SEMA SIMAI Springer Series 21

Maira Aguiar · Carlos Braumann
Bob W. Kooi · Andrea Pugliese
Nico Stollenwerk · Ezio Venturino
Editors

Current Trends in Dynamical Systems in Biology and Natural Sciences

SeMA SIMAI SOCIETÀ ITALIANA DI MATEMATICA APPLICATA E INDUSTRIALE
Springer

SEMA SIMAI Springer Series

Volume 21

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Current Trends in Dynamical Systems in Biology and Natural Sciences

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ISSN 2199-3041 ISSN 2199-305X (electronic)
SEMA SIMAI Springer Series
ISBN 978-3-030-41119-0 ISBN 978-3-030-41120-6 (eBook)
https://doi.org/10.1007/978-3-030-41120-6

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The Ninth Edition of the International Workshop "Dynamical Systems Applied to Biology and Natural Sciences (DSABNS)" was held at the Department of Mathematics of the University of Torino, Italy, from February 7th to 9th, 2018. The workshop program included both theoretical methods and practical applications, covering research topics in population dynamics, epidemiology of infectious diseases, eco-epidemiology, molecular and antigenic evolution, and methodological topics in the natural sciences and mathematics.

Since 2010, the DSABNS workshop, which was upgraded to a conference in 2019, has been organized by the Mathematical Biology Group of the Center for Mathematics, Fundamental Applications and Operations Research (CMAF-cIO) of Lisbon University, in collaboration with professors and researchers from Portugal, Italy, and the Netherlands. From 2010 to 2015, the event was held at Lisbon University, during which time it acquired a broad organizational structure and attracted an increasing number of participants. From 2016 to 2017, the workshop was held in Évora (Portugal) and it then moved to Italy, being held in Torino in 2018 and Naples in 2019. As a traditional "no registration fee" scientific event, the DSABNS attracts researchers and students from different countries around the world who draw on their own funding to attend and present their recent scientific results. A book of abstracts (with ISBN number) is also published at the end of each event.

The Ninth DSABNS 2018 in Torino attracted the participation of 133 delegates from 30 countries. There were 13 plenary talks, 10 invited talks, 58 contributed talks, and a poster session. In this book, we have collected papers based on the research topics presented during DSABNS 2018, centering mainly on topics involving ecology and epidemiology but even touching on waste recycling and a genetic application. Some contributions also involve the application of numerical techniques to problems of structured populations.

In ecology, the contributions range from a theoretical investigation aimed at reconstructing the interactions of populations from a niche theory to other issues as the study of suitable techniques for the assessment of the patterns generated by invasive species in the spatiotemporal domain.

In the former case, chapter “Modelling Ecological Systems from a Niche Theory to Lotka-Volterra Equations”, the concept of fitness landscape allows a stochastic description of species dynamics and the introduction of the notion of fitness potential for the evolution of a mutual ecosystem. Feasibility of its thermodynamic equilibrium, whose distribution is a multinomial negative distribution, is provided by the study of a master equation. In chapter “Accurate Recognition of Spatial Patterns Arising in Spatio-Temporal Dynamics of Invasive Species”, it is remarked that being able to distinguish between the patchy spatial density patterns and continuous front spatial density patterns is essential for the implementation of control measures against invasive species. A model consisting of two integro-difference equations is proposed to investigate various spatial density distributions. With it, several topological characteristics are generated, among which it is found that the number of objects in the visual image of a spatial distribution offers the most reliable conclusion for distinguishing between continuous and patchy spatial structures. The two most relevant features of the monitoring protocol are found, namely the threshold density value and the number of sampling locations.

More abstract problems related to population theory are studied in the next two chapters. In chapter “Collocation Techniques for Structured Populations Modeled by Delay Equations”, an improved numerical scheme is proposed based on piecewise polynomial collocation to reduce delay systems to systems of ordinary differential equations or to approximate a periodic solution. For realistic models of structured populations, the proposed method substantially improves performances in comparison with the existing ones that rely on an external ordinary differential equations solver. Its adaptability for the computation of periodic solutions is demonstrated.

A view differing from the classical predator-prey models is taken in chapter “Herding Induced by Encounter Rate, with Predator Pressure Influencing Prey Response”, where the effects of herding are investigated, observing that populations living together have less than well-mixed interactions. A range of models is thus obtained for a single population, specifically hyperbolic models which exhibit intermediate growths between the exponential and the logistic ones. In the context of Lotka-Volterra intermingling populations, this formulation stabilizes coexistence. For predators, predation pressure is reduced, as well as access to resources. The latter is modeled via a reduction in carrying capacity with increasing predator pressure, while predator escape is formulated in terms of the degree of herding. The latter is the stronger, the larger the predator pressure becomes. Hopf bifurcations are possible, leading to stable limit cycles for specialist predators and unstable ones when generalist predators are considered.

Still in the context of ecology, in chapter “Harvesting Policies with Stepwise Effort and Logistic Growth in a Random Environment” constant and variable effort harvesting policies to maximize the expected total discounted profit are investigated over a finite horizon in the presence of stochastic fluctuations naturally occurring in real-life ecological situations. Due to the inapplicability and other shortcomings of the optimal variable effort policy, constant effort policies were considered. They are easy to implement, have no such shortcomings, and surprisingly provide a profit that is only slightly lower. The paper then studies variable effort stepwise strategies, where the effort is kept constant over one or two years and then updated. These stepwise policies are easy to implement at the cost of reducing the already low profit advantage of the optimal variable effort strategy.

In chapter “Mathematical Modeling of the Population Dynamics of Long-Lived Raptor Species: Application to Eurasian Black Vulture Colonies”, a stochastic approach is also employed for the investigation of the population dynamics of raptor species. The long-lived Eurasian black vulture colonies are examined via discrete-time branching models, identified by time rather than by generation. A distinguishing feature in the population is the consideration of the coexistence of individuals from different generations. The most informative reproductive parameters are estimated in a non-parametric statistical setting using a Bayesian estimation procedure. Real data coming from the region of Extremadura (Spain) are used in the simulations. Specifically, the colonies used for the sampling represent two of the largest breeding colonies worldwide. They are located in the National Park of Monfragüe and in the Sierra San Pedro.

Control theory is also employed for waste recycling in chapter “On the Role of Inhibition Processes in Modeling Control Strategies for Composting Plants”, in particular for the composting process of biocells. It allows optimization of the ways to provide air when inhibition due to a high concentration of oxygen occurs, thereby guaranteeing that the aerobic biodegradation process proceeds smoothly. Special attention is devoted to the assessment of the minimal cost of the control policy thus devised.

A further application of control is presented in chapter “Optimal Control of Invasive Species with Budget Constraint: Qualitative Analysis and Numerical Approximation”. It concerns the optimal removal of invasive species, addressing the best temporal resource allocation strategy to achieve it. The optimality system in the state and control variables is derived, and phase-space analysis is used to provide qualitative insights about the behavior of the optimal solution. In particular, a practical situation involving plants is considered. The problem is reduced to a boundary-valued nearly-Hamiltonian system which is solved by suitable exponential Lawson symplectic approximations. An application to a real plant ground-reclaiming case is finally provided.

Control theory also represents the link with the second part of the contributions, describing investigations performed in the domain of epidemiology. A stabilization problem for an epidemic model, described by a reaction-diffusion system with feedback, is considered in chapter “A Shape Optimization Problem Concerning the Regional Control of a Class of Spatially Structured Epidemics: Sufficiency Conditions”, where sanitation measures are envisaged. The main aim is the assessment of control programs administered only in a given subdomain of the region of interest that induce an effective disease eradication in the whole habitat. The sufficient optimality conditions are obtained and an approximate conceptual algorithm is discussed.

Vaccination, as an explicit disease control measure accounting for people’s behavior, is considered in chapter “The Interplay Between Voluntary Vaccination and Reduction of Risky Behavior: A General Behavior-Implicit SIR Model for Vaccine Preventable Infections". Two broad classes of behavior-implicit SIR models are reviewed: prevalence-dependent vaccination and prevalence-dependent contact rate. Then behavior-dependent and nonlinear and linear forces of infection are set in a general framework that also encompasses epidemic memory. These two different issues are here combined in a single unified approach that allows an assessment of the complicated interplay between the different behavioral responses due to various epidemiological parameters. As a result, sustained oscillations of vaccine coverage, risky behavior, and infection prevalence are obtained.

In epidemiology, a fundamental concept is the disease basic reproduction number R_0. In the presence of parameter uncertainties, the sensitivity estimation of the stochastic model is allowed by suitable numerical methods using polynomial chaos expansions. Evaluation of Sobol indices by polynomial chaos-based methods are presented in chapter "PC-Based Sensitivity Analysis of the Basic Reproduction Number of Population and Epidemic Models", showing how R_0 is affected by varying the input parameters. The newly developed computational model for R_0 introduced here allows for the efficient and versatile treatment of rather complex epidemic models.

Finally, an application to genetics is presented in chapter "Linear Dynamics of mRNA Expression and Hormone Concentration Levels in Primary Cultures of Bovine Granulosa Cells". The Gene Regulatory Matrices technique is here generalized to encompass also hormones, specifically estradiol (E2) and progesterone (P4), by constructing a directed weighted graph to model the interactions of several mRNA encoding enzymes. This allows the calculation of hormone concentration from the concentration of mRNA. This approach had previously been attempted only via differential equations, which are, however, limited by the need for accurate knowledge of the decay rates of hormones and mRNA. The novel technique with Gene and Hormone Regulatory Matrices allows estimation of the concentration on the whole network by using only a subset of its nodes. The models are constructed from data obtained in experiments providing gene expression and hormone concentration levels for primary bovine granulosa cells.

The collection of selected papers presented in this SEMA SIMAI Springer Series followed the traditionally rigorous reviewing standards of journals that are traditional to this series. The authors are indebted and express their thanks to Luca Formaggia and SIMAI for the kind invitation to contribute to this series.

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Évora, Portugal — Carlos Braumann
Amsterdam, The Netherlands — Bob W. Kooi
Trento, Italy — Andrea Pugliese
Lisbon, Portugal — Nico Stollenwerk
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May 2019

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About the Editors

Maira Aguiar is a biologist who has also trained in the mathematical modeling of biological systems, with emphasis on nonlinear dynamics, bifurcation analysis, and biostatistics. Her research investigates problems in public health epidemiology, focusing on the dynamics of vector-borne diseases. She has authored more than 35 papers and is frequently invited as a plenary speaker at international scientific meetings. Since 2018, she has been Vice President of the European Society for Mathematical and Theoretical Biology.

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Bob W. Kooi's main research interests concern interacting populations in Life Sciences—Ecology, Evolution, Epidemiology, and Biochemistry—using mathematical models based on physical/chemical processes at different organizational levels: at the individual level, the Dynamic Energy Budget model, and at higher levels, unstructured/physiologically structured populations and community and ecosystem models. The emphasis is on sensitivity, perturbation, bifurcation, and nonlinear dynamics analysis techniques.

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Nico Stollenwerk received his PhD in theoretical physics from the University of Clausthal. He previously worked at the Research Center Jülich, Germany and is currently the Principal Investigator of the Mathematical Biology group at CMAF, Lisbon University, where he has designed a Biomathematics PhD course. He is a coauthor of the book "Population Biology and Criticality" and has also coauthored many articles in international journals and more than 40 book chapters, as well as refereeing contributions in international congresses.

Ezio Venturino received his PhD in Applied Mathematics from SUNY at Stony Brook in 1984. He is currently Professor of Mathematics in the Department of Mathematics, University of Turin, Italy. He has visited a number of international institutions worldwide and has a wide scientific collaboration network. His earlier research focused on numerical analysis, mainly methods for integral equations, and he is currently engaged in research on nonlinear models for biological and ecological applications.

Harvesting Policies with Stepwise Effort and Logistic Growth in a Random Environment

Nuno M. Brites and Carlos A. Braumann

Abstract Recently, we have developed optimal harvesting policies based on profit optimization in random varying environments. Namely, we have considered a logistic stochastic differential equation growth model, with the purpose of discussing the use of variable versus constant effort harvesting policies in terms of the expected accumulated discounted profit during a finite time interval. Using realistic parameters, we have concluded that there is only a slight reduction in profit when choosing the applicable constant effort policy instead of the variable effort policy, which presents strong disadvantages. Here, we apply a logistic growth model and a more general profit structure to present alternative policies based on variable effort, named stepwise policies, where the harvesting effort is determined, under the optimal variable effort policy, at the beginning of each year (or of each biennium) but is kept constant during that year (biennium). Replacing the optimal variable effort policy by these stepwise non-optimal policies has the advantage of applicability but, at best, considerably reduces the already small profit advantage the optimal variable effort policy has over the optimal constant effort sustainable policy.

Keywords Fisheries management · Stochastic differential equations · Profit optimization · Stepwise effort · Logistic growth

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M. Aguiar et al. (eds.), *Current Trends in Dynamical Systems in Biology and Natural Sciences*, SEMA SIMAI Springer Series 21,
https://doi.org/10.1007/978-3-030-41120-6_5

1 Introduction

Stochastic optimal control methods have been applied to derive optimal harvesting policies in a randomly varying environment [1, 9]. Since the population size experiences random fluctuations, it cannot be kept at an equilibrium size. Therefore, the fishing effort, $E(t)$, must be adjusted at every instant, so that the size of the population does not go above some threshold value. So, the optimal harvesting effort will have sudden frequent transitions between maximum or high harvesting efforts and low or null harvesting efforts. These transitions in effort are not compatible with the logistics of fisheries. Besides, the period of low or no harvesting poses social and economical undesirable implications (intermittent unemployment is just one of them). In addition to such shortcomings, these optimal policies require the knowledge of the population size at every instant, to define the appropriate level of effort. The estimation of the population size is a difficult, costly, time consuming and inaccurate task. Therefore, these policies should be considered unacceptable and inapplicable.

Braumann [3–5] has considered Stochastic Differential Equation (SDE) harvesting growth models with a constant fishing effort, $E(t) \equiv E$. For a large class of models it was found that, taking a constant fishing effort, there is, under mild conditions, a stochastic sustainable behaviour. Namely, the probability distribution of the population size at time t will converge, as $t \to +\infty$, to an equilibrium probability distribution (the so-called stationary or steady-state distribution) having a probability density function (the so-called stationary density). For the logistic and the Gompertz models, the stationary density was found, and the effort E that optimizes the steady-state yield was determined. The issue of profit optimization, however, was not addressed.

In [6] we have determined the constant effort that maximizes the expected profit per unit time at the steady-state for the specific case of the logistic model. One might think that a constant effort policy would result in a substantial profit reduction compared with the optimal variable effort policy, but we have shown that this is not the case. This new policy, rather than switching between large and small or null fishing effort, keeps a constant effort and is therefore compatible with the logistics of fisheries. Furthermore, this alternative policy does not require knowledge of the population size.

Since the optimal variable effort policy is not applicable, we present here for the logistic model, intermediate sub-optimal policies, named stepwise policies, where the harvesting effort is determined at the beginning of each year (or of each biennium) under the optimal variable effort policy and kept constant during that year (biennium). These policies are not optimal, but have the advantage of being applicable, since the changes in effort are less frequent and compatible with the fishing activity. Furthermore, although we still need to keep estimating the fish stock size, we do not need to do it so often. Replacing the optimal variable effort policy by these stepwise policies has the advantage of applicability but, at best, considerably reduces the already small profit advantage the optimal variable effort policy has over the optimal constant effort policy. In some cases, the optimal sustainable policy even outperforms these stepwise policies in terms of profit (although it might seem so at first glance, there is no contradiction). In [6] a linear price structure was considered. We now generalize that structure, and take a more realistic approach, by considering a quadratic form for the price function.

Section 2 presents the variable effort harvesting policy by applying a dynamic programming method. In Sect. 3 we present the sustainable approach based on constant effort. Section 4 shows an application for the Pacific halibut (*Hippoglossus hippoglossus*) with realistic biological and fishing parameters in which the stepwise effort policies are compared with the others by using numerical and Monte Carlo methods. We end up, in Sect. 5, with the conclusions.

2 Variable Effort Optimal Policy

In a random environment the dynamics of a population subject to harvesting and following a logistic growth model can be described, as in [6], by the SDE

$$\mathrm{d}X(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\mathrm{d}t - H(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t), \quad X(0) = x > 0, \tag{1}$$

where $X(t)$ is the population size at time t, measured as biomass or as number of individuals, r is the population intrinsic growth rate, K is the environment carrying capacity, $H(t)$ is the harvesting rate, σ measures the strength of environmental fluctuations, $W(t)$ is a standard Wiener process and $x > 0$ is the population size at time 0, which we assume known.

We choose the harvesting rate $H(t)$ as

$$H(t) = qE(t)X(t),$$

which is the most traditional form (see, for instance, [7–9]), where $q > 0$ is a constant representing the fraction of biomass harvested per unit of effort and per unit time and $E(t)$ corresponds to the effort exerted on the population at time instant t. We assume $E(t)$ to be non-anticipating, i.e., it only depends on information available up to time t (included) and to be constrained as

$$0 \leq E_{min} \leq E(t) \leq E_{max} < \infty. \tag{2}$$

The profit per unit time can be defined as the difference between sales revenues and fishing costs, i.e.,

$$P(t) := R(t) - C(t) = p(H(t))H(t) - c(E(t))E(t), \tag{3}$$

where $R(t)$ are the revenues per unit time from the harvested fish, $C(t)$ is the cost per unit time derived from fishing with effort $E(t)$, $p(H(t))$ denotes the price per unit yield and $c(E(t))$ is the cost per unit effort. We assume that the unit prices and costs have, respectively, the form

$$p(H(t)) = p_1 - p_2 H(t) \quad \text{and} \quad c(E(t)) = c_1 + c_2 E(t),$$

where $p_1 \geq 0$, $p_2 \geq 0$, $c_1 \geq 0$ and $c_2 > 0$ are constants, slightly generalizing the price structure appearing in [6]. Thus, (3) becomes

$$P(t) = \big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2) E(t)\big) E(t).$$

Given the stochastic nature of $X(t)$, we work with the expected profit per unit time

$$\mathbb{E}\left[P(t)\right] = \mathbb{E}\Big[\big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2) E(t)\big) E(t)\Big]. \tag{4}$$

For our purposes, harvesting begins at the time instant $t = 0$ and the corresponding population size is $X(0) = x > 0$. Furthermore, harvesting continues up to the time horizon $T < +\infty$ and we work with the profit present value, i.e., future profits are discounted by a rate $\delta > 0$ accounting for interest rate and cost of opportunity losses and for other social rates. For a time t in the time interval $[0, T]$, we define

$$J(y, t) := \mathbb{E}\left[\int_t^T e^{-\delta(\tau - t)} P(\tau) \mathrm{d}\tau \,\middle|\, X(t) = y\right], \tag{5}$$

which is the expected discounted future profits when the population size at that time is y.

The determination of the optimal variable effort harvesting policy is in fact an optimal control problem (OCP), and consists in maximizing the expected accumulated discounted profit per unite time during a finite time interval, i.e., for $0 \leq \tau \leq T$,

$$V^* := J^*(x, 0) = \max_{E(\tau)} \mathbb{E}_x \left[\int_0^T e^{-\delta\tau} P(\tau) \mathrm{d}\tau\right], \tag{6}$$

subject to (1), (2) and to the boundary condition $J(X(T), T) = 0$, obtained from (5).

The above OCP can be solved by the stochastic dynamic programming theory through Bellman's principle of optimality (as in [2]). The associated Hamilton-Jacobi-Bellman (HJB) equation (see [2, 10]) is

$$-\frac{\partial J^*(X(t),t)}{\partial t} = \max_{E(t)} \Bigg\{ \big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\big)E(t) - \delta J^*(X(t),t) + \frac{\partial J^*(X(t),t)}{\partial X(t)} \left(r\left(1 - \frac{X(t)}{K}\right) - qE(t)\right) X(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)} \sigma^2 X^2(t) \Bigg\}. \tag{7}$$

The optimal variable effort is obtained from the HJB equation (7). Let D be a function that represents the control switching term present in (7), that is,

$$D(E) = \big(p_1 q X(t) - c_1 - (p_2 q^2 X^2(t) + c_2)E(t)\big)E(t) - \frac{\partial J^*(X(t),t)}{\partial X(t)} qE(t)X(t), \tag{8}$$

and denote by $E^*_{free}(t)$ the unconstrained effort resulting from the maximization carried out in Eq. (8). Thus, $E^*_{free}(t)$ is obtained by solving the equation $dD(E)/dE = 0$ with respect to E, which gives

$$E^*_{free}(t) = \frac{\left(p_1 - \frac{\partial J^*(X(t),t)}{\partial X(t)}\right) qX(t) - c_1}{2\left(p_2 q^2 X^2(t) + c_2\right)}. \tag{9}$$

Representing the constrained optimal effort by $E^*(t)$ and replacing $E(t)$ by $E^*(t)$ in Eq. (7) yields the maximized HJB

$$-\frac{\partial J^*(X(t),t)}{\partial t} = (p_1 q X(t) - c_1)E^*(t) - (p_2 q^2 X^2(t) + c_2)E^{*2}(t) - \delta J^*(X(t),t) + \frac{\partial J^*(X(t),t)}{\partial X(t)} \left(r\left(1 - \frac{X(t)}{K}\right) - qE^*(t)\right) X(t) + \frac{1}{2}\frac{\partial^2 J^*(X(t),t)}{\partial X^2(t)} \sigma^2 X^2(t), \tag{10}$$

where the effort is given by

$$E^*(t) = \begin{cases} E_{min}, & if \quad E^*_{free}(t) < E_{min} \\ E^*_{free}(t), & if \quad E_{min} \leq E^*_{free}(t) \leq E_{max} \\ E_{max}, & if \quad E^*_{free}(t) > E_{max}, \end{cases}$$

with $E^*_{free}(t)$, given by (9), being the unconstrained effort (see [8]).

In summary, to determine the optimal variable effort policy, that is, to determine the values $J^*(x, 0)$ and $E^*(t)$, we need to solve (9) and (10) subject to the growth dynamic given by Eq. (1), and the boundary and initial conditions given above. We have obtained the solutions of (9) and (10) with numerical methods using a Crank-Nicolson discretization scheme (as in [6, 10]).

3 Constant Effort Optimal Policy

To apply a constant effort policy, one considers a particular case of Eq. (1) with $E(t) \equiv E$, that is,

$$\mathrm{d}X(t) = rX(t)\left(1 - \frac{X(t)}{K}\right)\mathrm{d}t - qEX(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t), \quad X(0) = x, \tag{11}$$

with the assumption $r - qE > \sigma^2/2$ to avoid almost sure extinction (see [4]).

From [6], and references therein, we know that the solution of the SDE (11) exists, is unique and is a homogeneous diffusion process. In addition, there exists a stationary distribution for the population size, i.e., an equilibrium probability distribution, with probability density function $f(X) = \frac{1}{\Gamma(\rho)}\alpha^\rho X^{\rho-1}e^{-\alpha X}$ (where $\Gamma(\cdot)$ represents the Gamma function, $\rho = \frac{2(r-qE)}{\sigma^2} - 1$ and $\alpha = \frac{2r}{K\sigma^2}$), towards which the distribution of the population size converges as $t \to \infty$. We have denoted by X_∞ the random variable with density f. It has mean value

$$\mathbb{E}[X_\infty] = K\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right). \tag{12}$$

and

$$\mathbb{E}[X_\infty^2] = \frac{(\rho+1)\rho}{\alpha^2} = K\left(1 - \frac{qE}{r}\right)\mathbb{E}[X_\infty]. \tag{13}$$

The expected sustainable profit per unit time (incorporating a generalization of the price structure presented in [6]) is

$$\begin{aligned}
\mathbb{E}\left[P_\infty\right] &= \mathbb{E}\left[(p_1 - p_2 H_\infty)H_\infty - (c_1 + c_2 E)E\right] \\
&= \mathbb{E}\left[(p_1 - p_2 qEX_\infty)qEX_\infty - (c_1 + c_2)E^2\right] \\
&= \left(p_1 qK\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right) - c_1\right)E \\
&\quad - \left(p_2 q^2 K^2\left(1 - \frac{qE}{r} - \frac{\sigma^2}{2r}\right)\left(1 - \frac{qE}{r}\right) + c_2\right)E^2,
\end{aligned} \tag{14}$$

and the steady-state optimization problem becomes

$$\max_E \mathbb{E}[P_\infty] = \left(p_1 q K\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)-c_1\right)E - \left(p_2 q^2 K^2\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)\left(1-\frac{qE}{r}\right)+c_2\right)E^2.$$

If there is a maximum in the admissible range $0 \le E < \frac{r-\sigma^2/2}{q}$, the optimization problem consists in solving the cubic equation $d\mathbb{E}[P_\infty]/dE = 0$, so that the solution satisfies $d^2\mathbb{E}[P_\infty]/dE^2 < 0$. The resulting optimal sustainable effort, E^{**}, is then solution of the equation

$$p_1 q K\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)-c_1-\frac{p_1 K q^2}{r}E -2E\left(p_2 q^2 K^2\left(1-\frac{qE}{r}\right)\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)+c_2\right) -E^2\left(p_2 q^2 K^2\left(-\frac{q}{r}\right)\left(1-\frac{qE}{r}-\frac{\sigma^2}{2r}\right)+p_2 q^2 K^2\left(1-\frac{qE}{r}\right)\left(-\frac{q}{r}\right)\right)=0.$$

The correspondent optimal expected sustainable profit per unit time, $\mathbb{E}[P_\infty^{**}]$, is

$$\mathbb{E}[P_\infty^{**}] = \left(p_1 q K\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)-c_1\right)E^{**} - \left(p_2 q^2 K^2\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right)\left(1-\frac{qE^{**}}{r}\right)+c_2\right)E^{**2}. \tag{15}$$

Finally, at the steady-state, the mean value of the population under the optimal effort E^{**} is

$$\mathbb{E}[X_\infty^{**}] = K\left(1-\frac{qE^{**}}{r}-\frac{\sigma^2}{2r}\right). \tag{16}$$

Note that the equations presented in [6] correspond to the particular case $p_2 = 0$.

4 Comparison of Policies

In [6] we have presented comparisons between the variable effort optimal policy and the constant effort optimal sustainable policy in terms of the effort and profit values and the population size. To perform these comparisons we have used a linear price structure, but here we will apply the quadratic structure using (4) and (15).

We notice that the profit values given by (6) and (15) can not be directly compared since the optimal policy yields the optimal expected accumulated discounted profit, V^*, over a finite time horizon and the optimal sustainable policy yields the optimal expected profit per unit time, $\mathbb{E}[P_\infty^{**}]$, for a large time horizon $T \to +\infty$. However, both profits can be compared by defining the profit per unit time values ($*$ refers to the optimal policy and $**$ refers to the optimal constant effort sustainable policy)

$$P^*(t) := (p_1 q X(t) - c_1) E^*(t) - (p_2 q^2 X^2(t) + c_2) E^{*^2}(t),$$
$$P^{**}(t) := (p_1 q X(t) - c_1) E^{**} - (p_2 q^2 X^2(t) + c_2) E^{**^2},$$

and using the following quantities of interest:

1. **Expected accumulated discounted profit** in the interval $[0, T]$:

$$V^* := \mathbb{E}_x \left[\int_0^T e^{-\delta\tau} P^*(\tau) \mathrm{d}\tau \right], \quad V^{**} := \mathbb{E}_x \left[\int_0^T e^{-\delta\tau} P^{**}(\tau) \mathrm{d}\tau \right]. \tag{17}$$

2. **Expected accumulated undiscounted profit** in the interval $[0, T]$:

$$V_u^* = \mathbb{E}_x \left[\int_0^T P^*(\tau) \mathrm{d}\tau \right], \quad V_u^{**} = \mathbb{E}_x \left[\int_0^T P^{**}(\tau) \mathrm{d}\tau \right]. \tag{18}$$

3. **Average expected profit per unit time (average weighted by the discount factors)**:

$$\overline{P^*} = \frac{V^*}{\int_0^T e^{-\delta\tau} \mathrm{d}\tau}, \quad \overline{P^{**}} = \frac{V^{**}}{\int_0^T e^{-\delta\tau} \mathrm{d}\tau}. \tag{19}$$

4. **Average expected profit per unit time (unweighted average)**:

$$\overline{P_u^*} = \frac{V_u^*}{T}, \quad \overline{P_u^{**}} = \frac{V_u^{**}}{T}. \tag{20}$$

The above values were computed by performing 1000 Monte Carlo simulations and using a set of parameter values (r, K, q, p_1, c_1 and c_2) from the Pacific halibut (*Hippoglossus hippoglossus*) stock found in [7, 8]. Other parameters with no information (E_{min}, E_{max}, σ, x, p_2 and δ) where chosen at reasonable values and the time horizon was set at $T = 50$ years. The complete set of parameter values is listed in Table 1.

Table 1 Parameter values used in the simulations. The Standardized Fishing Unit (SFU) measure is defined in [8]

Item	Description	Values	Units
r	Intrinsic growth rate	0.71	year^{-1}
K	Carrying capacity	$80.5 \cdot 10^6$	kg
q	Catchability coefficient	$3.30 \cdot 10^{-6}$	$\text{SFU}^{-1}\text{year}^{-1}$
E_{min}	Minimum fishing effort	0	SFU
E_{max}	Maximum fishing effort	$0.7r/q$	SFU
σ	Strength of environmental fluctuations	0.2	$\text{year}^{-1/2}$
x	Initial population size	$0.5K$	kg
δ	Discount factor	0.05	year^{-1}
p_1	Linear price parameter	1.59	$\$\text{kg}^{-1}$
p_2	Quadratic price parameter	$5 \cdot 10^{-9}$	$\$\text{year} \cdot \text{kg}^{-2}$
c_1	Linear cost parameter	$96 \cdot 10^{-6}$	$\$\text{SFU}^{-1}\text{year}^{-1}$
c_2	Quadratic cost parameter	$0.10 \cdot 10^{-6}$	$\$\text{SFU}^{-2}\text{year}^{-1}$
T	Time horizon	50	year

Table 2 Numerical comparison between policies of the expected profits 1. to 4. (see expressions (17) to (20)). The percent relative difference between the two policies is denoted by Δ. Besides the expected values, we also present the standard deviations (sd). Units are in million dollars for 1. and 2. and in million dollars per year for 3. and 4.

		Profit value	sd		Profit value	sd	$\Delta(\%)$
1.	V^*	391.082	34.396	V^{**}	378.514	31.865	−3.2
2.	V_u^*	1064.048	80.030	V_u^{**}	1025.457	80.777	−3.6
3.	$\overline{P^*}$	21.303	1.874	$\overline{P^{**}}$	20.618	1.736	−3.2
4.	$\overline{P_u^*}$	21.281	1.601	$\overline{P_u^{**}}$	20.509	1.616	−3.6

For the variable effort policies, the determination of the expected profit values (17) to (20) was based on a Crank-Nicolson discretization scheme (see [6, 10]) using a time and state space grid designed with $n = 150$ intervals for time (corresponding to a time step $\Delta t = 4$ months) and with $m = 75$ intervals for the state space (corresponding to a space step $\Delta x = 21.47 \cdot 10^5$ kg, with $x_{max} = 2K$). The resulting profit values are shown in Table 2, where the left side refers to the optimal variable effort policy, the right side refers to the optimal constant effort policy, and the last column indicates the percent loss in the profit value when using the second policy instead of the first. For each profit value, the respective standard deviation value is also shown.

In the first line of Table 2 appears the expected accumulated discounted profits (17), V^* and V^{**}, over the time horizon $T = 50$ years. One can see that the second policy implies a reduction in the expected profit of only 3.2% compared to the first policy. Assuming a null value for the depreciation rate, i.e., $\delta = 0$, implies a 3.6% expected profit reduction when comparing the expected accumulated undiscounted profits (18). The percent reductions are the same for the corresponding profits per year (19) and (20), obtained by taking time averages of these quantities over the 50 year horizon. The standard deviation values, which measure the variability across the simulated trajectories, are very similar for both policies, with the optimal sustainable policy having a slightly lower variability for the discounted profit and an opposite behaviour for the undiscounted profit.

The observed profit reductions that occur when considering a constant effort instead of a variable effort are quite small. Moreover, applying a constant effort policy, drives the fishery manager to maintain across time the same number of vessels, hooks or number of hours worked (just to name a few possibilities). Of course, this is extremely advantageous in terms of implementation, and avoids out-of-model costs such as the purchase of new equipment to sustain increased effort periods or payment of unemployment benefits during effort reduction periods.

Figure 1 shows what will happen when applying the optimal variable effort harvesting policy (left side) and the optimal constant effort sustainable policy (right side), in terms of the evolution, from time $t = 0$ to time $t = T = 50$ years, of the expected population sizes (top), optimal efforts (middle) and profits per unit time (bottom).

The thin lines of Fig. 1 show what the harvester would typically observe, i.e., one randomly chosen trajectory corresponding to a possible particular environmental behaviour. The thicker lines represent averages taken over all the simulated trajectories (the one effectively seen and all the others that might have occurred). The dashed lines on the right show the exact values at the steady-state for the population and profit given, respectively, by (16) and (15).

From the harvesters point of view (thin lines), the two policies behave quite differently. In fact, while the optimal sustainable effort E^{**} is constant across time (regardless of the population size), the optimal variable effort $E^*(t)$ changes quite frequently and abruptly, according to the population dynamics, having periods of null effort (meaning that the fishery is closed) and periods with maximum effort (which may involve extra out-of-model costs such as investment in backup equipment or hiring of extra employees not trained in fishing). This sudden and frequent changes in effort are not compatible with the fishing activity, since fishermen cannot accommodate frequent and abrupt changes on the number of vessels, number of gears, number of hours at the sea, among others. In addition, since the population size keeps varying, influenced by the random fluctuations of environmental conditions, a constant evaluation of its size is required.

Besides looking at the variation of the effort over time, it is also interesting to look at the time variability experienced by the harvester on the the profit per unit time. If we look at the thin lines at the bottom of Fig. 1 (corresponding to the environmental conditions randomly selected), we see that the optimal policy has frequent periods of zero profit (the periods of zero effort) and a much larger profit variability over time.

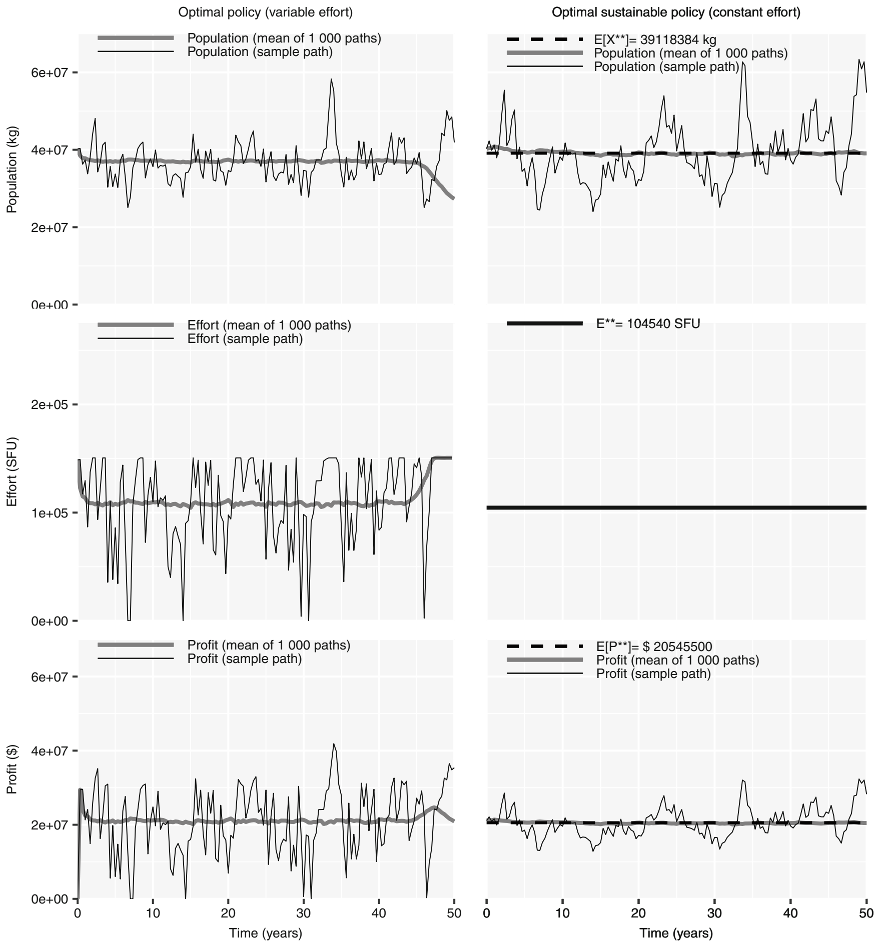

Fig. 1 Mean and randomly chosen sample path for the population, the effort and the profit per unit time. The optimal variable effort policy is on the left side and the optimal constant effort sustainable policy is on the right side

Another disadvantage of the optimal variable effort policy is the exhibition of a possibly dangerous effect near the time horizon, implying a considerable drop of the average population size (see solid line on top left), corresponding to an increase on the average effort (see solid line on middle left). This final effort increase is quite natural. Since “there is no tomorrow”, it is better profitwise to harvest as much as is profitable “now”, without worrying about stock preservation for near future fishing.

With the optimal sustainable policy, population size is driven to an equilibrium probability distribution with an average population size higher than the one of the variable effort policy. With the constant effort policy, there is no decay of the expected population size near the end of the time horizon.

So, the optimal policy leads to a highly variable effort, with occurrence of periods of zero effort and periods of harvesting at maximum effort rates, which imply frequent and abrupt changes on the number of vessels and gears, number of working hours and number of fishermen in activity, among others. Thus, the optimal effort policy is not applicable. We present now a sub-optimal policy, named stepwise policy, based on the variable effort obtained from the optimal policy, but where the effort is kept constant during periods of duration τ, say one or two years. We use $\tau = v\Delta t$ (v is a positive integer) to be a multiple of the time step, Δt, used in the numerical computations and in the Monte Carlo simulations. Therefore, in this stepwise policy, for time t in the period $[l\tau, (l+1)\tau[\;=\; [t_{lv}, t_{(l+1)v}[$, we keep the effort $E^*_{step}(t) = E^*(l\tau)$ constant and equal to the effort of the optimal policy at the beginning of the period. We are aware that this policy is not optimal nor stepwise optimal, but has however the advantage of being applicable, in contrast with the optimal policy.

We have focused the study on two scenarios: one with constant effort during periods of one year (annual), denoted by S_a scenario, and the other with constant effort during periods of two years (biennial), denoted by S_b scenario. For the optimal sustainable policy, the effort remains unchanged and it is constant for all time instants, as before.

For scenario S_a, we chose $\Delta t = 4$ months $= 4/12$ years and set the effort constant during periods of 1 year, i.e., during 3 consecutive time instants ($v = 3$). Similarly, for scenario S_b, we kept the effort constant during periods of 2 years, i.e., we set the effort constant during 6 time instants ($v = 6$). The case $v = 1$ corresponds to the discretization required to solve the HJB equation concerning the previous comparisons between the optimal effort policy and the optimal sustainable policy.

The first and second columns of Table 3 present, for each scenario, the resulting profit values (17)–(20) and their standard deviation values, respectively. The third column shows the percent relative difference between the policy based on stepwise effort and the optimal variable policy presented before (see values in Table 2). Similarly, the percent relative difference between the policy based on stepwise effort and the optimal constant policy (see values in Table 2) is shown in the last column.

From Table 3 one can see that, for the scenario S_a, the stepwise and applicable policy gives only slightly lower profit values when compared with the inapplicable variable effort policy (-1.0% and -0.8%, respectively for discounted and undiscounted profits). However, comparing the stepwise policy with the (also applicable) optimal sustainable policy increases the profit values ($+2.2\%$ and $+2.8\%$, respectively for discounted and undiscounted profits). When the stepwise

Table 3 Expected discounted and undiscounted profit values for the stepwise scenarios S_a (annual periods) and S_b (biennial). Besides the expected values, we also present the standard deviations. The percent relative difference between the stepwise policy and the variable effort policy is denoted by Δ^* and the percent relative difference between the stepwise policy and the constant effort policy is denoted by Δ^{**}. Currency values are in million dollars for V^*_{step} and $V^*_{step,u}$, and million dollars per year for $\overline{P^*_{step}}$ and $\overline{P^*_{step,u}}$

	V^*_{step}	sd	$\Delta^*(\%)$	$\Delta^{**}(\%)$
S_a	387.215	34.687	−1.0	+2.2
S_b	376.844	35.255	−3.6	−0.4
	$V^*_{step,u}$	sd	$\Delta^*(\%)$	$\Delta^{**}(\%)$
S_a	1055.103	81.068	−0.8	+2.8
S_b	1029.606	82.899	−3.2	+0.4
	$\overline{P^*_{step}}$	sd	$\Delta^*(\%)$	$\Delta^{**}(\%)$
S_a	21.092	1.889	−1.0	+2.2
S_b	20.527	1.920	−3.6	−0.4
	$\overline{P^*_{step,u}}$	sd	$\Delta^*(\%)$	$\Delta^{**}(\%)$
S_a	21.102	1.621	−0.8	+2.8
S_b	20.592	1.658	−3.6	−0.4

effort is applied during a longer biennial periods (scenario S_b), the profit differences with the optimal effort policy are higher than in the S_a scenario, resulting in profit reductions of -3.6% and -3.2%, respectively for discounted and undiscounted profits. On the contrary, applying the stepwise effort policy instead of the optimal sustainable policy will reduce the profit in -0.4%. In summary, we can conclude that, choosing the applicable policy with stepwise effort causes slight profit losses in comparison with the inapplicable variable effort policy and can be, sometimes, even more profitable than the constant effort policy. The comparison of policies in terms of the profits per year gives differences similar to the accumulated profit differences, since the profits per year are proportional to the accumulated profits.

Figure 2 shows the mean and the randomly chosen sample path for the population, the effort and the profit per unit time for both stepwise policies, S_a on the left and S_b on the right. For both scenarios we can see some periods where the population and profit sample paths variability increases in relation to the variable effort policy (see left side of Fig. 1). The increase in variability is more pronounced when we compare the stepwise policy with the constant effort policy (see right side of Fig. 1). Looking at the thick lines of Figs. 1 and 2, corresponding to the mean of the 1000 sample paths, we notice a similar behaviour in terms of variability. At the center part of Fig. 2 one can check the stepwise effort, for both the sample path and the mean of all simulated paths. Their depicted lines in a form of staircase lend the name to the policy: stepwise policy.

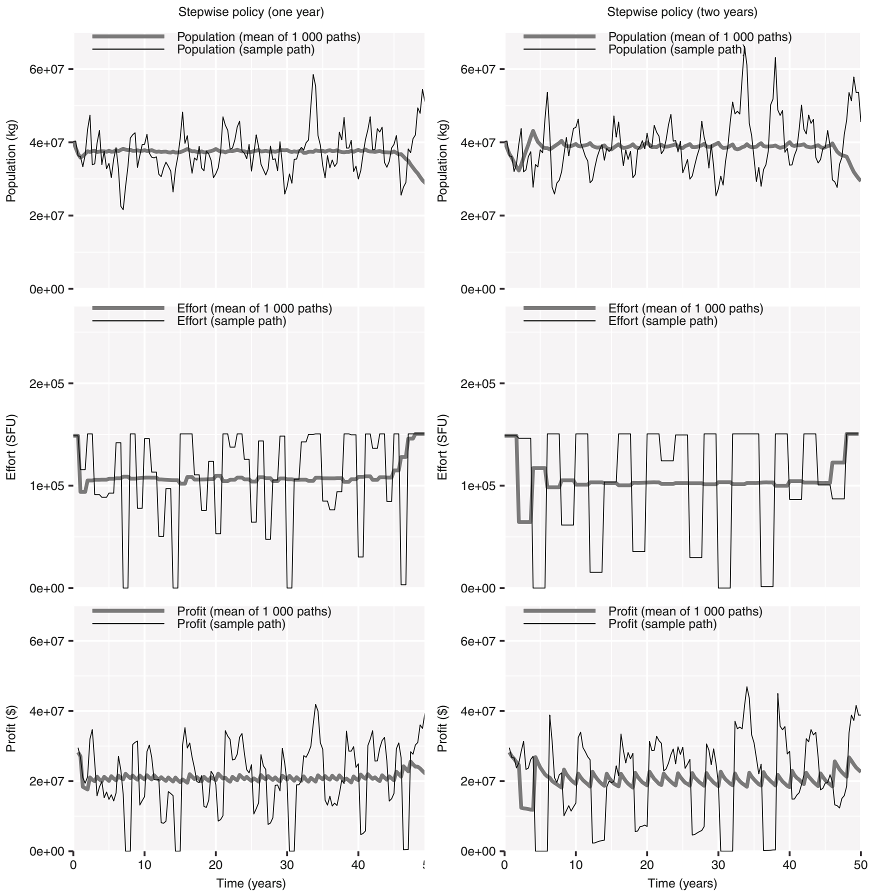

Fig. 2 Mean and randomly chosen sample path for the population, the effort and the profit per unit time. The stepwise policy (with one year steps, S_a) is on the left side and the stepwise policy (with two years steps, S_b) is on the right side

5 Conclusions

In this work we have presented numerical comparisons between the optimal policy with variable effort, the suboptimal policy with stepwise effort and the optimal sustainable policy with constant effort. The comparisons were realized in terms of four profit quantities: the expected accumulated discounted profit in a finite time interval, the expected accumulated undiscounted profit in a finite time interval, the average expected profit per unit time weighted by the discount factors and the unweighted average expected profit per unit time.

To obtain the profit values we have performed 1000 Monte Carlo simulations using a Crank-Nicolson discretization scheme in time and space of the HJB equation and an Euler scheme for the population paths. To compute the simulations we have applied the logistic model to realistic data with parameters from the Pacific halibut (*Hippoglossus hippoglossus*).

The profit differences between the two optimal policies are quite small. Also, we have seen that the optimal policy has frequent strong changes in effort, including periods of null effort, posing serious logistic applicability problems, producing social burdens and out-of-model costs (such as unemployment compensations) and leading to a great instability in the profit earned by the harvester. Furthermore, unlike the optimal variable effort policy, in the optimal constant effort policy there is no need to keep adjusting the effort to the randomly varying population size, and so there is no need to determine the size of the population at all times. This is a great advantage, since the estimation of the population size is a difficult, costly, time consuming and inaccurate task. The optimal policy also can create a possibly dangerous effect near the time horizon implying, on average, a considerable drop on the population size. On the contrary, the optimal sustainable policy does not have these shortcomings, is very easy to implement and drives the population to a stochastic equilibrium.

Since the optimal policy in not applicable, we have presented sub-optimal policies, named stepwise policies, based on variable effort but with periods of constant effort. These policies are not optimal, but have the advantage of being applicable, since the changes on effort are not so frequent and can be compatible with the fishing activity. Furthermore, although we still need to keep estimating the fish stock size, we do not need to do it so often. Replacing the optimal variable effort policy by these stepwise policies has the advantage of applicability but, at best, considerably reduces the already small advantages they have over the optimal sustainable policy. In some cases, the much easier to implement optimal constant effort policy even outperforms these stepwise policies in terms of profit. The stepwise policies share with the optimal variable effort policy the disadvantage of having periods of null or low fishing and periods of fishing at the highest rate, with the corresponding social implications and out-of-models costs.

Similar work on Gompertz and other population growth models and other population data is under way.

Acknowledgements The helpful and valuable comments of an anonymous referee are gratefully acknowledged. Nuno M. Brites was partially supported by the Project CEMAPRE/REM—UIDB/05069/2020—financed by FCT/MCTES through national funds. Both authors are members of the Centro de Investigação em Matemática e Aplicações, Universidade de Évora, funded by National Funds through Fundação para a Ciência e a Tecnologia, under the project UID/04674/2020.

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