**Universidade de Évora - Escola de Ciências e Tecnologia**

Mestrado em Modelação Estatística e Análise de Dados

Área de especialização | Modelação Estatística e Análise de Dados

Dissertação

# Volatility Forecast for the Brazilian Stock Market (Bovespa Index).

João Paulo Lázaro Alter de Freitas Lins Santos

Orientador(es) | Manuel Joaquim Piteira Minhoto
Anabela Afonso
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A dissertação foi objeto de apreciação e discussão pública pelo seguinte júri nomeado pelo Diretor da Escola de Ciências e Tecnologia:

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Évora 2021

**Previsão de Volatilidade para o Mercado de Ações Brasileiro (Índice BOVESPA)**


# Resumo

A volatilidade é uma medida de risco que, a partir da análise do comportamento de um ativo durante um determinado período, indica a velocidade de variação entre a queda e a subida. Ativos altamente voláteis apresentam oscilações rápidas e que podem acontecer de forma muito acentuada. Estudos sobre volatilidade como orientação de investimentos e instrumento de classificação de risco têm sido uma estratégia amplamente utilizada no mercado de capitais. O presente trabalho tem como objetivo analisar e prever a volatilidade do Índice BOVESPA (**IBOV**) empregando modelos da família ARCH/GARCH, bem como se existe uma relação de longo, curto prazo (ou ambos) entre o IBOV e os indicadores macroeconômicos auxiliares usando modelos VAR/VEC/ARDL. As séries analisadas são dados diárias não sequenciais de 2 de janeiro de 2019 a 30 de abril de 2020 (322 registros), e o período de previsão é de 5 de maio de 2020 a 10 de julho de 2020 (42 registros). As seguintes variáveis foram utilizadas como indicadores auxiliares: o preço do barril do petróleo Brent em dólares americanos (**Brent**); o índice de diferença entre a taxa de retorno dos títulos brasileiros e a taxa oferecida pelos títulos do Tesouro Norte-Americano (**EMBI**); e a taxa de câmbio entre o dólar norte-americano e o real brasileiro (**BRL_USD**). Os resultados sugerem a utilização do modelo IGARCH(1,1) com erros GED e indicam que existe uma relação de curto prazo quando o IBOV é a variável dependente, embora o modelo ARDL tenha sido considerado insatisfatório.



**Palavras-chave**: GARCH, volatilidade, previsão, indicadores, macroeconometria.

**Volatility Forecast for the Brazilian Stock Market (BOVESPA Index)**

# Abstract

Volatility is a measure of risk that, based on the behavior analysis of an asset during a certain period, indicates the speed at which it varies between falling and rising. Highly volatile assets show rapid oscillations that can happen very sharply. Studies on volatility as investment guidance and risk classification instrument have been a strategy widely used in the capital market. The present work aims to analyze and predict the volatility of the BOVESPA Index (**IBOV**) employing ARCH/GARCH family models, as well as whether there is a long-term, short-term (or both) relationship between the IBOV and the auxiliary macroeconomics indicators using VAR/VEC/ARDL models. The series analyzed are non-sequential daily information from January 2, 2019, to April 30, 2020 (322 registers), and the forecast period is from May 5, 2020, to July 10, 2020 (42 registers). The following variables were used as auxiliary indicators: the price of a barrel of Brent oil in US dollars (**Brent**); the index of difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the North American Treasury (**EMBI**); and the exchange rate between the US dollar and the Brazilian real (**BRL_USD**). The results suggest the use of the IGARCH model (1,1) with GED errors and indicate that there is a short-term relationship when IBOV is the dependent variable, although the ARDL model was considered non-satisfactory.

**Keywords:** GARCH, volatility, forecast, indicators, macro econometrics.

# Summary

# Figures

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# 1. Introduction

In the last decades, there has been an outbreak in the number of investors seeking greater profitability, as well as trying to change their resource allocations beyond traditional investments, such as savings, for example. The market itself has created several investment options, satisfying both cautious and aggressive investors. Among these options, there are stocks, derivatives, fixed income securities, multimarket funds, private equity funds, private pension plans, gold, antiques, real estate, among others.

The term 'market' can be understood as the process by which people interested in selling a product or service meet people interested in buying that same product or service. Both buyers and sellers, after an analysis of their alternatives, carry out the transaction that best meets their needs, establishing an equilibrium price in a process known as the law of supply and demand (Oliveira & Pacheco, 2017).

Due to the advent of the pandemic caused by COVID-19, the current economic scenario is once more a volatile market with constant fluctuations, very difficult to make decisions, and much more opportunistic in terms of the search for a financial asset to be traded. The market was again "contaminated" with intense discussions about risk profile, which strategic position to take in the short-, medium-, or long-term, and when to "stop" a possible loss or gain to switch positions.

Considering the uncertainties created in moments with high volatility, risk analysis is increasingly necessary for making investment decisions and for managing projects/portfolios; and a good understanding of the past scenario can offer future opportunities as stated by William Sharp, in 1990:

> "*Although it is always perilous to assume that the future will be like the past, it is at least instructive to find out what the past was like. Experience suggests that for predicting future values, historic data appear to be quite useful with respect to standard deviations, reasonably useful for correlations, and virtually useless for expected returns*".

## 1.1. Theme Background and Justifications

Theoretically, the volatility of an asset represents changes in its prices due to several factors related to the performance of the issuing company and the economic situation. Factors related to the performance of companies are, for example, good/bad news about individual corporations, regarding their organizational, administrative, or economic-financial aspects, which can be, more specifically, the internal and external competition, the advent of substitute products, costs, and supply of inputs, environmental regulation, changes in taxation and changes in the company's management. The economic factors that can influence an asset's volatility include inflation, interest, and exchange rates; in addition to the legal/institutional aspect, oil prices, recession, and world growth.

The market uses volatility "to try to predict prices in future settlement contracts, that is, a measure of expected changes in prices based on changes recorded in the past. Thus, volatility indicates what the average price change of an object asset would be if the market repeated past variations" (Costa, 2002).

Forecasting volatility is particularly important in the Brazilian market, as "there is empirical evidence that Brazilian assets and indices are considerably more volatile than their North American, European, and Japanese counterparts" (Duarte, Pinheiro, & Heil, 1996).

## 1.2. Formulation of the Problem and Objectives

The adequate forecast of volatility allows agents operating in the market to obtain competitive advantages, as they anticipate market movement in scenarios with high instability, for example.

The main objective of the present study is to predict the volatility of the BOVESPA Index (IBOV) through ARCH/GARCH family models while using daily macroeconomic indicators, namely, the price of a barrel of Brent oil in US dollars; the index of difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the US Treasury; and the exchange rate between the US dollar and the Brazilian real as regressors for the IBOV mean.

As specific or secondary objectives, the statistical properties and predictive capacity of models, as well as whether there is a long-term, short-term (or both) relationship between the variables under consideration will be evaluated through VAR/VEC/ARDL models.

During the elaboration, this work generated the following questions:

- ✓ How is it possible to model daily BOVESPA Index volatility?
- ✓ Which selected macroeconomic indicators can influence the daily IBOV?
- ✓ Is there a causality relationship between the variables under consideration?

These questions motivated the emergence of the present work and, therefore, will be answered here.

## 2. Literature

The literature involving the use of statistical modeling to analyze volatility and the relationship between macroeconomic indicators, such as GDP, interest rates, commodities, inflation, currency parity, among others, is quite extensive and widespread over the years.

### 2.1. Literature Regarding Volatility

Gaio, Pesanha, Oliveira, & Ázara (2007) empirically analyzed the volatility of the BOVESPA index returns between January 03, 2000, and December 29, 2005, using Autoregressive Conditional Heteroskedastic (ARCH) models. The empirical results suggested strong signs of persistence and asymmetry in the volatility of the series' returns. In addition, all models of the estimated ARCH performed well, highlighting the Gaussian Exponential Generalized ARCH(1,1) (EGARCH(1,1)) model, with the best fit considering the quality criteria.

Jubert, Paixão, Monte, & de Lima (2008) used GARCH, Threshold ARCH (TARCH), and EGARCH models to analyze the volatility pattern of Brazilian stock indexes between 2006 and 2007, namely the BOVESPA Index (IBOVESPA) and the sectoral indexes: Electric Energy Index (IEE), Corporate Sustainability Index (ISE), Industrial Sector Index (INDX) and the Telecommunications Index (ITEL). According to the authors, "asymmetric models showed the leverage effect in which negative returns are more associated with clusters of volatility, that is, negative shocks cause greater instability in the stock market". The empirical conclusions corroborated the theoretical expectations since the symmetric and asymmetric models have reasonably similar estimates.

Abanto-Valle, Migon, & Lachos (2012) utilized a "stochastic volatility in mean (SVM) model using the class of symmetric scale mixtures of normal (SMN) distributions. The SMN distributions form a class of symmetric thick-tailed distributions that includes the normal one as a special case, providing a robust alternative to estimation in SVM models in the absence of normality”. The method was illustrated by analyzing daily stock return data from IBOVESPA. According to the model selection criteria, as well as out-of-sample forecasting, the authors discovered that the SVM model with slash distribution provides a significant improvement in model fit, as well as a prediction for the IBOVESPA data over the usual normal model.

Ceretta, da Costa, Righi, & Müller (2013) investigated “intraday data on a 10-minute interval and compared the major market index in South America, the IBOVESPA, and sync up with the S&P 500 in New York”. The main target was to “determine differences of volatility, in the Brazilian index, before and after the opening bell in New York”. The authors utilized the GARCH approach “matching up times that both markets were open and comparing to the hours that the Brazilian index was trading alone, without the direct influence of one of the North American main indexes, the S&P 500”. As a result, the authors were “able to disclose that this difference in volatility exists”.

Oliveira & Andrade (2013) compared the Bayesian estimates obtained for the ARCH family process parameters with the normal and Student distributions for the conditional distribution of the BOVESPA Index return series. According to the authors, the results showed that “the Bayesian approach provided satisfactory estimates and that the GARCH process with Student $t$-distribution adjusted better to the data”.

Maciel, Gomide, Ballini, & Yager (2013) performed a “nonlinear approach for realized volatility forecasting with jumps using a simplified evolving fuzzy system based on the concepts of data clouds”. The approach offered “an alternative nonparametric form of fuzzy rule antecedents that reflects the real data distribution without requiring any explicit aggregation operations or membership functions, thus providing a more autonomous and efficient algorithm. Empirical results based on the Brazilian Stock Market Index reveal the high potential of the evolving cloud-based fuzzy approach in modeling time-varying realized volatility with jump components, outperforming a traditional benchmark based on linear regression, as well as alternative evolving fuzzy systems”.

Kumar & Maheswaran (2014) provided a “framework to model and forecast daily volatility” based on the “additive bias-corrected extreme value volatility estimator (the Add RS estimator). Using the opening, high, low, and closing prices of S&P 500, CAC 40, IBOVESPA, and S&P CNX Nifty indices, the authors found that the logarithm of the Add RS estimator is approximately Gaussian and that a simple linear Gaussian long memory model can be applied to forecast the logarithm of the Add RS estimator. The forecast evaluation analysis indicated that “the conditional Add RS estimator provides better forecasts of realized volatility than alternative range-based and return-based models”.

Kumari & Mahakud (2015) empirically analyzed the link between macroeconomic volatility and stock market volatility in the Indian Stock Market between July 1996 and March 2013 with the aid of VAR and ARCH modeling, in which the conditional volatility was extracted using models of univariate autoregressive conditional heteroskedasticity. The results suggested the above-mentioned linkage.

Rotta & Valls Pereira, (2016) evaluated the financial contagion between stock market returns employing regime-switching dynamic correlation (RSDC) and using daily data from 1 February 2003 to 20 September 2012 with indices in the United States (S&P 500), the United Kingdom (FTSE100), Brazil (IBOVESPA), and South Korea (KOSPI). The authors modified the original RSDC model, introducing the Glosten-Jagannathen-Runkle GARCH models with normal residuals (GJR-GARCH-N) or t-Student residuals (GJR-GARCH-t), on the equation of conditional univariate variances, in order to capture the asymmetric effects in volatility and heavy tails. The study “compared the methodology with those frequently found in literature, and the model RSDC with two regimes was defined as the most appropriate for the selected sample with t-Student distribution in the disturbances. The adapted RSDC model can be used to detect contagion – considering the definition of financial contagion from the World Bank called very restrictive – with the help of the empirical exercise”.

Bruhn, Calegario, Campos, & Sáfadi (2016) investigated the effects of political and macroeconomic events on the volatility of foreign direct investment (FDI) flows received by the Brazilian economy through ARCH and VAR models and intervention analysis. The results suggested that political and macroeconomic events affect the volatility of FDI flows during the analyzed period. The results indicated "a negative effect of the corruption scandal known as the "*Mensalão scandal*[1]" and a positive effect of the "subprime crisis" and the payment of the Brazilian external debt on the volatility of Brazilian FDI inflows". The results of the VAR model indicated that FDI inflows "had a positive effect on future FDI inflows in the following three-months period".

Horta & Ziegelmann (2018) considered the "stochastic evolution of the probability density functions (PDFs) of IBOVESPA intraday returns over business days, in a functional time series framework". The authors found evidence that "the dynamic structure of the PDFs reduces to a vector process lying in a two-dimensional space". First, the authors provided "further insights into the finite-dimensional decomposition of the curve process: it is shown that its evolution can be interpreted as a dynamic dispersion-symmetry shift". Second, the authors presented "an application to realized volatility forecasting, with a forecasting ability that is comparable to those of Heterogeneous Auto-Regressive (HAR) realized volatility models in the model confidence set framework".

Maluf & Asano (2019) intended to "verify which models for the Value at Risk (VaR), among those that do not consider conditional volatility (Extreme Values Theory and the traditional Historical Simulation), and those that do consider it (GARCH and IGARCH)" are adequate for IBOVESPA. The authors implemented "backtesting of adherence and the independence of first and higher orders, over forecast horizons of 1 and 10 days. The results showed that only GARCH family models were adequate". Thus, the authors recommended the utilization of internal risk models based on conditional volatility in order to minimize the occurrence of violation clusters".

Reghin & Lopes (2019) compared the VaR calculation using the parametric method (Exponentially Weighted Moving Average (EWMA) and GARCH) against the use of

[1] "The Mensalão scandal (a Brazilian neologism for big monthly payments) has drawn headlines throughout the country since 2005. Money was allegedly pulled from slush funds of the governing Workers' Party (Partido dos Trabalhadores, PT) and dealt out to Brazilian Congressmen in order to sway votes" (Damgaard, 2015).

feedforward neural networks and Long Short-Term Memory (LSTM) recurrent networks regarding the IBOVESPA. The result showed that "LSTM networks had a better performance when comparing the exception rate generated by the entire model. When analyzing periods of crisis or abrupt changes in behavior, LSTM, and Feedforward networks were less efficient in predicting VaR compared to the parametric method".

Bezerra & Albuquerque (2019) employed machine learning techniques to forecast financial volatility. The authors implemented "a standard Support Vector Regression model with Gaussian and Morlet wavelet kernels on daily returns of two stock market indexes: USA (SP&500) and Brazil (IBOVESPA) over the period 2008 to 2016. The random walk, GARCH(1,1), and GJR(1,1) on the skewed t-Student distribution serve as comparison models by using Mean Squared Error (MSE) and the Diebold-Mariano test. The empirical analysis suggests that the SVR can beat the random walk model in the USA (S&P 500) and Brazilian (IBOVESPA) markets at a one-period ahead forecasting horizon".

Shaik & Maheswaran (2020) propose an unbiased robust volatility estimator, the Add Extreme Value Robust Volatility Estimator (AEVRVE) which, according to the authors, is 2-3 times more efficient than the Classical Robust Volatility Estimator (CRVE) based on extreme values of asset prices. The authors used "a procedure to remove the downward bias present in the data even without increasing the number of steps in the stock price path", as well as performed Monte Carlo simulations to "show the properties of unbiasedness and efficiency regarding the empirical data based on global stock indices, namely CAC 40, DOW, IBOVESPA, NIKKEI, S&P 500 and SET 50".

Vartanian (2020) studied the "contagion of volatility between the commodity prices and the IBOVESPA, through a multivariate GARCH model, to verify the possibility of diversification of investments. The research hypothesized that there is a strong relationship between the commodity prices and the IBOVESPA index, with the presence of the contagion effect, which inhibits the diversification of investments between the stocks that make up the index and the commodities on the international scene. The results partially corroborated the hypothesis formulated, since it was possible to observe a strong increase in conditional covariance between the two variables during the international financial crisis. On the other hand, the conditional correlation between the IBOVESPA and the commodity prices showed that the relationship between the variables was relatively small in the periods before and after the 2008 crisis, which suggests that concomitant investments in commodities and the IBOVESPA constitute a risk diversification strategy, contrary to what is commonly supposed".

## 2.2. Literature Regarding the Influence of Indicators

**EMBI**

Casotti (2010) analyzed the presence of a common factor affecting the volatilities of the exchange rate between Brazilian and North American currencies, the BOVESPA Index (IBOVESPA) and the Emerging Market Bond Index + Brazil (EMBI+ Brazil). The work intended to extract this latent factor applying the quasi-maximum likelihood estimator into an adapted stochastic volatility model using daily data between 2001 and 2009. The estimation required "the Kalman filtering on its diffuse version, once it is supposed that the volatilities follow a non-stationary process. The results indicated the presence of one common factor driving the mentioned volatilities, confirming the expectations".

Triandafil, Brezeanu, & Huidumac (2011) studied "the CEE (Central and Eastern Europe) countries volatility captured by the exchange rate dynamics". The volatility was analyzed from the perspective of the permanent and transitory dimensions. The authors concluded that "volatility is of long-term nature in the CEE countries, with a certain degree of peculiarity in terms of shock reaction. The authors also conducted an analysis on the key determinants of the exchange rate volatility wherein the following variables originating in financial markets were selected - EMBI spreads, Central Bank interest rate - as well as macroeconomic fundamentals - inflation, CROI index - in order to identify factors by which volatility pattern can be depicted. The key result of the research points toward a deep correlation of the exchange rate volatility between the CEE countries and the Euro Zone, implying the necessity to develop strong financial management strategies at the macroeconomic level, capable of annihilating the transmission belt crisis mechanisms".

Montes & Tiberto (2012) considered the "empirical evidence about (i) the influence of macroeconomic variables and economic policies on country risk (EMBI) and (ii) the influence of macroeconomic variables and country risk on the main Brazilian index of the stock market (IBOVESPA)". The period of analysis runs from December 2001 to September 2010. The study analyzed "the role that macroeconomic fundamentals plays, but also the role that the credibility of the regime of inflation targeting and the reputation of the central bank play in lessening country risk and in the improvement of the stock market performance. The empirical evidence was obtained through the application of ordinary least squares (OLS), generalized method of moments (GMM), and GMM systems. The results suggested that monetary policy and public debt management, as well as credibility and reputation affect country risk and the performance of the Brazilian stock market".

Villalba Padilla & Flores-Ortega (2014) studied the variance behavior of three variables: (a) the main Mexican stock market index (CPI), (b) the Emerging Markets Bond Index for Mexico (EMBI) as country risk indicator and, (c) the price of the Mexican crude oil basket exports mix (MEZCLA) that included three oil types (Olmeca, Istmo, and Maya). The study included daily data between December 31, 2001, and July 15, 2013. The results of the Granger causality test indicated that there is an interrelation between the three series. In the case of the IPC and the EMBI, it was found that there is causality from the IPC to the EMBI from the second day of lag, likewise, the findings indicated that the EMBI does not cause the IPC. On the other hand, it was verified that the causality is bidirectional between the IPC and the MEZCLA, considering that the MEZCLA causes the IPC in delay 2. Finally, the relationship between the EMBI and the MEZCLA is that the first series does cause the second, but not vice versa.

**Exchange Rate**

Ali, Ziaei, & Anwar (2012) studied the "impact of the global crisis over intertwining between exchange rates and stock prices is examined in Brazil. Average nominal exchange with US Dollar and BOVESPA stock exchange index on weekly basis covering the period from May 5, 2003, to September 6, 2011, is used for analysis. Furthermore, data were divided into three sub-periods i.e., pre-crisis, crisis period and post-crisis period Results of the unit root test revealed that data of both markets were found to be non-stationer)' and integrated at order one. The Johansen cointegration test is applied to investigate the movement of exchange rates and stock prices during three sub-periods. The results show that no proof of cointegration is found during the pre-crisis period but only single cointegration is found during and post-crisis period. Thereafter, the Granger causality test is applied which postulates that bilateral causality is found between exchange rates and stock prices in the pre-crisis period. It can be suggested with the help of results that both series are affected by each other in the short run. During crisis and post-crisis periods suggested that stock prices are significantly Granger cause to exchange rates".

Garcia (2015) evaluated the impact of macroeconomic factors on the flow of foreign investment in the Brazilian stock market for the period from 2002 to 2014. After performing the unit root test, the analysis considered vector autoregressive models to construct Granger causality tests, as well as the evaluation of impulse response functions and variance decomposition. The results indicated that the IBOVESPA, the R$/U$S exchange rate, the S&P 500, the US interest rate, the commodities prices, and previous lags of the foreign investment have the most influence in the reversal of foreign capital on the Brazilian Stock Market.

Duy (2016) examined the variability of the Vietnamese Stock Exchange between 2005 and 2014 using VECM modeling and with the aid of secondary indicators, such as the consumer price; industrial production; interest rate; exchange rate of the local currency over the US dollar; retail oil prices; gross domestic product and gold prices. The author highlighted the individualized impact of each indicator on the variable of interest.

**Oil**

Da Silva, Bertella, & Magner Pereira (2014) "investigated empirically if national macroeconomic variables (industrial production, inflation, real interest rate, domestic credit risk, and real exchange rate) and international ones (index of the American stock exchange, the U.S. interest rate and the price of oil) can explain the behavior of long and/or short-term index of the São Paulo Stock Exchange (BOVESPA) from 1995 to 2007, with monthly data. An econometric methodology of error correction models was applied resulting in evidence of a positive long-term relationship with the BOVESPA index to the American stock market and price of oil and negative for the U.S interest rate and credit risk. From a dynamic point of view, the BOVESPA index was identified as the only adjustment variable for deviations from a long-term relationship and significant temporal precedence for most variables, indicating the possibility of previous assessment of the future behavior of the Brazilian stock index".

Bagchi (2017) examined the "dynamic relationship between crude oil price volatility and stock markets in the emerging economies like BRIC (Brazil, Russia, India, and China) countries in the context of sharp continuous fall in the crude oil price in recent times. The author adopted an Asymmetric Power ARCH (APARCH) model which considers long memory behavior, speed of market information, asymmetries, and leverage effects. "Findings: For BOVESPA, MICEX, BSE Sensex, and crude oil there is an asymmetric response of volatilities to positive and negative shocks and a negative correlation exists between returns and volatility indicating that negative information will create greater volatility. However, for the Shanghai Composite positive information has a greater effect on stock price volatility in comparison to negative information. The study results also suggest the presence of long memory behavior and persistent volatility clustering phenomenon amongst crude oil price and stock markets of the BRIC countries. Originality/value: First, the author has considered crude oil prices up to January 31, 2016, so that the study can reflect the impact of the declining trend of crude oil prices on the stock indices which is also regarded as "new oil price shock" to measure the volatility between crude oil price and stock market indices of BRIC countries. Second, the volatility is captured by the APARCH model which considers long memory behavior, speed of market information, asymmetries, and leverage effects".

Gçdek (2017) analyzed the "impact of oil price changes on the Russian (RTS), Brazilian

(BOVESPA), and Norwegian (OSEAX) stock exchange index level" during the period from the beginning of July 2014 to the end of June 2017. "The analysis used an econometric model built in accordance with the Engel-Granger methodology. The results of this analysis showed that the impact of oil prices on financial markets of crude oil exporter countries in the period was very varied. That impact was most visible in the case of Russia, somewhat weaker in the case of Brazil (in both cases, the fall in oil prices affected the value of the index). That impact was not determined in the case of Norway".

Ferreira, Pereira, & Silva (2020) evaluated "how WTI (West Texas Intermediate) oil price shocks are related to the Brazilian economy as a whole, but also with each of the listed companies in IBOVESPA, searching for the relationships with different economic activities. Based on the Detrended Cross-Correlation Analysis coefficient", the authors "concluded unsurprisingly that the most affected sectors are those most related to the use of oil. However, another important result is the significant correlation between oil price shocks and the returns of the financial sector, showing this particular sector's exposure to oil, i.e., this is one of the sectors most correlated with oil returns. This is relevant not only for investors but also for authorities because possible future oil shocks could have a high impact on the Brazilian financial sector".

## 3. Analyzed Data

The data under consideration vary from January 2, 2019, to July 10, 2020, and reflects the 364 closing prices[2] of the Brazilian Stock Exchange (B3). The data was splitted into two subsets: **in-sample** (January 2, 2019, to April 30, 2020, containing 322 registers) and **out-of-sample** (May 5, 2020, to July 10, 2020, containing 42 registers). The first period denotes the information used for estimating the initial parameters and models selection, while the second period represents the information employed to evaluate forecast performance. Data analysis was performed using IBM SPSS Statistics, E-Views, Stata, and R. In all procedures, a significance level of 5% was adopted. The database comprises four variables and 1,288 unique observations that come from different sources, namely:

- BOVESPA Index (**IBOV**): Brazilian Stock Exchange (B3), b3.com.br;

- Price of the barrel of **Brent** oil in US dollars (barrel/dollar): Europe Brent Spot Price, eia.gov/dnav/pet/hist/RBRTED.htm;

- Exchange rate, for purchase, between the Brazilian real and the US dollar (**BRL_USD**): Institute of Applied Economic Research (Instituto de Pesquisa Econômica Aplicada), ipeadata.gov.br/Default.aspx;

- Emerging Markets Bond Index Plus (**EMBI+**): JP Morgan, jpmorgan.com.

EMBI+ is an index based on bonds (debt securities) issued by emerging countries. It shows the financial returns obtained each day by a selected portfolio of securities from these countries. The unit of measurement is the base point. Ten basis points are equivalent to a tenth of 1%. The observations show the difference between the return rate on emerging country bonds and the return rate offered by bonds issued by the US Treasury. This difference is the spread, also known as the sovereign spread, and was

[2] The sample ignores days without activity in the Brazilian stock exchange, for example holidays and weekends.

created to classify only countries that present a high level of risk according to rating agencies and that have issued bonds with a minimum value of US$ 500 million, with a maturity of at least 2.5 years.

### BOVESPA Index (IBOV or IBOVESPA)

The BOVESPA Index is the most important indicator of the average performance of the Brazilian Stock Market, as it portrays the behavior of the main assets traded in Brazil. This market has more than 300 companies listed on the Country's Stock Exchange, and the index currently comprises 75 stocks[3] (Appendix 1 - Composition of the BOVESPA Index). The index reflects not only the changes in share prices but also the impact of the distribution of earnings, being considered an indicator that assesses the total return of its component shares.

This indicator is the current value, in the local currency, of a theoretical portfolio of shares constituted on January 02, 1968, with a base value of 100 points, grounded on a hypothetical application.

The criteria for choosing the stocks that comprise the portfolio are based, above all, on their participation in the volume of business and their presence in the trading sessions.

**Representativeness of the IBOVESPA:**

✓ **In Terms of Liquidity**:

The stocks included in the theoretical portfolio of the BOVESPA Index account for more than 80% of the number of trades and financial volume verified in the spot market (round lots) of the Brazilian Stock Exchange (B3, 2021a).

✓ **In Terms of Stock Market Capitalization**:

[3] Four companies account for 38.30% of the total participation in the current IBOV, namely: Vale (10.15%), Petrobras (10.37%), Itaú (10.19%) and Bradesco (7.58%).

The companies present in the theoretical portfolio of the BOVESPA Index are responsible, on average, for approximately 70% of the sum of the stock market capitalization of all companies with tradable assets.

The BOVESPA Index is nothing more than a weighted sum of the stock prices included in its theoretical portfolio. The stocks are weighted by the market value of the company, i.e., the price of the stock times the number of shares outstanding. Therefore, the BOVESPA index is calculated using the following formula:

$$T_t = \sum_{i=1}^{n} P_{it} \, Q_{it}$$

where:

· $T_t$ is the BOVESPA Index at instant $t$;

· $n$ is the total number of stocks comprising the theoretical portfolio;

· $P_{it}$ is the last price of stock $i$ at instant $t$;

· $Q_{it}$ is the theoretical quantity of stock $i$ in the portfolio at instant $t$.

The theoretical portfolio of IBOVESPA is valid for four months and has its selection updated in the periods from January to April, May to August, and September to December. The new list of stocks that are part of the index takes effect on the first Monday of the first month of each period, that is, in January, or May, or September.

This frequent reassessment makes it possible to add or exclude the stock index that started to meet or failed to respect some of the pre-established criteria. According to the methodology available at b3.com.br, eligible stocks which may be selected as IBOV constituents are those that meet the following cumulative criteria:

**(i)** Being amongst the eligible stocks which, as measured in descending order by individual Tradability Ratio (IN) over a period comprising the three (3) previous portfolio cycles, collectively account for 85.0% of the total of such metric;

**(ii)** Having actively traded in at least 95.0% of the trading sessions held over a period comprising the three previous portfolio cycles;

**(iii)** Having accounted for no less than 0.1% of the overall value traded on the cash equity market (round lots) over a period comprising the three previous portfolio cycles;

**(iv)** Not being classified as a penny stock, whose weighted average value during the term of the theoretical portfolio before the rebalancing, disregarding the last day of that period, was less than R$ 1.00 (one real).

It should be highlighted that these requirements were the same for 50 years, from 1968 when the IBOVESPA was created. However, in 2014, there were some modifications[4] and the calculation of the index was changed. The stocks that meet the criteria required by the Stock Exchange are organized according to the Tradability Ratio (IN), which is "calculated to consider 1/3 of a component's share of the overall number of trades and 2/3 of the component's share of the overall value traded on the cash equity market" (B3, 2021b).

[4] No company could represent more than 20% in the composition of the IBOVESPA and that the shares that make up this theoretical portfolio must correspond, cumulatively, to 85% of the negotiations that occurred in the period.

# 4. Theoretical Grounding

Some econometric concepts are extremely important for understanding the study in this area and will be described individually. Those concepts are return (arithmetic return and geometric return), risk (market risk, operational risk, credit risk, and legal risk), volatility (historical volatility and implied volatility), and conditional heteroskedasticity.

## 4.1. Return

It is defined as a price variation, or index, that occurred in a certain period. It can also be understood as capital appreciation at the end of the investment horizon. Without considering dividends, the returns can be calculated in two ways:

✓ **Arithmetic return**: if the asset price at period $t$ is denoted by $P_t$, the return $R_t$ between instants $t$-$1$ and $t$, the arithmetic return is:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1 \quad \Rightarrow \quad 1 + R_t = \frac{P_t}{P_{t-1}}$$

✓ **Geometric Return**: logarithmic or log-return ($r_t$) is defined as the log of the ratio between successive prices and can be defined as:

$$r_t = log\left(\frac{P_t}{P_{t-x}}\right) = log\left(\frac{P_t}{P_{t-1}}\right) + log\left(\frac{P_{t-1}}{P_{t-2}}\right) + \cdots + log\left(\frac{P_{t-x+1}}{P_{t-x}}\right)$$

If the variations are small, it can be proved through Taylor series expansion that the geometric and arithmetic returns are similar:

$$r_t = log(1 + R_t) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} R_1^k = R_t - \frac{1}{2}R_t^2 + \frac{1}{3}R_t^3 - \cdots$$

If $R_t$ is small, the parcels with values of $R_t^2$ and higher-order will tend to zero, which will result in:

$$r_t \approx R_t$$

The log-return is more often employed due to its statistical properties, such as stationarity (Chapter 5.1) and ergodicity[5]. "The advantage of log-return is that its distribution can never lead to a negative price, which is consistent with the idea that this variable is always greater or equal to zero" (Jorion, 2004). Other relevant information refers to the use of the log-return in the comparison of assets in different measures, due to the converging properties due to the application of the logarithm.

### 4.2. Risk

Risk can be generally defined as "the possibility of occurring unexpected results" (La Rocque & Lowenkron, 2004). Therefore, market risk can be defined as a result of the chances of losses/gains resulting from possible changes in the prices of an Institution's active or passive variables.

In finance theory, the relationship between risk and returns plays a significant role, and many theoretical models are utilized to study this linkage. For example, the Capital Asset Pricing Model (CAPM[6]) is "widely used in applications, such as estimating the cost of equity capital for firms and evaluating the performance of managed portfolios. The attraction of the CAPM is its powerfully simple logic and intuitively pleasing predictions about how to measure risk and about the relation between expected return and risk"

---

[5] "The ergodic hypothesis is a key analytical device of equilibrium statistical mechanics. It underlies the assumption that the time average and the expectation value of an observable are the same" (Peters, 2019).

[6] "The Capital Asset Pricing Model (CAPM) for a security is a linear relationship between the expected excess return of the security and the expected excess return of the market. It was developed by William Sharpe, John Lintner and Jan Mossin. It is a useful framework to discuss idiosyncratic and systematic risk. The security market line is a powerful graphical construct of the CAPM. While the CAPM has strong underlying assumptions, recent research has relaxed many of these assumptions" (Bhattacharya, 2016).

(Fama & French, 2004).

Financial risks can be classified into four categories: market risk, operational risk, credit risk, and legal risk (Duarte M. A., 1996). A representation of the types of financial risks can be observed in the diagram of Figure 1.

- ✓ **Market Risk**: "depends on the behavior of the asset's price under market conditions. In order to understand and measure possible losses due to market fluctuations, it is important to identify and quantify, as accurately as possible, the volatilities and correlations of the factors that impact the asset's price dynamics".

- ✓ **Operational Risk**: "related to possible losses as a result of inadequate systems and/or controls, management failures, and human errors".

- ✓ **Credit Risk**: "linked with possible losses when one of the contractors does not honor its commitments. The losses here are related to the values that will no longer be received".

- ✓ **Legal Risk**: "associated with possible losses when a contract cannot be legally supported. This category includes the risks of losses due to insufficient documentation, insolvency, illegality, lack of representation, and/or authority on the part of a negotiator, *et cetera*".

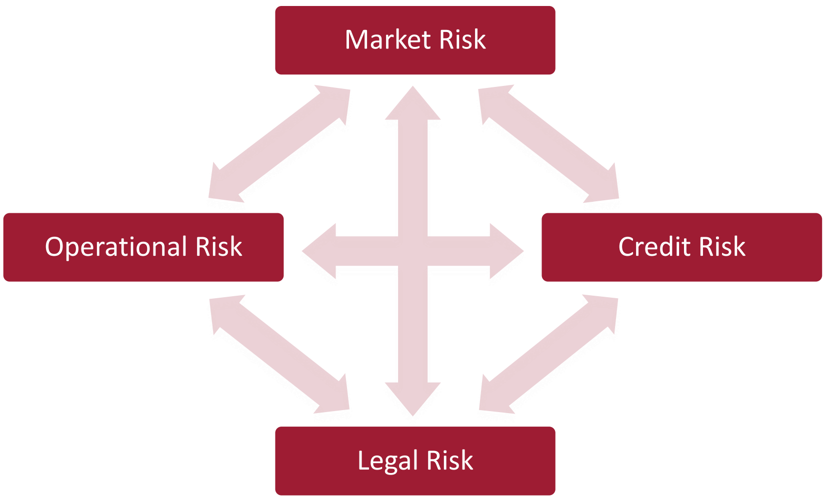



**Figure 1 - Types of Financial Risks**
- adapted from (Duarte M. A., 1996) -

## 4.3. Volatility

Volatility is a crucial management tool regarding the frequency/intensity of fluctuations in assets price and uncertainty associated with the financial investment in a determined period. When the volatility of a financial asset increases, so does the expected variation of the asset value, as well as its risk. On the other hand, lower volatility indicates small variations in the asset value. "Empirical results show that the volatility of a stock is much higher when the exchange is open than when it is closed. This suggests that, to some extent, trading itself causes stock price volatility" (Hull, 2013).

Measuring volatility is not like measuring price because, while the price is something that can be observed at the current instant of time (directly), instantaneous volatility is not observable and requires price changes over time.

There are two approaches to estimate the volatility (Hull, 2013):

**(i)** **Historical Volatility (HV)** - is a dispersion measure of returns on a particular asset or market index over a predetermined period. Usually, it is quantified as the standard deviation of financial instrument´s returns in the given period (the larger the standard deviation value, the greater the volatility).

The standard deviation is the most common, but not the only, way to estimate volatility. Especially, the historical volatility, also referred to as the asset's actual or realized volatility, can be modeled using ARCH/GARCH family models (Chapter 6.5).

It is important to highlight that, although volatility and standard deviation are statistics that quantify fluctuations (herein, of the returns), these concepts should not be interpreted as synonyms. While the second can be applied more broadly (weight, height, quantity, volume, counts, *et cetera*), volatility can be understood as "the standard deviation over time".

For a sample of observed returns, the historical volatility (HV) on day $t$ is calculated as follow (Lahmiri, 2017):

$$HV_t = \left[\left(\frac{1}{n-1}\right)\sum_{t=1}^{n}(R_t - \mu_R)^2\right]^{1/2}$$

where:

· $R_t$ is the return (for example, the logarithmic return) observed on time $t$;

· $\mu_R$ is the mean of the observed returns; and

· $n$ is the sample size.

**(ii)** **Implied Volatility (IV)** - also known as projected volatility, refers to the market's perception of how volatile a stock is likely to be in the future (the expected magnitude of a stock's future price changes, as implied by the stock's option prices). The IV is represented as an annualized percentage.

The critical question is to "examine whether volatility implied in the structured warrant's or option's price is an unbiased forecast of future volatility of the underlying asset over the option's maturity" (Samsudin, Mohamad, & Sifat, 2021). In this context, and considering all financial activities (risk management, pricing of derivatives and hedging, portfolio selection, *et cetera*) there is a need to forecast volatility.

It is important to highlight that the asymmetry of volatility, known as the leverage effect, is greater when returns are negative if compared to positive returns. This phenomenon is more accentuated during large declines, indicating that volatility is sensitive to reductions in the average asset price.

**Why predict volatility?**

Volatility in asset prices can affect the level of investment, as holders (investors) would be more afraid to make new investments in countries that are excessively volatile. Holders could interpret this volatility as an increase in risk in the return on assets and, thus, tend to seek lower-risk assets in other markets.

In addition to the opportunity to make big gains, the excess of volatility can also increase the cost of companies when financing an expansion project, as well as making the IPO[7] unfeasible since, in a situation of high volatility, investors will tend to prefer shares in well-known and large companies. This makes new issues of shares or IPOs by unknown or small firms difficult, or even such behavior would be nullifying the institutional function of the stock exchange.

### 4.4. Conditional Heteroscedasticity

In statistics, if the variance of a data set remains constant over time, the series is considered homoscedastic (Bollerslev, 1986), otherwise is considered heteroscedastic. In this context, when a time series presents a non-constant (fluctuating) variance over time is related to previous periods (for example, daily) of a time series, this series is considered conditional heteroskedastic.

The specific behavior of time-varying variance can be observed, for example, in the series of financial returns, which is expressed by the existence of volatility clustering over time, that is "periods in which they exhibit wide swings for an extended period followed by a period of comparative tranquility" (Gujarati & Porter, 2008).

Since close-in-time returns tend to show similar levels of variability, shocks occurring at time $t$ will influence the volatility during $n$ periods ahead, explaining the persistence in volatility. On the other hand, in the long run, assuming that the unconditional variance is finite, periods of high volatility tend to be followed by periods of less volatility, and *vice versa*. This characteristic of volatility is known as mean reversion (Mandelbrot, 1963).

For example, elections tend to generate momentary instabilities, which are reflected in more unstable/volatile indicators at specific instants in the analyzed time series. This

[7] Initial Public Offering (IPO): process of transition from a private company to a public company becoming listed on the Stock Exchange by offering shares in a new issuance. It allows a company to raise capital while open new investment opportunities and attract new investors to the market.

leads to thinking about a model whose variance strongly depends on its previous values and the values of recent returns (Leblang & Mukherjee, 2004). These models are referred to in the literature as conditional heteroscedasticity models. In these models, the unconditional variance remains constant, what changes is the conditional variance of the series.

## 5. Stylized Facts of the Financial Series

Stylized facts are "statistical properties that appear to be present in many empirical asset returns. It is important to be aware of them because when building models that are supposed to represent asset price dynamics, the models must be able to capture/replicate these properties" (Lewinson, 2020).

When looking at a projected graph of series of financial returns (arithmetic or geometric), it is possible to observe that they vary around a breakeven point, the series' unconditional mean. In addition, the returns tend to demonstrate an excess of kurtosis (leptokurtic curve); negative asymmetry (mean < median < mode) with large negative returns occurring more frequently than large positive ones; shows a small degree of similarity between observations at successive time intervals; and last, but not least, the series tends to present the volatility clustering effect.

Financial series have specific behaviors, prices are usually non-stationary, non-normality distributed, and have a well-defined relationship between current and past values, while returns are typically stationary, also non-normality distributed, and do not tend to depend on past values.

Before moving forward, it is crucial to present two concepts that are also important to better understand the stylized facts of financial series: **stochastic process** and **white noise**.

"The word 'stochastic' comes from the Greek word 'stokhos' meaning "a bull's eye." The outcome of throwing darts on a dart board is a stochastic process, that is, a process fraught with misses" (Gujarati & Porter, 2008). In a more formal context, a stochastic process can be defined as a collection of random variables ordered in time $\{Y_t, t \in T\}$, where $T$ is an ordered set of indices.

The white noise is a stochastic process where random variables have a mean equal to zero, constant variance, and the covariances are null. In a more formal context, let a sequence $\{Y_t\}_{t=+\infty}^{-\infty}$, if $E(Y_t) = 0, \forall\ t$; $Var(Y_t) = \sigma_Y^2, \forall\ t$; $E(Y_t Y_{t-j}) = 0$, for $j \neq 0$, the process $Y_t$ is said to be white noise. When white noise presents normal distribution, $Y_t \sim N(0, \sigma_Y^2)$, then it is called a Gaussian white noise.

## 5.1. Stationarity

"A time series is stationary if its mean and variance do not vary systematically over time" (Gujarati & Porter, 2008). A time series $Y_t$, with $E(Y_t^2) < \infty$, is **weakly stationary** if $E(Y_t)$ is independent of $t$ and $Cov(Y_t, Y_{t-h}) = E\big[\big(Y_t - E(Y_t)\big)\big(Y_{t-h} - E(Y_{t-h})\big)\big]$ is independent of $t$ for each $h$ (Brockwell & Davis, 2016).

Regression models that involve non-stationary time series can generate inconsistent results. If the stationarity condition is not satisfied (e.g., the process presents trend or seasonality), the raw data can be transformed using, for example, differencing, Box-Cox, or other transformation techniques. The present work will utilize the first one.

Differencing transformation: If $Y_t$ denotes a time series that is non-stationary regarding the mean, then the differenced series can be written as $\Delta Y_t = Y_t - Y_{t-k}$, where $k = 1, 2, 3, \ldots$ represents the number of lags used to transform a non-stationary process into a stationary process. For example, if $Y$ become stationary using $k = 1$, it can be said that the first difference series $\Delta^1 Y_t$, or simply $\Delta Y_t$, is a stationary process, wherein $\Delta$ is known as the difference operator.

The difference operator ($\Delta$) is linked to the lag operator ($L$). Since $L^1 Y_t = Y_{t-1}$, $L^2 Y_t = Y_{t-2}, \ldots,\ L^k Y_t = Y_{t-k}$, the difference $\Delta Y_t$ can be written as $\Delta Y_t = Y_t - Y_{t-k} = Y_t - L^k Y_t = (1 - L^k) Y_t$. The lag operator will be useful to explain the stationary and invertibility conditions (Chapter 6.1).

**Examining Stationarity**

To investigate stationarity, it is possible to utilize graphs, such as a simple plot of the series to verify the presence of a trend and/or seasonal component, as well as Hodrick-

Prescott filter; and tests such as Augmented Dickey-Fuller, Phillips-Perron, and KPSS.

The Hodrick-Prescott filter is a tool named after Robert Hodrick and Edward Prescott that additionally decomposed an original time series ($Y_t$) into a non-stationary long-trend (growth) component ($G_t$) and a stationary short-term cyclical component ($C_t$). The HP filter equation can be expressed as $Y_t = G_t + C_t,\ t = 1, 2, 3, \dots$ (Hodrick & Prescott, 1997). The authors isolated $G_t$ from $Y_t$ employing a minimization problem:

$$min\ [\{G_t\}_{t=-1}^{T}] \left\{ \sum_{t=1}^{T} (Y_t - G_t)^2 + \lambda \sum_{t=1}^{T} (G_t - G_{t-1}) - (G_{t-1} - G_{t-2})^2 \right\}$$

where $\lambda$ is a smooth parameter.

Although being very popular and widely employed, there are some drawbacks of the above-mentioned filter (Hamilton, 2018). In this context, the presence/absence of stationarity must be analyzed in conjunction with specific tests, such as ADF, PP, and KPSS.

The Augmented Dickey-Fuller (ADF) test was created by David Dickey and Wayne Fuller and is used to analyze if the series has a unit root[8] (i.e., the series is not stationary). It consists of estimating the following regression (Gujarati & Porter, 2008) [*apud* (Fuller, 1999)]:

$$\Delta Y_t = \beta_0 + \beta_1 t + \delta Y_{t-1} + \sum_{i=1}^{m} \alpha_i\, \Delta Y_{t-i} + \varepsilon_t$$

where:

· $Y_t$ is the value of the time series at time $t$ and $\Delta Y_t = Y_t - Y_{t-1}$, $\Delta Y_{t-1} = Y_{t-1} - Y_{t-2}$, $\Delta Y_{t-2} = Y_{t-2} - Y_{t-3}$, and so on;

· $\beta_0$ is the intercept, also called the series drift;

[8] The terms nonstationary, random walk, and unit root can be treated as synonymous (Gujarati & Porter, 2008).

· $\beta_1$ is the trend coefficient;

· $\delta$ is the unit root presence coefficient;

· $t$ is the time or trend variable;

· $m$ is the number of lags taken in the series; and

· $\varepsilon_t$ is a pure white noise error term.

The null hypothesis is given by $H_0$: $\delta = 0$ (the series has a unit root and, therefore, is not stationary) and the test statistic is given by:

$$DF = \frac{\hat{\delta}}{s(\hat{\delta})}$$

where $s(\hat{\delta})$ is the standard error of $\hat{\delta}$. The DF statistic has a specific distribution known as the Dickey-Fuller table.

The Phillips-Perron (PP) test was created by Peter Phillips and Pierre Perron and is a generalization of the Dickey-Fuller test, with the same null hypothesis, for the cases in which the errors $\varepsilon_t$ are correlated and, possibly, heteroscedastic. The PP test is more robust than the ADF test and "use nonparametric statistical methods to take care of the serial correlation in the error terms without adding lagged difference terms" (Gujarati & Porter, 2008). The PP test (considering the same regression of the ADF test) utilizes the following test statistic (Phillips & Perron, 1988):

$$Z = n\hat{\delta}_n - \frac{n\hat{\sigma}^2}{2s_n^2}\left(\hat{\lambda}_n^2 - \hat{\gamma}_{0,n}\right)$$

in which:

$$\hat{\gamma}_{j,n} = \frac{1}{n}\sum_{i=1+j}^{m} r_i\, r_{i-j} \quad ; \quad \hat{\lambda}_n^2 = \hat{\gamma}_{0,n} + 2\sum_{j=1}^{q}\left(1 - \frac{j}{q+1}\right)\hat{\gamma}_{j,n} \quad ; \quad s_n^2 = \frac{1}{n-k}\sum_{i=1}^{n} r_i^2$$

where:

· $r_i$ is the residual in $y_t$ using least-squares estimators;

· $k$ is the number of covariates in the regression; and

· $q$ is the number of lags used to calculate $\hat{\lambda}_n^2$.

If the process is not correlated, we have null covariance and, in this case, $\hat{\lambda}_n^2 = \hat{\gamma}_{0,n}$. If the process is not heteroscedastic, we have that $\delta = 1/n$ and then $Z$ is given by $Z = {}^{\hat{\delta}}\!/_{s(\hat{\delta})}$.

The KPSS test was created by Denis Kwiatkowski, Peter Phillips, Peter Schmidt, and Yongcheol Shin, and the acronym KPSS test is due to their names. The null hypothesis of this test is the same as the **alternative** hypotheses of the Dickey-Fuller and Phillips-Perron tests, i.e., $H_0$: the series has no unit root and, therefore, is stationary.

The test decomposes the time series $(Y_t)$ into three parts: a random walk $(R_t)$, a deterministic trend $(\beta_t)$, and a stationary error $(\varepsilon_t)$ with the additional regression equation: $Y_t = R_t + \beta_t t + \varepsilon_t$. Considering an addition regression wherein $\varepsilon_t$ is modeled with the three above-mentioned components as regressors plus an intercept, the test statistic for the KPSS test is given by Kwiatkowski, Phillips, Schmidt, & Shin (1992):

$$KPSS = \left(n^{-2}\sum_{t=1}^{n} {S_t}^2\right) / \ \hat{\sigma}_\varepsilon^2$$

where:

· $S_t$ is the sum of residuals of the addition regression;

· $n$ is the number of observations of the addition regression;

· $\hat{\sigma}_\varepsilon^2$ is an estimative for the variance of errors of the addition regression.

In summary, the three tests (Augmented Dickey-Fuller, Phillips-Perron, and KPSS) can be used in conjunction to investigate the absence/presence of unit root.

## 5.2. Autocorrelation

Autocorrelation (or serial correlation) can be understood as the degree of association between $Y$ at instant $t$ ($Y_t$) and its past values ($Y_{t-k}$), and its future ($Y_{t+k}$) values, separated by the same interval $k$, measuring the dependency relationship among observations. It can be studied through the ordering of values in time (time series) or space (spatial data). The autocorrelation coefficient at lag $k$ for a **stationary** time series is (Montgomery, Jennings, & Kulahci, 2015):

$$\rho_k = Cor(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)} \times \sqrt{Var(Y_{t-k})}} = \frac{\gamma_k}{\gamma_0}$$

where:

· $Cov(Y_t, Y_{t-k}) = E[(Y_t - \mu_Y)(Y_{t-k} - \mu_Y)] = \gamma_k$ is the autocovariance at lag $k$;

· $k = 0, 1, 2, \ldots$;

· $\rho_0 = 1$;

· $-1 \leq \rho_k \leq 1$; and $\rho_k = \rho_{-k}$;

· The plot of $\rho_k$ over $k = 0, 1, 2, \ldots$ is called the **correlogram**.

Since autocovariance and autocorrelation are measures of linear dependence between $Y_t$ and $Y_{t-k}$ and taking into account that $Var(Y_t)$ and $Var(Y_{t-k})$ are positive, $Cov(Y_t, Y_{t-k})$ and $Cor(Y_t, Y_{t-k})$ always have the same sign. In this context, if $Cov(Y_t, Y_{t-k}) > 0 \implies Cor(Y_t, Y_{t-k}) > 0$, then $Y_t$ and $Y_{t-k}$ move in the same direction and are positively correlated. On the other hand, if $Cov(Y_t, Y_{t-k}) < 0 \Leftrightarrow Cor(Y_t, Y_{t-k}) < 0$, then $Y_t$ and $Y_{t-k}$ move in the opposite direction and are negatively correlated.

If $Y_t \perp Y_{t-k} \implies Cor(Y_t, Y_{t-k}) = 0$, but zero correlation does not imply independence. If $Cor(Y_t, Y_{t-k}) \neq 0$, then there is a linear relationship between $Y_t$ and $Y_{t-k}$ and we can say that $Y_t$ and $Y_{t-k}$ are autocorrelated.

**Autocorrelation Function (ACF)**

The collection of the values of $\rho_k$ ($k = 0, 1, 2, \ldots$) is called the autocorrelation function (ACF). The analysis of the autocorrelation function is a standard procedure for investigating the dependence implied by a time series model since it measures the length and memory of a process, that is, the extent to which the value at time $t$ depends on that at time $t - k$.

**Partial Autocorrelation Function (PACF)**

The partial autocorrelation measures the correlation between time series observations that are separated by $k$ lags, after adjusting for correlations at intermediate lags (i.e., lag less than $k$). In other words, "partial autocorrelation is the correlation between $Y_t$ and $Y_{t-k}$ after removing the effect of the intermediate Y's" (Gujarati & Porter, 2008).

**Examining Autocorrelation**

ACF and PACF plots (correlograms) are useful tools when identifying the number of parameters of autoregressive (AR) and moving average (MA) models (Chapter 6.1). If the ACF decays exponentially (with or without signal alternation) and the PACF presents an abrupt cut after lag $p$, then a process $Y_t$ is an AR($p$). On the other hand, if the ACF has an abrupt cut after lag $q$ while the PACF shows a gradually decaying (with or without signal alternation), then $Y_t$ is an MA($q$). Least but not last, if the ACF and PACF decay exponentially (with or without signal alternation) after lag $p$ and $q$ (respectively), then the process is an ARMA $(p, q)$.

In addition, the autocorrelation function and the partial autocorrelation function support the evaluation regarding the presence/absence of autocorrelation involving the residual of an estimated model (Chapter 8).

The presence of autocorrelations of estimated residuals must be examined since an adequate model should present residuals that are approximately white noise. To check the overall acceptability of the residual autocorrelations, the present work utilizes the

Breusch-Godfrey (BG) test, named for Trevor Breusch and Leslie Godfrey.

According to the literature, the BG test is more powerful when compared to traditional tests such as Durbin-Watson and Ljung-Box (Maddala & Lahiri, 2009). In summary, the BG test is indicated for endogenous and exogenous variables (Verbeek, 2017) and "allows for: (1) non-stochastic regressors, such as the lagged values of the dependent variable; (2) higher-order autoregressive schemes; and (3) simple or higher-order moving averages of white noise error terms" (Gujarati & Porter, 2008).

In this context, the BG Test can detect autocorrelation up to any predesignated order $m$ lag (Gujarati & Porter, 2008). With the null hypothesis that $H_0$: $\rho_1 = \rho_2 = \cdots = \rho_k = 0$, it is (essentially) a test of lack of fit: if the autocorrelations of the residuals are very small, the model does not show 'significant lack of fit'. The test is carried out as follows (Verbeek, 2017):

1 - Run an OLS regression $Y_t = \beta_0 + \beta_1 X_{t1} + \cdots + \beta_k X_{tk} + u_t$ and collect its residuals ($\hat{u}_t$);

2 - Use $\hat{u}_t$ from step 1 as a dependent variable, and run an auxiliary regression $\hat{u}_t = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k + \hat{\rho}_1 \hat{u}_{t-1} + \hat{\rho}_2 \hat{u}_{t-2} + \cdots + \hat{\rho}_k \hat{u}_{t-k} + \varepsilon_t$.

3 - Based on the $R^2$ of the regression of step 2, calculate the LM (Lagrange Multiplier) statistic:

$$BG_{LM} = (n - k)R^2 \sim \chi^2_k$$

where $n$ is the original sample size and $k$ is the number of lags.

If the null hypothesis is rejected, there is at least one $\rho_i$ significantly different from zero and the autocorrelation phenomenon is present.

## 5.3. Causality

Confusion between the terms “correlation” and “causality” underlies many misunderstandings since correlation does not necessarily imply causality. While the correlation measures and quantify the degree of association between variables, it is harder to determine whether one variable causes another variable. In other words, “the existence of a relationship between variables does not prove causality or the direction of influence” (Gujarati & Porter, 2008).

**Examining Causality**

Clive Granger developed the Granger’s Causality test on time series data that provides evidence if variable $X$ causes $Y$, i.e., $X$ contains useful information for predicting $Y$. The test is based on the following OLS regression models (Gujarati & Porter, 2008):

$$Y_t = \alpha_0 + \sum_{j=1}^{n} \alpha_j Y_{t-j} + \sum_{i=1}^{n} \beta_i X_{t-i} + \varepsilon_{1t}$$

$$X_t = \lambda_0 + \sum_{i=1}^{n} \lambda_i X_{t-i} + \sum_{j=1}^{n} \delta_j Y_{t-j} + \varepsilon_{2t}$$

The null hypothesis is that the $X$ does not ‘Granger-cause’ $Y$ in the first regression ($H_0\colon \beta_1 = \beta_2 = \cdots = \beta_i = 0$) and that $Y$ does not ‘Granger-cause’ $X$ in the second regression ($H_0 = \delta_1 = \delta_2 = \cdots = \delta_{i1} = 0$).

It should be highlighted that Granger’s Causality test assumes that and that $\varepsilon_{1t}$ and $\varepsilon_{2t}$ are uncorrelated. The test also assumes that both the $X$ and $Y$ time series are stationary. If this is not the case, an appropriate transformation, such as differencing, must first be employed before using Granger’s Causality test. The steps involved in the Granger causality test are as follows (Gujarati & Porter, 2008):

1. Regress current $Y$ on all lagged $Y$ terms and other variables, if any, but do not include the lagged $X$ variables in this regression. Collect the restricted residual sum of squares

($RSS_1$) from this regression.

2. Run the regression including the lagged $X$ terms. Collect the unrestricted residual sum of squares ($RSS_2$) from this regression.

3. To test the null hypothesis that $H_0: \beta_1 = \beta_2 = \cdots = \beta_i = 0$ (lagged $X$ terms do not belong in the regression) apply the $F$ test given by:

$$F = \frac{(RSS_1 - RSS_2)/_m}{RSS_2/_{(n-k)}} \sim F_{m,n-k}$$

where $m$ is the to the number of lagged $X$ terms, and $k$ is the number of parameters estimated in the unrestricted regression.

4. If $F_{obs} > F_{m,n-k,1-\alpha}$, where $F_{m,n-k,1-\alpha}$ is the quantile of probability $1-\alpha$ of the $F_{m,n-k}$ distribution, the null hypothesis is rejected. Meaning that the lagged $X$ can be used to explain $Y$. This is another way of saying that $X$ Granger-causes $Y$.

5. Steps 1 to 5 can be repeated to test whether $Y$ Granger-causes $X$.

The causality analysis will be employed in the present work to better understand the relationship between the variables under consideration (IBOV, Brent, EMBI, and BRL_USD).

## 5.4. Cointegration

Since many time series datasets move together fluctuating around a long-run equilibrium, the concept of cointegration is widely used in econometrics in order to compare financial series and investigate whether there is any kind of long-term relationship between them, making it possible to improve the investments decision-making, optimizing the results and minimizing losses (Maia, 2011).

Integration is intrinsically linked to stationarity. If we have a non-stationary process $Y_t$ that must be differenced $k$ times to achieve stationarity, then $Y_t$ is designated as an integrated process of order $k$, or simply $I(k)$. In this context, if $Y_t$ is stationary, then it is called integrated of order $0$, or simply $I(0)$.

The concept of cointegration "applies when two series are $I(1)$, but a linear combination of them is $I(0)$; in this case, the regression of one on the other is not spurious, but instead tells us something about the long-run relationship between them" (Wooldridge, 2015). In other words, considering two series, integrated of order one, $Y_t$ and $X_t$, and a supposed linear relationship between them is reflected in the proposition that there exist some value $\beta$ such that $Y_t - \beta X_t$ is $I(0)$, although $Y_t$ and $X_t$ are both $I(1)$. In this circumstance, it is said that $Y_t$ and $X_t$ are cointegrated, and that they share a common trend (Verbeek, 2017).

**Examining Cointegration**

To determine whether two or more time series are cointegrated, the following hypotheses must be verified: (i) The two series are stationary of order $I(0)$; (ii) There is at least one linear combination of time series to which the residual of the regression between them is stationary (Banerjee, Dolado, Galbraith, & Hendry, 1993).

With these hypotheses satisfied, it can be said that there is a cointegration relationship between the time series studied. The present work will use Johansen's Cointegration test[9] and Bounds test.

The first test, named after Søren Johansen, is an asymptotic approach that analyzes the cointegrating relationships between numerous non-stationary time series. By checking the presence of cointegration of various time series it is possible to avoid issues generated when errors are carried forward to the next steps of the investigation. There are two types of Johansen's test: Trace ($\lambda_{trace}$ statistic); and Maximum Eigenvalue ($\lambda_{max}$

[9] "The details of the Johansen procedure are very complex. Further details can be found in (Johansen & Juselius, 1990); (Johansen, 1991); (Banerjee, Dolado, Galbraith, & Hendry, 1993); (Hamilton, 1994); (Johansen, 1995); (Stewart & Gill, 1998); and (Verbeek, 2017).

statistic) (Johansen & Juselius, 1990).

The general null hypothesis for both tests is elaborated in order to check the **existing** number of cointegrated vectors ($k$) over the **maximum** number of cointegrated vectors ($n$). In summary, the null hypothesis is $H_0$: $k \leq n$, where $n = 0, 1, \dots, p - 1$; and $p$ denotes the number of variables used. The difference between Trace and Maximum Eigenvalue lies in the alternative hypothesis (Verbeek, 2017):

| | | |
|---|---|---|
| **Trace Test** | $H_0: k \leq n$ | $H_1: k > n$ |
| **Maximum Eigenvalue Test** | $H_0: k \leq n$ | $H_1: k = n + 1$ |

Although being possible to verify the number of cointegration vectors, the main goal of the tests is to check the presence/absence of cointegration, such information can be extracted by using $n = 0$ in the null hypothesis. If $H_0$: $k \leq 0$ is rejected, there is a cointegration relationship in the estimated model.

✓ Trace's alternative hypothesis is that the number of cointegrating relationships is at least one. The $\lambda_{trace}$ statistic can be calculated as:

$$\lambda_{trace} = -T \sum_{i=n+1}^{r} ln(1 - \hat{\lambda}_i)$$

where $T$ is the number of observations; $\hat{\lambda}_i$ is the $i$-th largest estimated eigenvalue (matrix of stationary variables of the estimated model); and the eigenvalues are in decreasing order.

✓ The Maximum Eigenvalue's alternative hypothesis uses $n + 1$. Rejecting the null hypothesis means that there is only one combination of the non-stationary variables that give a stationary process. The $\lambda_{max}$ statistic can be calculated as:

$$\lambda_{max} = -T\, ln(1 - \hat{\lambda}_{n+1})$$

Trace and Maximum Eigenvalue tests are likelihood ratio tests that do not present the typical Chi-square distributions. The appropriate distributions for both tests are multivariate extensions of the Dickey-Fuller distributions (Verbeek, 2017).

The Bounds test, introduced by Pesaran, Shin, & Smith (2001), is an important instrument for exploring cointegration. This test is intrinsically connected to the Autoregressive Distributed Lag Model (ARDL) models (Chapter 6.4), whose specification can estimate long-term and short-term cointegration relationships at the same equation.

This test analyzes the joint significance of the regressors linked to the long-run terms, wherein the hull hypothesis ($H_0$: there is no cointegration equation) is rejected when the calculated $F$-statistic value of the test is greater than the upper critical bound value ($I(1)$ bound), meaning that there is a long-run relationship between the variables. On the other hand, if the calculated $F$-statistic is inferior to the lower critical bound value ($I(0)$ bound), there is no long-run relationship between the variables. If the calculated value of the test statistic falls between the lower bound value and the upper bound value, the test is considered inconclusive.

For a full understanding of the mathematical framework of this test, check (Pesaran, Shin, & Smith, 2001), (Wooldridge, 2015), and (Wong & Hook, 2018).

The presence/absence of cointegration is linked to the "usability" of Vector Autoregressive (VAR) models (Chapter 6.2), Vector Error Correction (VEC) models (Chapter 6.3), and ARDL models. In summary, if the variables are cointegrated, we can construct short-run (VAR) and long-run (VEC) models. On the other hand, if variables are not cointegrated, we can construct only the short-run (VAR) models. (Wooldridge, 2015).

## 6. Statistical Methodology

Econometrics means 'economic measure', and economic models "should be explicitly designed to incorporate randomness; stochastic errors should not be simply added to deterministic models to make them random. Once we acknowledge that an economic model is a probability model, it follows naturally that an appropriate tool way to quantify, estimate, and conduct inferences about the economy is through the powerful theory of mathematical statistics." (Hansen, 2015).

The review, understanding, and analysis of the empirical characteristics of the financial series are of dominant importance since the models that try to reproduce the behavior of the time series must be constructed considering these characteristics.

Once the generic model has been identified, the maximum likelihood (efficient) estimates are obtained for the parameters. Then, the model is diagnosed, the series are analyzed regarding the residuals. Once the proposed model is 'accepted', it proceeds to the forecasting phase, otherwise, the analysis of the residuals helps to indicate the new current model (new approach). Figure 2 below presents a schematic for the data modeling steps.

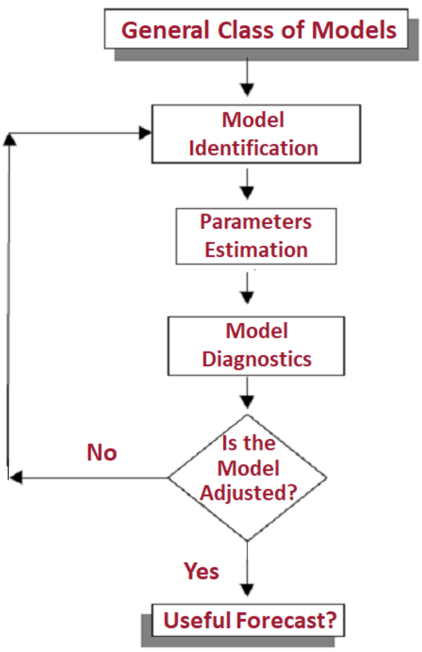



**Figure 2 - Modeling Diagram**
- adapted from (Lustosa, Mesquita, Quelhas, & Oliveira, 2008) -

## 6.1. Autoregressive and Moving Average Processes

In order to better clarify the ideas behind conditional heteroscedasticity and understand the VAR, VEC, ARDL, and ARCH/GARCH (Chapter 6.2 to Chapter 6.5) models used herein, it is important to introduce autoregressive (AR) process, moving average (MA) process, autoregressive and moving average (ARMA) process and Box–Jenkins Methodology.

### 6.1.1. Autoregressive (AR) Process

If $Y_t$ represents a process wherein the value of $Y$ at time $t$ depends on its value in the previous period plus a random term, $Y_t$ can be modeled as (Gujarati & Porter, 2008):

$$Y_t = \alpha + \phi_1 Y_{t-1} + \varepsilon_t$$

where $\alpha$ is a constant; $\phi_1 \neq 0$ is a parameter; and $\varepsilon_t$ is an uncorrelated random error term with zero mean and constant variance (i.e., white noise). It can be said that $Y_t$ follows a first-order autoregressive, AR$(1)$, process.

Still according to Gujarati & Porter (2008), this model says that the "value of $Y$ at time $t$ is simply some proportion (= $\phi_1$) of its value at time $(t-1)$ plus a random shock or disturbance at time $t$". In addition, if the value of $Y$ at time $t$ depends on its value in the previous two time periods, then, $Y_t$ follows a second-order autoregressive, AR$(2)$, process, and can be written as:

$$Y_t = \alpha + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$$

where $\phi_2 \neq 0$.

In general, a $p^{th}$-order autoregressive, AR$(p)$, process can be expressed as:

$$\text{AR}(p): \quad Y_t = \alpha + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \varepsilon_t \quad \text{or} \quad Y_t = \alpha + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t$$

where $\phi_p \neq 0$.

**Stationarity Conditions**

An AR(1) process is said to be stationary if $|\phi_1| < 1$. An AR(2) process is said to be stationary if $|\phi_2| < 1$; $\phi_2 + \phi_1 < 1$; and $\phi_2 - \phi_1 < 1$. Taking into account the lag operator $(L)$ mentioned in Chapter 5.1, the AR($p$) equation can be rewritten as $\phi_p(L)Y_t = \alpha + \varepsilon_t$, where $\phi_p(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ is the $p$-degree polynomial. In this context, an AR($p$) process is said to be stationary if all the roots (solutions) of the characteristic equation $\phi_p(L) = 0$ lie outside the unit circle (all exceed 1 in absolute value). The stationary conditions ensure that the mean and variance of the process $Y_t$ do not change over time (Brockwell & Davis, 2016).

An AR($p$) process is always invertible, i.e., $Y_t$ can be explained by its past $Y_{t-k}$ values (Brockwell & Davis, 2016).

### 6.1.2. Moving Average (MA) Process

If $Y_t$ represents a process wherein the value of $Y$ at time $t$ is equal to a constant plus a moving average of the current and past error terms, $Y_t$ can be modeled as (Montgomery, Jennings, & Kulahci, 2015):

$$Y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

where $\alpha$ is a constant; $\theta_1 \neq 0$ is a parameter; and $\varepsilon_t$ is a white noise. Then, it can be said that $Y_t$ follows a first-order moving average, MA(1), process. Similarly, a second-order moving average, MA(2), process can be written as:

$$Y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$$

where $\theta_2 \neq 0$.

More generally, a $q^{th}$-order moving average, MA($q$), process can be expressed as:

$$\text{MA}(q): \quad Y_t = \alpha + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \quad \text{or} \quad Y_t = \alpha + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

where $\theta_q \neq 0$.

**Invertibility Conditions**

If the MA equation can be rewritten in AR form of infinite order, it is possible to say that the process $Y_t$ is invertible, that is, we can use an AR($p$) to explain an MA($q$) (Montgomery, Jennings, & Kulahci, 2015). An MA(1) process is said to be invertible if $|\theta_1| < 1$. An MA(2) process is said to be invertible if $|\theta_2| < 1$; $\theta_2 + \theta_1 < 1$; and $\theta_2 - \theta_1 < 1$. Taking into account the lag operator ($L$), the MA($q$) equation can be rewritten as $Y_t = \alpha + \theta_q(L)\varepsilon_t$, where $\theta_q(L) = 1 - \theta_1 L - \cdots - \theta_q L^q$ is the $q$-degree polynomial. In this context, an MA($q$) process is said to be invertibility if all the roots of the characteristic equation $\theta(L) = 0$ lie outside the unit circle.

An MA($q$) process is always stationary since its mean and variance are finite and constant, and the autocorrelation function is finite and does not depend on $t$.

### 6.1.3. Autoregressive and Moving Average (ARMA) Process

In the case of $Y_t$ combines AR and MA characteristics, it can be designated as an ARMA process. For example, if $Y_t$ presents one autoregressive and one moving average term, it follows an ARMA(1,1) process and can be written as (Montgomery, Jennings, & Kulahci, 2015):

$$Y_t = \alpha + \phi_1 Y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

where $\alpha$ is a constant; $\phi_1 \neq 0$ and $\theta_1 \neq 0$ are parameters; and $\varepsilon_t$ is a white noise.

Likewise, in an ARMA$(p, q)$, there will be $p$ autoregressive and $q$ moving average terms, and can be expressed as:

$$\text{ARMA}(p, q): \; Y_t = \alpha + \sum_{i=1}^{p} \phi_i Y_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

where $\phi_p \neq 0$ and $\theta_q \neq 0$.

The stationarity and invertibility conditions discussed above are also valid for ARMA$(p, q)$ processes. Again, taking into account the lag operator $(L)$, the ARMA$(p, q)$ equation can be rewritten as $\phi_p(L)Y_t = \alpha + \theta_q(L)\varepsilon_t$, where $\phi_p(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\theta_q(L) = 1 - \theta_1 L - \cdots - \theta_q L^q$. An ARMA$(p, q)$ process will be stationary if all the roots of $\phi_p(L) = 0$ lie outside the unit circle (AR$(p)$ component) and will be invertible if all the roots of $\theta_q(L) = 0$ lie outside the unit circle (MA$(q)$ component).

An especial variation of the above-mentioned process is the Autoregressive Integrated Moving Average (ARIMA) model, which is used on time series that present trends, also known as integrated series. In summary, if the process $Y_t$ (with trend) is integrated of order $d$ [i.e., it is $I(d)$], we can differentiate it $d$ times, obtaining a process without the trend component [i.e., it is $I(0)$]. The ARIMA$(p, d, q)$ model incorporates the autoregressive term $(p)$, the moving average term $(q)$, and the number of times $(d)$ that the original series $I(d)$ must be differenced before it presents no trend $I(0)$. Intuitively, if $d = 0$, ARIMA$(p, d, q)$ = ARMA$(p, q)$.

### 6.1.4. Box–Jenkins (BJ) Methodology

This iterative approach created by George Box and Gwilym Jenkins arrived in order to identify if a time series follows an AR$(p)$, MA$(q)$, ARMA$(p, q)$ or ARIMA$(p, d, q)$, as well as what is the value of $p$, $q$, and $d$. The methodology consists of 4 steps:

**Step 1**: **Identification** of the parameter $d$ (if necessary, transform a non-stationary time series into a stationary time series) and the appropriate values of $p$, $q$, since it derives from the behavior of the ACF and PACF. In the AR$(p)$ process, the ACF decays exponentially (or has a damped sine wave pattern), while the PACF presents an abrupt cut after lag $p$. Regarding the MA$(q)$ process, the ACF presents an abrupt cut after lag $q$ while the PACF decays exponentially (or has a damped sine wave pattern). In the ARMA$(p, q)$ process, the ACF decays exponentially after lag $q$ and the PACF decays exponentially after lag $p$ (Gujarati & Porter, 2008).

For a better understanding, Figure 3 shows examples of ACF and PACF of AR$(1)$, MA$(1)$, and ARMA$(1,1)$ models.

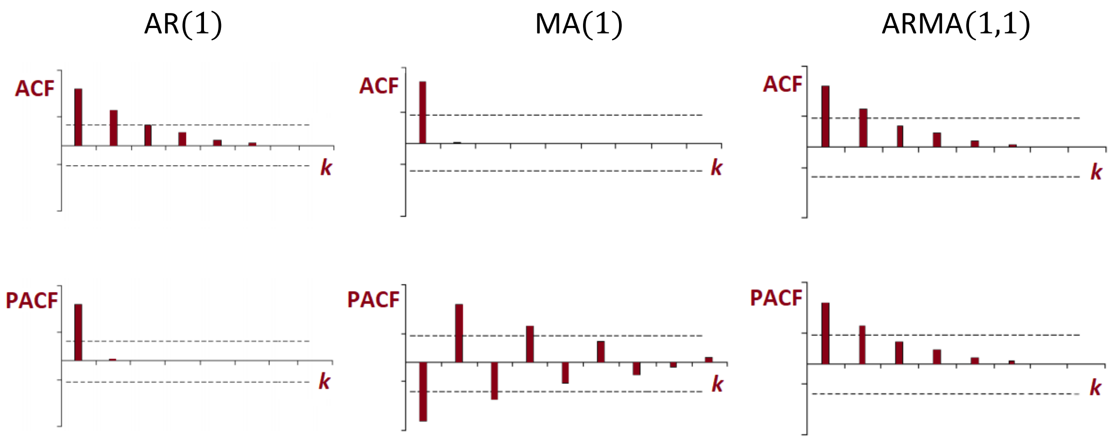



**Figure 3 - Examples of AR(1), MA(1), and ARMA(1,1)**
- adapted from (Gujarati & Porter, 2008) -

**Step 2**: **Estimation** of the parameters $p$ and $q$ included in the model. The estimation method used can be, for example, the maximum likelihood.

**Step 3**: **Diagnostics** of the chosen model in order to check if it properly fits the data. Verify the statistical significance of the parameters estimated, as well as the stationary and invertibility conditions. In addition, check if the residuals are white noise and the presence of outliers.

**Step 4**: Forecasting using the BJ methodology usually presents accurate forecasts that "are more reliable than those obtained from traditional econometric modeling, particularly for short-term forecasts" (Gujarati & Porter, 2008). Figure 4 represents the

BJ methodology.

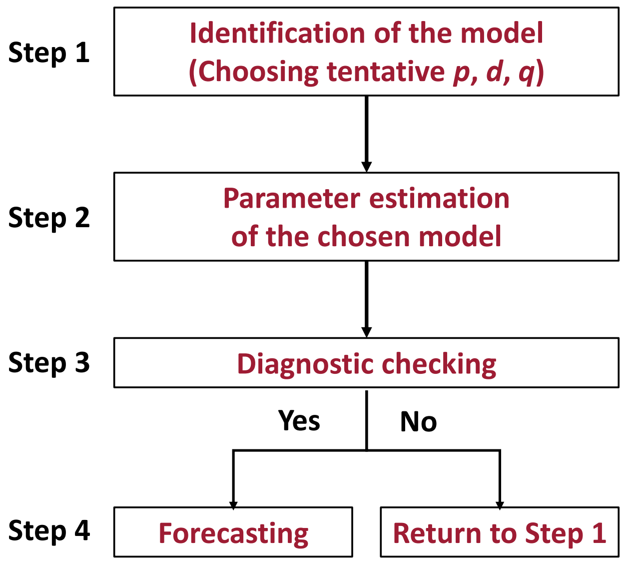



**Figure 4 - Box-Jenkins Methodology**
- source: (Gujarati & Porter, 2008) -

## 6.2. Vector Autoregressive Model (VAR)

The autoregressive moving average (ARMA) models can be extended to the multivariate case, wherein two or more time series are individually modeled "as a linear function of past values of all variables, plus disturbances that have zero means given all past values of the observed variables" (Wooldridge, 2015). This approach is known as Vector Autoregressive (VAR) model.

A VAR model "describes the dynamic evolution of a number of variables from their common history. If we consider two variables, $Y_t$ and $X_t$, the VAR consists of two equations" (Verbeek, 2017). Considering that all variables in the VAR system are endogenous (there are no exogenous variables), the first-order VAR with two variables would be defined as:

$$Y_t = \delta_1 + \theta_{11} Y_{t-1} + \theta_{12} X_{t-1} + \varepsilon_{1t}$$

$$X_t = \delta_2 + \theta_{21} Y_{t-1} + \theta_{22} X_{t-1} + \varepsilon_{2t}$$

where $\varepsilon_{1t}$ and $\varepsilon_{2t}$ are white noise processes that may be correlated. For example, if the parameter $\theta_{12}$ is not equal to zero, we can say that the past values of $X$ help to explain $Y$. The two equations above can be written as a system:

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} + \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

If we denote $\vec{Y}_t = \begin{pmatrix} Y_t \\ X_t \end{pmatrix}$ as a vector of two variables, $\vec{\delta} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ as a vector of two intercepts, $\Theta_1 = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ is a $2 \times 2$ matrix of coefficients, and $\vec{\varepsilon} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$ as a vector of white noise terms, the system can be written as:

$$\vec{Y}_t = \vec{\delta} + \Theta_1 \vec{Y}_{t-1} + \vec{\varepsilon}_t$$

which can be understood as a dimensional case of a first-order autoregressive process. In this context, a VAR($p$) model for $k$-dimensional vector $\vec{Y}_t$ can be written as:

$$\vec{Y}_t = \vec{\delta} + \Theta_1 \vec{Y}_{t-1} + \cdots + \Theta_p \vec{Y}_{t-p} + \vec{\varepsilon}_t \qquad \text{or} \qquad \vec{Y}_t = \vec{\delta} + \sum_{i=1}^{p} \Theta_i \vec{Y}_{t-i} + \vec{\varepsilon}_t$$

where:

· $\vec{\delta}$ is a vector of intercepts;

· $p$ is the optimal lag length;

· $\Theta_i$ is a $k \times k$ matrix of short-run coefficients; and

· $\vec{\varepsilon}_t$ is a $k$-dimensional vector of white noise terms.

The VAR models make it possible to analyze the effect of variation over time for a given variable on the others. In addition, there is no need to impose any initial restriction of causality between the variables, which can be tested by the Granger's Causality test, neither it is necessary to assume any long-term relationship between the variables, which can be examined by Johansen's Cointegration test (Maia, 2011).

It should be emphasized that VAR models are constructed only if the variables are integrated of order one, $I(1)$, that is, stationary after the first difference. In addition, VAR variables have the same lag length in all equations. Selecting the lag length in a VAR($p$) model is not always easy, and univariate ACF and PACF will not help. For a VAR($p$) the ACF matrix will exhibit a mixture of exponential decay and a damped sine wave pattern (Montgomery, Jennings, & Kulahci, 2015). Verbeek (2017) suggests the use of information criteria, like Akaike Information Criterion (AIC) and/or Bayesian information criterion (BIC), to select the lag length of a VAR model. Both criteria will be detailed in Chapter 7.

## 6.3. Vector Error Correction Model (VEC)

If the analyzed processes are non-stationary, for example, $I(1)$, the VAR model must be estimated in first differences. However, the result obtained through first differences is not always reasonable (Maia, 2011) [*apud* (Harvey, 1989)], "leading to spurious regression, in which estimators and test statistics are misleading" (Verbeek, 2017).

In order to overcome the above-mentioned drawbacks, the Vector Error Correction (VEC) model uses cointegration analysis to estimate models when variables do not show stationarity. In summary, if a cointegration relationship between variables is detected, the VEC model can be considered as a VAR model with cointegration restrictions (Zou, 2018).

The existence of cointegration (presence of a long-run relationship between variables) also has implications for the short-run behavior of the $I(1)$ variables "because there has to be some mechanism that drives the variables to their long-run equilibrium relationship. This mechanism is modeled by an **error-correction mechanism**, in which the 'equilibrium error' also drives the short-run dynamics of the series" (Verbeek, 2017).

If the first difference operator ($\Delta\vec{Y}_t = \vec{Y}_t - \vec{Y}_{t-1}$) is applied to the VAR($p$) model expressed above, and a cointegration term is added to the equation (also known as error-correction term), the resulting model is a VEC model that can be written as:

$$\Delta\vec{Y}_t = \vec{\delta} + \sum_{i=1}^{p-1} \Theta_i \Delta\vec{Y}_{t-i} + \lambda_i ECT_{t-1} + \vec{\varepsilon}_t$$

where:

· $\Delta\vec{Y}_t$ is a vector with $k$ variables;

· $\vec{\delta}$ is a vector of intercepts;

· $p - 1$ is the optimal lag length (remembering that $\vec{Y}_t$ was differentiated);

· $\Theta_i$ is a $k \times k$ matrix of short-run coefficients;

· $\lambda_i$ is the speed of adjustment parameter with a negative sign;

· $ECT_{t-1}$ is the error correction term. For further details how $ECT_{t-1}$ is calculated and incorporated into the VEC model, check (Verbeek, 2017); and

· $\vec{\varepsilon}_t$ is a $k$-dimensional vector of white noise terms.

It should be highlighted that like the VAR model, the VEC approach assumes that all variables are endogenous (there are no exogenous variables) and the lag length is selected using information criteria. In addition, if there is a possibility that the multivariate time series are cointegrated, the literature suggests that the VEC should be used instead of the VAR.

## 6.4. Autoregressive Distributed Lag Model (ARDL)

While VAR models are strictly for endogenous variables, the Autoregressive Distributed Lag Model employs a combination of endogenous and exogenous variables. In this approach, the dependent variable is expressed by its own lagged values plus the current and lagged values of auxiliary variables. In addition, the ARDL model can be specified if variables are integrated in a different order, that is, a model with original variables, $I(0)$, and variables in first difference, $I(1)$ (Pesaran, Shin, & Smith, 2001).

The ARDL model presents others benefits. First, this approach is relatively more efficient regardless of the sample size (can be estimated with small and finite sample data sizes). Second, it generates unbiased long-run estimates for the analysis. Third, the ARDL model can, simultaneously, estimate long-term and short-term cointegration relationships (Pesaran, Shin, & Smith, 2001).

Starting from a model with one lag on both dependents and auxiliary variables, the ARDL(1,1) can be expressed as:

$$Y_t = \delta + \alpha_1 Y_{t-1} + \beta_0 X_t + \beta_1 X_{t-1} + \varepsilon_t$$

In this context, an ARDL($p, q_1, \ldots, q_k$) model can be expressed as:

$$Y_t = \delta + \sum_{i=1}^{p} \alpha_i Y_{t-i} + \sum_{l=1}^{k} \sum_{i=0}^{q_l} \beta_{l,i} X_{l,t-i} + \varepsilon_t$$

where:

· $\delta$ is the intercept;

· $p \geq 1$ is the number of used lags of $Y_t$;

· $k$ is the number of auxiliary variables;

· $q_l$ is the number of used lags for the auxiliary variables $X_{l,t}$;

· $\alpha_i$ is the coefficient associated with the $i$ lags of $Y_t$;

· $\beta_{l,i}$ are the coefficients associated with the $i$ lags of regressors $X_{l,t}$; and

· $\varepsilon_{it}$ is a white noise.

For this model to be reasonable, the lag coefficients, $\beta_{l,i}$, must tend to zero as $i \rightarrow \infty$. This does not mean that, for example, $\beta_{l,2}$ has reduced magnitude when compared to $\beta_{l,1}$; it only means that the impact of $X_{l,t-i}$ on $Y_t$ must eventually become small as $i$ gets large. "In most applications, this makes economic sense as well: the distant past of $X$

should be less important for explaining $Y$ than the recent past of $X$” (Wooldridge, 2015). Like in VAR and VECM models, the lag length selection employs information criteria.

For additional details regarding the ARDL models, check (Kripfganz & Schneider, 2018), (Pesaran, Shin, & Smith, 2001), and (Wooldridge, 2015).

## 6.5. ARCH/GARCH Models

Several practical cases in financial time series (such as stock prices, inflation rates, and foreign exchange rates) infringe the assumption of constant variance due to the presence of volatility. Linear trend models, exponential smoothers, or even ARIMA models would fail to fit well, as all assume homoscedasticity of the error (Verbeek, 2017).

The potential existence of autocorrelation in the variance $\sigma^2$ at time $t$ with its values lagged one or more periods was the motivation behind the Autoregressive Conditional Heteroscedasticity (ARCH) model, in which the error variance is related to the squared error term in the previous term, as well as the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model, in which the error variance is related to squared error terms several periods in the past (Gujarati & Porter, 2008).

Remembering that the **autoregressive** describes the feedback mechanism that allows the incorporation of past observations into the model. While **conditional** implies that the current observations depend on the immediately previous observations, and **heteroscedasticity** means fluctuation of the variance over time.

Before describing the ARCH, GARCH, IGARCH, and EGARCH models, it is important to determine if this approach can be employed. In other words, verify if the data can be modeled through ARCH/GARCH models.

**Examining Heteroscedasticity**

The present work employed the ARCH LM test which is similar to the Breusch-Godfrey (BG) test for autocorrelation. The null hypothesis ($H_0$: there is no ARCH factor when considering the residuals up to order $k$) must be rejected before modeling through ARCH/GARCH since ignoring ARCH effects may result in loss of efficiency. Steps of the ARCH LM test:

1 - Run the regression of the model using OLS and collect the residuals $\varepsilon$.

2 - Square the residuals $\varepsilon$ and run the following secondary regression: $\varepsilon_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \cdots + \beta_k X_{t-k} + u_t$, where $k$ is the number of lags included in this secondary regression. The appropriate number of lags can either be determined by the span of the data (i.e., four if we have quarterly data) or by Information Criteria;

3 - Based on the $R^2$ of the regression of step 2, compute the $T \times R^2$ statistic, where $T$ is the number of observations. This test statistic follows, asymptotically, a Chi-squared distribution with $k$ degrees of freedom.

If the null hypothesis is rejected (i.e., $\chi_{obs}^2 > \chi_{k,1-\alpha}^2$), ARCH/GARCH approach can be used.

### 6.5.1. ARCH Model

In 1982, Robert Engle, winner of the 2003 Nobel Memorial Prize in Economic Sciences, considered that it was possible to build a parametric model in which the variance would be conditioned by an algebraic equation, modeling not only the average but also the conditioned variance (Engle, 1982). In summary, “while conventional time series and econometric models operate under an assumption of constant variance, the ARCH process allows the conditional variance to change over time as a function of past errors leaving the unconditional variance constant” (Bollerslev, 1986).

The estimation of ARCH models cannot be done through OLS, as this method minimizes the sum of squares of the residuals, which depends only on the parameters of the mean equation and not on the conditional variance. In this context, the estimation of ARCH models is made through maximum likelihood, under the hypothesis that the errors are conditionally distributed (based on past errors).

Starting from a simple time series regression model for a process $Y_t$ expressed by its past values ($Y_{t-1}$) plus a white noise error term ($\varepsilon_t$). The model could be expressed as:

$$Y_t = \beta Y_{t-1} + \varepsilon_t$$

in which $\varepsilon_t$ is independent and identically distributed with zero mean and variance $\sigma_t^2$ ($\varepsilon_t \overset{iid}{\sim} N(0, \sigma_t^2)$). If exogenous variables ($X_t$) are used to explain $Y_t$, the equation can be redrafted as $Y_t = \beta X_t + \varepsilon_t$.

If we let the distribution of the error term be conditionally normal, such as $\varepsilon_t | I_{t-1} \sim N(0, \sigma_t^2)$, in which $I_{t-1}$ represents the information available at time $t-1$. The conditional variance $\sigma_t^2$ can be described as a function of a constant term ($b_0$) and the lagged squared error term ($\varepsilon_{t-1}^2$) (Verbeek, 2017):

$$\sigma_t^2 = b_0 + b_1 \varepsilon_{t-1}^2$$

where $b_0 > 0$; $0 \leq b_1 < 1$; and $u_t$ is the error term. Here, $b_0$ and $b_1$ must be positive to ensure a positive variance, $b_1 < 1$ guarantees the stationary condition, otherwise, $\sigma_t^2$ continues to increase over time (explosive process). If $b_1 = 0$, the process does not exhibit time-varying volatility.

This is the representation of the one-lag ARCH process, ARCH(1). Of course, the conditional variance can also depend on more than one lag. And the number of lagged periods of the squared error indicates the structure of the ARCH model:

ARCH(1): $\sigma_t^2 = b_0 + b_1 \varepsilon_{t-1}^2$

ARCH(2): $\sigma_t^2 = b_0 + b_1\varepsilon_{t-1}^2 + b_2\varepsilon_{t-2}^2$

ARCH($q$): $\sigma_t^2 = b_0 + b_1\varepsilon_{t-1}^2 + b_2\varepsilon_{t-2}^2 + \cdots + b_q\varepsilon_{t-q}^2$

The conditional variance is a linear function of the past $q$ squared innovations (the term 'innovation' is used instead of the 'residual' and expresses the unpredictable part of a financial series) in the ARCH($q$) model (Bollerslev, 1986).

Again, when exogenous variables ($X_t$) are used to explain $Y_t$, i.e., $Y_t = \beta X_t + \varepsilon_t$, and an ARCH($q$) effect is introduced in the model, the mean and the conditional variance of a dependent variable are:

| **Mean Equation** | **Conditional Variance Equation** |
|---|---|
| $Y_t = \beta X_t + \varepsilon_t$ | $\sigma_t^2 = b_0 + \sum_{i=1}^{q} b_i\varepsilon_{t-i}^2$ |

The conditional variance depends on the square errors of the regression in order $q$. It is possible to prove that, in the ARCH model above, the conditional variance tends to converge to a constant. This constant represents the unconditional variance given by:

$$\sigma_\infty^2 = \frac{b_0}{1 - \sum_{i=1}^{q} b_i}$$

The restriction of the sum of the coefficients $b_i$ to be less than 1 is to ensure that the model has stationary covariance.

This model, however, has some limitations in its assumptions. The main ones are highlighted by (Marques, 2017) [*apud* (Brooks & Lee, 1997)]:

✓ Absence of a methodology to properly establish the maximum lags to capture the volatility of the process.

✓ Possible non-parsimonious model due to the need for a high number of lags to capture all the dependencies of the conditioned variance.

✓ The more parameters that are introduced in the model, the greater the probability of obtaining estimated negative coefficients (possibility of violating the non-negativity restrictions).

### 6.5.2. GARCH Model

In order to overcome some of the limitations expressed above, Tim Bollerslev (1986) proposed a generalization of the ARCH model that resembles the ARMA approach, naming it as **Generalized Autoregressive Conditional Heteroscedasticity** (GARCH) model. Here, the conditional variance depends not only on the squared random shocks that occurred in the immediately preceding $i$ moments but also on the conditional variances of the immediately preceding $t$ moments. The mean equation remains unchanged and the GARCH($p, q$) regarding the conditional variance is:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} b_i \, \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

where $q$ is the ARCH process degree with $b_i$ parameters; and $p$ is the GARCH process degree with $\beta_j$ parameters.

If $p = 0$, then the GARCH($0, q$) model is equivalent to the ARCH($q$) model. In addition, the following conditions must be met:

$$\alpha_0 > 0;\ b_i \geq 0\ (i = 1,2, \dots, q);\ \beta_j \geq 0\ (j = 1,2, \dots, p);\ \text{and}\ \sum (b_i + \beta_j) < 1.$$

Although the structure of GARCH models and ARMA models are very similar, there is an important distinction between the methods. While the returns are dependent only on the returns from previous periods in the ARMA model, ARCH/GARCH models consider that the returns also depend on the variance (volatility) observed in the past, as well as the errors associated with the previous process (Bollerslev, 1986).

In common, the ARMA and GARCH models assume that the markets are efficient, and the stock returns follow stochastic processes, and, therefore, are unpredictable in the medium- and long-term. In addition, for both, the simple stochastic models assume that the returns are not correlated and whose average is zero (Wooldridge, 2015). The GARCH model for conditional variance can be considered as an ARMA process in the squared innovations, although not in the variations as the equations might seem to suggest.

The feedback effect provided by the introduction of lagged conditional variances makes the GARCH model more parsimonious (fewer parameters) when compared to the ARMA model. The great advantage of this model is that to estimate a certain parameter, it considers different weights for each observation in the time series, giving greater weight to the most recent ones.

The GARCH model allows the presence of autoregressive components and moving averages in the heteroscedastic variance of financial assets. Although GARCH models have a wide range of applications, combining several characteristics observed in time series, and produces satisfactory results, there are limitations (Marques, 2017) [*apud* (Brooks & Lee, 1997)]:

✓ It is required that the series presents a conditional variance that fluctuates over time.

✓ Those models often fail to capture highly unexpected events that can lead to important structural change, as well as irregular phenomena such as successive drops, and their implications.

✓ Asymmetric effects on the volatility, for example, ‘good news’ (tranquility, with decreasing volatility) or ‘bad news’ (turbulence, with increasing volatility) cannot be modeled by the GARCH approach.

As the asymmetric effect is one of the limitations of the ARCH and GARCH models, under these conditions, it will be better to consider a model in which the volatility presents asymmetric reactions for positive or negative values of the residual variance.

### 6.5.3. EGARCH Model

The **Exponential Generalized Autoregressive Conditional Heteroscedastic** (EGARCH) Model presented by Nelson (1991) is an extension of the GARCH family and considers the above-mentioned asymmetry in the variance, which allows capturing the effect of more intense innovations in a given period than in the other.

The EGARCH model is used for the conditional variance of innovations with leverage conditions (negative correlation between present returns and future volatility) (Nelson, 1991). In this model, the leverage effect is exponential, and it is no longer necessary to impose non-negativity restrictions on parameters $b_i$ and $\beta_j$ as it was imposed in the GARCH approach. The leverage effect can be analyzed by the parameter $\gamma_i$, which is related to the symmetry/asymmetric shock (Pinho, Valente, Madaleno, & Vieira, 2012).

In order to "relax" the positive coefficients restriction in the GARCH model, Nelson (1991) uses $ln(\sigma_t^2)$. Starting from the following equation:

$$ln(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{q} \alpha_i g(\varepsilon_{t-i}) + \sum_{j=1}^{p} \beta_j \, ln(\sigma_{t-j}^2)$$

$$\text{with:} \quad g(\varepsilon_t) = \begin{cases} (\theta + \gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) \text{ if } 0 < \varepsilon_t < \infty \\ (\theta - \gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) \text{ if } -\infty < \varepsilon_t \leq 0 \end{cases}$$

where $\sigma_t^2$ is the conditional variance; $\alpha_0$ is the intercept; $\theta$ and $\gamma$ are constants related to positive $(\theta + \gamma)$ and negative $(\theta - \gamma)$ shocks, $\varepsilon_t$ is independent and identically distributed with zero mean; $\varepsilon_{t-i}$ is the error observed on instant $t - i$.

The function $g(\varepsilon_t)$ allows the model to respond differently to negative and positive variations in $Y_t$, another improvement over the GARCH approach.

Nelson (1991) used ${\varepsilon_{t-i}}/{(\sigma_{t-i}^2)^{1/2}}$ and, without going deep on the mathematic approach (see (Castaño, 2010)), the equation was redrafted to present an EGARCH$(p, q)$ model:

$$ln(\sigma_t^2) = \alpha_0 \quad + \quad \sum_{i=1}^{q} \eta_i \left| \frac{\varepsilon_{t-i}}{(\sigma_{t-i}^2)^{1/2}} \right| \quad + \quad \sum_{i=1}^{q} \gamma_i \left| \frac{\varepsilon_{t-i}}{(\sigma_{t-i}^2)^{1/2}} \right| \quad + \quad \sum_{j=1}^{p} \beta_j \, ln(\sigma_{t-j}^2)$$

where $\eta_i$ is a parameter related to the ARCH effect; $\gamma_i$ is the parameter of asymmetric effect; and $\beta_i$ is a parameter related to the GARCH effect.

✓ if $\gamma_i > 0$, a positive shock would decrease the volatility of returns.

✓ if $\gamma_1 = \gamma_2 = \cdots = \gamma_i = 0$, a positive shock will have a similar effect on the volatility of returns as a negative shock of the same magnitude (the model is symmetric: identical to a GARCH model).

✓ if $\gamma_i < 0$, a negative shock would increase the volatility of returns.

Since the EGARCH is presented in logarithms, the variance will always be positive, which is an advantageous factor. However, since the expected future variance beyond a period cannot be analytically calculated is a disadvantage of this approach (Pinho, Valente, Madaleno, & Vieira, 2012).

### 6.5.4. IGARCH Model

The **Integrated Generalized Autoregressive Conditional Heteroscedasticity** (IGARCH) model is identical to the GARCH model with the restriction that the sum of the coefficients is equal to one. Nelson (1991) provides theoretical foundations for estimating IGARCH models for financial data using GARCH specifications:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} b_i \, \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2$$

with the restriction of:

$$\sum_{i=1}^{q} b_i + \sum_{j=1}^{p} \beta_j = 1$$

In this model, the conditional variance is then an integrated process, meaning that a shock in the time series influences, or remains important, for a long period of forecasts. Many financial series have this characteristic.

Although such a specification seems rather restrictive, several empirical studies indicate that the IGARCH(1,1) model has shown great efficiency to model financial data (e.g., Caporale, Spagnolo, & Pittis (2003), Gaio, Pesanha, Oliveira, & Ázara (2007); Ali, Ziaei, & Anwar (2012); Ali (2013); Marques (2017); Bezerra & Albuquerque (2019); Hernandez & Al Janabi (2020).

## 7. Model Selection

When we talk about returns, the coefficient of determination ($R^2$) of models is very low and sometimes negative, taking away some of the intrinsic intuition of the analysis. Frequently, the mean is modeled to analyze the conditional variance. Therefore, the coefficient $R^2$ should not be considered to verify the quality of the model since it fits the model to the mean.

Given a set of time series models that present significant parameters and that were approved in the evaluation steps related to the quality of the residuals (autocorrelation and heteroskedasticity), the information criteria can help to select, among these models, the one that simultaneously minimizes the discrepancy (between the models and the data) and the complexity of the model (number of parameters).

To choose the “best” model among those that have been adjusted, the following criteria can be used, namely AIC, BIC (or SIC), MAPE, and RMSE. Herein, the AIC e BIC criteria are used under the in-sample analyzes, while MAPE and RMSE are utilized on the out-of-sample evaluation.

### Akaike Information Criterion (AIC)

Named after Hirotugu Akaike, AIC is a tool often used to choose the optimal specification of a regression equation for non-nested alternative models. The Akaike Information Criterion balances the trade-offs between the goodness of fit of a candidate model and its complexity, considering the number of parameters estimated and how well the model "fits" the data (Moffatt, 2019).

Given a set of data, several models can be used, but the “best” is the one with the smallest AIC. Its formula can be described as (Wagenmakers & Farrell, 2004) [*apud* (Schwarz, 1978)]:

$$\text{AIC}_m = -2\ log(L_m) + 2k$$

where:

· $L_m$ is the maximum log-likelihood for the candidate model $m$; and

· $k$ is the number of estimated coefficients (including the intercept).

### Bayesian information criterion (BIC)

Also known as Schwarz Information Criterion (SIC) or Schwarz-Bayesian information criterion (SBIC), it was developed by Gideon E. Schwarz. This criterion is similar to the AIC with the characteristic of imposing a greater penalty for including additional coefficients to be estimated. BIC can be calculated as:

$$\text{BIC}_m = -2\ log(L_m) + 2k \log(n)$$

where:

· $L_m$ is the maximum log-likelihood for the candidate model $m$;

· $k$ is the number of estimated coefficients (including the intercept); and

· $n$ is the number of observations that enter into the likelihood calculation.

Although the equations of AIC and BIC look very similar, they originate from different backgrounds. The first calculate a measure of the distance between the probability density generated by the model and data. While the second assumes that the true generation model is in the set of candidate models, measuring the degree of belief that a certain model is the real data-generating model (Wagenmakers & Farrell, 2004).

Asymptotically, the AIC tends to over-identify the models while the BIC tends to correctly identify the (consistent) model.

### Mean Absolute Percentage Error (MAPE)

The MAPE is a statistical measure of how accurate a forecast system is. The predictive capacity measures are based on the forecast error: $e_t = Y_t - \hat{Y}_t$.

The Mean Absolute Percentage Error indicates the average absolute percent error of the predictions over the entire test set. The lower the value of MAPE, the more accurate is the forecasting. The formula for calculating the MAPE of an estimator is defined as (Bowerman, O'Connell, & Koehler, 2005):

$$\text{MAPE} = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{Y_t - \hat{Y}_t}{Y_t}\right| \times 100$$

where:

· $Y_i$ is the observed (actual) value;

· $\hat{Y}_i$ is the predicted value;

· $n$ is the number of predicted observations.

### Root Mean Square Error (RMSE)

RMSE of an estimator of a population parameter is the square root of the mean square error (MSE). Root Mean Square Error can be understood as a performance measure to check how well a regression fits the data under consideration. RMSE assigns greater weight to errors with larger absolute values when compared to errors with smaller absolute values (Chai & Draxler, 2014). The formula for calculating the RMSE is (Bowerman, O'Connell, & Koehler, 2005):

$$\text{RMSE} = \left[\frac{1}{n}\sum_{t=1}^{n}(Y_t - \hat{Y}_t)^2\right]^{1/2}$$

where:

· $Y_i$ is the observed (actual) value;

· $\widehat{Y}_i$ is the predicted value;

· $n$ is the number of predicted observations.

While MAPE and RMSE have both been used to evaluate models' performance, there is no consensus on the most appropriate metric for models' errors. When the data analyzed presents errors that are expected to be Gaussian, the RMSE tends to be more appropriate to represent model performance when compared to MAPE (Chai & Draxler, 2014).

## 8. Results

In order to predict volatility and adjust a model for the economic situation regarding the BOVESPA Index (**IBOV**) during the COVID's "first wave", the present study considered the daily IBOV as a dependent variable, and the exchange rate between the US dollar and the Brazilian real (**BRL_USD**), the price of a barrel of Brent oil in US dollars (**Brent**), and the index of difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the North American Treasury (**EMBI**) as auxiliary variables. The data was separated into two samples: in-sample (322 registers) and out-of-sample (42 registers).

### 8.1. In-Sample Analysis

To understand the "behavior" of each time series, the descriptive statistics of the four series under consideration must be analyzed. The most prominent attributes of the data are summarized in Table 1, wherein it is possible to observe that the IBOV and Brent present a heavier left tail distribution (negative skewness), while BRL_USD and EMBI have a heavier right tail (positive skewness). Besides, the four series are leptokurtic since they are more concentrated around the mean when compared to the normal distribution.

**Table 1 - Descriptive Statistics (n = 322)**

| Statistic | IBOV | BRL_USD | Brent | EMBI |
|---|---|---|---|---|
| **Minimum** | 63,569.62 | 3.651 | 13.770 | 189 |
| **First Quartile** | 95,059.92 | 3.850 | 58.802 | 229.25 |
| **Median** | 100,052.15 | 4.033 | 62.470 | 243 |
| **Mean** | 99,639.84 | 4.129 | 58.744 | 257.174 |
| **Third Quartile** | 106,050.58 | 4.181 | 66.432 | 253.75 |
| **Maximum** | 119,527.60 | 5.650 | 74.940 | 476 |
| **Std. Error (Mean)** | 618.2001 | 0.024 | 0.787 | 3.199 |
| **CV** | 0.1113 | 0.103 | 0.238 | 0.223 |
| **Variance** | 1.23E+08 | 0.181 | 198.243 | 3,296.150 |
| **Skewness** | -0.7796 | 1.776 | -1.998 | 2.389 |
| **Kurtosis** | 0.86129 | 2.619 | 3.060 | 4.979 |

It should be highlighted that the coefficients of variation (CV) are greater than 0.1, which means that the time series present enough variability to be used in tests, models, and proposed analyzes.

The scatterplot matrix (Figure 5) shows the histograms of the four variables in the main diagonal of the matrix, corroborating the analysis concerning kurtosis and asymmetry. Furthermore, the graphic shows the systematic increment and/or decrement between each pair of variables and the trend of their relationship, wherein the BOVESPA Index has a negative relationship with BRL_USD and EMBI while showing a positive association with Brent.

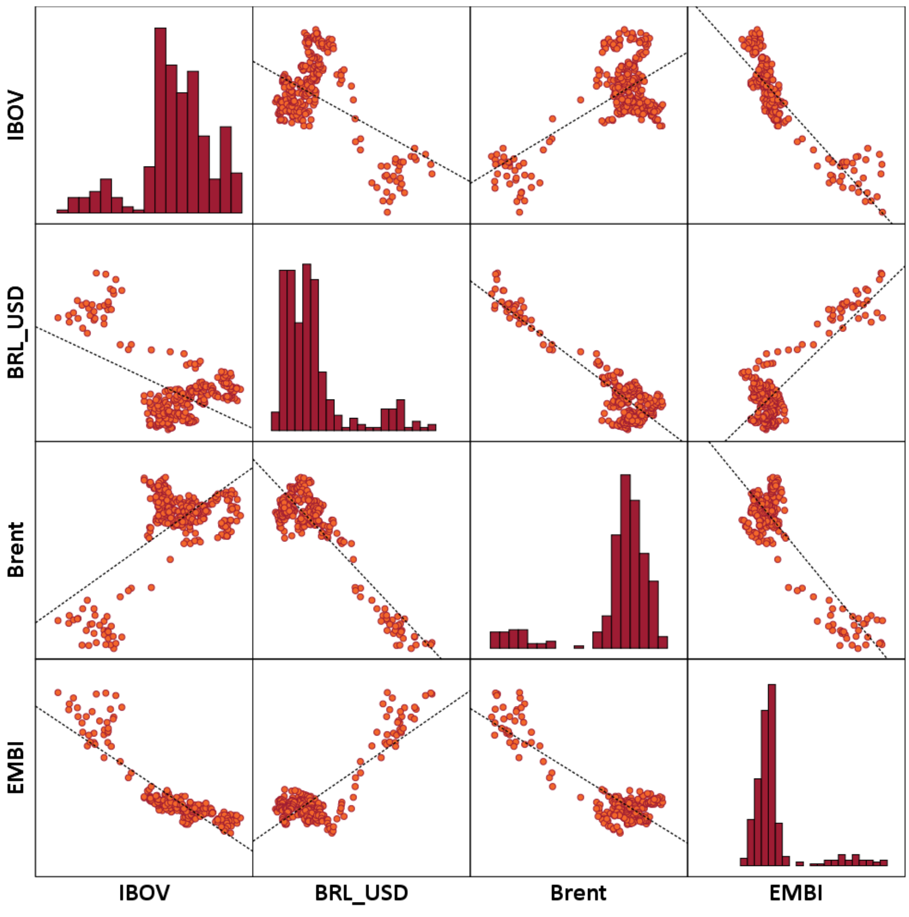



**Figure 5 - Scatterplot Matrix of the Four Series**

The correlation matrix (expressed with Pearson's correlation coefficient) is exhibited in Table 2 below, in which EMBI has a high negative linear correlation (-0.859) with the IBOV, indicating that the Stock Exchange reacts negatively to the positive evolution of the risk associated with the Brazilian economy. The same reasoning can be applied to the exchange rate between the Brazilian Real and the US Dollar (USD_BRL) due to a negative linear correlation (-0.508) with the BOVESPA Index. The only process that presents a positive linear correlation with the IBOV is the price of a barrel of Brent oil in US$ (0.656).

**Table 2 - Pearson's Correlation Coefficients**

| | IBOV | BRL_USD | Brent | EMBI |
|---|---|---|---|---|
| **IBOV** | 1.000 | | | |
| **BRL_USD** | -0.508 | 1.000 | | |
| **Brent** | 0.656 | -0.896 | 1.000 | |
| **EMBI** | -0.859 | 0.821 | -0.868 | 1.0000 |

Since the data is not in the same magnitude, the variables were standardized[10] in order to compare their amplitude (range) and analyze the presence of outliers. After the transformation, the box plots of the standardized variables show that the indicators have a similar range. In addition, IBOV and Brent present lower influential points (outliers) while BRL_USD and EMBI have upper influential points as can be seen in Figure 6:

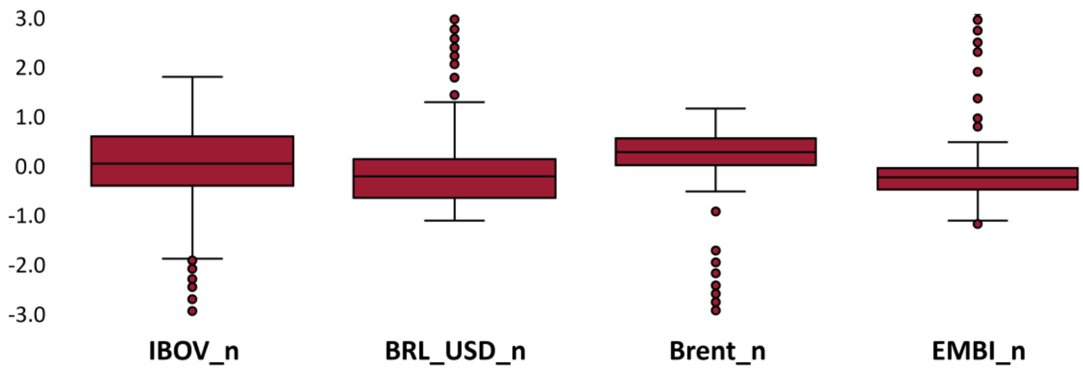



**Figure 6 - Box plot (Standardized Variables)**

The historical series of the BOVESPA Index, together with its daily returns, is shown in Figure 7 below, where it is possible to verify the sharp drop related to the global

[10] Standardization is the process of transforming original data ($X$), wherein the new variable ($Z$) present a mean of zero ($\mu = 0$) and a standard deviation of one ($\sigma = 1$). The data can be standardized with the following formula:

$$Z_i = \frac{X_i - \mu_X}{\sigma_X} \sim N(0,1)$$

economic ramifications from Covid-19's escalation. At the end of March 2020, the IBOV reached 63 thousand points, a value previously recorded only in July 2017. The series of returns shows the presence of high volatility in March 2020, with an emphasis on the 15.99% retraction observed on day 12 of that month and subsequent recovery of 13.02%. During this month, the circuit breaker, a mechanism that suspends trading when the IBOVESPA registers significant drops from 10%, was triggered six times, reflecting concerns about the effects of the COVID-19 pandemic. The graphs of the other variables can be found in Appendix 2 - Time Series, Return and Return2 of the Auxiliary Variables.

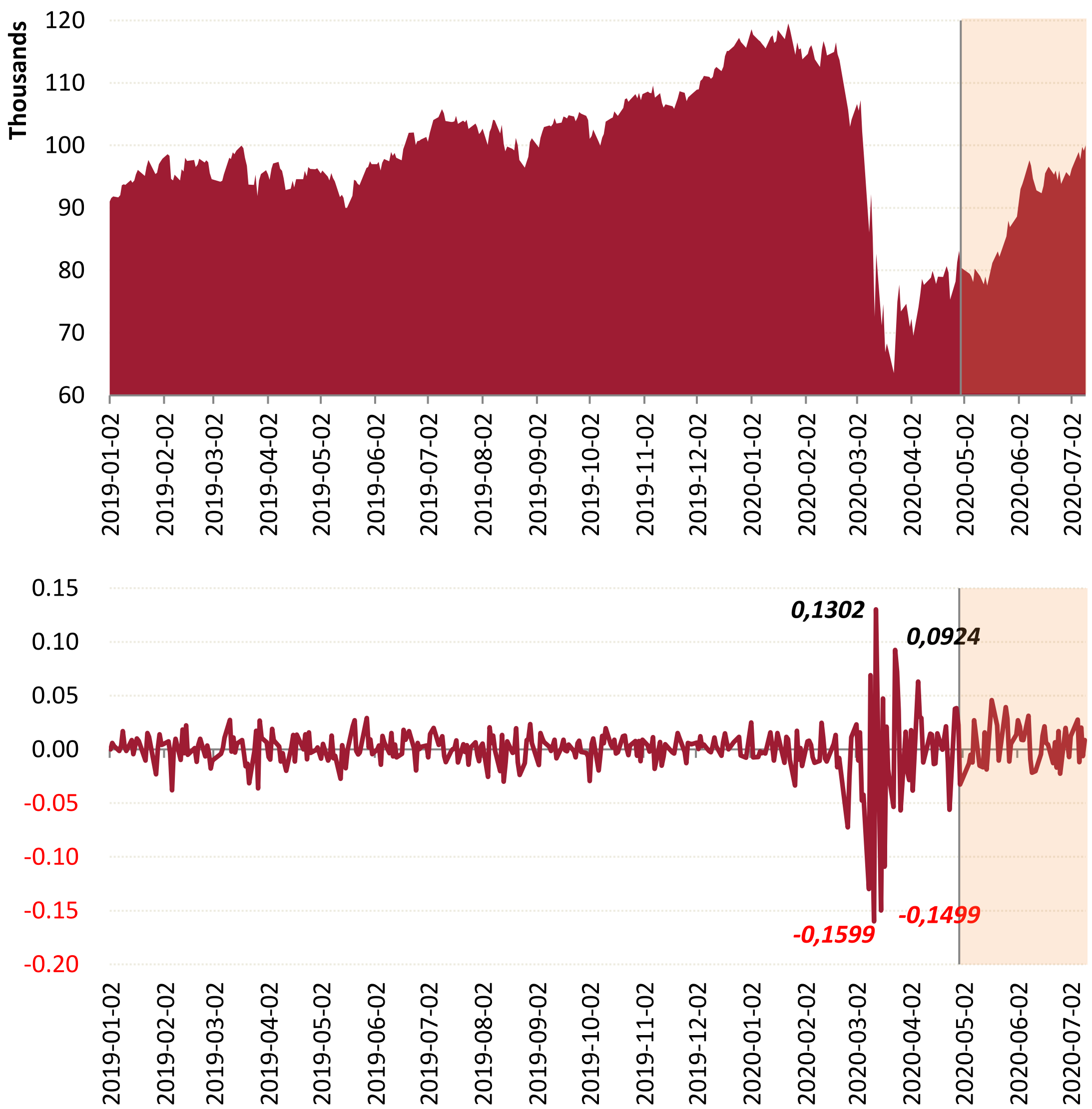



**Figure 7 - IBOV Time Series and IBOV Returns**
(orange region represents the out-of-sample)

Figure 7 can give a false effect that the volatility of the BOVESPA Index is only due to the pandemic of COVID-19. In the intention to verify this possibility, Figure 8 below shows data of the squared return, excluding information obtained since February 2020. As can be observed, the volatility of the time series under analysis is a recurrent phenomenon, independent of the advent of the global pandemic.

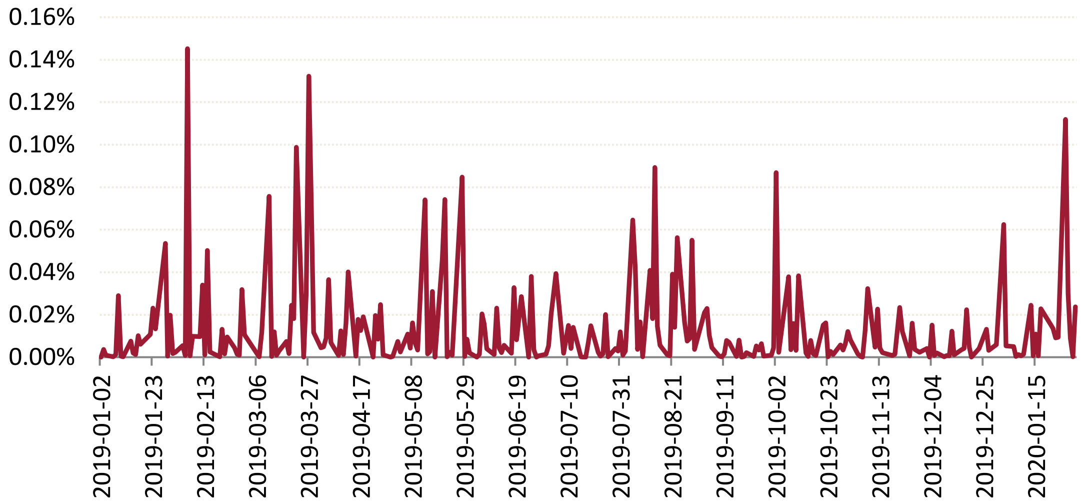



**Figure 8 - Squared Returns of IBOV**

The Kolmogorov-Smirnov and Jarque-Bera Tests ($H_0$: the series follows a normal distribution), as expected, reject the null hypothesis of normality for the original series under study as can be seen in Table 3 below.

**Table 3 - Normality Tests**

| | Kolmogorov-Smirnov | | Jarque-Bera | |
|---|---|---|---|---|
| | *statistic* | *p-value* | *statistic* | *p-value* |
| **IBOV** | 0.132 | 0.000 | 41.546 | 0.000 |
| **BRL_USD** | 0.218 | 0.000 | 255.593 | 0.000 |
| **Brent** | 0.268 | 0.000 | 332.569 | 0.000 |
| **EMBI** | 0.311 | 0.000 | 623.546 | 0.000 |

The unit root tests were used to verify whether the data series have a stationary pattern. The tests used were: Augmented Dickey-Fuller (ADF); Phillips-Perron (PP) (both have the same null hypothesis, $H_0$: the series has a unit root (not stationary)); and KPSS ($H_0$: the series does not have a unit root (it is stationary)). Table 4 and Table 5 presents the results of these tests for the original series (in level) and the first differences (differences at lag

1, denoted as "d_variable"[11]). For the KPSS test, instead of the p-value, is presented the 5% critical value.

**Table 4 - Stationary Tests (original series)**

| | | ADF | | PP | | | KPSS | |
|---|---|---|---|---|---|---|---|---|
| | | *no trend* | *trend* | *no trend* | *trend* | | *no trend* | *trend* |
| **IBOV** | statistic | -0.977 | -0.897 | -1.448 | -1.341 | statistic | 0.252 | 0.237 |
| | *p-value* | *0.762* | *0.954* | *0.559* | *0.876* | *crit 5%* | *0.463* | *0.146* |
| **BRL_USD** | statistic | 1.845 | -0.274 | 1.800 | -0.242 | statistic | 1.449 | 0.321 |
| | *p-value* | *1.000* | *0.991* | *1.000* | *0.992* | *crit 5%* | *0.463* | *0.146* |
| **Brent** | statistic | 0.915 | -0.903 | 0.922 | -0.874 | statistic | 0.956 | 0.288 |
| | *p-value* | *0.996* | *0.953* | *0.996* | *0.956* | *crit 5%* | *0.463* | *0.146* |
| **EMBI** | statistic | 0.034 | -0.575 | -0.639 | -1.264 | statistic | 0.508 | 0.275 |
| | *p-value* | *0.960* | *0.979* | *0.858* | *0.895* | *crit 5%* | *0.463* | *0.146* |

**Table 5 - Stationary Tests (first difference of the series)**

| | | ADF | | PP | | | KPSS | |
|---|---|---|---|---|---|---|---|---|
| | | *no trend* | *trend* | *no trend* | *trend* | | *no trend* | *trend* |
| **D_IBOV** | statistic | -21.725 | -21.823 | -21.328 | -21.414 | statistic | 0.235 | 0.074 |
| | *p-value* | *0.000* | *0.000* | *0.000* | *0.000* | *crit 5%* | *0.463* | *0.146* |
| **D_BRL_USD** | statistic | -18.144 | -18.420 | -18.146 | -18.419 | statistic | 0.602 | 0.101 |
| | *p-value* | *0.000* | *0.000* | *0.000* | *0.000* | *crit 5%* | *0.463* | *0.146* |
| **D_Brent** | statistic | -16.773 | -17.120 | -16.755 | -17.104 | statistic | 0.692 | 0.079 |
| | *p-value* | *0.000* | *0.000* | *0.000* | *0.000* | *crit 5%* | *0.463* | *0.146* |
| **D_EMBI** | statistic | -9.218 | -9.420 | -17.126 | -17.313 | statistic | 0.280 | 0.050 |
| | *p-value* | *0.000* | *0.000* | *0.000* | *0.000* | *crit 5%* | *0.463* | *0.146* |

The ADF and PP tests do not reject the null hypothesis of the presence of a unit root for the individual series (for all: p-value > 0.5) while rejecting the hypothesis for the first differences of the series (for all: p-value < 0.001). In summary, the results point out the absence of stationarity for the series in level and presence of stationarity for the first differences of the series.

The KPSS test, on the other hand, when comparing the critical values observed with the tabulated statistics (if $KPSS_{obs} > KPSS_{tab}$, $H_0$ is rejected) rejects the null hypothesis of

[11] Remembering that difference at lag 1 means $\Delta Y_t = Y_t - Y_{t-1}$. The first difference is expressed as "d_variable" herein due to the notation used while constructing the equation when using E-Views.

stationarity for the original series (in level) BRL_USD, Brent, and EMBI, but was inconclusive for the IBOV series, since the tests (trend and no trend) presented divergent results.

Regarding the series in the first difference, the result was inconclusive for BRL_USD and Brent. In order to elucidate the presence/absence of stationarity, the present study employed the Hodrick-Prescott Filter for all analyzed series (Appendix 3 - Hodrick-Prescott Filters), where it was clear that all series in level are non-stationary, and that their correspondents in first difference are stationary.

Intending to better understand the relationship between the variables under consideration, Granger's Causality test was employed, which can be interpreted as a way for verifying whether a time series ($X$) helps to forecast other time series ($Y$), or vice versa. If $X$ contributes to forecasting $Y$, then the unrestricted model with the past values of $X$ will add relevant (significant) information to forecast $Y$.

The variables in the first differences were used since the Granger's Causality test requires that the tested indicators be stationary (a fact corroborated by the ADF, PP, and KPSS tests). The results suggest that:

- ✓ the exchange rate between the US dollar and the Brazilian real (BRL_USD) presents a causality effect over the BOVESPA index (p-value = 0.001), and an inverse relationship is also significant (p-value = 0.021).

- ✓ the price of a barrel of Brent oil in US dollars (Brent) has no causality effect over the IBOV (p-value = 0.489), but the opposite relationship is observed (p-value = 0.053).

- ✓ the index regarding the difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the North American Treasury (EMBI) offers a causality effect over the BOVESPA index (p-value = 0.052), and an inverse relationship is not significant (p-value = 0.477).

In summary, BRL_USD and EMBI would be relevant in the BOVESPA Index modeling, while Brent could be dropped. The tests were performed with other lags and the results were different, so the present analysis will not discard the Brent regressor based only on the Granger's Causality test since the intrinsic relationship between Brent and the other variables (high correlation) could generate a more adequate model. The outputs of the tests with two lags can be seen in Table 6.

**Table 6 - Granger's Causality Tests**

Pairwise Granger Causality Tests
Date: 07/22/20 Time: 02:27
Sample: 1 322
Lags: 2

| Null Hypothesis: | Obs | F-Statistic | Prob. |
|---|---|---|---|
| D(BRL_USD) does not Granger Cause D(IBOV) | 319 | 7.06088 | 0.0010 |
| D(IBOV) does not Granger Cause D(BRL_USD) | | 3.89110 | 0.0214 |

Pairwise Granger Causality Tests
Date: 07/22/20 Time: 02:28
Sample: 1 322
Lags: 2

| Null Hypothesis: | Obs | F-Statistic | Prob. |
|---|---|---|---|
| D(BRENT) does not Granger Cause D(IBOV) | 319 | 0.71660 | 0.4892 |
| D(IBOV) does not Granger Cause D(BRENT) | | 2.96023 | 0.0533 |

Pairwise Granger Causality Tests
Date: 07/22/20 Time: 02:29
Sample: 1 322
Lags: 2

| Null Hypothesis: | Obs | F-Statistic | Prob. |
|---|---|---|---|
| D(EMBI) does not Granger Cause D(IBOV) | 319 | 2.98788 | 0.0518 |
| D(IBOV) does not Granger Cause D(EMBI) | | 0.74198 | 0.4770 |

In order to verify whether the variables have a long-term equilibrium relationship, that is, that they will move together in time and that the difference between them will be stable (i.e., stationary), the present study utilized the Johansen's Cointegration test ($H_0$: the number of cointegration vectors is $\leq n$, wherein $n = 0, 1, \ldots, p-1$; and $p$ denotes the number of variables used in the test).

According to the tests, there is statistical evidence (Trace and Maximum Eigenvalue statistics) for the presence of cointegration equations for the four variables analyzed, more details can be extracted from Table 7 below.

**Table 7 - Cointegration Test**

Sample (adjusted): 5 322
Included observations: 318 after adjustments
Trend assumption: Linear deterministic trend
Series: IBOV BRL_USD BRENT EMBI
Lags interval (in first differences): 1 to 3

Unrestricted Cointegration Rank Test (Trace)

| Hypothesized No. of CE(s) | Eigenvalue | Trace Statistic | 0.05 Critical Value | Prob.** |
|---|---|---|---|---|
| None * | 0.120588 | 63.37935 | 47.85613 | 0.0009 |
| At most 1 | 0.032076 | 22.51584 | 29.79707 | 0.2707 |
| At most 2 | 0.026457 | 12.14852 | 15.49471 | 0.1499 |
| At most 3 | 0.011325 | 3.621898 | 3.841466 | 0.0570 |

Trace test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

Unrestricted Cointegration Rank Test (Maximum Eigenvalue)

| Hypothesized No. of CE(s) | Eigenvalue | Max-Eigen Statistic | 0.05 Critical Value | Prob.** |
|---|---|---|---|---|
| None * | 0.120588 | 40.86351 | 27.58434 | 0.0006 |
| At most 1 | 0.032076 | 10.36731 | 21.13162 | 0.7096 |
| At most 2 | 0.026457 | 8.526627 | 14.26460 | 0.3277 |
| At most 3 | 0.011325 | 3.621898 | 3.841466 | 0.0570 |

Max-eigenvalue test indicates 1 cointegrating eqn(s) at the 0.05 level
* denotes rejection of the hypothesis at the 0.05 level
**MacKinnon-Haug-Michelis (1999) p-values

The results indicated that the null hypothesis of no cointegration equation must be rejected (line "None *": both p-value < 0.001; Table 7). In parallel, the null hypothesis of the presence of at most one cointegration equation is not rejected (line "At most 1": both p-value > 0.2). Consequently, it is concluded that there is cointegration between the variables under deliberation and that there is some long-term relationship between the four variables considered.

Since the variable to be modeled is the BOVESPA Index, it is necessary to assess whether there is cointegration when IBOV is the dependent variable. In short, check if the IBOV is cointegrated with the regressors (BRL_USD, Brent, and EMBI) to, in this context, indicate which model should be applied.

Following the results observed in the Bounds test, if there is cointegration, it is possible to specify both short-term (ARDL) or long-term (VEC) models. Otherwise, only the short-term model (ARDL) should be estimated. Using a significance level of 5%, the Bounds test results do not reject the null hypothesis ($H_0$: there is no long-term relationship (lack of cointegration)) since the $F$-statistic and $T$-statistic (absolute values) are lower than the critical values of $I(0)$. In this context, there is no statistical evidence for the presence of cointegration when IBOV is the dependent variable as can be observed in Table 8.

**Table 8 - Bounds Test (F-statistic & T-statistic)**

| F-Bounds Test | | Null Hypothesis: No levels relationship | | |
|---|---|---|---|---|
| Test Statistic | Value | Signif. | I(0) | I(1) |
| | | | Asymptotic: n=1000 | |
| F-statistic | 1.172782 | 10% | 2.72 | 3.77 |
| k | 3 | 5% | 3.23 | 4.35 |
| | | 2.5% | 3.69 | 4.89 |
| | | 1% | 4.29 | 5.61 |
| Actual Sample Size | 319 | | Finite Sample: n=80 | |
| | | 10% | 2.823 | 3.885 |
| | | 5% | 3.363 | 4.515 |
| | | 1% | 4.568 | 5.96 |

| t-Bounds Test | | Null Hypothesis: No levels relationship | | |
|---|---|---|---|---|
| Test Statistic | Value | Signif. | I(0) | I(1) |
| t-statistic | -0.430801 | 10% | -2.57 | -3.46 |
| | | 5% | -2.86 | -3.78 |
| | | 2.5% | -3.13 | -4.05 |
| | | 1% | -3.43 | -4.37 |

In light of the foregoing, it is concluded that the best approach to analyze the BOVESPA Index through the selected regressors is the ARDL (short-term model).

When analyzing the cointegration equation, the signs of the coefficients must be interpreted in reverse, i.e., on average (*ceteris paribus*), the value of the US dollar over the Brazilian real (BRL_USD) presents a positive impact on the BOVESPA Index, while the oil price in US dollars (Brent) and the return rate on Brazilian bonds (EMBI) have negative impacts on the IBOV. Table 9 shows the estimated cointegration equation.

**Table 9 - Cointegration Equation**

| 1 Cointegrating Equation(s): | Log likelihood | -3804.300 | |
|---|---|---|---|
| Normalized cointegrating coefficients (standard error in parentheses) | | | |
| IBOV | BRL_USD | BRENT | EMBI |
| 1.000000 | -19635.97 | 260.0010 | 366.7011 |
| | (3541.66) | (125.163) | (24.5080) |
| Adjustment coefficients (standard error in parentheses) | | | |
| D(IBOV) | 0.130766 | | |
| | (0.02662) | | |
| D(BRL_USD) | -2.30E-06 | | |
| | (5.2E-07) | | |
| D(BRENT) | 4.39E-05 | | |
| | (2.3E-05) | | |
| D(EMBI) | -0.000864 | | |
| | (0.00014) | | |

Concerning the optimal number of lags to be used for the ARDL models, the selection criteria did not converge to a single choice, since each criterion points to the use of different lags (asterisks) as can be seen in Table 10. The present study used the BIC (here SC) criterion since a smaller number of parameters will generate a more parsimonious model. It should be highlighted that Table 10 shows the fours variables as endogenous due to the absence of cointegration when IBOV is the dependent variable (Table 8).

**Table 10 - Lags Number Selection**

VAR Lag Order Selection Criteria
Endogenous variables: IBOV BRL_USD BRENT EMBI
Exogenous variables: C
Date: 07/20/20 Time: 13:10
Sample: 1 322
Included observations: 314

| Lag | LogL | LR | FPE | AIC | SC | HQ |
|---|---|---|---|---|---|---|
| 0 | -5738.368 | NA | 9.01e+10 | 36.57559 | 36.62336 | 36.59468 |
| 1 | -3835.062 | 3745.998 | 542007.7 | 24.55453 | 24.79335* | 24.64996 |
| 2 | -3799.063 | 69.93367 | 477208.0 | 24.42715 | 24.85702 | 24.59892 |
| 3 | -3769.613 | 56.46144 | 438083.9 | 24.34148 | 24.96240 | 24.58959* |
| 4 | -3749.412 | 38.21497 | 426616.6 | 24.31473 | 25.12670 | 24.63917 |
| 5 | -3739.473 | 18.54842 | 443575.2 | 24.35333 | 25.35635 | 24.75412 |
| 6 | -3721.405 | 33.25858 | 438012.6 | 24.34016 | 25.53423 | 24.81729 |
| 7 | -3686.376 | 63.58819 | 388309.5* | 24.21895* | 25.60408 | 24.77243 |
| 8 | -3671.579 | 26.48268* | 391691.1 | 24.22662 | 25.80280 | 24.85643 |

* indicates lag order selected by the criterion
LR: sequential modified LR test statistic (each test at 5% level)
FPE: Final prediction error
AIC: Akaike information criterion
SC: Schwarz information criterion
HQ: Hannan-Quinn information criterion

In summary, the short-term ARDL model can be estimated by OLS with the differences of the original variables containing one lag. As described in Table 11, the estimated model will be:

$$\Delta IBOV_t = -77.636 - 0.088 \times \Delta IBOV_{t-1} + 7686.559 \times \Delta BRL_{USD\,t-1} + 18.254 \times \Delta BRENT_{t-1} + 9.884 \times \Delta EMBI_{t-1}$$

**Table 11 - ARDL(1,1,1,1) Model**

Dependent Variable: D(IBOV)
Method: Least Squares
Date: 07/26/20 Time: 21:25
Sample (adjusted): 3 322
Included observations: 320 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | -77.63603 | 116.1531 | -0.668394 | 0.5044 |
| D(IBOV(-1)) | -0.088114 | 0.083969 | -1.049367 | 0.2948 |
| D(BRL_USD(-1)) | 7686.559 | 3598.999 | 2.135749 | 0.0335 |
| D(BRENT(-1)) | 18.25410 | 76.60447 | 0.238290 | 0.8118 |
| D(EMBI(-1)) | 9.884195 | 16.40551 | 0.602492 | 0.5473 |

| | | | |
|---|---|---|---|
| R-squared | 0.058683 | Mean dependent var | -34.55738 |
| Adjusted R-squared | 0.046729 | S.D. dependent var | 2104.154 |
| S.E. of regression | 2054.403 | Akaike info criterion | 18.10886 |
| Sum squared resid | 1.33E+09 | Schwarz criterion | 18.16774 |
| Log likelihood | -2892.418 | Hannan-Quinn criter. | 18.13237 |
| F-statistic | 4.909354 | Durbin-Watson stat | 1.918188 |
| Prob(F-statistic) | 0.000748 | | |

Unfortunately, the model under consideration is not satisfactory since the diagnostics step pointed out that only one parameter is statistically significant, as can be seen in Table 11, in which the first difference of the exchange rate between the US dollar and the Brazilian Real presents a p-value lower than 0.05.

In addition, the Breusch-Godfrey test rejects the null hypothesis of no serial correlation of the residuals, as can be observed in

Table 12. Last but not less important, the results presented in Table 13 corroborate the lack of fit of the model estimated since the null hypothesis of homoscedastic residuals is rejected. Although the model has some stability (Figure 9).

**Table 12 - Autocorrelation of the Residuals - ARDL**

Breusch-Godfrey Serial Correlation LM Test:

| | | | |
|---|---|---|---|
| F-statistic | 9.737712 | Prob. F(1,314) | 0.0020 |
| Obs*R-squared | 9.625285 | Prob. Chi-Square(1) | 0.0019 |

**Table 13 - Heteroscedasticity of the Residuals - ARDL**

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 38.71924 | Prob. F(1,317) | 0.0000 |
| Obs*R-squared | 34.72243 | Prob. Chi-Square(1) | 0.0000 |

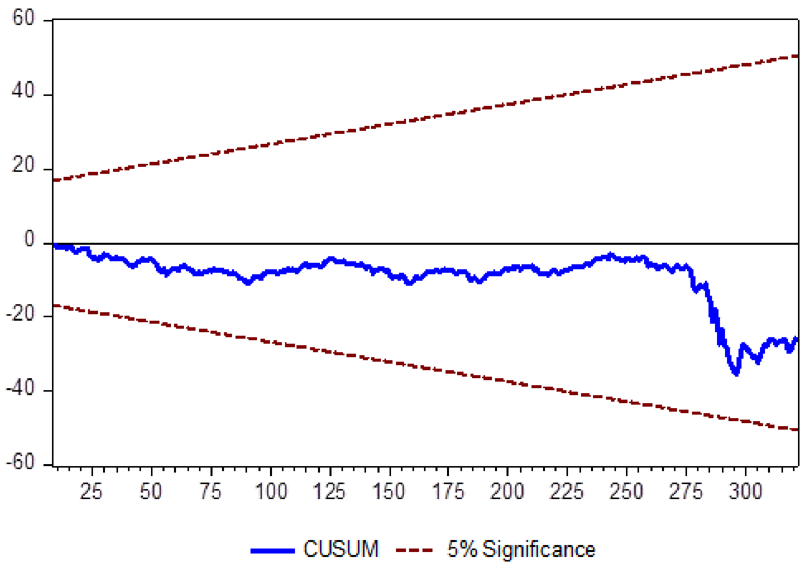



**Figure 9 - Model Stability - ARDL**

Models containing only BRL_USD / EMBI and Brent / EMBI as regressors were equally unsatisfactory according to the diagnostic steps. In this context, the ADRL modeling will not be appreciated in the forecasts stage and adjustment quality diagnostics.

Furthermore, the present study performed the ARCH/GARCH modeling with the log-returns of the variables, represented as R_IBOV, R_USD_BRL, R_Brent, and R_EMBI. First, the adequacy of this approach was assessed through linear regression using R_IBOV as a dependent variable and R_USD_BRL, R_Brent, and R_EMBI as auxiliary variables. Subsequently, the analysis concerning the presence/absence of conditional heteroscedasticity and autocorrelation in the residuals of the generated model was conducted. The linear model estimated is shown in Table 14.

**Table 14 - ARCH/GARCH (Pre-diagnoses)**

Dependent Variable: R_IBOV
Method: Least Squares
Date: 07/30/20 Time: 00:30
Sample (adjusted): 2 322
Included observations: 321 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000807 | 0.000910 | 0.886881 | 0.3758 |
| R_BRL_USD | -0.367560 | 0.121863 | -3.016175 | 0.0028 |
| R_BRENT | 0.055116 | 0.020532 | 2.684349 | 0.0076 |
| R_EMBI | -0.457956 | 0.033236 | -13.77901 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.576808 | Mean dependent var | -0.000382 |
| Adjusted R-squared | 0.572803 | S.D. dependent var | 0.024709 |
| S.E. of regression | 0.016150 | Akaike info criterion | -5.401419 |
| Sum squared resid | 0.082680 | Schwarz criterion | -5.354423 |
| Log likelihood | 870.9277 | Hannan-Quinn criter. | -5.382654 |
| F-statistic | 144.0231 | Durbin-Watson stat | 2.282640 |
| Prob(F-statistic) | 0.000000 | | |

The ARCH LM test indicates that the ARCH approach can be employed for modeling the IBOV returns since the LM test rejects the null hypothesis ($H_0$: there is no ARCH factor when considering the residuals) at a significance level of 5% as can be seen in the first portion of Table 15.

**Table 15 - ARCH LM Test (Pre-diagnoses)**

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 40.93996 | Prob. F(1,318) | 0.0000 |
| Obs*R-squared | 36.49855 | Prob. Chi-Square(1) | 0.0000 |

Test Equation:
Dependent Variable: RESID^2
Method: Least Squares
Date: 07/30/20 Time: 00:30
Sample (adjusted): 3 322
Included observations: 320 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000171 | 3.45E-05 | 4.946845 | 0.0000 |
| RESID^2(-1) | 0.354848 | 0.055459 | 6.398434 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.114058 | Mean dependent var | 0.000258 |
| Adjusted R-squared | 0.111272 | S.D. dependent var | 0.000601 |

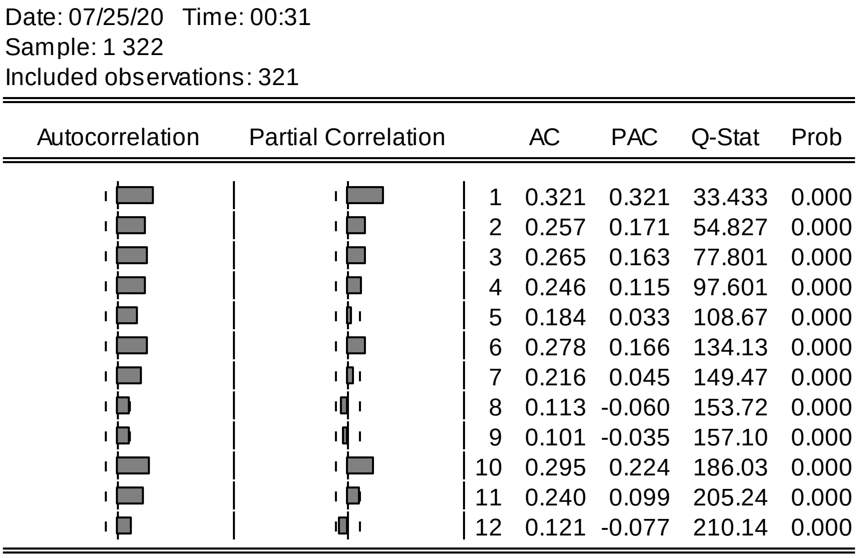
| | | | |
|---|---|---|---|
| S.E. of regression | 0.000566 | Akaike info criterion | -12.10934 |
| Sum squared resid | 0.000102 | Schwarz criterion | -12.08579 |
| Log likelihood | 1939.495 | Hannan-Quinn criter. | -12.09994 |
| F-statistic | 40.93996 | Durbin-Watson stat | 2.007313 |
| Prob(F-statistic) | 0.000000 | | |

The second portion of Table 15 shows that the squared residuals at time $t$ (here RESID^2) can be modeled using the squared residuals at time $t - 1$ (here RESID^2(-1)) as a regressor (statistically significant with p-value < 0.001). This indicates the presence of autocorrelation regarding the residuals estimated.

The correlogram of the residuals and squared residuals (Figure 10), which presents the autocorrelation (ACF, here represented as “AC”) and partial autocorrelation (PACF, here represented as “PAC”), corroborates the presence of autocorrelation involving the residuals estimated by the linear model (p-values < 0.05).

Date: 07/25/20 Time: 00:31
Sample: 1 322
Included observations: 321

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|---|---|
| | | 1 | -0.163 | -0.163 | 8.5919 | 0.003 |
| | | 2 | -0.013 | -0.040 | 8.6449 | 0.013 |
| | | 3 | -0.029 | -0.039 | 8.9159 | 0.030 |
| | | 4 | 0.085 | 0.075 | 11.259 | 0.024 |
| | | 5 | -0.041 | -0.016 | 11.801 | 0.038 |
| | | 6 | -0.114 | -0.124 | 16.072 | 0.013 |
| | | 7 | 0.134 | 0.102 | 22.011 | 0.003 |
| | | 8 | -0.171 | -0.156 | 31.697 | 0.000 |
| | | 9 | -0.026 | -0.076 | 31.923 | 0.000 |
| | | 10 | 0.014 | 0.016 | 31.991 | 0.000 |
| | | 11 | -0.054 | -0.099 | 32.958 | 0.001 |
| | | 12 | -0.040 | -0.046 | 33.481 | 0.001 |



Date: 07/25/20 Time: 00:31
Sample: 1 322
Included observations: 321

| | AC | PAC | Q-Stat | Prob |
|---|---|---|---|---|
| 1 | 0.321 | 0.321 | 33.433 | 0.000 |
| 2 | 0.257 | 0.171 | 54.827 | 0.000 |
| 3 | 0.265 | 0.163 | 77.801 | 0.000 |
| 4 | 0.246 | 0.115 | 97.601 | 0.000 |
| 5 | 0.184 | 0.033 | 108.67 | 0.000 |
| 6 | 0.278 | 0.166 | 134.13 | 0.000 |
| 7 | 0.216 | 0.045 | 149.47 | 0.000 |
| 8 | 0.113 | -0.060 | 153.72 | 0.000 |
| 9 | 0.101 | -0.035 | 157.10 | 0.000 |
| 10 | 0.295 | 0.224 | 186.03 | 0.000 |
| 11 | 0.240 | 0.099 | 205.24 | 0.000 |
| 12 | 0.121 | -0.077 | 210.14 | 0.000 |

**Figure 10 - Correlogram (Pre-diagnoses)**

As expected, the Jarque-Bera test strongly rejects the null hypothesis of the returns being normally distributed (p-value < 0.001) (Table 16).

**Table 16 - Descriptive Statistics of the Returns (n = 321)**

| Statistic | R_IBOV | R_BRL_USD | R_Brent | R_EMBI |
|---|---|---|---|---|
| **Minimum** | -0.15994 | -0.03057 | -0.25639 | -0.14328 |
| **Maximum** | 0.13023 | 0.043813 | 0.301613 | 0.194935 |
| **Mean** | -0.00038 | 0.00106 | -0.00341 | 0.00133 |
| **Std. Error (Mean)** | 0.0247 | 0.0089 | 0.0474 | 0.0341 |
| **Skewness** | -1.7030 | 0.568701 | -0.9236 | 1.362025 |
| **Kurtosis** | 17.9999 | 7.0496 | 17.0659 | 11.7171 |
| | | | | |
| **Jarque-Bera** | 3164.478 | 236.6440 | 2691.880 | 1115.595 |
| **Probability** | 0.000000 | 0.000000 | 0.000000 | 0.000000 |

Since the log-returns have an ARCH factor, the modeling can be conducted properly. Several models were tested with different residual distributions (Normal, Student's *t* and Generalized (GED)), in which those with significant coefficients, higher log-likelihood, lower BIC, absence of heteroscedasticity and autocorrelation of the residuals using ARCH LM test (with 1 lag and 36 lags[12]), were selected for the fit quality and

[12] The literature indicates that many lags (here 36) also must be used while performing the ARCH LM test during ARCH/GARCH modeling when daily data is analyzed.

forecast stage.

The individual diagnoses involving four candidate models using three different residual distributions are presented in Table 17.

Table 17 - Adherent Models - ARCH/GARCH

| | *residual distribution type* | | |
|---|---|---|---|
| **GARCH(1,1)** | **Normal** | **Student's *t*** | **GED** |
| Is ARCH coefficient significant? | Yes | Yes | Yes |
| Is GARCH coefficient significant? | Yes | Yes | Yes |
| Log-likelihood | 939.960 | 944.465 | 944.053 |
| BIC | -5.731 | -5.741 | -5.739 |
| Is there heteroscedasticity? (lag 1) | No | No | No |
| Is there heteroscedasticity? (lag 36) | No | No | No |
| Are the residues autocorrelated? | No | No | No |
| Are the squared residues autocorrelated? | No | No | No |

| | *residual distribution type* | | |
|---|---|---|---|
| **GARCH(1,2)** | **Normal** | **Student's *t*** | **GED** |
| Is ARCH coefficient significant? | Yes | Yes | Yes |
| Are GARCH coefficients significant? | Yes | Yes | Yes |
| Log-likelihood | 940.606 | 944.888 | 944.691 |
| BIC | -5.717 | -5.725 | -5.724 |
| Is there heteroscedasticity? (lag 1) | No | No | No |
| Is there heteroscedasticity? (lag 36) | No | No | No |
| Are the residues autocorrelated? | No | No | No |
| Are the squared residues autocorrelated? | No | No | No |

| | *residual distribution type* | | |
|---|---|---|---|
| **EGARCH(1,1)** | **Normal** | **Student's *t*** | **GED** |
| Is ARCH coefficient significant? | Yes | Yes | Yes |
| Is GARCH coefficient significant? | Yes | Yes | Yes |
| Log-likelihood | 937.513 | 942.685 | 942.341 |
| BIC | -5.715 | -5.729 | -5.727 |
| Is there heteroscedasticity? (lag 1) | No | No | No |
| Is there heteroscedasticity? (lag 36) | No | No | No |
| Are the residues autocorrelated? | No | No | No |
| Are the squared residues autocorrelated? | No | No | No |

| | *residual distribution type* | | |
|---|---|---|---|
| **IGARCH(1,1)** | **Normal** | **Student's *t*** | **GED** |
| Is ARCH coefficient significant? | Yes | Yes | Yes |
| Is GARCH coefficient significant? | Yes | Yes | Yes |
| Log-likelihood | 930.132 | 940.454 | 940.072 |
| BIC | -5.705 | -5.752 | -5.749 |

| | | | |
|---|---|---|---|
| Is there heteroscedasticity? (lag 1) | No | No | No |
| Is there heteroscedasticity? (lag 36) | No | No | No |
| Are the residues autocorrelated? | Yes | Yes | No |
| Are the squared residues autocorrelated? | - | - | No |

The models containing residuals with Student's $t$ distribution showed better results in terms of log-likelihood and BIC in the GARCH(1,1), GARCH(1,2) and EGARCH(1,1), while IGARCH(1,1) with generalized residuals (GED) was the only one approved in the criteria mentioned in Table 17 above. Those models will be evaluated and compared to determine the “best” model in the out-of-sample analysis section below.

## 8.2. Out-of-Sample Analysis (Volatility Forecast)

In order to assess the in-sample models selected for the R_IBOV modeling, the indexes achieved in the MAPE and RMSE were compared for the “best” models using the forecast period containing 42 registers (out-of-sample). The results related to the selection criteria can be seen in Table 18 below.

**Table 18 - Estimated Models and Selection Criteria**

| Model | Residual Type | MAPE | RMSE |
|---|---|---|---|
| GARCH(1,1) | Student's T | 116.721 | 0.0159 |
| GARCH(1,2) | Student's T | 116.642 | **0.0158** |
| EGARCH(1,1) | Student's T | 117.449 | 0.0159 |
| IGARCH(1,1) | GED | **114.636** | 0.0162 |

As can be seen, the selection criteria did not converge to a single model for the BOVESPA Index. The literature, despite classifying this disagreement as a rare event, guides the choice for the IGARCH(1,1) model since the MAPE criterion is notoriously most used in this type of analysis, in addition to the distribution of the errors not being Gaussian (see (Chai & Draxler, 2014)). The model with the best fit is described below and Table 19 demonstrates that the parameters are statically significant (p-values < 0.05).

$$R\_IBOV_t = 0.0007 - 0.3258 \times R\_BRL\_USD_t + 0,0767 \times R\_Brent_t - 0,2337 \times R\_EMBI_t$$

$$\sigma_t^2 = 0.0845 \times \varepsilon_{t-1}^2 + 0.9154 \times \sigma_{t-1}^2$$

**Table 19 - Best Model - R_IBOV**

Dependent Variable: R_IBOV
Method: ML ARCH - Generalized error distribution (GED) (BFGS / Marquardt steps)
Date: 07/25/20 Time: 00:30
Sample (adjusted): 2 322
Included observations: 321 after adjustments
Convergence achieved after 29 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(5)*RESID(-1)^2 + (1 - C(5))*GARCH(-1)

| Variable | Coefficient | Std. Error | z-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.000776 | 0.000525 | 1.478903 | 0.1392 |
| R_BRL_USD | -0.325812 | 0.077588 | -4.199255 | 0.0000 |
| R_BRENT | 0.076734 | 0.021576 | 3.556494 | 0.0004 |
| R_EMBI | -0.233749 | 0.024687 | -9.468434 | 0.0000 |
| Variance Equation | | | | |
| RESID(-1)^2 | 0.084550 | 0.021298 | 3.969877 | 0.0001 |
| GARCH(-1) | 0.915450 | 0.021298 | 42.98327 | 0.0000 |
| GED PARAMETER | 1.464341 | 0.148630 | 9.852234 | 0.0000 |

| | | | |
|---|---|---|---|
| R-squared | 0.483500 | Mean dependent var | -0.000382 |
| Adjusted R-squared | 0.478612 | S.D. dependent var | 0.024709 |
| S.E. of regression | 0.017842 | Akaike info criterion | -5.819768 |
| Sum squared resid | 0.100910 | Schwarz criterion | -5.749273 |
| Log likelihood | 940.0727 | Hannan-Quinn criter. | -5.791621 |
| Durbin-Watson stat | 2.508307 | | |

It should be highlighted that the sum of the ARCH and GARCH parameters (0.0845 and 0.9154) is equal to one, which indicates that volatility shocks have a persistent effect on the conditional variance. In other words, the conditional variance does not converge on a constant unconditional variance in the long run.

After defining that the IGARCH(1,1) with GED errors is the most appropriate model for the volatility of the BOVESPA Index returns according to the in-sample collected – and before analyzing its capability to predict the said indicator using the out-of-sample containing 42 registers –, it is important to verify whether the estimated model presents stable and non-correlated residuals.

In this context, the residual diagnostics employing the ARCH LM test using 1 lag (Table 20) and 36 lags (Table 21) was conducted (remembering that the literature indicates that a high length lag structure must also be used when working with daily data).

The results using 1 lag indicates that the residuals are homoscedasticity since the null hypothesis ($H_0$: there is no ARCH factor when considering the residuals) was not rejected (p-value > 0.05) as described in the first portion of Table 20.

The second portion of Table 20 shows that the squared residuals at time $t$ cannot be modeled using the squared residuals at time $t-1$ as a regressor (parameter not statistically significant with p-value > 0.05). This indicates the absence of autocorrelation regarding the residuals estimated by the IGARCH(1,1) model.

**Table 20 - ARCH LM Test with 1 lag (Residuals - IGARCH(1,1))**

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 0.101982 | Prob. F(1,318) | 0.7497 |
| Obs*R-squared | 0.102590 | Prob. Chi-Square(1) | 0.7487 |

Test Equation:
Dependent Variable: WGT_RESID^2
Method: Least Squares
Date: 07/25/20 Time: 00:30
Sample (adjusted): 3 322
Included observations: 320 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 1.257368 | 0.140098 | 8.974945 | 0.0000 |
| WGT_RESID^2(-1) | -0.017904 | 0.056063 | -0.319346 | 0.7497 |

| | | | |
|---|---|---|---|
| R-squared | 0.000321 | Mean dependent var | 1.235380 |
| Adjusted R-squared | -0.002823 | S.D. dependent var | 2.179518 |
| S.E. of regression | 2.182592 | Akaike info criterion | 4.405134 |
| Sum squared resid | 1514.859 | Schwarz criterion | 4.428686 |
| Log likelihood | -702.8214 | Hannan-Quinn criter. | 4.414538 |
| F-statistic | 0.101982 | Durbin-Watson stat | 1.995368 |
| Prob(F-statistic) | 0.749674 | | |

The same analysis was conducted using 36 lags and the results held. The ARCH LM test with 36 lags indicates that the residuals are homoscedasticity (p-value > 0.005) as described in the first portion of Table 21. The second portion of Table 21 shows that the squared residuals at time $t$ cannot be modeled using the squared residuals at time $t-1$ to $t-36$ (parameters not statistically significant with p-value > 0.05), which indicates the presence of autocorrelation regarding the residuals estimated.

**Table 21 - ARCH LM Test with 36 lags (Residuals - IGARCH(1,1))**

Heteroskedasticity Test: ARCH

| | | | |
|---|---|---|---|
| F-statistic | 0.776291 | Prob. F(36,248) | 0.8179 |
| Obs*R-squared | 28.86338 | Prob. Chi-Square(36) | 0.7950 |

Test Equation:
Dependent Variable: WGT_RESID^2
Method: Least Squares
Date: 07/25/20 Time: 00:30
Sample (adjusted): 38 322
Included observations: 285 after adjustments

| Variable | Coefficient | Std. Error | t-Statistic | Prob. |
|---|---|---|---|---|
| C | 0.974419 | 0.427378 | 2.279992 | 0.0235 |
| WGT_RESID^2(-1) | -0.028928 | 0.063487 | -0.455646 | 0.6490 |
| WGT_RESID^2(-2) | 0.032745 | 0.062953 | 0.520151 | 0.6034 |
| WGT_RESID^2(-3) | 0.032254 | 0.062971 | 0.512196 | 0.6090 |
| WGT_RESID^2(-4) | 0.022431 | 0.062979 | 0.356163 | 0.7220 |
| WGT_RESID^2(-5) | 0.017675 | 0.063020 | 0.280463 | 0.7794 |
| WGT_RESID^2(-6) | 0.021183 | 0.063063 | 0.335902 | 0.7372 |
| WGT_RESID^2(-7) | -0.012963 | 0.062922 | -0.206020 | 0.8369 |
| WGT_RESID^2(-8) | 0.099841 | 0.062732 | 1.591537 | 0.1128 |
| WGT_RESID^2(-9) | 0.034144 | 0.062962 | 0.542289 | 0.5881 |
| WGT_RESID^2(-10) | 0.053728 | 0.062936 | 0.853699 | 0.3941 |
| WGT_RESID^2(-11) | 0.115647 | 0.062990 | 1.835952 | 0.0676 |
| WGT_RESID^2(-12) | -0.045786 | 0.063369 | -0.722528 | 0.4707 |
| WGT_RESID^2(-13) | 0.062010 | 0.063180 | 0.981488 | 0.3273 |
| WGT_RESID^2(-14) | -0.009381 | 0.061455 | -0.152644 | 0.8788 |
| WGT_RESID^2(-15) | -0.036635 | 0.061065 | -0.599936 | 0.5491 |
| WGT_RESID^2(-16) | 0.022900 | 0.060888 | 0.376100 | 0.7072 |
| WGT_RESID^2(-17) | -0.010881 | 0.060991 | -0.178397 | 0.8586 |
| WGT_RESID^2(-18) | -0.060297 | 0.060953 | -0.989238 | 0.3235 |
| WGT_RESID^2(-19) | -0.002784 | 0.060846 | -0.045749 | 0.9635 |
| WGT_RESID^2(-20) | 0.001644 | 0.060818 | 0.027028 | 0.9785 |
| WGT_RESID^2(-21) | -0.102286 | 0.059102 | -1.730686 | 0.0848 |
| WGT_RESID^2(-22) | -0.109686 | 0.059401 | -1.846527 | 0.0660 |
| WGT_RESID^2(-23) | -0.034090 | 0.059829 | -0.569793 | 0.5693 |
| WGT_RESID^2(-24) | -0.082993 | 0.059642 | -1.391521 | 0.1653 |
| WGT_RESID^2(-25) | -0.029470 | 0.059855 | -0.492357 | 0.6229 |
| WGT_RESID^2(-26) | -0.025816 | 0.059486 | -0.433993 | 0.6647 |
| WGT_RESID^2(-27) | 0.014883 | 0.059457 | 0.250314 | 0.8026 |
| WGT_RESID^2(-28) | 0.043132 | 0.059420 | 0.725885 | 0.4686 |
| WGT_RESID^2(-29) | 0.031841 | 0.059230 | 0.537586 | 0.5913 |
| WGT_RESID^2(-30) | 0.071589 | 0.059140 | 1.210508 | 0.2272 |
| WGT_RESID^2(-31) | -0.003108 | 0.059270 | -0.052436 | 0.9582 |
| WGT_RESID^2(-32) | 0.001867 | 0.059340 | 0.031459 | 0.9749 |
| WGT_RESID^2(-33) | -0.009893 | 0.059808 | -0.165417 | 0.8688 |

| | | | | |
|---|---|---|---|---|
| WGT_RESID^2(-34) | 0.004472 | 0.061219 | 0.073052 | 0.9418 |
| WGT_RESID^2(-35) | 0.135372 | 0.061740 | 2.192628 | 0.0293 |
| WGT_RESID^2(-36) | 0.007483 | 0.062026 | 0.120647 | 0.9041 |

| | | | |
|---|---|---|---|
| R-squared | 0.101275 | Mean dependent var | 1.228990 |
| Adjusted R-squared | -0.029185 | S.D. dependent var | 2.159129 |
| S.E. of regression | 2.190409 | Akaike info criterion | 4.526642 |
| Sum squared resid | 1189.877 | Schwarz criterion | 5.000825 |
| Log likelihood | -608.0465 | Hannan-Quinn criter. | 4.716731 |
| F-statistic | 0.776291 | Durbin-Watson stat | 1.998121 |
| Prob(F-statistic) | 0.817866 | | |

In addition, as can be seen in Figure 11, the correlogram of the residual and squared residuals of this model present values with low magnitude (close to zero), which are in accordance with the large p-values observed. Thus, the diagnoses obtainable from the correlogram below corroborates that the residuals are not autocorrelated.

| Autocorrelation | Partial Correlation | | AC | PAC | Q-Stat | Prob* |
|---|---|---|---|---|---|---|
| | | 1 | -0.018 | -0.018 | 0.1037 | 0.747 |
| | | 2 | 0.048 | 0.048 | 0.8507 | 0.654 |
| | | 3 | 0.044 | 0.046 | 1.4808 | 0.687 |
| | | 4 | 0.002 | 0.002 | 1.4827 | 0.830 |
| | | 5 | 0.029 | 0.025 | 1.7572 | 0.882 |
| | | 6 | 0.032 | 0.031 | 2.0951 | 0.911 |
| | | 7 | 0.044 | 0.042 | 2.7269 | 0.909 |
| | | 8 | 0.082 | 0.079 | 4.9322 | 0.765 |
| | | 9 | 0.028 | 0.025 | 5.1940 | 0.817 |
| | | 10 | 0.017 | 0.007 | 5.2908 | 0.871 |
| | | 11 | 0.081 | 0.072 | 7.5071 | 0.757 |
| | | 12 | -0.060 | -0.063 | 8.7068 | 0.728 |
| | | 13 | 0.060 | 0.044 | 9.9146 | 0.701 |
| | | 14 | -0.013 | -0.020 | 9.9710 | 0.764 |
| | | 15 | -0.059 | -0.070 | 11.138 | 0.743 |
| | | 16 | 0.023 | 0.006 | 11.324 | 0.789 |
| | | 17 | -0.028 | -0.029 | 11.586 | 0.825 |
| | | 18 | -0.070 | -0.079 | 13.284 | 0.774 |
| | | 19 | 0.011 | -0.000 | 13.325 | 0.822 |
| | | 20 | -0.012 | -0.000 | 13.375 | 0.861 |
| | | 21 | -0.078 | -0.080 | 15.500 | 0.797 |
| | | 22 | -0.093 | -0.098 | 18.526 | 0.674 |
| | | 23 | -0.056 | -0.035 | 19.627 | 0.664 |
| | | 24 | -0.068 | -0.067 | 21.246 | 0.624 |
| | | 25 | -0.047 | -0.023 | 22.016 | 0.635 |
| | | 26 | -0.042 | -0.018 | 22.632 | 0.654 |
| | | 27 | 0.021 | 0.030 | 22.787 | 0.696 |
| | | 28 | 0.015 | 0.052 | 22.867 | 0.740 |
| | | 29 | 0.000 | 0.036 | 22.867 | 0.783 |
| | | 30 | 0.057 | 0.080 | 24.025 | 0.771 |
| | | 31 | -0.039 | 0.002 | 24.562 | 0.787 |
| | | 32 | -0.020 | 0.003 | 24.704 | 0.818 |
| | | 33 | -0.020 | -0.010 | 24.855 | 0.845 |
| | | 34 | -0.027 | -0.013 | 25.121 | 0.865 |
| | | 35 | 0.108 | 0.120 | 29.376 | 0.736 |
| | | 36 | 0.006 | -0.005 | 29.389 | 0.774 |

**Figure 11 - Correlogram of Residuals - IGARCH(1,1)**

Finally, Figure 12 illustrates a visual comparison between the observed returns and the returns adjusted by the IGARCH(1,1) model. It should be emphasized that the orange area represents the out-of-sample observations.

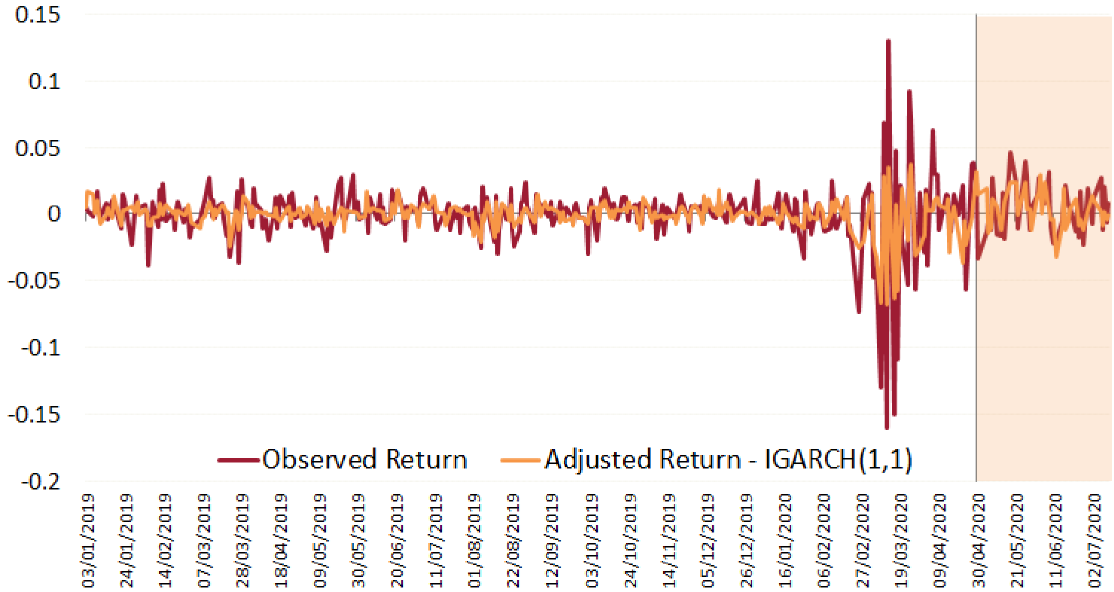



**Figure 12 - Observed Return x Adjusted Return - IGARCH(1,1)**

After choosing the model with the best fit, a 95.0% confidence interval for the IBOV return was elaborated to verify the coverage of the estimated model over the actual observations, that is, observe the number of points that remain within the confidence interval for the series in-sample and out-of-sample of the BOVESPA Index returns. Figure 13 exhibits the confidence interval.

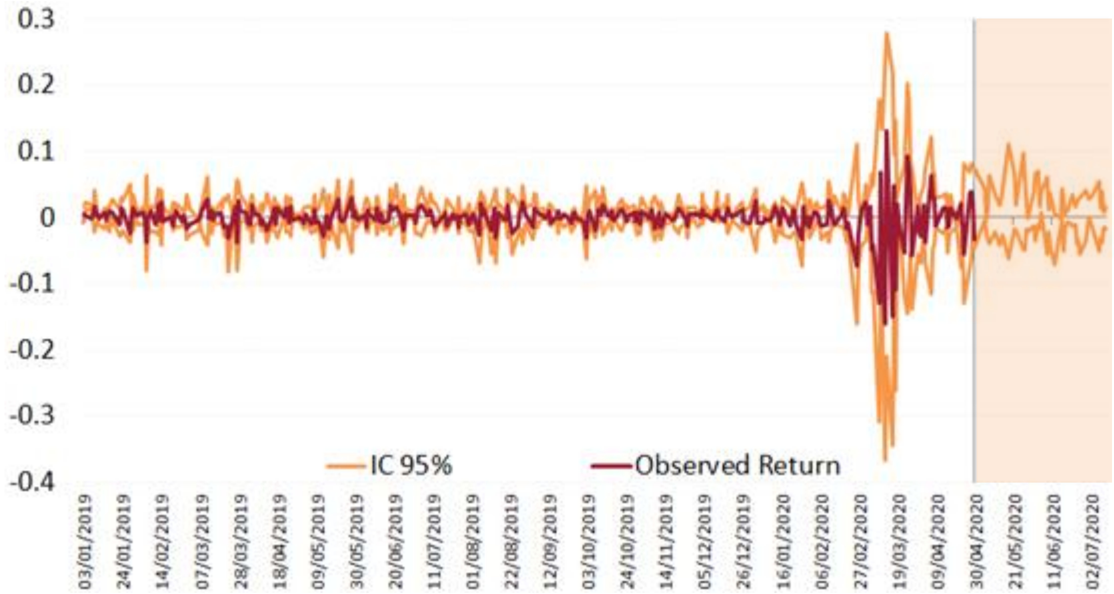



**Figure 13 - 95.0% Confidence Interval - IGARCH(1,1)**

The graphic analysis regarding the 95.0% confidence interval was implemented only with the out-of-sample registers to facilitate the assessment of the fit quality and can be seen in Figure 14:

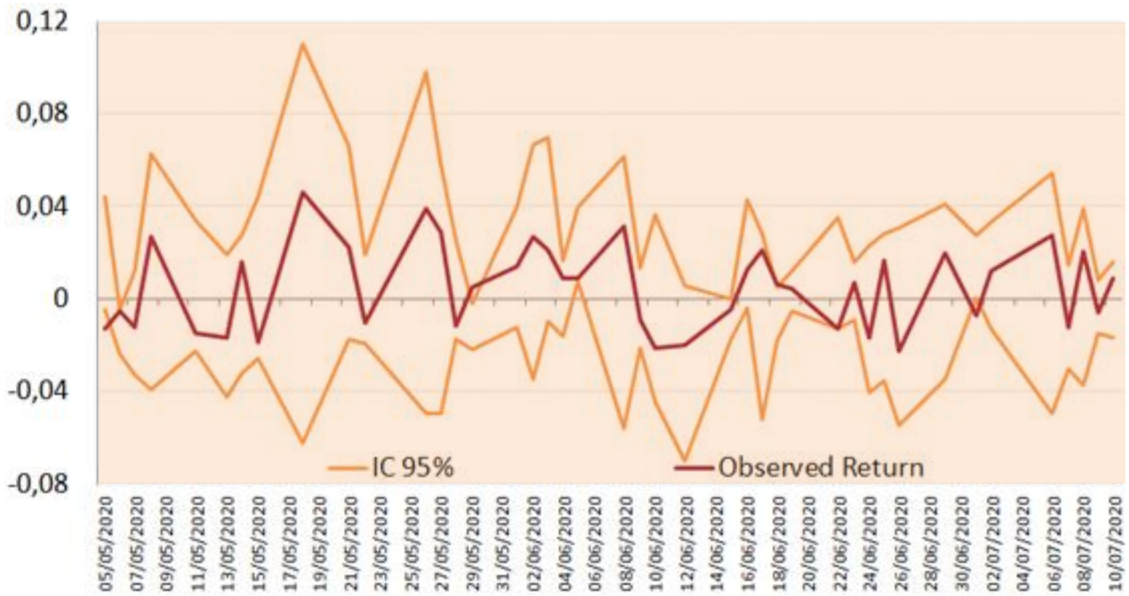



**Figure 14 - 95.0% Confidence Interval (out-of-sample) - IGARCH(1,1)**

Despite the increase in volatility observed at the end of February/2020, the IGARCH(1,1) model with GED residuals presented sufficient quality to cover more than 82.0% of the in-sample observations. The coverage of the out-of-sample was even greater, reaching 90.0% of the observations contained in this section as shown in Table 22 below.

**Table 22 - Probability Coverage (IGARCH(1,1))**

| | In-sample | out-of-sample | Total |
|---|---|---|---|
| Covered Points | 266 | 38 | 304 |
| Number of Observations | 321 | 42 | 363 |
| **Coverage Probability** | **82.87%** | **90.48%** | **83.75%** |

## 9. Conclusions

The volatility significantly influences stakeholders in making decisions about investing and how much to invest. Within this context, it is extremely important to estimate and forecast the volatility of financial assets and indices, especially in the composition of investment portfolios because investors are not only interested in the average returns of stocks but also their risk. When it is possible to measure, with the greatest precision, whether the variance of an asset will be greater or lesser in a given period, the risk and the exposure to large losses can be better controlled.

The criteria for estimating and comparing models were based on the significance of the parameters, values of AIC/BIC, and the absence of heteroscedasticity and autocorrelation of the residuals. Thus, MAPE and RMSE were employed during the choice of the best model and consequent extrapolation of the sample in order to verify the fit quality during the estimation.

The results suggested that *ceteris paribus* the exchange rate between the US dollar and the Brazilian real (BRL_USD), as well as the index regarding the difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the North American Treasury (EMBI), would be relevant in the BOVESPA Index modeling, while the price of a barrel of Brent oil in US dollars (Brent) would not be pertinent for such analysis.

Correspondingly, according to the results, the Autoregressive Distributed Lag Model (short-term model) was the most appropriate approach for modeling the BOVESPA Index since there is no statistical evidence for the presence of cointegration when IBOV is the dependent variable. Unfortunately, the above-mentioned was not satisfactory since the diagnostics step highlighted that only one parameter was statistically significant (the first difference of the exchange rate between the US dollar and the Brazilian real) presenting a p-value lower than 0.05. In addition, the null hypothesis of no serial correlation of the residuals was rejected, as well as the null hypothesis of homoscedastic, corroborating the lack of fit of the model estimated although it had some stability.

Regarding the causality, the exchange rate between the US dollar and the Brazilian real (BRL_USD) presents a causality effect over the BOVESPA index and an inverse relationship is also significant. In addition, the price of a barrel of Brent oil in US dollars (Brent) has no causality effect over the IBOV, but the opposite relationship is observed. Finally, the index regarding the difference between the return rate on Brazilian bonds and the rate offered by bonds issued by the North American Treasury (EMBI) offers a causality effect over the BOVESPA index and an inverse relationship is not significant.

Among the models tested concerning volatility, the IGARCH(1,1) with GED residuals showed the best fit quality. This model provided a behavioral analysis of the samples, providing satisfactory short-term forecasts with a coverage probability greater than 82,0% during the sample replicating process (out-of-sample).

## 9.1. Limitations of the Study

The sample did not include information before 2019 since there were declaration errors on weekends and holidays, such as trade registers on December 25 and December 31.

The daylight-saving time in Brazil remained unchanged between 1985 and 2019 and no system has been prepared for the end of this event. There were a series of errors with time zone, opening and closing times of exchanges, and data streaming. It was necessary to do a total reset on the Brazilian calendar, the data still presents some problems, and the system did not recover all holidays (investing.com support team, 2020).

Considering the above-mentioned issue, the present analysis was limited to 364 registers: January 2, 2019, to April 30, 2020, containing 322 registers (**in-sample**); and May 5, 2010, to July 10, 2020, containing 42 registers (**out-of-sample**).

## 9.2. Future Study Recommendations

Still linked to the sample issue, the study can be replicated using a larger sample to corroborate the statistical evidence for the presence of cointegration when IBOV is the dependent variable.

In addition, a larger sample could point out a different range of ARCH/GARCH family models to analyze the IBOV volatility with the support of the auxiliary variables (BRENT, BRL_USD, and EMBI).

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# Appendix 1 - Composition of the BOVESPA Index

Table 23 - Current Composition of the IBOVESPA (May to August - 2020)

| Code | Stock | Type | Theoretical quantity | Part. (%) |
|---|---|---|---|---|
| ABEV3 | AMBEV S/A | ON | 4.354.228.928 | 3,395 |
| AZUL4 | AZUL | PN N2 | 317.471.474 | 0,38 |
| B3SA3 | B3 | ON NM | 2.046.021.644 | 5,405 |
| BBAS3 | BRASIL | ON NM | 1.418.466.803 | 2,78 |
| BBDC3 | BRADESCO | ON N1 | 1.253.093.907 | 1,513 |
| BBDC4 | BRADESCO | PN N1 | 4.261.649.634 | 5,612 |
| BBSE3 | BBSEGURIDADE | ON NM | 671.601.167 | 1,226 |
| BEEF3 | MINERVA | ON NM | 259.260.688 | 0,219 |
| BPAC11 | BTGP BANCO | UNT N2 | 201.102.231 | 0,585 |
| BRAP4 | BRADESPAR | PN N1 | 222.089.397 | 0,456 |
| BRDT3 | PETROBRAS BR | ON NM | 728.125.000 | 0,978 |
| BRFS3 | BRF SA | ON NM | 811.416.229 | 1,082 |
| BRKM5 | BRASKEM | PNA N1 | 264.632.416 | 0,385 |
| BRML3 | BR MALLS PAR | ON NM | 843.313.591 | 0,582 |
| BTOW3 | B2W DIGITAL | ON NM | 194.807.150 | 0,981 |
| CCRO3 | CCR SA | ON NM | 1.115.695.556 | 0,947 |
| CIEL3 | CIELO | ON NM | 1.118.386.806 | 0,312 |
| CMIG4 | CEMIG | PN N1 | 966.005.862 | 0,632 |
| COGN3 | COGNA ON | ON NM | 1.829.322.117 | 0,697 |
| CPFE3 | CPFL ENERGIA | ON NM | 170.209.781 | 0,34 |
| CRFB3 | CARREFOUR BR | ON NM | 384.888.219 | 0,529 |
| CSAN3 | COSAN | ON NM | 153.417.832 | 0,637 |
| CSNA3 | SID NACIONAL | ON | 642.387.288 | 0,396 |
| CVCB3 | CVC BRASIL | ON NM | 145.617.442 | 0,138 |
| CYRE3 | CYRELA REALT | ON NM | 281.154.098 | 0,308 |
| ECOR3 | ECORODOVIAS | ON NM | 171.079.276 | 0,123 |
| EGIE3 | ENGIE BRASIL | ON NM | 254.813.401 | 0,685 |
| ELET3 | ELETROBRAS | ON N1 | 358.028.908 | 0,597 |
| ELET6 | ELETROBRAS | PNB N1 | 240.022.548 | 0,459 |
| EMBR3 | EMBRAER | ON NM | 736.171.905 | 0,438 |
| ENBR3 | ENERGIAS BR | ON NM | 295.402.225 | 0,345 |
| ENGI11 | ENERGISA | UNT N2 | 250.709.436 | 0,753 |
| EQTL3 | EQUATORIAL | ON NM | 1.010.186.085 | 1,273 |
| FLRY3 | FLEURY | ON NM | 305.126.422 | 0,478 |
| GGBR4 | GERDAU | PN N1 | 1.031.246.473 | 0,833 |
| GNDI3 | INTERMEDICA | ON NM | 458.380.483 | 1,728 |
| GOAU4 | GERDAU MET | PN N1 | 662.644.908 | 0,235 |
| GOLL4 | GOL | PN N2 | 134.613.917 | 0,115 |
| HAPV3 | HAPVIDA | ON ED NM | 217.060.254 | 0,783 |
| HGTX3 | CIA HERING | ON NM | 126.302.831 | 0,13 |
| HYPE3 | HYPERA | ON NM | 407.518.048 | 0,815 |

| | | | | |
|---|---|---|---|---|
| IGTA3 | IGUATEMI | ON NM | 86.913.923 | 0,197 |
| IRBR3 | IRBBRASIL RE | ON NM | 923.377.994 | 0,649 |
| ITSA4 | ITAUSA | PN N1 | 4.494.029.326 | 2,781 |
| ITUB4 | ITAUUNIBANCO | PN ED N1 | 4.738.562.684 | 7,414 |
| JBSS3 | JBS | ON ED NM | 1.620.646.499 | 2,657 |
| KLBN11 | KLABIN S/A | UNT N2 | 637.772.642 | 0,781 |
| LAME4 | LOJAS AMERIC | PN N1 | 696.190.451 | 1,191 |
| LREN3 | LOJAS RENNER | ON ED NM | 785.308.019 | 2,073 |
| MGLU3 | MAGAZ LUIZA | ON NM | 661.834.080 | 2,262 |
| MRFG3 | MARFRIG | ON NM | 423.143.092 | 0,374 |
| MRVE3 | MRV | ON NM | 294.706.737 | 0,307 |
| MULT3 | MULTIPLAN | ON N2 | 270.279.854 | 0,388 |
| NTCO3 | GRUPO NATURA | ON NM | 671.197.455 | 1,639 |
| PCAR3 | P.ACUCAR-CBD | ON ED NM | 156.575.731 | 0,707 |
| PETR3 | PETROBRAS | ON N2 | 2.949.857.480 | 3,783 |
| PETR4 | PETROBRAS | PN N2 | 4.520.185.835 | 5,61 |
| QUAL3 | QUALICORP | ON ED NM | 281.558.308 | 0,502 |
| RADL3 | RAIADROGASIL | ON NM | 207.425.426 | 1,494 |
| RAIL3 | RUMO S.A. | ON NM | 1.053.753.059 | 1,434 |
| RENT3 | LOCALIZA | ON NM | 593.233.155 | 1,395 |
| SANB11 | SANTANDER BR | UNT | 362.227.661 | 0,673 |
| SBSP3 | SABESP | ON EJ NM | 339.982.576 | 0,937 |
| SULA11 | SUL AMERICA | UNT N2 | 278.742.752 | 0,858 |
| SUZB3 | SUZANO S.A. | ON NM | 725.859.318 | 1,967 |
| TAEE11 | TAESA | UNT N2 | 218.568.274 | 0,413 |
| TIMP3 | TIM PART S/A | ON NM | 807.711.660 | 0,709 |
| TOTS3 | TOTVS | ON EDB NM | 403.397.655 | 0,547 |
| UGPA3 | ULTRAPAR | ON NM | 1.086.422.872 | 1,083 |
| USIM5 | USIMINAS | PNA ED N1 | 513.781.576 | 0,17 |
| VALE3 | VALE | ON NM | 3.292.010.807 | 10,155 |
| VIVT4 | TELEF BRASIL | PN | 415.131.868 | 1,303 |
| VVAR3 | VIAVAREJO | ON NM | 1.146.662.628 | 0,724 |
| WEGE3 | WEG | ON NM | 689.271.972 | 1,893 |
| YDUQ3 | YDUQS PART | ON ED NM | 299.667.897 | 0,625 |
| **Total Theoretical Quantity** | | | **67.959.663.646** | **100** |

## Appendix 2 - Time Series, Return and Return$^2$ of the Auxiliary Variables

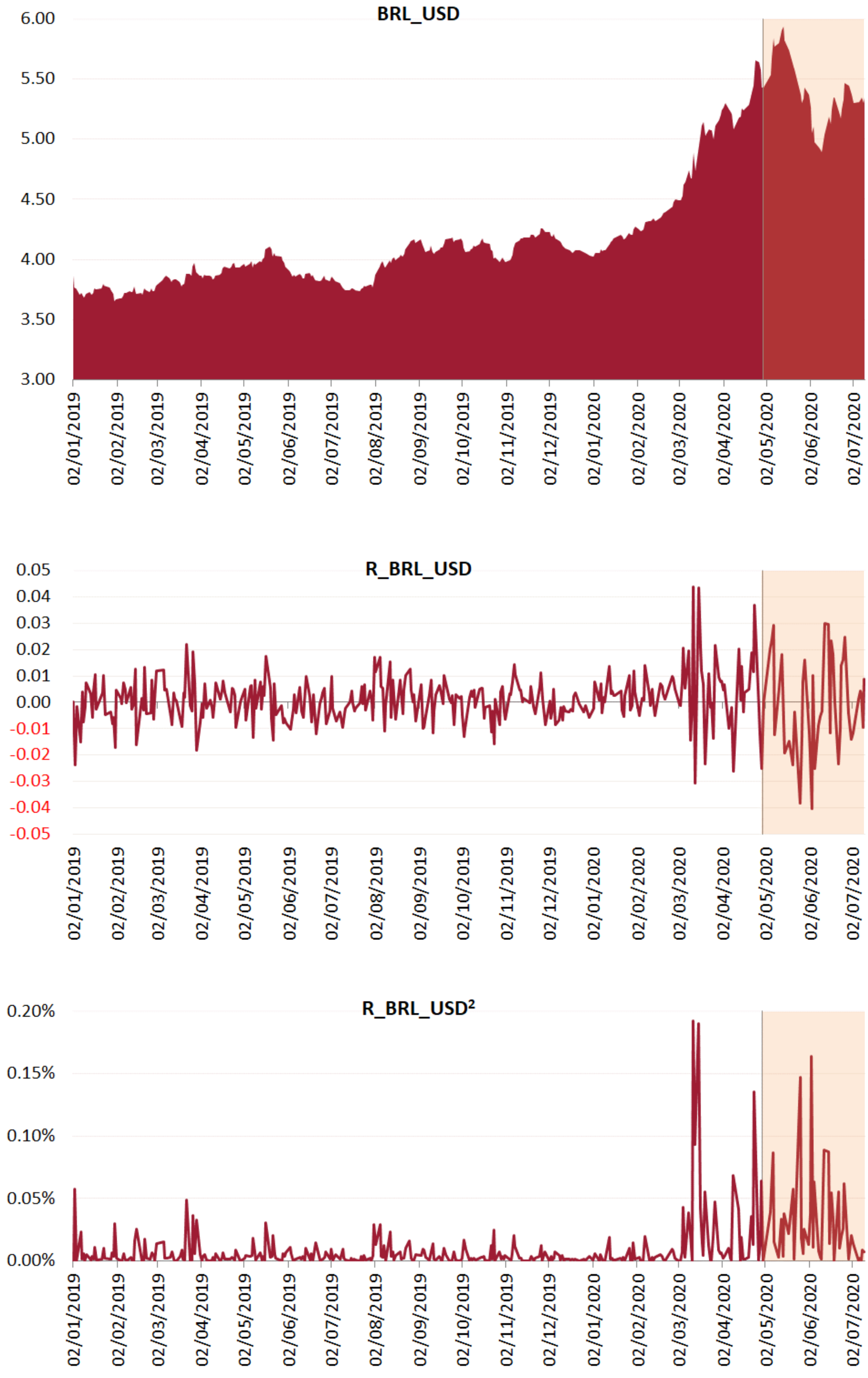



Figure 15 - BRL_USD Daily Time Series

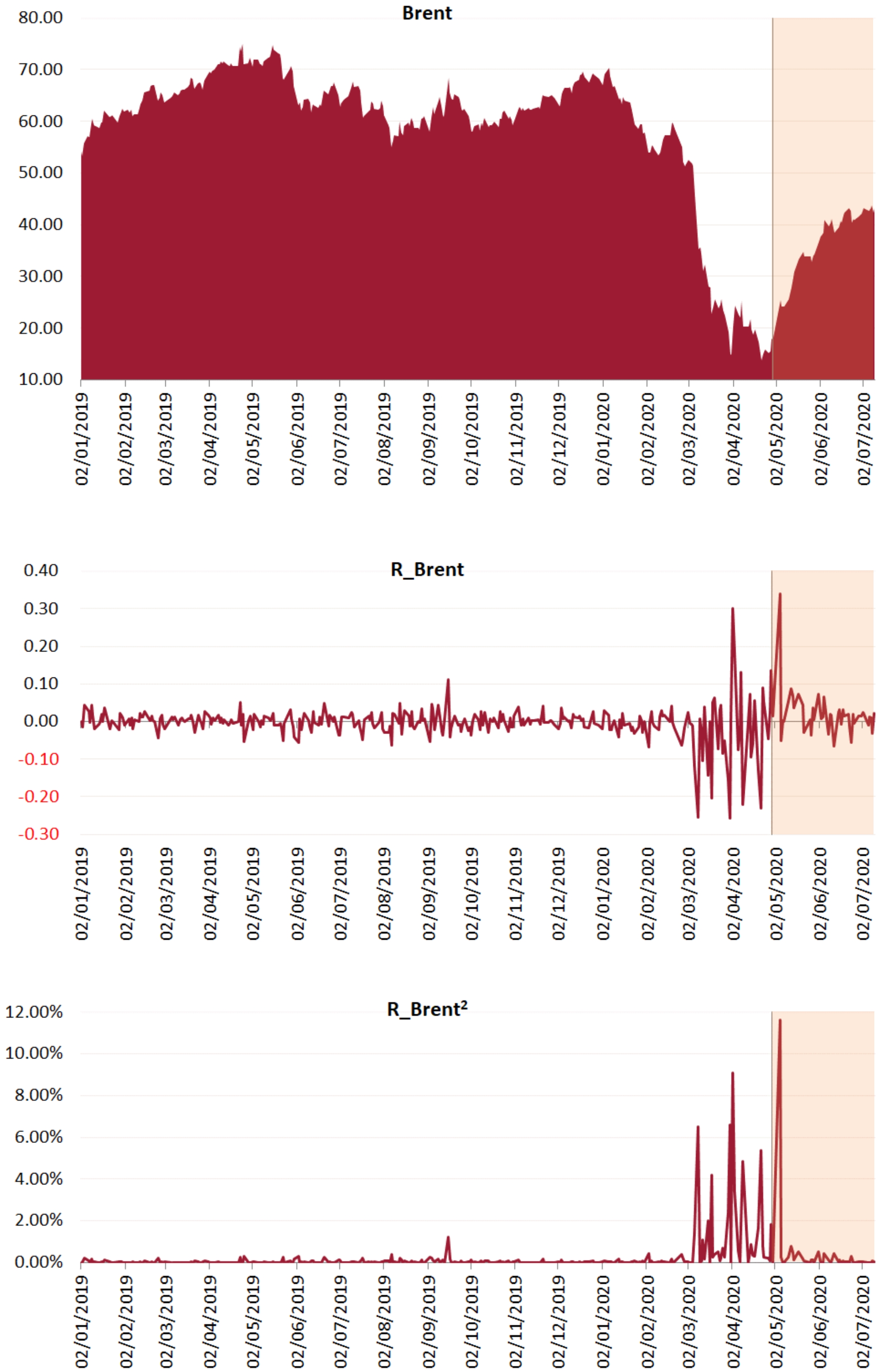



Figure 16 - Brent Daily Time Series

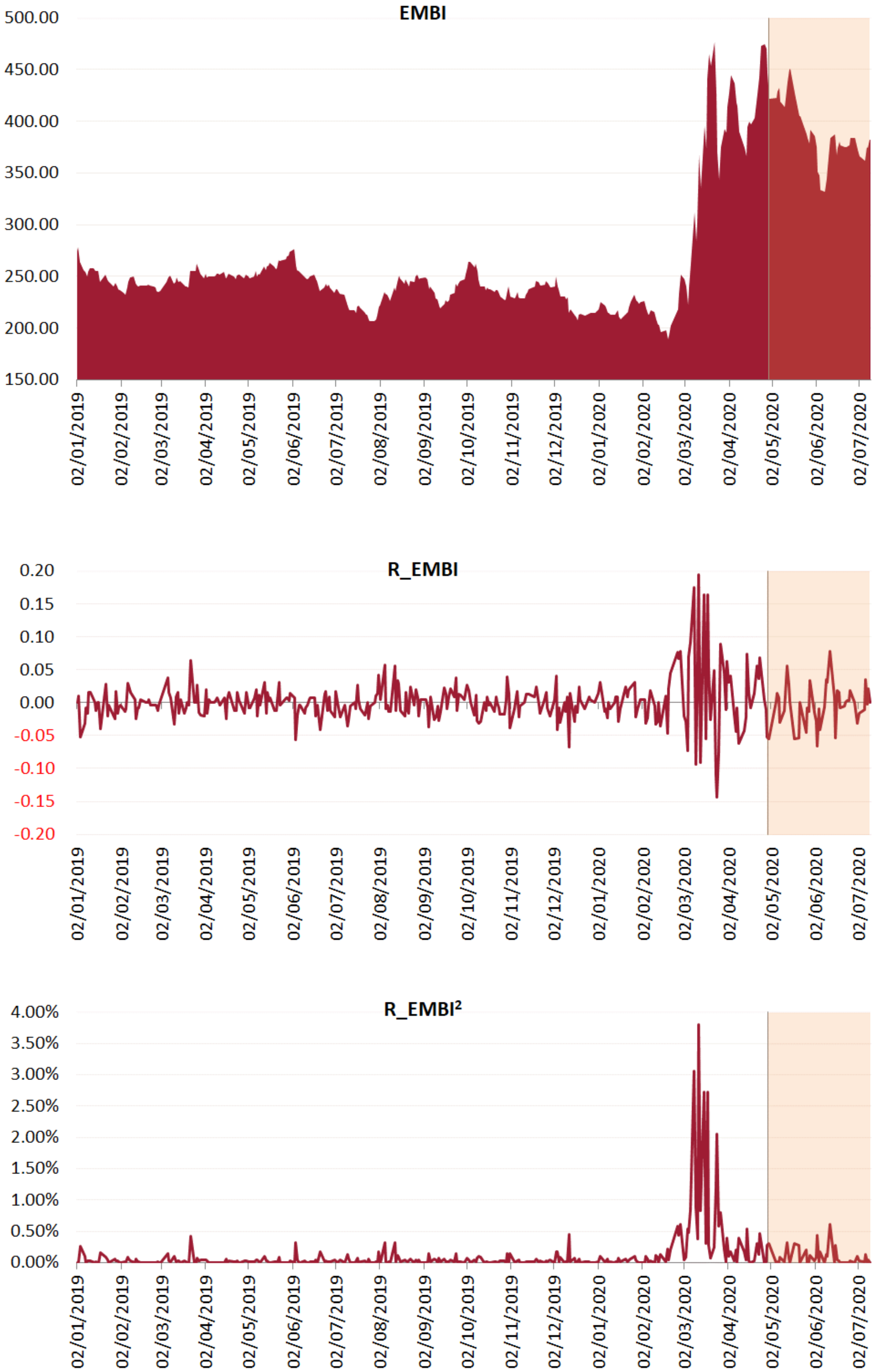
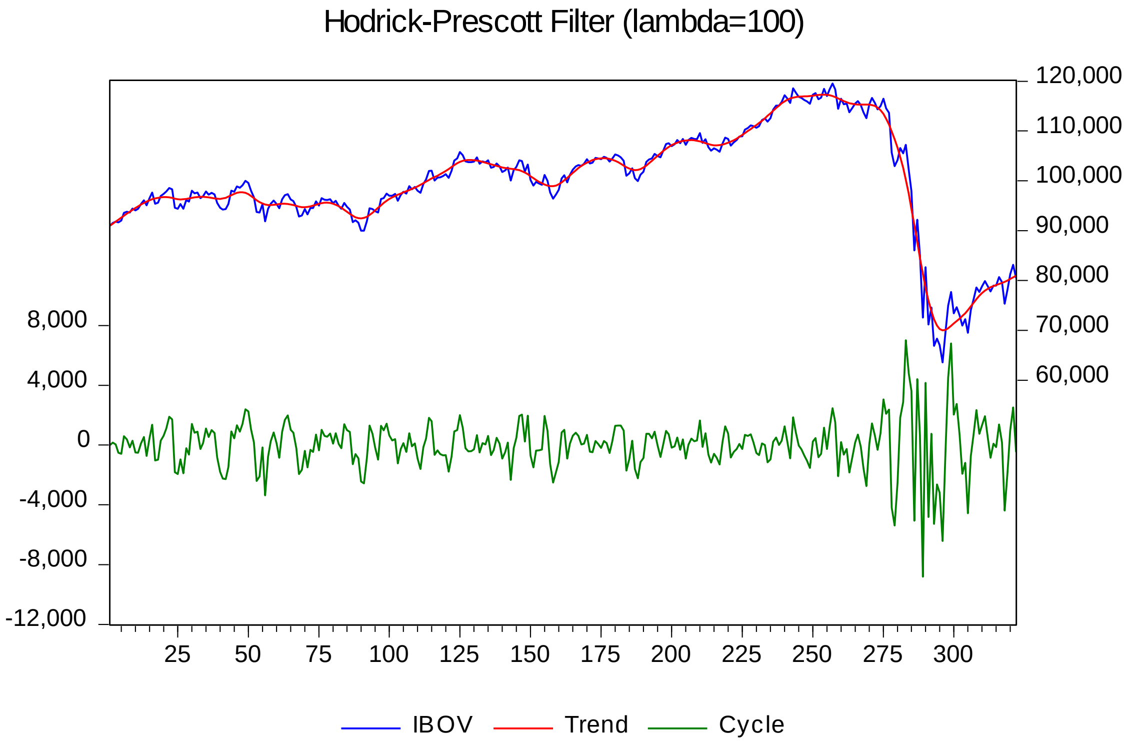



**Figure 17 - EMBI Daily Time Series**

## Appendix 3 - Hodrick-Prescott Filters



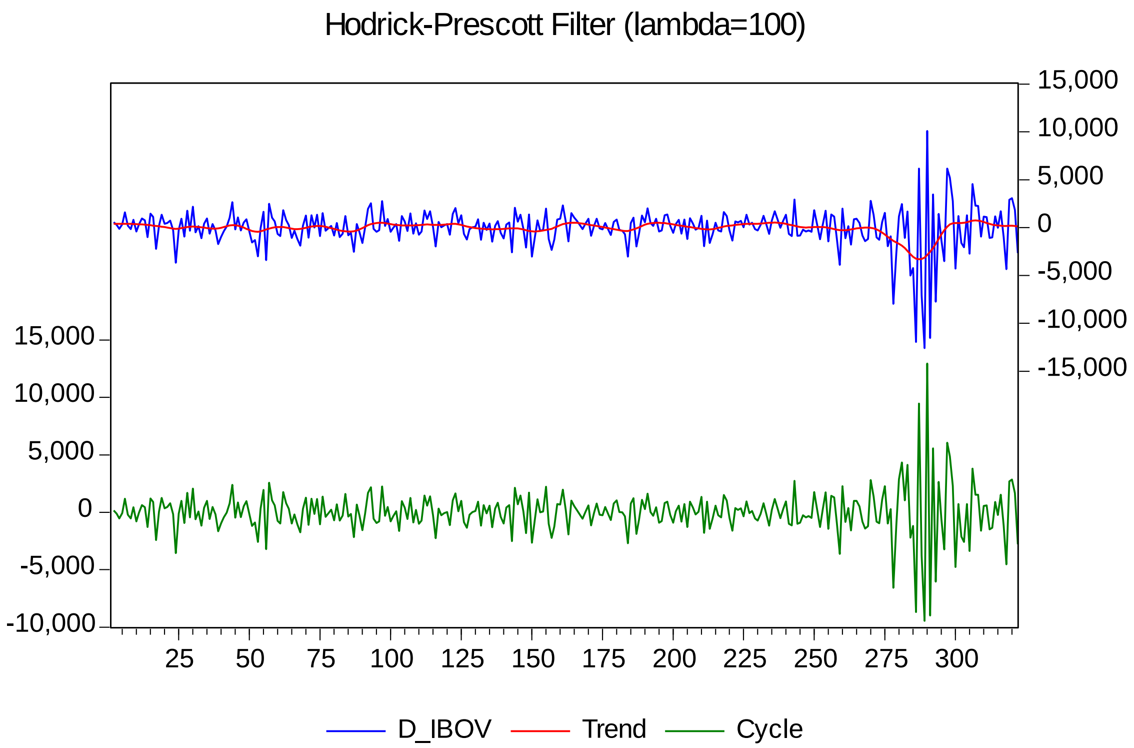
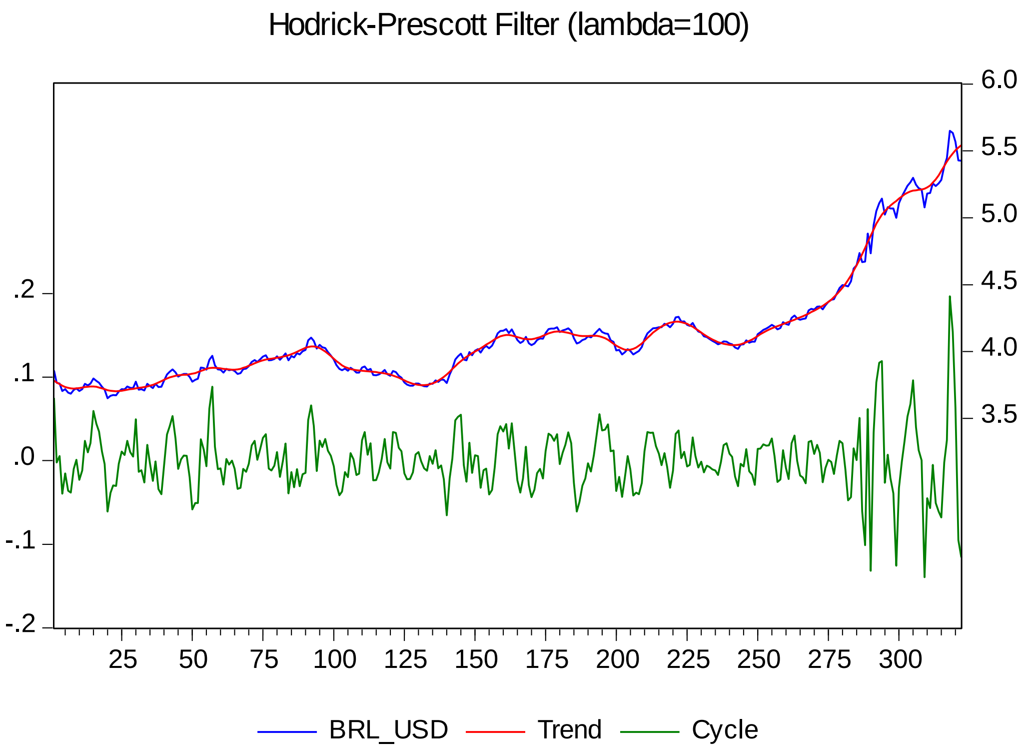




Figure 18 - Hodrick-Prescott Filter (IBOV and D_IBOV)



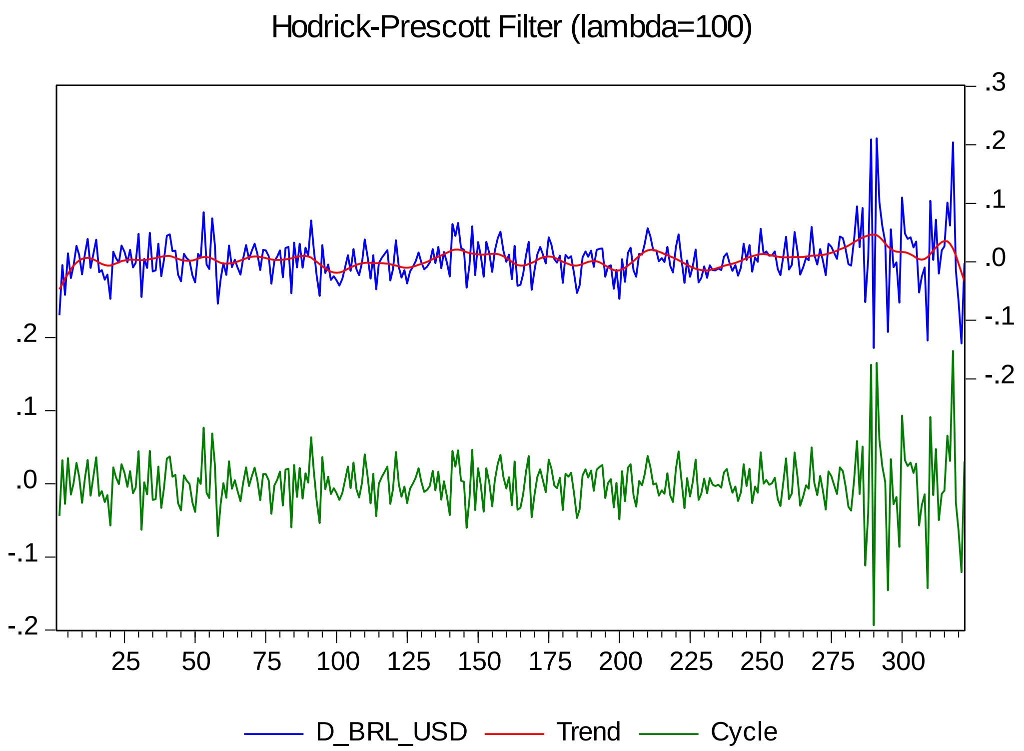
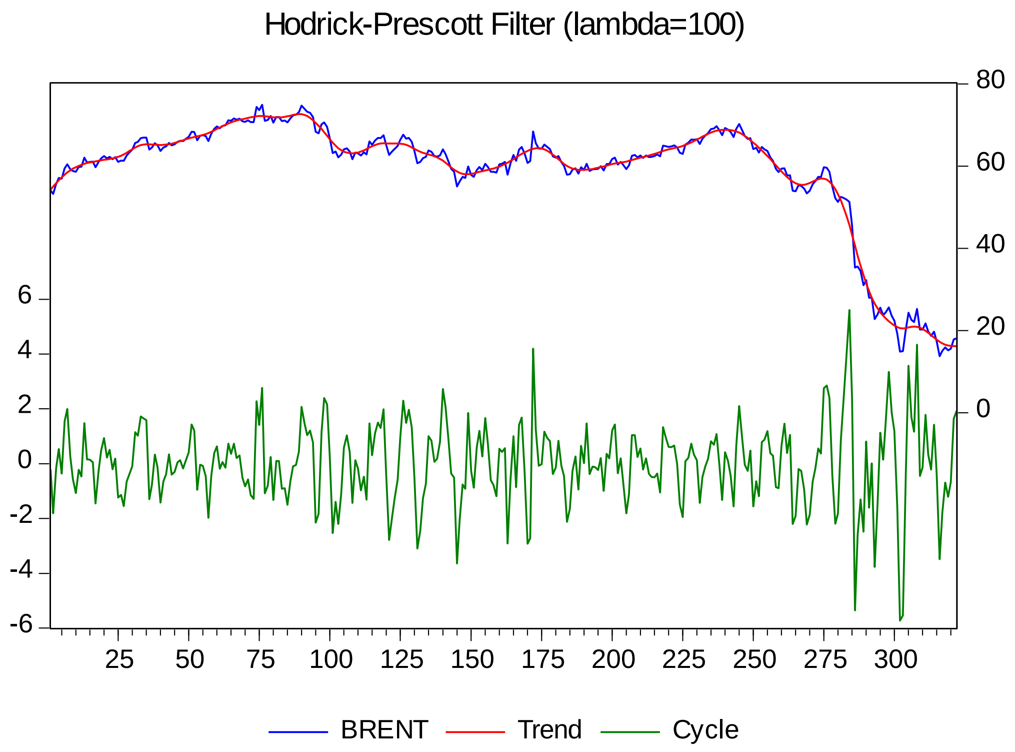




Figure 19 - Hodrick-Prescott Filter (BRL_USD and D_BRL_USD)



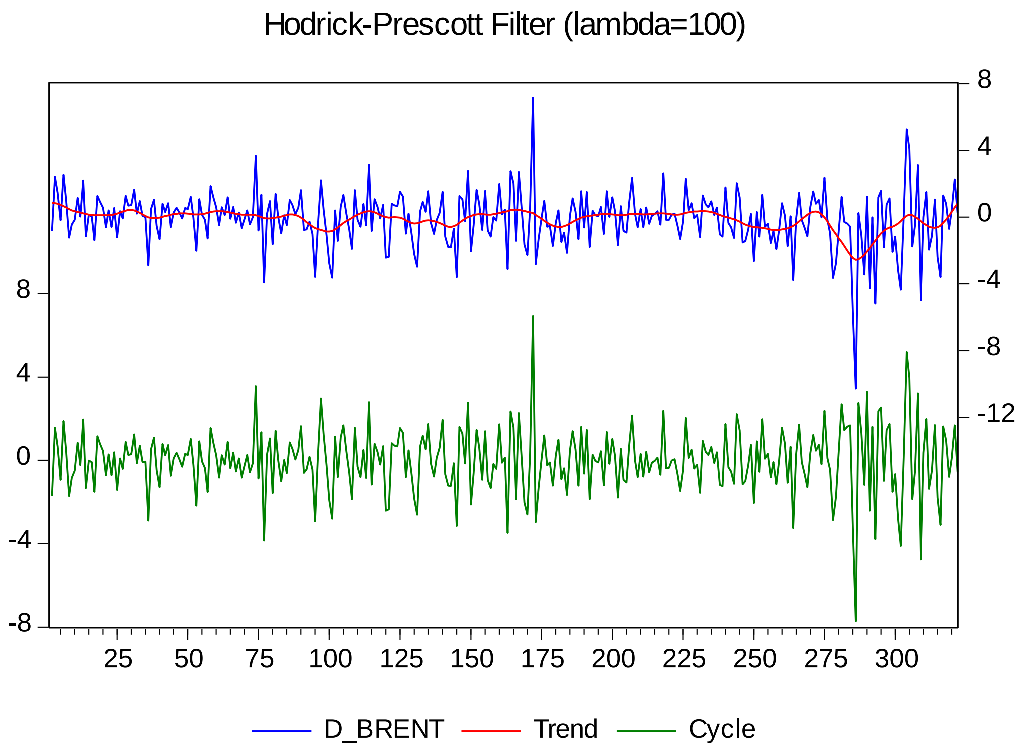
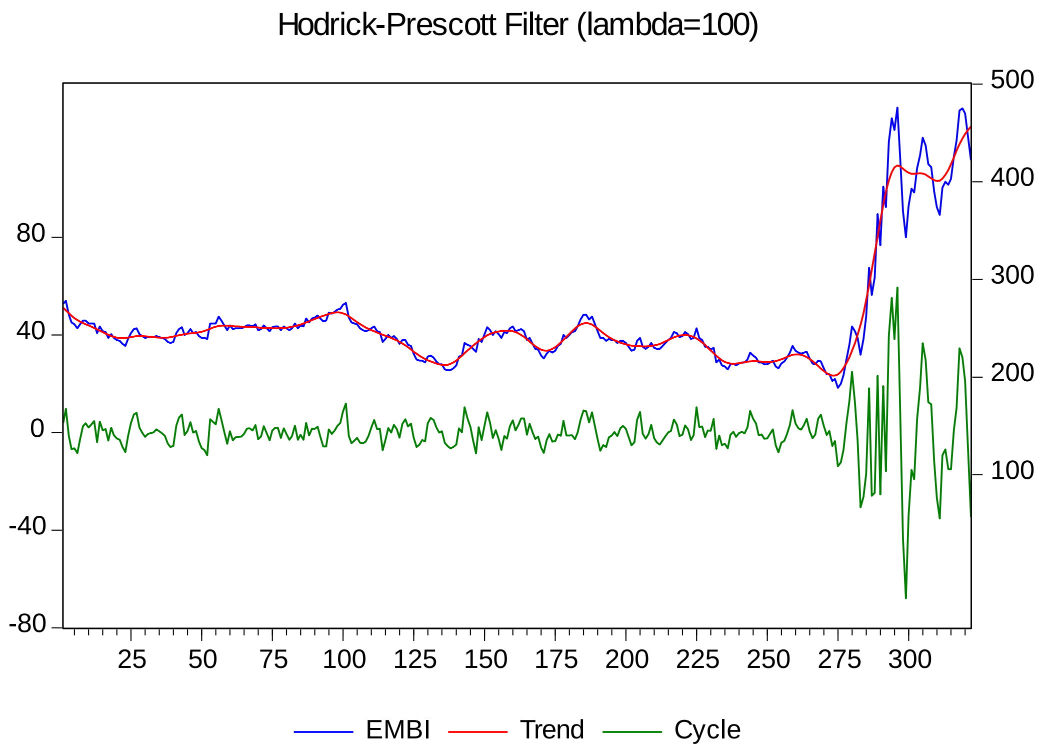




**Figure 20 - Hodrick-Prescott Filter (Brent and D_Brent)**



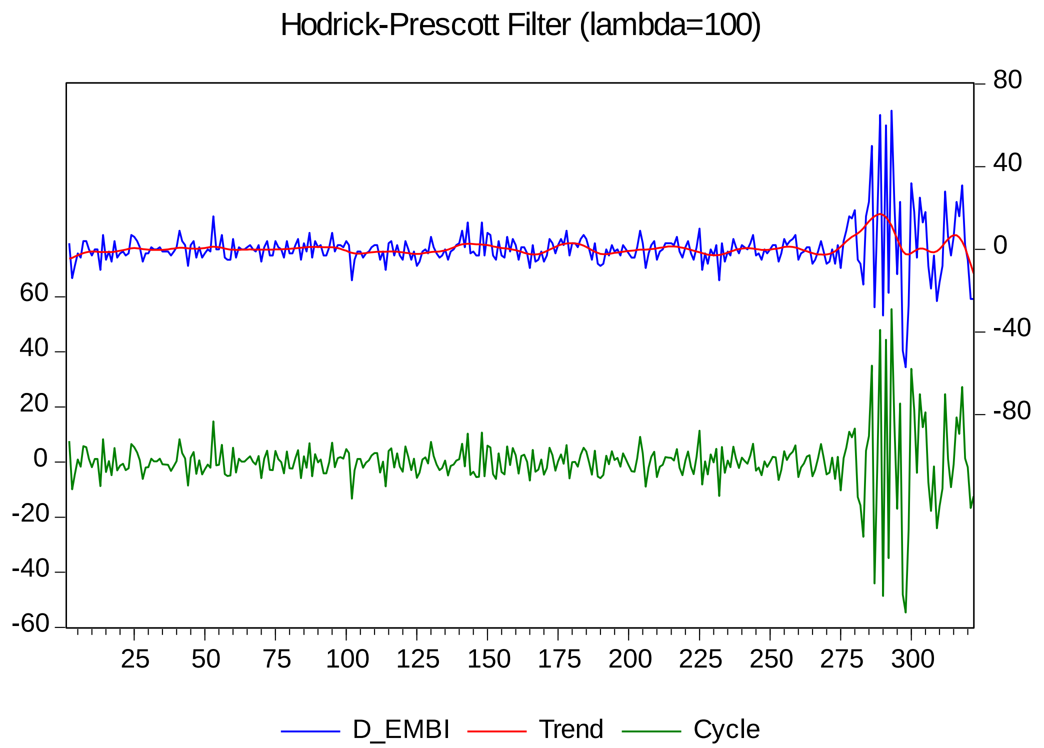




**Figure 21 - Hodrick-Prescott Filter (EMBI and D_EMBI)**