desirable since it provokes serious social and economic problems. The presence of Allee effects should be checked since they may, depending on their strength, have a considerable impact on effort values and in designing appropriate fishing policies.

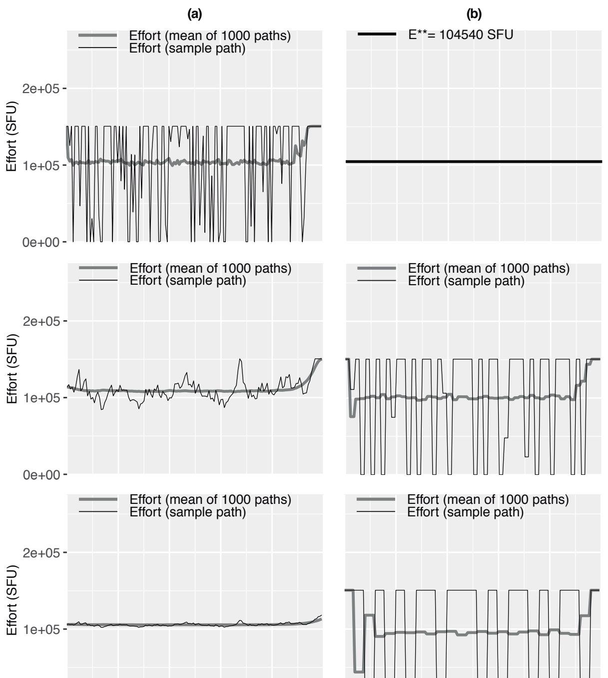

Figure 2. Effort mean (thick lines) and effort of a randomly chosen sample path (thin lines) for the logistic model for: the optimal policy with variable effort (first row (a)), the optimal sustainable policy with constant effort (first row (b)), the optimal penalized policy ($\varepsilon = 0.001$) with variable effort (second row (a)), the optimal penalized policy ($\varepsilon = 0.01$) with variable effort (third row (a)), the optimal policy with stepwise effort with 1-year steps (second row (b)) and the optimal policy with stepwise effort with 2-year steps (third row (b)).

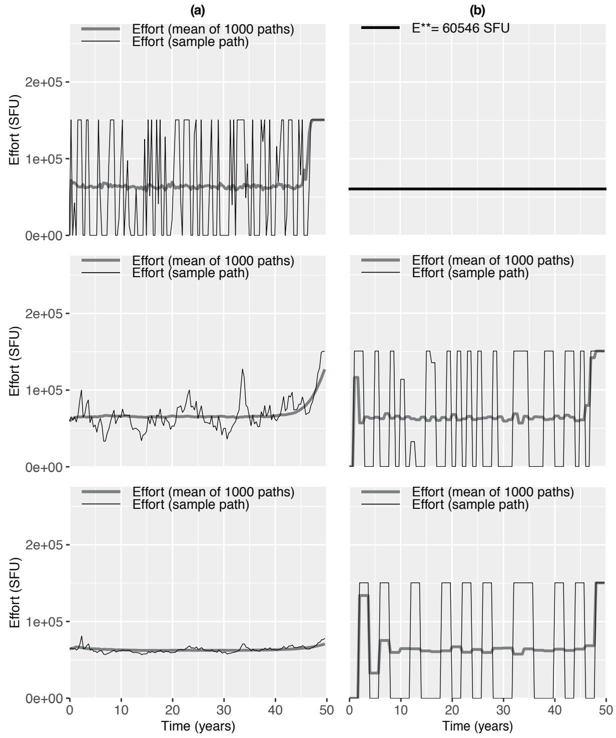

Figure 3. Effort mean (thick lines) and effort of a randomly chosen sample path (thin lines) for the logistic model with Allee effects with $A = -0.75K$ for: the optimal policy with variable effort (first row (a)), the optimal sustainable policy with constant effort (first row (b)), the optimal penalized policy ($\varepsilon = 0.001$) with variable effort (second row (a)), the optimal penalized policy ($\varepsilon = 0.01$) with variable effort (third row (a)), the optimal policy with stepwise effort with 1-year steps (second row (b)) and the optimal policy with stepwise effort with 2-year steps (third row (b)).

4. Conclusions

In this work, although the stochastic models presented (logistic and logistic-like with Allee effects) can be suitable for harvested problems in general, due to data availability, we illustrate an application in fisheries using realistic biological and economic parameters. We have presented numerical profit and effort comparisons between harvesting policies based on constant, variable (with and without penalization), and stepwise effort. For that, we have performed simulations with a discretization scheme in time and space of the HJB equation and an Euler scheme for the population paths.

The optimal policy with variable effort poses serious applicability and social problems, leads to great instability in the profit, and can create a possibly dangerous effect near the time horizon. On the contrary, the optimal sustainable policy does not have these shortcomings, is very easy to implement, drives the population to a stochastic equilibrium, and avoids the need for frequent estimation of population size.

One way to eliminate the social problems posed by the optimal variable effort policy is to incorporate a penalization in the form of an artificial running energy cost, designed to tame the abrupt changes in effort. Unfortunately, the major problems of applicability and need to estimate the population size at each time instant do persist.

Since the optimal policy is not applicable, we have presented sub-optimal policies, the stepwise policies. These policies share some disadvantages with the optimal variable

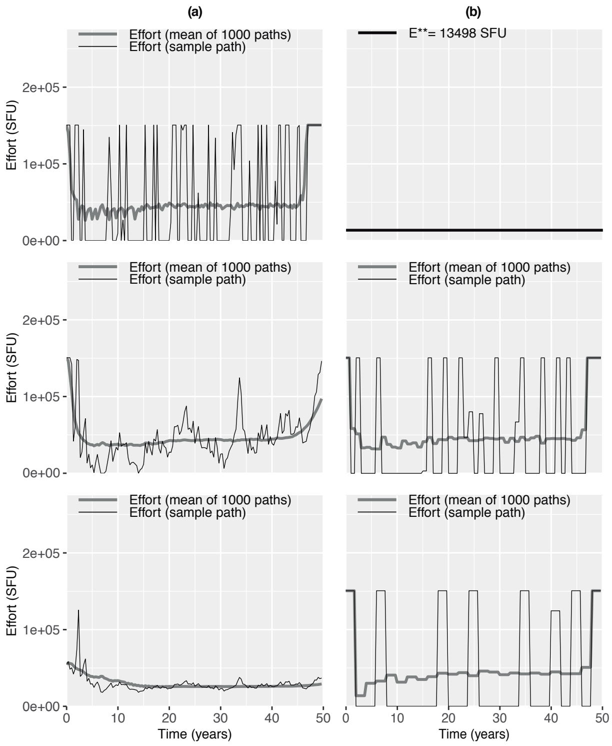

Figure 4. Effort mean (thick lines) and effort of a randomly chosen sample path (thin lines) for the logistic model with Allee effects with $A = -0.10K$ for: the optimal policy with variable effort (first row (a)), the optimal sustainable policy with constant effort (first row (b)), the optimal penalized policy ($\varepsilon = 0.001$) with variable effort (second row (a)), the optimal penalized policy ($\varepsilon = 0.01$) with variable effort (third row (a)), the optimal policy with stepwise effort with 1-year steps (second row (b)) and the optimal policy with stepwise effort with 2-year steps (third row (b)).

effort policy but have the advantage of being applicable. Furthermore, although we still need to keep estimating the fish stock size, we do not need to do it so often.

In the absence of Allee effects, the profit differences among these policies are relatively small. When Allee effects are present, their influence, in comparison with the pure logistic model, depends on the Allee intensity parameter A. When A increases (approaching zero), the Allee effects become more influential and imply large reductions in profit, as well as large profit differences among the different policies. So, although the logistic model (without Allee effects) is the common paradigm in fishery applications, the possible presence of Allee effects should be assessed and, depending on their strength, taken into account in the design of the fishing policy.

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References

[1] Brites NM, Braumann CA. Stochastic differential equations harvesting policies: Allee effects, logistic-like growth and profit optimization. Appl Stochastic Models Bus Ind. 2020;36:825–835.

[2] Carlos C, Braumann CA. General population growth models with Allee effects in a random environment. Ecological Complexity. 2017;30:26–33.

[3] Dennis B. Allee effects in stochastic populations. Oikos. 2002;96(3):389–401.

[4] Allee WC. Principles of Animal Ecology. Philadelphia, Saunders; 1949.

[5] Brites NM, Braumann CA. Harvesting policies with stepwise effort and logistic growth in a random environment. In: Ventorino E, Aguiar M.A.F., Stollenwek N, Braumann CA, Kooi B, Pugliese A, editors. Dynamical Systems in Biology and Natural Sciences: 2020; Berlin: SEMA SIMAI Springer Series; 21. p. 95–110.

[6] Brites NM, Braumann CA. Fisheries management in random environments: Comparison of harvesting policies for the logistic model. Fisheries Research. 2017;195:238–246.

[7] Brites NM, Braumann CA. Fisheries management in randomly varying environments: Comparison of constant, variable and penalized efforts policies for the Gompertz model. Fisheries Research. 2019;216:196–203.

[8] Brites NM, Braumann CA. Harvesting in a random varying environment: optimal, stepwise and sustainable policies for the Gompertz model. Statistics Opt. Inform. Comput. 2019;7:533–544.

[9] Bellman R. Dynamic Programming. New Jersey, Princeton University Press; 1957.

[10] Fleming W, Soner H. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag New York; 2006.

[11] Hanson FB, Ryan D. Optimal harvesting with both population and price dynamics. Math. Biosci. 1998;148(2):129–146.

[12] Brites NM. Stochastic differential equation harvesting models: sustainable policies and profit optimization [PhD thesis]: Universidade de Évora; 2017.

[13] Braumann CA. Stochastic differential equation models of fisheries in an uncertain world: extinction probabilities, optimal fishing effort, and parameter estimation. In: Capasso V, Grosso E, Paveri-Fontana SL, editors. Mathematics in Biology and Medicine: 1985; Berlin: Springer; p. 201–206.

[14] Braumann CA. Introduction to Stochastic Differential Equations with Applications to Modelling in Biology and Finance. New York: John Wiley & Sons, Inc.; 2019.

[15] Iolov A, Ditlevsen S, Longtin A. Stochastic optimal control of single neuron spike trains. J. Neural Eng. NIH. 2014;11(4):1–22.