A FOURTH-ORDER BVP OF STURM-LIOUVILLE TYPE WITH ASYMMETRIC UNBOUNDED NONLINEARITIES

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It is obtained an existence and location result for the fourth-order boundary value problem of Sturm-Liouville type $u^{(iv)}(t) = f(t,u(t),u'(t),u''(t),u'''(t))$ for $t \in [0,1]$; $u(0) = u(1) = A$; $k_1u'''(0) - k_2u''(0) = 0$; $k_3u'''(1) + k_4u''(1) = 0$, where $f : [0,1] \times R^4 \to R$ is a continuous function and $A, k_i \in R$, for $1 \le i \le 4$, are such that $k_1, k_3 > 0$, $k_2, k_4 \ge 0$. We assume that f verifies a one-sided Nagumo-type growth condition which allows an asymmetric unbounded behavior on the nonlinearity. The arguments make use of an a priori estimate on the third derivative of a class of solutions, the lower and upper solutions method and degree theory.

1. Introduction

In this paper it is considered the fourth-order fully nonlinear differential equation

$$u^{(iv)}(t) = f\left(t,u(t),u'(t),u''(t),u'''(t)\right) \quad \text{for } t \in I = [0,1], \tag{1.1}$$

with the Sturm-Liouville boundary conditions

$$\begin{gathered} u(0) = u(1) = A, \\ k_1u'''(0) - k_2u''(0) = 0, \qquad k_3u'''(1) + k_4u''(1) = 0, \end{gathered} \tag{1.2}$$

where $A, k_1, k_2, k_3, k_4 \in R$ are such that $k_1, k_3 > 0$, $k_2, k_4 \ge 0$, and $f : [a,b] \times R^4 \to R$ is a continuous function verifying one-sided Nagumo-type growth assumption.

This problem generalizes the classical beam equation and models the study of the bending of an elastic beam simply supported [8, 9, 11].

As far as we know it is the first time, in fourth-order problems, that the nonlinearity f is assumed to satisfy a growth condition from above but no restriction from below. This asymmetric type of unboundedness is allowed since f verifies one-sided Nagumo-type

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condition, that is, there exists a positive continuous function φ such that

$$f(t,x_0,x_1,x_2,x_3) \le \varphi(|x_3|), \quad \forall (t,x_0,x_1,x_2,x_3) \in E, \tag{1.3}$$

on some given subset $E \subset I \times R^4$, and $\int_0^{+\infty}(s/\varphi(s))ds = +\infty$.

Some boundedness of Nagumo-type plays a key role in these results because, as it is known for second-order boundary value problems, the existence of well-ordered lower and upper solutions, by itself, is not sufficient to ensure the existence of solutions (see [10, 15]).

When a one-sided Nagumo-type condition is assumed, the situation becomes more delicate since this condition does not provide a priori estimates for the third-order derivative of all solutions of (1.1) which is usually the key point for studying this sort of problem, as it can be seen in [2, 3, 13, 14].

However, it is still possible to establish a priori bounds for classes S_η of solutions of (1.1) (see Lemma 2.2). More precisely, if we define for $\eta \ge 0$

$$S_\eta = \{u \text{ solution of } (1.1) : u'''(0) \le \eta,\ u'''(1) \ge -\eta\}, \tag{1.4}$$

we prove that there is $r > 0$ such that if $u \in S_\eta$, then it satisfies $\|u'''\|_\infty < r$.

The existence and location of a solution for problem (1.1)-(1.2) (see Theorem 3.1) are established by using the method of lower and upper solutions to obtain a priori estimations on a class of solution and some derivatives, which allow us to define an open set where the topological degree is well defined [12].

This kind of arguments was suggested by [1] for second-order boundary value problems and by [4–7] for higher-order separated boundary value problems.

2. Preliminaries

In this section we will introduce the main concepts that we will use throughout this paper. Given $y,z \in C(I)$ such that $y \le z$ in I, we denote

$$[y,z] := \{x \in C(I) : y(t) \le x(t) \le z(t),\ \forall t \in I\}. \tag{2.1}$$

In order to obtain an a priori bound for the third-order derivative $u'''(t)$ of a class of solutions of problem (1.1)-(1.2), we will introduce the concept of one-sided Nagumo-type growth condition.

Definition 2.1. Given a subset $E \subset I \times R^4$, a function $f : I \times R^4 \to R$ is said to satisfy a one-sided Nagumo-type condition in E if there exists, for some $a > 0$, $\varphi \in C(R_0^+, [a,+\infty))$ such that

$$f(t,x_0,x_1,x_2,x_3) \le \varphi(|x_3|), \quad \forall (t,x_0,x_1,x_2,x_3) \in E, \tag{2.2}$$

with

$$\int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty. \tag{2.3}$$

This asymmetric growth condition will be an important tool in the proof of next lemma.

Lemma 2.2. *Consider, for $i = 0,1,2$, the functions $\gamma_i, \Gamma_i \in C(I,R)$ such that $\gamma_i(t) \le \Gamma_i(t)$, for all $t \in I$, and define the set*

$$E = \{(t,x_0,x_1,x_2,x_3) \in I \times R^4 : \gamma_i(t) \le x_i \le \Gamma_i(t),\ i = 0,1,2\}. \tag{2.4}$$

Let $\varphi : R_0^+ \to [a,+\infty)$, for some $a > 0$, be a continuous function such that

$$\int_\eta^{+\infty} \frac{s}{\varphi(s)} ds > \max_{t\in I} \Gamma_2(t) - \min_{t\in I} \gamma_1(t), \tag{2.5}$$

where $\eta \ge 0$ is given by $\eta = \max\{\Gamma_2(0) - \gamma_2(1), \Gamma_2(1) - \gamma_2(0)\}$.

Then there is $r > 0$ (depending only on φ, γ_2, and Γ_2), such that, for every continuous function $f : I \times R^4 \to R$ satisfying one-sided Nagumo-type condition and every solution $u(t)$ of (1.1) verifying

$$u'''(0) \le \eta, \quad u'''(1) \ge -\eta, \tag{2.6}$$

$$u^{(i)}(t) \in [\gamma_i, \Gamma_i] \quad \text{for } i = 0,1,2,\ \forall t \in I, \tag{2.7}$$

satisfies

$$\|u'''\|_\infty < r. \tag{2.8}$$

Proof. The proof follows the arguments used in [7] and the technique suggested in [13] for fourth-order boundary value problems. □

This lemma still holds if condition (2.2) is replaced by

$$f(t,x_0,x_1,x_2,x_3) \ge -\varphi(|x_3|), \quad \forall (t,x_0,x_1,x_2,x_3) \in E, \tag{2.9}$$

and (2.7) by $u'''(0) \ge -\eta$, $u'''(1) \le \eta$.

Lower and upper solutions for problem (1.1)-(1.2) must be defined as a pair of functions, in the following way.

Definition 2.3. Consider $A, k_i \in R$, for $1 \le i \le 4$, such that $k_1, k_3 > 0$ and $k_2, k_4 \ge 0$. The functions $\alpha, \beta \in C^4(I)$ satisfying

$$\alpha(t) \le \beta(t), \quad \alpha'(t) \ge \beta'(t), \quad \alpha''(t) \le \beta''(t), \quad \forall t \in I, \tag{2.10}$$

define a pair of lower and upper solutions of problem (1.1)-(1.2) if the following conditions are verified:

(i) $\alpha^{(iv)}(t) \ge f(t,\alpha(t),\alpha'(t),\alpha''(t),\alpha'''(t))$,

$$\alpha(1) \le A, \quad k_1\alpha'''(0) - k_2\alpha''(0) \ge 0, \quad k_3\alpha'''(1) + k_4\alpha''(1) \le 0; \tag{2.11}$$

(ii) $\beta^{(iv)}(t) \le f(t,\beta(t),\beta'(t),\beta''(t),\beta'''(t))$,

$$\beta(1) \ge A, \quad k_1\beta'''(0) - k_2\beta''(0) \le 0, \quad k_3\beta'''(1) + k_4\beta''(1) \ge 0; \tag{2.12}$$

(iii) $\alpha'(1) - \beta'(1) \ge \max\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\}$.

Remark 2.4. (a) Condition (iii) is optimal and cannot be removed, as it will be proved forward (see counterexample).

(b) If the maximum refereed in (iii) is nonnegative, that is,

$$\alpha'(1)-\beta'(1)\geq \max\{\beta(0)-\beta(1),\alpha(1)-\alpha(0),0\}, \tag{2.13}$$

then assumption (2.10) can be replaced by $\alpha''(t)\leq\beta''(t)$ in I, since the other inequalities can be deduced by integration.

3. Existence and location results

The main result of this work is the following existence and location theorem.

Theorem 3.1. *Assume that there exists a pair of lower and upper solutions of problem (1.1)-(1.2), $\alpha(t)$ and $\beta(t)$, respectively. Consider the set*

$$E_1=\left\{\begin{matrix}(t,x_0,x_1,x_2,x_3)\in I\times R^4:\alpha(t)\leq x_0\leq\beta(t),\\ \alpha'(t)\geq x_1\geq\beta'(t),\ \alpha''(t)\leq x_2\leq\beta''(t)\end{matrix}\right\}, \tag{3.1}$$

and let $f:I\times R^4\to R$ be a continuous function such that:

(a) *f satisfies the one-sided Nagumo-type condition in E_1;*

(b) *for $(t,x_2,x_3)\in I\times R^2$, $\alpha(t)\leq x_0\leq\beta(t)$ and $\alpha'(t)\geq x_1\geq\beta'(t)$*

$$f(t,\alpha,\alpha',x_2,x_3)\geq f(t,x_0,x_1,x_2,x_3)\geq f(t,\beta,\beta',x_2,x_3). \tag{3.2}$$

Then problem (1.1)-(1.2) has at least one solution $u(t)\in C^4(I)$ that satisfies

$$u\in[\alpha,\beta],\quad u'\in[\beta',\alpha'],\quad u''\in[\alpha'',\beta''],\quad \forall t\in I. \tag{3.3}$$

Proof. For $\lambda\in[0,1]$, consider the homotopic equation

$$u^{(iv)}(t)=\lambda f(t,\xi_0(t,u(t)),\xi_1(t,u'(t)),\xi_2(t,u''(t)),u'''(t))+u''(t)-\lambda\xi_2(t,u''(t)), \tag{3.4}$$

where $\xi_i:I\times R\to R$ are the auxiliary continuous functions defined by

$$\begin{aligned}\xi_i(t,x_i)&=\max\{\alpha^{(i)}(t),\min\{x_i,\beta^{(i)}(t)\}\}\quad\text{for } i=0,2,\\ \xi_1(t,x_1)&=\max\{\beta'(t),\min\{x_1,\alpha'(t)\}\}\end{aligned} \tag{3.5}$$

with the boundary conditions

$$\begin{gathered}u(0)=u(1)=\lambda A,\\ u'''(0)=\lambda\left(\frac{k_2}{k_1}\right)u''(0),\qquad u'''(0)=-\lambda\left(\frac{k_4}{k_3}\right)u''(1).\end{gathered} \tag{3.6}$$

Take $r_1 > 0$ large enough such that, for every $t \in I$,

$$-r_1 < \alpha''(t) \le \beta''(t) < r_1,$$
$$f(t,\alpha(t),\alpha'(t),\alpha''(t),0) - r_1 - \alpha''(t) < 0, \tag{3.7}$$
$$f(t,\beta(t),\beta'(t),\beta''(t),0) + r_1 - \beta''(t) > 0. \tag{3.8}$$

The proof is deduced from the following four steps.

Step 1. Every solution $u(t)$ of problem (3.4)-(3.6) satisfies $|u^{(i)}(t)| < r_1$, for every $t \in I$ and $i = 0,1,2$, independently of $\lambda \in [0,1]$.

Assume, by contradiction, that the above estimate does not hold for $i = 2$. So, for $\lambda \in [0,1]$, there exist $t \in I$ and a solution u of (3.4)-(3.6) such that $|u''(t)| \ge r_1$. In the case $u''(t) \ge r_1$ define

$$u''(t_0) := \max_{t\in I} u''(t) \ge r_1. \tag{3.9}$$

If $t_0 \in (0,1)$, then $u'''(t_0) = 0$ and $u^{(iv)}(t_0) \le 0$. For $\lambda \in [0,1]$, by (3.2) and (3.8), the following contradiction is obtained:

$$\begin{aligned} 0 &\ge u^{(iv)}(t_0) \\ &\ge \lambda f(t_0,\beta(t_0),\beta'(t_0),\beta''(t_0),0) + u''(t_0) - \lambda\beta''(t_0) \\ &= \lambda[f(t_0,\beta(t_0),\beta'(t_0),\beta''(t_0),0) + r_1 - \beta''(t_0)] + u''(t_0) - \lambda r_1 > 0. \end{aligned} \tag{3.10}$$

So $t_0 \notin (0,1)$. If $t_0 = 0$, for $\lambda \in [0,1]$, we obtain, by (3.6),

$$0 \ge u'''(0) = \lambda\left(\frac{k_2}{k_1}\right)u''(0) \ge \lambda\left(\frac{k_2}{k_1}\right)r_1 \ge 0. \tag{3.11}$$

Thus $u'''(0) = 0$ and $u^{(iv)}(0) \le 0$. Replacing in the above computations t_0 by 0, it can be proved that $t_0 \ne 0$. For $t_0 = 1$ the technique is similar and so $u''(t) < r_1$, for every $t \in I$. The case $u''(t) \le -r_1$ follows analogous arguments and then $|u''(t)| < r_1$, for all $t \in I$.

By (3.4), there exists $\xi \in (0,1)$ such that $u'(\xi) = 0$. Then, integrating on $[\xi,t]$ first and then on $[0,t]$, we obtain

$$|u'(t)| = \left|\int_\xi^t u''(s)ds\right| < r_1|t-\xi| \le r_1, \qquad |u(t)| = \left|\int_0^t u'(s)ds\right| < r_1 t \le r_1. \tag{3.12}$$

Step 2. There is $r_2 > 0$ such that, for every solution $u(t)$ of problem (3.4)-(3.6), $|u'''(t)| < r_2$ in I, independently of $\lambda \in [0,1]$.

Consider the set

$$E_{r_1} = \{(t,x_0,x_1,x_2,x_3) \in I \times R^4 : -r_1 \le x_i \le r_1,\ i = 0,1,2\}, \tag{3.13}$$

and, for $\lambda \in [0,1]$, the function $F_\lambda : E_{r_1} \to R$ is given by

$$F_\lambda(t,x_0,x_1,x_2,x_3) = \lambda f(t,\xi_0(t,x_0),\xi_1(t,x_1),\xi_2(t,x_2),x_3) + x_2 - \lambda\xi_2(t,x_2). \tag{3.14}$$

As

$$F_\lambda(t,x_0,x_1,x_2,x_3) \le \lambda\varphi(|x_3|) + r_1 - \lambda\alpha''(t) \le \varphi(|x_3|) + 2r_1, \tag{3.15}$$

then F_λ satisfies one-sided Nagumo-type condition in E_{r_1} with φ replaced by $\overline{\varphi} := 2r_1 + \varphi(t)$, independently of $\lambda \in [0,1]$. By (3.6) and Step 1, we have

$$\begin{aligned} u'''(0) &= \lambda\left(\frac{k_2}{k_1}\right)u''(0) \le \lambda\left(\frac{k_2}{k_1}\right)r_1 \le \left(\frac{k_2}{k_1}\right)r_1 \le \rho, \\ u'''(1) &= -\lambda\left(\frac{k_4}{k_3}\right)u''(1) \ge -\lambda\left(\frac{k_4}{k_3}\right)r_1 \ge -\left(\frac{k_4}{k_3}\right)r_1 \ge -\rho. \end{aligned} \tag{3.16}$$

So, applying Lemma 2.2 with $\gamma_i(t) \equiv -r_1$, $\Gamma_i(t) \equiv r_1$, for $i = 0,1,2$, and

$$\rho := \max\left\{\left(\frac{k_2}{k_1}\right)r_1, \left(\frac{k_4}{k_3}\right)r_1\right\}, \tag{3.17}$$

there is $r_2 > 0$ such that $|u'''(t)| < r_2$, for all $t \in I$. As r_1 and φ do not depend on λ, then r_2 is independent of λ.

Step 3. For $\lambda = 1$, problem (3.4)-(3.6) has at least a solution $u_1(t)$.

Define the operators $\mathcal{L} : C^4(I) \subset C^3(I) \to C(I) \times R^4$ by

$$\mathcal{L}u = (u^{(iv)} - u''(t), u(0), u(1), u''(0), u''(1)) \tag{3.18}$$

and, for $\lambda \in [0,1]$, $\mathcal{N}_\lambda : C^3(I) \to C(I) \times R^4$ by

$$\begin{aligned} \mathcal{N}_\lambda u = \Big(&\lambda f(t,\xi_0(t,u(t)),\xi_1(t,u'(t)),\xi_2(t,u''(t)),u'''(t)) \\ &- \lambda\xi_2(t,u''(t)), \lambda A, \lambda A, \lambda\left(\frac{k_2}{k_1}\right)u''(0), -\lambda\left(\frac{k_4}{k_3}\right)u''(1)\Big). \end{aligned} \tag{3.19}$$

As $\mathcal{L}$ has a compact inverse, we can define the completely continuous operator $\mathcal{T}_\lambda : (C^3(I),R) \to (C^3(I),R)$ by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u). \tag{3.20}$$

For r_2 given by Step 2, consider the set

$$\Omega = \left\{x \in C^3(I) : \|x^{(i)}\|_\infty < r_1,\ i = 0,1,2,\ \|x'''\|_\infty < r_2\right\}. \tag{3.21}$$

By Steps 1 and 2, for every u solution of (3.4)–(3.6), $u \notin \partial\Omega$ and so the degree $d(I - \mathcal{T}_\lambda,\Omega,0)$ is well defined, for every $\lambda \in [0,1]$. By the invariance under homotopy,

$$d(I - \mathcal{T}_0,\Omega,0) = d(I - \mathcal{T}_1,\Omega,0). \tag{3.22}$$

Since the equation $x = \mathcal{T}_0(x)$, equivalent to the problem

$$\begin{gathered} u^{(iv)}(t) - u''(t) = 0, \\ u(0) = u(1) = u'''(0) = u'''(1) = 0, \end{gathered} \tag{3.23}$$

has only the trivial solution, then $d(I-\mathcal{T}_0,\Omega,0)=\pm 1$. Therefore, by degree theory, the equation $x=\mathcal{T}_1(x)$ has at least one solution. That is, the problem

$$u^{(iv)}(t)=f\left(t,\xi_0(t,u),\xi_1(t,u'),\xi_2(t,u''),u'''(t)\right)+u''(t)-\xi_2(t,u'') \tag{3.24}$$

with the boundary condition (1.2) has at least one solution $u_1(t)$ in Ω.

Step 4. The function $u_1(t)$ is a solution of problem (1.1)-(1.2).

We remark that this statement holds if $u_1(t)$ verifies (3.3). Assume, by contradiction, that there is $t\in I$ such that $u_1''(t)>\beta''(t)$ and define

$$(u_1-\beta)''(t_1):=\max_{t\in I}\left\{(u_1-\beta)''(t)\right\}>0. \tag{3.25}$$

If $t_1\in(0,1)$, then $u_1'''(t_1)=\beta'''(t_1)$ and $u_1^{(iv)}(t_1)\le\beta^{(iv)}(t_1)$. By (b) and (ii), the following contradiction is achieved:

$$\begin{aligned}u_1^{(iv)}(t_1)&\ge f\left(t_1,\beta(t_1),\beta'(t_1),\beta''(t_1),\beta'''(t_1)\right)+u_1''(t_1)-\beta''(t_1)\\&>f\left(t_1,\beta(t_1),\beta'(t_1),\beta''(t_1),\beta'''(t_1)\right)\ge\beta^{(iv)}(t_1).\end{aligned} \tag{3.26}$$

If $t_1=0$, then $(u_1-\beta)'''(0)\le 0$ so, by (3.6) and Definition 2.1,

$$0\ge u_1'''(0)-\beta'''(0)=\frac{\left[k_2u_1''(0)-k_1\beta'''(0)\right]}{k_1}\ge\left(\frac{k_2}{k_1}\right)\left[u_1''(0)-\beta''(0)\right]\ge 0. \tag{3.27}$$

Thus $u_1'''(0)=\beta'''(0)$ and $u_1^{(iv)}(0)\le\beta^{(iv)}(0)$. Therefore, replacing in the above inequality t_1 by 0 a contradiction is obtained. By similar arguments it can be proved that $t_1\neq 1$ and so $u_1''(t)\le\beta''(t)$, for every $t\in I$. Using an analogous technique, we prove that $\alpha''(t)\le u_1''(t)$, for all $t\in I$. So $u_1''\in[\alpha'',\beta'']$. Then, by integration and (iii), we have

$$\beta'(1)\le\alpha(0)-\alpha(1)+\alpha'(1)=\int_0^1\int_t^1\alpha''(s)\,ds\,dt\le\int_0^1\int_t^1 u_1''(s)\,ds\,dt=u_1'(1). \tag{3.28}$$

As $(\beta-u_1)'(t)$ is nondecreasing, then $\beta'(t)-u_1'(t)\le\beta'(1)-u_1'(1)\le 0$, for every $t\in I$. By the monotony of $(\beta-u_1)(t)$ and (ii), we have $0\le\beta(1)-u_1(1)\le\beta(t)-u_1(t)$, for all $t\in I$. The inequalities $u_1'(t)\le\alpha'(t)$ and $u_1(t)\ge\alpha(t)$, for all $t\in I$, can be deduced in a similar way. □

If f satisfies the reversed one-sided Nagumo-type condition (2.2), then Theorem 3.1 still holds.

Moreover, if in Definition 2.3 we consider the following new assumptions:

$$\alpha(t)\le\beta(t),\quad \alpha'(t)\le\beta'(t),\quad \alpha''(t)\le\beta''(t),\quad \forall t\in I, \tag{3.29}$$

the initial value inequalities $\alpha(0)\le A$, $\beta(0)\ge A$, and

(iii′) $\alpha'(0)-\beta'(0)\le\min\{\beta(0)-\beta(1),\alpha(1)-\alpha(0)\}$,
then Theorem 3.1 remains true for

$$E_2=\left\{\begin{array}{c}(t,x_0,x_1,x_2,x_3)\in I\times R^4:\alpha(t)\le x_0\le\beta(t),\\ \alpha'(t)\le x_1\le\beta'(t),\ \alpha''(t)\le x_2\le\beta''(t)\end{array}\right\},\tag{3.30}$$

and f verifying

$$f(t,\alpha(t),\alpha'(t),x_2,x_3)\ge f(t,x_0,x_1,x_2,x_3)\ge f(t,\beta(t),\beta'(t),x_2,x_3)\tag{3.31}$$

for $(t,x_2,x_3)\in I\times R^2$, $\alpha(t)\le x_0\le\beta(t)$, $\alpha'(t)\le x_1\le\beta'(t)$.

4. Example and counterexample

Next example shows the applicability and improvement given by Theorem 3.1, since the nonlinearity considered does not satisfy the usual two-sided Nagumo condition.

Example 4.1. Consider the fully fourth-order differential equation

$$u^{(iv)}(t)=8-e^{u(t)}+[u'(t)-4][2-u''(t)]^2-|u'''(t)|^\theta,\quad t\in I,\tag{4.1}$$

where $\theta>2$, with the boundary conditions of Sturm-Liouville type

$$u(0)=u(1)=0,\qquad u'''(0)-2u''(0)=0,\qquad u'''(1)+u''(1)=0.\tag{4.2}$$

It is easy to see that the continuous functions $\alpha,\beta:I\to R$ given by

$$\alpha(t)=-t^2+3t-2,\qquad \beta(t)=t^2-3t+2\tag{4.3}$$

define a pair of lower and upper solutions for problem (4.1)-(4.2). On

$$E=\left\{\begin{array}{c}(t,x_0,x_1,x_2,x_3)\in I\times R^4:-t^2+3t-2\le x_0\le t^2-3t+2,\\ 3-2t\ge x_1\ge 2t-3,\ -2\le x_2\le 2\end{array}\right\},\tag{4.4}$$

the continuous function $f:E\to R$ given by

$$f(t,x_0,x_1,x_2,x_3)=8-e^{x_0}+(x_1-4)(2-x_2)^2-|x_3|^\theta,\tag{4.5}$$

verifies (3.2) and the one-sided Nagumo-type condition with $\varphi(x_3)\equiv 8-e^{-2}$.

Therefore, by Theorem 3.1, there is at least a solution $u(t)$ of problem (4.1)-(4.2) such that, for every $t\in I$,

$$-t^2+3t-2\le u(t)\le t^2-3t+2,\qquad 3-2t\ge u'(t)\ge 2t-3,\qquad -2\le u''(t)\le 2.\tag{4.6}$$

Notice that the nonlinearity f given by (4.5) does not verify the two-sided Nagumo-type condition. In fact, assume, by contradiction, that there is a positive continuous function φ verifying (2.3) and such that

$$|f(t,x_0,x_1,x_2,x_3)|\le\varphi(|x_3|),\quad\forall(t,x_0,x_1,x_2,x_3)\in E.\tag{4.7}$$

In particular, $-f(t,x_0,x_1,x_2,x_3) \leq \varphi(|x_3|)$, for every $(t,x_0,x_1,x_2,x_3) \in E$, and so, for $t \in [0,1]$, $x_0 = 2$, $x_1 = 2$, $x_2 = 0$, and $x_3 \in R$, we have

$$-f(t,2,2,0,x_3) = e^2 + |x_3|^\theta \leq \varphi(|x_3|). \tag{4.8}$$

As $\int_0^{+\infty}(s/(e^2+s^\theta))ds$, with $\theta > 2$, is finite, then we have the following contradiction:

$$+\infty > \int_0^{+\infty} \frac{s}{e^2+s^\theta} ds \geq \int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty. \tag{4.9}$$

Counterexample 4.2. We will show that assumption (iii) in Definition 2.3 cannot be removed. In fact, considering the fourth-order boundary value problem

$$\begin{gathered} u^{(iv)}(t) = -2u'''(t) + 3u''(t), \\ u(0) = u(1) = 0, \\ u'''(0) - u''(0) = 0, \qquad u'''(1) + 3u''(1) = 0, \end{gathered} \tag{4.10}$$

the functions $\alpha(t) = -(t-1)(3t-1)/3$, $\beta(t) = (1-t)(4-t)/3$ are lower and upper solutions of problem (4.10) but condition (iii) does not hold. As (4.10) has only the trivial solution $u(t) \equiv 0$, then condition (3.3) is not satisfied. In fact, $0 \equiv u(t) < \alpha(t) < \beta(t)$, for $t \in]1/3,1[$, and $0 \equiv u'(t) > \alpha'(t) > \beta'(t)$, for $t \in]2/3,1[$.

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