Sufficient Conditions for Oscillation of Fourth-Order Neutral Differential Equations with Distributed Deviating Arguments

Omar Bazighifan [1,2,†] , **Feliz Minhos** [3,*,†] and **Osama Moaaz** [4,†]

1   Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen; o.bazighifan@gmail.com
2   Department of Mathematics, Faculty of Education, Seiyun University, Hadhramout 50512, Yemen
3   Departamento de Matematica, Escola de Ciencias e Tecnologia, Centro de Investigacao em Matematica e Aplicacoes (CIMA), Instituto de Investigacao e Formacao Avancada, Universidade de Evora, Rua Romao Ramalho, 59, 7000-671 Evora, Portugal
4   Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; o_moaaz@mans.edu.eg
*   Correspondence: fminhos@uevora.pt
†   These authors contributed equally to this work.

Abstract: Some new sufficient conditions are established for the oscillation of fourth order neutral differential equations with continuously distributed delay of the form $\left(r\left(t\right) \left(N_x'''\left(t\right) \right)^\alpha \right)' + \int_a^b q\left(t,\vartheta\right) x^\beta\left(\delta\left(t,\vartheta\right) \right) d\vartheta = 0$, where $t \geq t_0$ and $N_x\left(t\right) := x\left(t\right) + p\left(t\right) x\left(\varphi\left(t\right) \right)$. An example is provided to show the importance of these results.

Keywords: fourth-order differential equations; neutral delay; oscillation

1. Introduction

The theory of differential equations is an adequate mathematical apparatus for the simulation of processes and phenomena observed in biotechnology, neural networks, physics etc, see [1]. One area of active research in recent times is to study the sufficient criterion for oscillation of delay differential equations, see [1–28].

In this work, we establish the asymptotic behavior of fourth-order neutral differential equation of the form

$$\left(r\left(t\right) \left(N_x'''\left(t\right) \right)^\alpha \right)' + \int_a^b q\left(t,\vartheta\right) x^\beta\left(\delta\left(t,\vartheta\right) \right) d\vartheta = 0, \tag{1}$$

where $t \geq t_0$ and $N_x\left(t\right) := x\left(t\right) + p\left(t\right) x\left(\varphi\left(t\right) \right)$. In this paper, we assume that:

A1: α and β are a quotient of odd positive integers and $\beta \geq \alpha$;

A2: $r, p \in C[t_0,\infty), r\left(t\right) > 0, r'\left(t\right) \geq 0$ and $\int^\infty r^{-1/\alpha}\left(s\right) \mathrm{d}s = \infty$;

A3: $q \in C\left([t_0,\infty) \times \left(a,b\right), \mathbb{R}\right), q\left(t,\vartheta\right) > 0, 0 \leq p\left(t\right) < p_0 < \infty$ and $q\left(t\right)$ is not identically zero for large t;

A4: $\varphi \in C^1[t_0,\infty), \delta \in p\left([t_0,\infty) \times \left(a,b\right), \mathbb{R}\right), \varphi'\left(t\right) > 0, \varphi\left(t\right) \leq t, \lim_{t\to\infty} \varphi\left(t\right) = \lim_{t\to\infty} \delta\left(t,\vartheta\right) = \infty$ and $\delta\left(t,\vartheta\right)$ has nondecreasing.

Definition 1. *The function* $x \in C^3[t_y,\infty), t_y \geq t_0$, *is called a solution of* (1), *if* $r\left(t\right) \left(N_x'''\left(t\right) \right)^\alpha \in C^1[t_y,\infty)$, *and* $x\left(t\right)$ *satisfies* (1) *on* $[t_y,\infty)$.

Definition 2. *A solution of* (1) *is called oscillatory if it has arbitrarily large zeros on* $[t_x, \infty)$, *and otherwise is called to be nonoscillatory.*

Definition 3. *The Equation* (1) *is called oscillatory if every its solutions are oscillatory.*

In the following, we discuss some important papers:

Chatzarakis et al. [9] proved the equation (1) where $\alpha = \beta$, is oscillatory, if

$$\int_{t_0}^{\infty} \left(\varpi(s) - \frac{2^{\alpha} r(s)}{\mu^{\alpha} s^{2\alpha} \rho^{\alpha}(s)} \left(\frac{\rho'(s)}{\alpha+1} \right)^{\alpha+1} \right) ds = \infty,$$

for some $\mu \in (0, 1)$ and

$$\int_{t_0}^{\infty} \left(\vartheta(s) \left(\int_t^{\infty} (Q(v))^{\frac{1}{\alpha}} r^{\frac{-1}{\alpha}}(v) \, dv \right) - \frac{\theta_+'^2(s)}{4\theta(s)} \right) ds = \infty,$$

where $\varpi(t) := k\rho(t) Q(t) (1 - p(\delta(t, a)))^{\alpha} (\delta(t, a) \backslash t)^{3\alpha}$ and $\rho, \theta \in C^1([v_0, \infty), (0, \infty))$.

Moaaz et al. in [19] extended the Riccati transformation to obtain new oscillatory criteria for (1) as condition

$$\int_{t_0}^{\infty} \left[\theta(s) Q(s) - \frac{1}{\lambda 4} \left(\frac{\theta'(s)}{\theta(s)} \right)^2 \right] ds = \infty,$$

where $\lambda \in (0, 1)$ and a function $\theta \in C^1([v_0, \infty), (0, \infty))$.

Authors in [24] studied oscillatory behavior of equation

$$N_x^{(n)}(t) + q(t) x(\delta(t)) = 0, \tag{2}$$

where n is even, they proved it oscillatory by using the Riccati transformation if either

$$\lim_{t \to \infty} \inf \int_{\varphi(t)}^{t} Q(s) \, ds > \frac{(n-1)!}{e}, \tag{3}$$

or

$$\lim_{t \to \infty} \sup \int_{\varphi(t)}^{t} Q(s) \, ds > (n-1)!,$$

where $Q(t) := \varphi^{n-1}(t) (1 - p(\varphi(t))) q(t)$.

Xing et al. [22] proved that the even-order differential equation

$$\left(r(t) \left(N_x^{(n-1)}(t) \right)^{\alpha} \right)' + q(t) x^{\beta}(\delta(t)) = 0,$$

is oscillatory, if

$$\left(\delta^{-1}(t) \right)' \geq \delta_0 > 0, \ \varphi'(t) \geq \varphi_0 > 0, \ \varphi^{-1}(\delta(t)) < t$$

and

$$\lim_{t \to \infty} \inf \int_{\varphi^{-1}(\delta(t))}^{t} \frac{\widehat{q}(s)}{r(s)} \left(s^{n-1} \right)^{\alpha} ds > \frac{\left(\frac{1}{\delta_0} + \frac{p_0^{\alpha}}{\delta_0 \varphi_0} \right)}{e \left((n-1)! \right)^{-\alpha}}, \tag{4}$$

where $\widehat{q}(t) := \min \left\{ q\left(\delta^{-1}(t)\right), q\left(\delta^{-1}(\varphi(t))\right) \right\}$ and n is even.

To prove this, we apply the previous results to the equation

$$(x(t) + px(\varphi t))^{(n)} + bx(\delta t) = 0, \ t \geq 1, \tag{5}$$

where $n = 4$, $p = 7/8$, $\varphi = 1/e$, $\delta = 1/e^2$ and $b = q_0/v^4$, we find:

1. By applying condition (3) in (5), we find

    $$q_0 > 3561.9.$$

2. By applying condition (4) in (5), we get

    $$q_0 > 3008.5.$$

Hence, [22] improved the results in [24].

Thus, the motivation in studying this paper is complement results in [9] and improve results [22,24].

By using the Riccati transformations, we establish a new oscillation criterion for a class of fourth-order neutral differential equations (1). An example is provided to illustrate the main results.

2. Some Auxiliary Lemmas

We shall employ the following lemmas

Lemma 1 ([3]). *Let $x \in C^n([t_0, \infty), (0, \infty))$. Assume that $x^{(n)}(t)$ is of fixed sign and not identically zero on $[t_0, \infty)$ and there exists a $t_1 \geq t_0$ such that $x^{(n-1)}(t) x^{(n)}(t) \leq 0$ for all $t \geq t_1$. If $\lim_{t \to \infty} x(t) \neq 0$, then for every $\mu \in (0, 1)$ there exists $t_\mu \geq t_1$ such that*

$$x(t) \geq \frac{\mu}{(n-1)!} t^{n-1} \left| x^{(n-1)}(t) \right| \text{ for } t \geq t_\mu.$$

Lemma 2 ([16]). *Let the function x satisfies $x^{(i)}(t) > 0$, $i = 0, 1, ..., n$, and $x^{(n+1)}(t) < 0$, then*

$$\frac{x(t)}{t^n/n!} \geq \frac{x'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 3 ([4]). *Assume that $x, v \geq 0$ and $\alpha \geq 1$ is a positive real number. Then*

$$(x + v)^\alpha \leq 2^{\alpha-1} (x^\alpha + v^\alpha)$$

and

$$(x + v)^\beta \leq x^\beta + v^\beta, \text{ for } \beta \leq 1.$$

Lemma 4 ([9]). *Assume that x is an eventually positive solution of (1). Then, there exist two possible cases:*

$$(\mathbf{S}_1) \; N_x^{(\kappa)}(t) > 0 \text{ for } \kappa = 0, 1, 2, 3;$$
$$(\mathbf{S}_2) \; N_x(t) > 0, \; N_x'(t) > 0, \; N_x''(t) < 0 \text{ and } N_x'''(t) > 0,$$

for $t \geq t_1$, where $t_1 \geq t_0$ is sufficiently large.

Notation 1. *We consider the following notations:*

$$p_1(t) = \frac{1}{p(\varphi^{-1}(t))}\left(1 - \frac{\left(\varphi^{-1}(\varphi^{-1}(t))\right)^3}{\left(\varphi^{-1}(t)\right)^3 p(\varphi^{-1}(\varphi^{-1}(t)))}\right),$$

$$p_2(t) = \frac{1}{p(\varphi^{-1}(t))}\left(1 - \frac{\left(\varphi^{-1}(\varphi^{-1}(t))\right)}{\left(\varphi^{-1}(t)\right) p(\varphi^{-1}(\varphi^{-1}(t)))}\right)$$

$$\Psi(t) = M_1^{\beta-\alpha}\theta(t)\int_a^b q(t,\vartheta)\, p_1^\beta(\delta(t,\vartheta))\, d\vartheta$$

$$\tilde{R}(t) = \int_a^b \left(\frac{\mu\left(\varphi^{-1}(\eta(t,\vartheta))\right)^3}{6}\right)^\beta q(t,\vartheta)\, p_1^\beta(\eta(t,\vartheta))\, r^{-\beta/\alpha}\left(\varphi^{-1}(\eta(t,\vartheta))\right) d\vartheta$$

$$R(t) = \int_t^\infty \left(\frac{1}{r(\varrho)}\int_\varrho^\infty\left(\int_a^b q(s,\vartheta)\left(\frac{\varphi^{-1}(\sigma(s,\vartheta))}{s}\right)^\beta d\vartheta\right) ds\right)^{1/\alpha} d\varrho,$$

and

$$\Phi(t) := p_2^{\beta/\alpha}\theta_1(t)\, M_2^{(\beta-\alpha)/\alpha}\int_t^\infty\left(\frac{1}{r(\varrho)}\int_\varrho^\infty\left(\int_a^b q(s,\vartheta)\left(\frac{\varphi^{-1}(\delta(s,\vartheta))}{s}\right)^\beta d\vartheta\right) ds\right)^{1/\alpha} d\varrho.$$

3. Main Results

In this part, we will discuss some oscillation criteria for Equation (1).

Lemma 5. *Assume that x is an eventually positive solution of (1) and*

$$\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^3 < \left(\varphi^{-1}(t)\right)^3 p\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right). \tag{6}$$

Then

$$x(t) \geq \frac{1}{p(\varphi^{-1}(t))}\left(N_x\left(\varphi^{-1}(t)\right) - \frac{1}{p(\varphi^{-1}(\varphi^{-1}(t)))}N_x\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)\right). \tag{7}$$

Proof. Let x be an eventually positive solution of (1) on $[t_0,\infty)$. From the definition of $z(t)$, we see that

$$p(t)\, x(\varphi(t)) = N_x(t) - x(t),$$

and so

$$p\left(\varphi^{-1}(t)\right)x(t) = N_x\left(\varphi^{-1}(t)\right) - x\left(\varphi^{-1}(t)\right).$$

Repeating the same process, we obtain

$$x(t) = \frac{1}{p(\varphi^{-1}(t))}\left(N_x\left(\varphi^{-1}(t)\right) - \left(\frac{N_x\left(\varphi^{-1}(\varphi^{-1}(t))\right)}{p(\varphi^{-1}(\varphi^{-1}(t)))} - \frac{x\left(\varphi^{-1}(\varphi^{-1}(t))\right)}{p(\varphi^{-1}(\varphi^{-1}(t)))}\right)\right),$$

which yields

$$x(t) \geq \frac{N_x\left(\varphi^{-1}(t)\right)}{p(\varphi^{-1}(t))} - \frac{1}{p(\varphi^{-1}(t))}\frac{N_x\left(\varphi^{-1}(\varphi^{-1}(t))\right)}{p(\varphi^{-1}(\varphi^{-1}(t)))}.$$

Thus, (7) holds. This completes the proof. □

Theorem 1. *Let $\delta(t) \leq \varphi(t)$ and (6) holds. If there exist positive functions $\theta, \theta_1 \in C^1([t_0, \infty), \mathbb{R})$ such that*

$$\int_{t_0}^{\infty} \left(\Psi(s) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{r\left(\varphi^{-1}(\delta(s,a))\right)(\theta'(s))^{\alpha+1}}{\left(\mu_1 \theta(s)\left(\varphi^{-1}(\delta(s,a))\right)'(\delta(s,a))'\left(\varphi^{-1}(\delta(s,a))\right)^2\right)^{\alpha}} \right) ds = \infty \qquad (8)$$

and

$$\int_{t_0}^{\infty} \left(\Phi(s) - \frac{(\theta_1'(s))^2}{4\theta_1(s)} \right) ds = \infty, \qquad (9)$$

for some $\mu_1 \in (0,1)$ and every $M_1, M_2 > 0$, then (1) is oscillatory.

Proof. Let x be a non-oscillatory solution of (1) on $[t_0, \infty)$. Without loss of generality, we can assume that x is eventually positive. It follows from Lemma 4 that there exist two possible cases $(\mathbf{S}_1)$ and $(\mathbf{S}_2)$.

Let $(\mathbf{S}_1)$ holds. From Lemma 2, we obtain $N_x(t) \geq \frac{1}{3} t N_x'(t)$ and hence the function $t^{-3} N_x(t)$ is nonincreasing, which with the fact that $\varphi^{-1}(t) \leq \varphi^{-1}\left(\varphi^{-1}(t)\right)$ gives

$$\left(\varphi^{-1}(t)\right)^3 N_x\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right) \leq \left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^3 N_x\left(\varphi^{-1}(t)\right). \qquad (10)$$

From (7) and (10), we get that

$$\begin{aligned}
x(t) &\geq \frac{N_x\left(\varphi^{-1}(t)\right)}{p\left(\varphi^{-1}(t)\right)} \left(1 - \frac{\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)^{n-1}}{\left(\varphi^{-1}(t)\right)^{n-1} p\left(\varphi^{-1}\left(\varphi^{-1}(t)\right)\right)} \right) \\
&\geq p_1(t) N_x\left(\varphi^{-1}(t)\right).
\end{aligned} \qquad (11)$$

From (1) and (11), we obtain

$$\left(r(t) \left(N_x'''(t)\right)^{\alpha} \right)' + \int_a^b q(t, \vartheta) p_1^{\beta}(\delta(t, \vartheta)) N_x^{\beta}\left(\varphi^{-1}(\delta(t, \vartheta))\right) d\vartheta \leq 0. \qquad (12)$$

Since $\delta(t, \xi)$ is nondecreasing with respect to s, we get $\delta(t, \vartheta) \geq \delta(t, a)$ for $\xi \in (a, b)$ and so

$$\left(r(t) \left(N_x'''(t)\right)^{\alpha} \right)' + N_x^{\beta}\left(\varphi^{-1}(\delta(t, a))\right) \int_a^b q(t, \vartheta) p_1^{\beta}(\delta(t, \vartheta)) d\vartheta \leq 0.$$

Next, we define a function ω by

$$\omega(t) := \theta(t) \frac{r(t) \left(N_x'''(t)\right)^{\alpha}}{N_x^{\alpha}\left(\varphi^{-1}(\delta(t, a))\right)} > 0.$$

Differentiating and using (12), we obtain

$$\begin{aligned}
\omega'(t) &\leq \frac{\theta'(t)}{\theta(t)} \omega(t) - \theta(t) N_x^{\beta-\alpha}\left(\varphi^{-1}(\delta(t, a))\right) \int_a^b q(t, \vartheta) p_1^{\beta}(\delta(t, \vartheta)) d\vartheta \\
&\quad - \alpha\theta(t) \frac{r(t) \left(N_x'''(t)\right)^{\alpha} \left(\varphi^{-1}(\delta(t, a))\right)' (\delta(t, a))' N_x'\left(\varphi^{-1}(\delta(t, a))\right)}{N_x^{\alpha+1}\left(\varphi^{-1}(\delta(t, a))\right)}.
\end{aligned} \qquad (13)$$

Recalling that $r(t) \left(N_x'''(t)\right)^{\alpha}$ is decreasing, we get

$$r\left(\varphi^{-1}(\delta(t, a))\right) \left(N_x'''\left(\varphi^{-1}(\delta(t, a))\right)\right)^{\alpha} \geq r(t) \left(N_x'''(t)\right)^{\alpha}.$$

This yields

$$\left(N_x''' \left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)\right)^{\alpha} \geq \frac{r\left(t\right)}{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)} \left(N_x'''\left(t\right)\right)^{\alpha}. \tag{14}$$

It follows from Lemma 1 that

$$N_x'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right) \geq \frac{\mu_1}{2}\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2 N_x'''\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right), \tag{15}$$

for all $\mu_1 \in (0,1)$. Thus, by (13)–(15), we get

$$\begin{aligned}\omega'\left(t\right) \quad \leq \quad & \frac{\theta'\left(t\right)}{\theta\left(t\right)}\omega\left(t\right) - \theta\left(t\right) N_x^{\beta-\alpha}\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)\int_a^b q\left(t,\vartheta\right) p_1^{\beta}\left(\delta\left(t,\vartheta\right)\right) d\vartheta \\ & -\alpha\theta\left(t\right)\frac{\mu_1}{2}\left(\frac{r\left(t\right)}{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)}\right)^{1/\alpha}\frac{r\left(t\right)\left(N_x'''\left(t\right)\right)^{\alpha+1}\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)'\left(\delta\left(t,a\right)\right)'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2}{N_x^{\alpha+1}\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)}\end{aligned}$$

Hence,

$$\begin{aligned}\omega'\left(t\right) \quad \leq \quad & \frac{\theta'\left(t\right)}{\theta\left(t\right)}\omega\left(t\right) - \theta\left(t\right) N_x^{\beta-\alpha}\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)\int_a^b q\left(t,\vartheta\right) p_1^{\beta}\left(\delta\left(t,\vartheta\right)\right) d\vartheta \\ & -\alpha\frac{\mu_1}{2}\left(\frac{r\left(t\right)}{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)}\right)^{1/\alpha}\frac{\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)'\left(\delta\left(t,a\right)\right)'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2}{\left(r\theta\right)^{1/\alpha}\left(t\right)}\omega^{\frac{\alpha+1}{\alpha}}\left(t\right).\end{aligned}$$

Since $N_x'\left(t\right) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that

$$N_x\left(t\right) > M, \tag{16}$$

for all $t \geq t_2$. Using the inequality

$$Ux - Vx^{(\beta+1)/\beta} \leq \frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{U^{\beta+1}}{V^{\beta}}, \quad V > 0,$$

with

$$U = \frac{\theta'\left(t\right)}{\theta\left(t\right)}, \quad V = \alpha\frac{\mu_1}{2}\left(\frac{r\left(t\right)}{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)}\right)^{1/\alpha}\frac{\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)'\left(\delta\left(t,a\right)\right)'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2}{\left(r\theta\right)^{1/\alpha}\left(t\right)}$$

and $x = \omega$, we get

$$\omega'\left(t\right) \leq -\Psi\left(t\right) + \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)\left(\theta'\left(t\right)\right)^{\alpha+1}}{\left(\mu_1\theta\left(t\right)\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)'\left(\delta\left(t,a\right)\right)'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2\right)^{\alpha}}.$$

This implies that

$$\int_{t_1}^t \left(\Psi\left(s\right) - \frac{2^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)\left(\theta'\left(t\right)\right)^{\alpha+1}}{\left(\mu_1\theta\left(t\right)\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)'\left(\delta\left(t,a\right)\right)'\left(\varphi^{-1}\left(\delta\left(t,a\right)\right)\right)^2\right)^{\alpha}}\right) ds \leq \omega\left(t_1\right),$$

which contradicts (8).

In the case where ($\mathbf{S}_2$) satisfies, by using Lemma 2, we find that

$$N_x\left(t\right) \geq tN_x'\left(t\right) \tag{17}$$

and hence $\left(t^{-1}N_x\left(t\right)\right)' \leq 0$. Therefore,

$$\varphi^{-1}\left(t\right)N_x\left(\varphi^{-1}\left(\varphi^{-1}\left(t\right)\right)\right) \leq \varphi^{-1}\left(\varphi^{-1}\left(t\right)\right)N_x\left(\varphi^{-1}\left(t\right)\right). \tag{18}$$

From (7) and (18), we have

$$\begin{aligned}
x\left(t\right) &\geq \frac{1}{p\left(\varphi^{-1}\left(t\right)\right)}\left(1 - \frac{\left(\varphi^{-1}\left(\varphi^{-1}\left(t\right)\right)\right)}{\left(\varphi^{-1}\left(t\right)\right)p\left(\varphi^{-1}\left(\varphi^{-1}\left(t\right)\right)\right)}\right)N_x\left(\varphi^{-1}\left(t\right)\right) \\
&= p_2\left(t\right)N_x\left(\varphi^{-1}\left(t\right)\right),
\end{aligned}$$

which with (1) gives

$$\left(r\left(t\right)\left(N_x'''\left(t\right)\right)^\alpha\right)' \leq -\int_a^b q\left(t,\vartheta\right)p_2^\beta\left(\delta\left(t,\vartheta\right)\right)N_x^\beta\left(\varphi^{-1}\left(\delta\left(t,\vartheta\right)\right)\right)d\vartheta.$$

Integrating this inequality from t to ϱ, we obtain

$$r\left(\varrho\right)\left(N_x'''\left(\varrho\right)\right)^\alpha - r\left(t\right)\left(N_x'''\left(t\right)\right)^\alpha \leq -\int_t^\varrho\left(\int_a^b q\left(t,\vartheta\right)p_2^\beta\left(\delta\left(t,\vartheta\right)\right)N_x^\beta\left(\varphi^{-1}\left(\delta\left(t,\vartheta\right)\right)\right)d\vartheta\right)ds. \tag{19}$$

From (17), we get that

$$N_x\left(\varphi^{-1}\left(\delta\left(t,\vartheta\right)\right)\right) \geq \frac{\varphi^{-1}\left(\delta\left(t,\vartheta\right)\right)}{t}N_x\left(t\right). \tag{20}$$

Letting $\varrho \to \infty$ in (19) and using (20), we obtain

$$r\left(t\right)\left(N_x'''\left(t\right)\right)^\alpha \geq p_2^\beta\left(\delta\left(t,a\right)\right)N_x^\beta\left(t\right)\int_t^\infty\left(\int_a^b q\left(s,\vartheta\right)\left(\frac{\varphi^{-1}\left(\delta\left(s,\vartheta\right)\right)}{s}\right)^\beta d\vartheta\right)ds.$$

Integrating this inequality again from t to ∞, we get

$$N_x''\left(t\right) \leq -p_2^{\beta/\alpha}N_x^{\beta/\alpha}\left(t\right)\int_t^\infty\left(\frac{1}{r\left(\varrho\right)}\int_\varrho^\infty\left(\int_a^b q\left(s,\vartheta\right)\left(\frac{\varphi^{-1}\left(\delta\left(s,\vartheta\right)\right)}{s}\right)^\beta d\vartheta\right)ds\right)^{1/\alpha}d\varrho, \tag{21}$$

for all $\mu_2 \in \left(0,1\right)$.

Now, we define

$$w\left(t\right) = \theta_1\left(t\right)\frac{N_x'\left(t\right)}{N_x\left(t\right)}.$$

Then $w\left(t\right) > 0$ for $t \geq t_1$. By differentiating w and using (21), we find

$$\begin{aligned}
w'\left(t\right) &= \frac{\theta_1'\left(t\right)}{\theta_1\left(t\right)}w\left(t\right) + \theta_1\left(t\right)\frac{N_x''\left(t\right)}{N_x\left(t\right)} - \theta_1\left(t\right)\left(\frac{N_x'\left(t\right)}{N_x\left(t\right)}\right)^2 \\
&\leq \frac{\theta_1'\left(t\right)}{\theta_1\left(t\right)}w\left(t\right) - \frac{1}{\theta_1\left(t\right)}w^2\left(t\right) \\
&\quad -p_2^{\beta/\alpha}\theta_1\left(t\right)N_x^{\beta/\alpha-1}\left(t\right)\int_t^\infty\left(\frac{1}{r\left(\varrho\right)}\int_\varrho^\infty\left(\int_a^b q\left(s,\vartheta\right)\left(\frac{\varphi^{-1}\left(\delta\left(s,\vartheta\right)\right)}{s}\right)^\beta d\vartheta\right)ds\right)^{1/\alpha}d\varrho.
\end{aligned}$$

Thus, we obtain

$$w'\left(t\right) \leq -\Phi\left(t\right) + \frac{\theta_1'\left(t\right)}{\theta_1\left(t\right)}w\left(t\right) - \frac{1}{\theta_1\left(t\right)}w^2\left(t\right),$$

and so

$$w'(t) \leq -\Phi(t) + \frac{(\theta_1'(t))^2}{4\theta_1(t)}.$$

Then, we get

$$\int_{t_1}^t \left(\Phi(s) - \frac{(\theta'(t))^2}{4\theta(t)} \right) ds \leq w(t_1),$$

which contradicts (9). This completes the proof.　□

Theorem 2. *Let*

$$\frac{\left(\varphi^{-1} \left(\varphi^{-1}(t) \right) \right)^{n-1}}{\left(\varphi^{-1}(t) \right)^{n-1} p \left(\varphi^{-1} \left(\varphi^{-1}(t) \right) \right)} \leq 1. \tag{22}$$

Suppose that there exist positive functions η, $\sigma \in p^1 \left([t_0, \infty), \mathbb{R} \right)$ satisfying

$$\eta(t) \leq \delta(t), \ \eta(t) < \varphi(t), \ \sigma(t) \leq \delta(t), \ \sigma(t) < \varphi(t), \ \sigma'(t) \geq 0 \text{ and } \lim_{t \to \infty} \eta(t) = \lim_{t \to \infty} \sigma(t) = \infty. \tag{23}$$

If the equations

$$\psi'(t) + \tilde{R}(t) \psi^{\beta/\alpha} \left(\varphi^{-1}(\eta(t,a)) \right) = 0 \tag{24}$$

and

$$\phi'(t) + p_2^{\beta/\alpha} \left(\varphi^{-1}(\sigma(t,a)) \right)^{\beta/\alpha} R(t) \phi^{\beta/\alpha} \left(\varphi^{-1}(\sigma(t,a)) \right) = 0 \tag{25}$$

are oscillatory, then (1) *is oscillatory.*

Proof. Let x be a non-oscillatory solution of (1) on $[t_0, \infty)$. Without loss of generality, we suppose that $x > 0$. From Lemma 4, we find there exist two possible cases ($\mathbf{S}_1$) and ($\mathbf{S}_2$).

Assume that Case ($\mathbf{S}_1$) holds. From Theorem 1, we get that (12) holds. Since $\eta(t) \leq \delta(t)$ and $z'(t) > 0$, we obtain

$$\left(r(t) \left(N_x'''(t) \right)^\alpha \right)' \leq - \int_a^b q(t,\vartheta) p_1^\beta(\eta(t,\vartheta)) N_x^\beta \left(\varphi^{-1}(\eta(t,\vartheta)) \right) d\vartheta. \tag{26}$$

Now, by using Lemma 1, we have

$$N_x(t) \geq \frac{\mu}{6} t^3 N_x'''(t). \tag{27}$$

for some $\mu \in (0,1)$. It follows from (26) and (27) that, for all $\mu \in (0,1)$,

$$\left(r(t) \left(N_x'''(t) \right)^\alpha \right)' + \int_a^b \left(\frac{\mu \left(\varphi^{-1}(\eta(t,\vartheta)) \right)^3}{6} \right)^\beta q(t,\vartheta) p_1^\beta(\eta(t,\vartheta)) \left(N_x''' \left(\varphi^{-1}(\eta(t,\vartheta)) \right) \right)^\beta d\vartheta \leq 0.$$

Thus, we choose

$$\psi(t) = r(t) \left(N_x'''(t) \right)^\alpha.$$

So, we find that ψ is a positive solution of the inequality

$$\psi'(t) + \tilde{R}(t) \psi^{\beta/\alpha} \left(\varphi^{-1}(\eta(t,a)) \right) \leq 0.$$

Using (see ([15] Theorem 1)), we see (24) also has a positive solution, a contradiction.

Suppose that Case ($\mathbf{S}_2$) holds. From Theorem 1, we get that (21) holds. Since $\sigma(t) \leq \delta(t)$ and $N_x'(t) > 0$, we have that

$$N_x'' (t) \leq -p_2^{\beta/\alpha} N_x^{\beta/\alpha} \left(\varphi^{-1} (\sigma (t,a)) \right) \int_t^\infty \left(\frac{1}{r (\varrho)} \int_\varrho^\infty \left(\int_a^b q (s, \vartheta) \left(\frac{\varphi^{-1} (\sigma (s, \vartheta))}{s} \right)^\beta d\vartheta \right) ds \right)^{1/\alpha} d\varrho, \quad (28)$$

Using Lemma 2, we get that

$$N_x (t) \geq t N_x' (t) . \quad (29)$$

From (18) and (29), we obtain

$$N_x'' (t) \leq -p_2^{\beta/\alpha} \left(N_x' \left(\varphi^{-1} (\sigma (t,a)) \right) \right)^{\beta/\alpha} \left(\varphi^{-1} (\sigma (t,a)) \right)^{\beta/\alpha} R (t) .$$

Now, we choose $\phi (t) := N_x' (t)$, thus, we find that ϕ is a positive solution of

$$\phi' (t) + p_2^{\beta/\alpha} \left(\varphi^{-1} (\sigma (t,a)) \right)^{\beta/\alpha} R (t) \phi^{\beta/\alpha} \left(\varphi^{-1} (\sigma (t,a)) \right) \leq 0. \quad (30)$$

Using (see ([15] Theorem 1)), we see (25) also has a positive solution, a contradiction. The proof is complete. $\square$

Example 1. *Consider the differential equation*

$$\left(\left[x (t) + \frac{1}{2} x \left(\frac{t}{3} \right) \right]''' \right)' + \int_0^1 \left(\frac{q_0}{t^4} \right) \vartheta x \left(\frac{t - \xi}{2} \right) d\vartheta = 0, \quad (31)$$

where $q_0 > 0$ is a constant. Let $\alpha = \beta = 1$, $r (t) = 1$, $p (t) = 1/2$, $\varphi (t) = t/3$, $\varphi^{-1} (t) = 3t$, $\delta (t, a) = t/2$, $q (t, \vartheta) = \left(q_0 \backslash t^4 \right) \vartheta$.

Thus, by using Theorem 1, then Equation (31) is oscillatory.

Remark 1. *By applying our results in (5), we see that our results improve [22,24].*

Remark 2. *One can easily see that the results obtained in [24] cannot be applied to conditions in Theorem 1, so our results are new.*

4. Conclusions

In this work, our method is based on using the Riccati transformations to get some oscillation criteria of (1). There are numerous results concerning the oscillation criteria of fourth order equations, which include various forms of criteria as Hille/Nehari, Philos, etc. This allows us to obtain also various criteria for the oscillation of (1). Further, we can try to get some oscillation criteria of (1) if $N_x (t) := x (t) - p (t) x (\varphi (t))$ in the future work.

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