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EXISTENCE AND LOCATION RESULT FOR A FOURTH ORDER BOUNDARY VALUE PROBLEM

F. Minhós[*,a], T. Gyulov[b] and A. I. Santos[a]

[a] Departamento de Matemática. Universidade de Évora
Centro de Investigação em Matemática e Aplicações da U.E. (CIMA-UE)
Rua Romão Ramalho, 59. 7000-671 Évora. PORTUGAL

[b] C.A.M.I., University of Rousse
8, " Studenska" Str., 7017 Rousse. BULGARIA

Abstract. In the present work we prove an existence and location result for the fourth order fully nonlinear equation

$$u^{(iv)} = f\left(t, u, u', u'', u'''\right), \quad 0 < t < 1,$$

with the Lidstone boundary conditions

$$u\left(0\right) = u''\left(0\right) = u\left(1\right) = u''\left(1\right) = 0,$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function satisfying a Nagumo type condition. The existence of at least a solution lying between a pair of well ordered lower and upper solutions is obtained using an *a priori* estimate, lower and upper solutions method and degree theory.

1. **Introduction.** In this work we apply the lower and upper solutions method to the fourth order fully nonlinear equation

$$u^{(iv)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t)\right), \quad 0 < t < 1, \tag{1}$$

with $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ a continuous function.

This equation can be used to model the deformations of an elastic beam and the type of boundary conditions considered depends on how the beam is supported at the two endpoints (see [10, 11] and the references therein). We consider the Lidstone boundary conditions

$$u\left(0\right) = u''\left(0\right) = u\left(1\right) = u''\left(1\right) = 0, \tag{2}$$

meaning that both endpoints are simply supported.

This problem has been studied by many authors using the variational formulation, in the cases where the nonlinearity depends only on u or u'' ([9, 10, 12]), the topological technique ([1, 2, 14]) or both ([4]). However, in all the referred papers there are no dependence on the odd-order derivatives. Recently several papers ([5, 6]) apply the lower and upper solutions method to the fully equation (1) with nonlinear boundary conditions.

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The present work follows the arguments used in [3] for second order, in [7] and [8] for third order and higher order and it is not cover by the results in [6] because we deal with a weaker assumption than the monotonicity of f (see (7)) and different definitions of lower and upper solutions. In fact we consider a reversed relation between α'' and β'' and the nonlinear boundary conditions for the lower and upper functions do not include our definition if they verify the relation

$$\alpha(0) < 0 < \beta(0) \quad \text{or} \quad \alpha'(1) < 0 < \beta'(1).$$

The paper is organized like it follows. In section 2 it is defined the well ordered lower and upper solutions, i.e., $\alpha(t) \leq \beta(t)$, for every $t \in [0,1]$. On the other hand, lower and upper solutions must be defined as a pair, that is, they can not be defined independently (see Definition 1, *(iii)*, and the Counter-example in last section).

A Nagumo-type condition will have an important role to prove an *a priori* bound for the third derivative and to obtain the existence result, in section 3, using the Leray-Schauder theory. The location part, inherent to the lower and upper solutions technique, may be useful, for instance, to prove the existence of positive solutions, indeed, it will be enough to consider a non-negative lower solution.

2. **Definitions and *a priori* bound.** The lower and upper solutions for problem (1)-(2) used in this paper require a "well ordered" relation in the second derivatives and a definition like a pair of functions.

Definition 1. The functions $\alpha, \beta \in C^4(]0,1[) \cap C^3([0,1])$ verifying

$$\alpha''(t) \leq \beta''(t), \quad \forall t \in [0,1], \tag{3}$$

define a pair of lower and upper solutions of problem (1)-(2) if the following conditions are satisfied:

(i) $\quad \alpha^{(iv)}(t) \geq f(t, \alpha(t), \alpha'(t), \alpha''(t), \alpha'''(t)),$

$\quad \beta^{(iv)}(t) \leq f(t, \beta(t), \beta'(t), \beta''(t), \beta'''(t));$

(ii) $\quad \alpha(0) \leq 0, \quad \alpha''(0) \leq 0, \quad \alpha''(1) \leq 0,$

$\quad \beta(0) \geq 0, \quad \beta''(0) \geq 0, \quad \beta''(1) \geq 0,$

(iii) $\quad \alpha'(0) - \beta'(0) \leq \min\{\beta(0) - \beta(1), \alpha(1) - \alpha(0), 0\}.$

Remark 1. a) Condition *(iii)* can not be removed, as it will be proved in the next section (See Counter-example).
b) By integration, from *(iii)* and (3), we obtain

$$\alpha(t) \leq \beta(t), \quad \alpha'(t) \leq \beta'(t), \quad \forall t \in [0,1],$$

i.e. lower and upper solutions and their first derivatives are well ordered, too.

To have an *a priori* estimate on u''' it must be defined a Nagumo-type growth condition.

Definition 2. Given a subset $E \subset [0,1] \times \mathbb{R}^4$, a continuous function $f : E \to \mathbb{R}$ is said to satisfy the Nagumo-type condition in E if there exists a real continuous function $h_E : \mathbb{R}_0^+ \to [a, +\infty[$, for some $a > 0$, such that

$$|f(t, x_0, x_1, x_2, x_3)| \leq h_E(|x_3|), \quad \forall (t, x_0, x_1, x_2, x_3) \in E, \tag{4}$$

with

$$\int_0^{+\infty} \frac{s}{h_E(s)} ds = +\infty. \tag{5}$$

Lemma 1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function, verifying Nagumo-type conditions (4) and (5) in*

$$E = \left\{ (t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \gamma_i(t) \le x_i \le \Gamma_i(t), \ i = 0,1,2 \right\},$$

where $\gamma_i(t)$ and $\Gamma_i(t)$ are continuous functions such that, for each $i = 0,1,2$ and every $t \in [0,1]$,

$$\gamma_i(t) \le \Gamma_i(t).$$

Then there is $r > 0$, such that every solution $u(t)$ of problem (1)-(2) verifying

$$\gamma_i(t) \le u^{(i)}(t) \le \Gamma_i(t),$$

for $i = 0,1,2$ and every $t \in [0,1]$, satisfies $\|u'''\|_\infty \le r$.

Proof. Define the non-negative real number

$$\eta := \max\{\Gamma_2(1) - \gamma_2(0), \Gamma_2(0) - \gamma_2(1)\}.$$

Take $r > \eta$ such that

$$\int_\eta^r \frac{s}{h_E(s)} ds \ge \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t). \tag{6}$$

Consider $u(t)$ a solution of (1)-(2) such that, for $i = 0,1,2$,

$$\gamma_i(t) \le u^{(i)}(t) \le \Gamma_i(t), \quad \forall t \in [0,1].$$

Suppose that $|u'''(t)| > \eta$, for every $t \in [0,1]$. If $u'''(t) > \eta$, for every $t \in [0,1]$, the following contradiction is achieved

$$\begin{aligned} \Gamma_2(1) - \gamma_2(0) &\ge u''(1) - u''(0) \\ &= \int_0^1 u'''(t)\, dt > \int_0^1 \eta \, dt \ge \Gamma_2(1) - \gamma_2(0). \end{aligned}$$

If $u'''(t) < -\eta$, for every $t \in [0,1]$, a similar contradiction is obtained. So, there exists $t \in [0,1]$, such that $|u'''(t)| \le \eta$.

If $|u'''(t)| \le \eta$, for every $t \in [0,1]$, it is enough to take $r := \eta$ to finish the proof. If not, suppose that there is $t \in [0,1]$ such that $u'''(t) > \eta$ and consider an interval $I = [t_0, t_1]$, (or $I = [t_1, t_0]$) such that $u'''(t_0) = \eta$ and $u'''(t) > \eta$ for $t \in I \setminus \{t_0\}$. Assume $I = [t_0, t_1]$ (the other case is analogous). Applying a convenient change of variable we have, by (4) and (6), for arbitrary $t_2 \in I \setminus \{t_0\}$

$$\begin{aligned} \int_{u'''(t_0)}^{u'''(t_2)} \frac{s}{h_E(s)} ds &= \int_{t_0}^{t_2} \frac{u'''(t)}{h_E(u'''(t))}\, u^{(iv)}(t)\, dt = \\ &= \int_{t_0}^{t_2} \frac{u'''(t)}{h_E(u'''(t))}\, f(t, u, u', u'', u''')\, dt \le \\ &\le \int_{t_0}^{t_2} u'''(t)\, dt = u''(t_2) - u''(t_0) \le \\ &\le \max_{t\in[0,1]} \Gamma_2(t) - \min_{t\in[0,1]} \gamma_2(t) \le \int_\eta^r \frac{s}{h_E(s)} ds. \end{aligned}$$

Then $u'''(t_2) \le r$ and so we have $u'''(t) \le r$, for every $t \in I$. Arguing as before in the intervals J, where $u'''(t) > \eta$, $t \in J$, we obtain that $u'''(t) \le r$, for every $t \in [0,1]$.

The proof of $u'''(t) \ge -r$, for every $t \in [0,1]$ such that $u'''(t) < -\eta$, follows similar steps. □

Remark 2. Observe that the estimation r depends only on the functions h_E, γ_2 and Γ_2 and it does not depend on the boundary conditions.

3. **Existence and location result.** The main result is an existence and location result, that is, it provides not only the existence of solution but also some information about the strip where the solution and some derivatives lie.

Theorem 1. *Suppose that there exists a pair of lower and upper solutions of (1)-(2), $\alpha(t)$ and $\beta(t)$, respectively. Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be a continuous function such that f satisfies Nagumo-type conditions (4) and (5) in*

$$E_* = \{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \le x_0 \le \beta(t), \\ \alpha'(t) \le x_1 \le \beta'(t), \ \alpha''(t) \le x_2 \le \beta''(t)\}.$$

If f verifies

$$\begin{aligned} f(t, \alpha(t), \alpha'(t), x_2, x_3) &\ge f(t, x_0, x_1, x_2, x_3) \ge \\ &\ge f(t, \beta(t), \beta'(t), x_2, x_3), \end{aligned} \tag{7}$$

for $(t, x_2, x_3) \in [0,1] \times \mathbb{R}^2$ and

$$(\alpha(t), \alpha'(t)) \le (x_0, x_1) \le (\beta(t), \beta'(t)),$$

where $(x_0, x_1) \le (y_0, y_1)$ means $x_0 \le y_0$ and $x_1 \le y_1$, then the problem (1)-(2) has at least a solution $u(t) \in C^4([0,1])$, satisfying

$$\alpha(t) \le u(t) \le \beta(t), \ \alpha'(t) \le u'(t) \le \beta'(t), \ \alpha''(t) \le u''(t) \le \beta''(t),$$

for $t \in [0,1]$.

Proof. Define the auxiliary continuous functions

$$\delta_i(t, x_i) = \begin{cases} \alpha^{(i)}(t) & if \quad x_i < \alpha^{(i)}(t) \\ x_i & if \quad \beta^{(i)}(t) \ge x_i \ge \alpha^{(i)}(t) \\ \beta^{(i)}(t) & if \quad x_i > \beta^{(i)}(t) \end{cases}, \quad i = 0, 1, 2.$$

For $\lambda \in [0,1]$, consider the homotopic equation

$$\begin{aligned} u^{(iv)}(t) &= \lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) + \\ &\quad + u''(t) - \lambda \delta_2(t, u''(t)), \end{aligned} \tag{8}$$

with the boundary conditions (2).

Take $r_1 > 0$ large enough such that, for every $t \in [0,1]$,

$$-r_1 < \alpha''(t) \le \beta''(t) < r_1, \tag{9}$$

$$f(t, \alpha(t), \alpha'(t), \alpha''(t), 0) - r_1 - \alpha''(t) < 0, \tag{10}$$

$$f(t, \beta(t), \beta'(t), \beta''(t), 0) + r_1 - \beta''(t) > 0. \tag{11}$$

Step 1. *Every solution $u(t)$ of problem (8)-(2) satisfies*

$$\left|u^{(i)}(t)\right| < r_1, \quad \forall t \in [0,1],$$

for $i = 0,1,2,$ *independently of* $\lambda \in [0,1]$.

Assume, by contradiction, that the above estimate does not hold for $i = 2$. So there exist $\lambda \in [0,1]$, $t \in [0,1]$ and a solution u of (8)-(2) such that $|u''(t)| \geq r_1$. In the case $u''(t) \geq r_1$ define

$$\max_{t\in[0,1]} u''(t) := u''(t_0) \geq r_1.$$

As $t_0 \in]0,1[$ then $u'''(t_0) = 0$ and $u^{(iv)}(t_0) \leq 0$. Then by (7) and (11), for $\lambda \in [0,1]$, the following contradiction is obtained

$$\begin{aligned}
0 &\geq u^{(iv)}(t_0) \\
&= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \delta_2(t_0, u''(t_0)), u'''(t_0)) \\
&\quad + u''(t_0) - \lambda\delta_2(t_0, u''(t_0)) \\
&= \lambda f(t_0, \delta_0(t_0, u(t_0)), \delta_1(t_0, u'(t_0)), \beta''(t_0), 0) + u''(t_0) - \lambda\beta''(t_0) \\
&\geq \lambda f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), 0) + u''(t_0) - \lambda\beta''(t_0) \\
&= \lambda[f(t_0, \beta(t_0), \beta'(t_0), \beta''(t_0), 0) + r_1 - \beta''(t_0)] + u''(t_0) - \lambda r_1 > 0.
\end{aligned}$$

The case $u''(t) \leq -r_1$, for all $t \in [0,1]$, yields also a similar contradiction. Therefore $|u''(t)| < r_1$, $\forall t \in [0,1]$.

By the boundary conditions (2) there exists $\xi \in]0,1[$, such that $u'(\xi) = 0$. Then by integration we obtain

$$|u'(t)| = \left|\int_\xi^t u''(s)\,ds\right| < r_1|t-\xi| \leq r_1,$$

and

$$|u(t)| = \left|\int_0^t u'(s)\,ds\right| < r_1 t \leq r_1.$$

Step 2. *There is* $r_2 > 0$ *such that, for every solution* $u(t)$ *of the problem (8)-(2)*

$$|u'''(t)| < r_2, \quad \forall t \in [0,1],$$

independently of $\lambda \in [0,1]$.

Consider the set

$$E_{r_1} = \left\{(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : -r_1 \leq x_i \leq r_1,\ i = 0,1,2\right\},$$

and, for $\lambda \in [0,1]$, the function $F_\lambda : E_{r_1} \to \mathbb{R}$ given by

$$F_\lambda(t, x_0, x_1, x_2, x_3) \quad = \quad \lambda f(t, \delta_0(t, x_0), \delta_1(t, x_1), \delta_2(t, x_2), x_3) + x_2 - \lambda\delta_2(t, x_2).$$

In the following we shall prove that the function F_λ satisfies Nagumo-type conditions (4) and (5) in E_{r_1} independently of $\lambda \in [0,1]$. Indeed, as f verifies (4) in E_*, then

$$\begin{aligned}
|F_\lambda(t, x_0, x_1, x_2, x_3)| &\leq |f(t, \delta_0(t, x_0), \delta_1(t, x_1), \delta_2(t, x_2), x_3)| + |x_2| + |\delta_2(t, x_2)| \\
&\leq h_{E_*}(|x_3|) + 2r_1.
\end{aligned}$$

So, defining $h_{E_{r_1}}(t) = 2r_1 + h_{E_*}(t)$ in $\mathbb{R}_0^+$, we have that F_λ verifies (4) with E and h_E replaced by E_{r_1} and $h_{E_{r_1}}$, respectively. The condition (5) is also verified since

$$\begin{aligned}\int_0^{+\infty} \frac{s}{h_{E_{r_1}}(s)} ds &= \int_0^{+\infty} \frac{s}{h_{E_*}(s) + 2r_1} ds \\ &\geq \frac{1}{1 + \frac{2r_1}{a}} \int_0^{+\infty} \frac{s}{h_{E_*}(s)} ds = +\infty.\end{aligned}$$

For r_1 given by Step 1, taking in Lemma 1

$$\gamma_i(t) \equiv -r_1 \text{ and } \Gamma_i(t) \equiv r_1, \text{ for } i = 0, 1, 2,$$

we can conclude that there is $r_2 > 0$ such that

$$|u'''(t)| < r_2, \quad \forall t \in [0, 1].$$

Since r_1 and $h_{E_{r_1}}$ do not depend on λ we observe that r_2 is also independent of λ.

Step 3. *For $\lambda = 1$, the problem (8)-(2) has at least a solution $u_1(t)$.*

Define the operators

$$\mathcal{L} : C^4([0, 1]) \subset C^3([0, 1]) \to C([0, 1]) \times \mathbb{R}^4$$

by

$$\mathcal{L}u = \left(u^{(iv)}, u(0), u''(0), u(1), u''(1)\right)$$

and, for $\lambda \in [0, 1]$, $\mathcal{N}_\lambda : C^3([0, 1]) \to C([0, 1]) \times \mathbb{R}^4$ by

$$\begin{aligned}\mathcal{N}_\lambda u &= (\lambda f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) \\ &\quad + u''(t) - \lambda \delta_2(t, u''(t)), 0, 0, 0, 0).\end{aligned}$$

As $\mathcal{L}$ has a compact inverse we can define the completely continuous operator

$$\mathcal{T}_\lambda : \left(C^3([0, 1]), \mathbb{R}\right) \to \left(C^3([0, 1]), \mathbb{R}\right)$$

by

$$\mathcal{T}_\lambda(u) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u).$$

For r_2 given by Step 2, consider the set

$$\Omega = \left\{x \in C^3([0, 1]) : \left\|x^{(i)}\right\|_\infty < r_1,\ i = 0, 1, 2,\ \|x'''\|_\infty < r_2\right\}.$$

By Steps 1 and 2, the degree $d(I - \mathcal{T}_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$ and by the invariance with respect to a homotopy,

$$d(I - \mathcal{T}_0, \Omega, 0) = d(I - \mathcal{T}_1, \Omega, 0).$$

The equation $x = \mathcal{T}_0(x)$ is equivalent to the problem

$$\begin{cases} u^{(iv)}(t) = u''(t), \\ u(0) = u''(0) = u(1) = u''(1) = 0, \end{cases}$$

and has only the trivial solution. Then by the degree theory,

$$d(I - \mathcal{T}_0, \Omega, 0) = \pm 1.$$

Therefore, the equation $x = \mathcal{T}_1(x)$ has at least one solution. That is, the problem composed by

$$\begin{aligned}u^{(iv)}(t) &= f(t, \delta_0(t, u(t)), \delta_1(t, u'(t)), \delta_2(t, u''(t)), u'''(t)) \\ &\quad + u''(t) - \delta_2(t, u''(t)),\end{aligned} \tag{12}$$

and the boundary conditions (2) has at least one solution $u_1(t)$ in Ω.

Step 4. *The function $u_1(t)$ is a solution of the problem (1)-(2).*

The proof will be finished if the above function $u_1(t)$ satisfies the inequalities

$$\alpha(t) \leq u_1(t) \leq \beta(t), \ \alpha'(t) \leq u_1'(t) \leq \beta'(t), \ \alpha''(t) \leq u_1''(t) \leq \beta''(t).$$

Assume, by contradiction, that there is $t \in [0,1]$ such that $u_1''(t) > \beta''(t)$ and define

$$\max_{t\in[0,1]} [u_1''(t) - \beta''(t)] := u_1''(t_2) - \beta''(t_2) > 0.$$

As $t_2 \in]0,1[$ then $u'''(t_2) = \beta'''(t_2)$ and

$$u^{(iv)}(t_2) \leq \beta^{(iv)}(t_2). \tag{13}$$

By (7), (12) and Definition 1 we obtain the following contradiction with (13):

$$\begin{aligned} u_1^{(iv)}(t_2) &= f(t_2, \delta_0(t_2, u_1(t_2)), \delta_1(t_2, u_1'(t_2)), \delta_2(t_2, u_1''(t_2)), u_1'''(t_2)) \\ &\quad + u_1''(t_2) - \lambda\delta_2(t_2, u_1''(t_2)) \\ &= f(t_2, \delta_0(t_2, u_1(t_2)), \delta_1(t_2, u_1'(t_2)), \beta''(t_2), \beta'''(t_2)) \\ &\quad + u_1''(t_2) - \beta''(t_2) \\ &\geq f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2), \beta'''(t_2)) + u_1''(t_2) - \beta''(t_2) \\ &> f(t_2, \beta(t_2), \beta'(t_2), \beta''(t_2), \beta'''(t_2)) \geq \beta^{(iv)}(t_2). \end{aligned}$$

Analogously, we prove that

$$\alpha''(t) \leq u_1''(t).$$

So, we have

$$\alpha''(t) \leq u_1''(t) \leq \beta''(t). \tag{14}$$

On the other hand, by (2),

$$\begin{aligned} 0 &= u_1(1) - u_1(0) = \int_0^1 u_1'(t)\,dt = \int_0^1 \left(u_1'(0) + \int_0^t u_1''(s)\,ds\right) dt \\ &= u_1'(0) + \int_0^1 \int_0^t u_1''(s)\,ds\ dt, \end{aligned}$$

that is

$$u_1'(0) = -\int_0^1 \int_0^t u_1''(s)\,ds\ dt. \tag{15}$$

Applying the same technique

$$-\int_0^1 \int_0^t \beta''(s)\,ds\ dt = -\int_0^1 \beta'(t)\ dt + \beta'(0) = \beta(0) - \beta(1) + \beta'(0)$$

and then by Definition 1 (iii), (14) and (15), we obtain

$$\begin{aligned} \alpha'(0) &\leq \beta'(0) - \beta(1) + \beta(0) \\ &= -\int_0^1 \int_0^t \beta''(s)\,ds\ dt \leq -\int_0^1 \int_0^t u_1''(s)\,ds\ dt = u_1'(0), \end{aligned}$$

and

$$\begin{aligned} \beta'(0) &\geq \alpha'(0) - \alpha(1) + \alpha(0) \\ &= -\int_0^1 \int_0^t \alpha''(s)\,ds\ dt \geq -\int_0^1 \int_0^t u_1''(s)\,ds\ dt = u_1'(0), \end{aligned}$$

i. e.

$$\alpha'(0) \leq u_1'(0) \leq \beta'(0). \tag{16}$$

Since, by (14), $\beta'(t) - u_1'(t)$ is nondecreasing then, by (16),

$$\beta'(t) - u_1'(t) \geq \beta'(0) - u_1'(0) \geq 0,$$

and, therefore, $\beta'(t) \geq u_1'(t)$ for every $t \in [0,1]$. By the monotony of $\beta(t) - u_1(t)$,

$$\beta(t) - u_1(t) \geq \beta(0) - u_1(0) = \beta(0) \geq 0$$

and so $\beta(t) \geq u_1(t)$ for every $t \in [0,1]$.

The inequalities $u_1'(t) \geq \alpha'(t)$ and $u_1(t) \geq \alpha(t)$, for every $t \in [0,1]$, can be proved by the same way. Then $u_1(t)$ is a solution of *(1)-(2)*. □

Example: Consider the fourth order boundary value problem

$$\begin{cases} u^{(iv)} = -e^{u} - (u')^3 + 2(u'')^5 + |u'''|^{\theta}, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{17}$$

with $\theta \in [0,2]$.The continuous function

$$f(t, x_0, x_1, x_2, x_3) = -e^{x_0} - x_1^3 + 2\ x_2^5 + |x_3|^{\theta}$$

verifies Nagumo-type conditions (4), (5) and assumption (7) on the set

$$\begin{aligned} E = \{ &(t, x_0, x_1, x_2, x_3) \in [0,1] \times \mathbb{R}^4 : \alpha(t) \leq x_0 \leq \beta(t), \\ &\alpha'(t) \leq x_1 \leq \beta'(t), \quad \alpha''(t) \leq x_2 \leq \beta''(t) \}, \end{aligned}$$

where α and β are defined by

$$\alpha(t) := -t^2 - t; \qquad \beta(t) := t^2 + t. \tag{18}$$

The functions α and β given by (18) define a pair of lower and upper solutions of (17). In fact,

$$\alpha''(t) \equiv -2 < \beta''(t) \equiv 2,\ \forall t \in [0,1],$$

$$f(t, \alpha, \alpha', \alpha'', \alpha''') \leq -e^{-2} - (-3)^3 + 2(-2)^5 \leq 0 \equiv \alpha^{(iv)}(t),$$

$$f(t, \beta, \beta', \beta'', \beta''') \geq -e^2 - 3^3 + 2^6 \geq 0 \equiv \beta^{(iv)}(t),$$

$$\alpha(0) = 0 = \beta(0)$$

and

$$\alpha'(0) - \beta'(0) = \min\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\} = -2.$$

Then, by Theorem 1, there is a solution $u(t)$ of (17) such that, for $t \in [0,1]$,

$$-t^2 - t \leq u(t) \leq t^2 + t, \quad -2t - 1 \leq u'(t) \leq 2t + 1, \quad -2 \leq u''(t) \leq 2,$$

that is, lying in the strip illustraded by Fig.1

Counter-example: To prove that the assumption (*iii*) in Definition 1 can not be avoided, we consider the fourth order problem

$$\begin{cases} u^{(iv)} = -2u' + 3u'', \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{19}$$

Defining

$$\alpha(t) := \frac{t(1-2t)}{4}, \quad \beta(t) := \frac{t(1+t)}{4}, \tag{20}$$

we have

$$\alpha''(t) \equiv -1 \leq \beta''(t) \equiv \frac{1}{2}, \ \ \alpha(0) = 0 = \beta(0),$$

$$\alpha^{(iv)} \equiv 0 \geq 2t - \frac{7}{2} = -2\alpha' + 3\alpha'', \quad \beta^{(iv)} \equiv 0 \leq 1 - t = -2\beta' + 3\beta''.$$

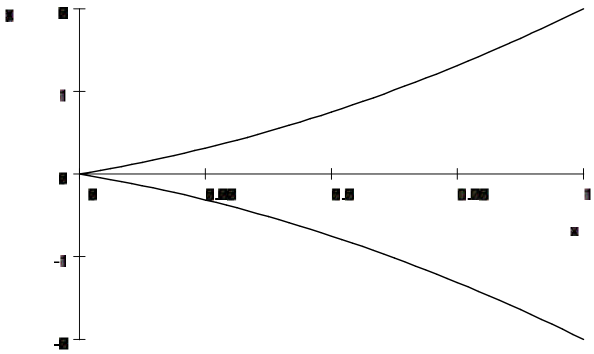

FIGURE 1.

Therefore α and β given by (20) are lower and upper solutions of (19) but assumption (*iii*) is not satisfied since

$$\alpha'(0) - \beta'(0) = 0 > \min\{\beta(0) - \beta(1), \alpha(1) - \alpha(0)\} = -\frac{1}{4}.$$

The problem (19) has only the zero solution $u(t) \equiv 0$ and, for $t \in \left]0, \frac{1}{2}\right[$, we have

$$0 = u(t) < \alpha(t) < \beta(t),$$

and for $t \in \left]0, \frac{1}{4}\right[$

$$0 = u'(t) < \alpha'(t) < \beta'(t),$$

that is the location given by Theorem 1 does not hold, as it can be seen in next figure.

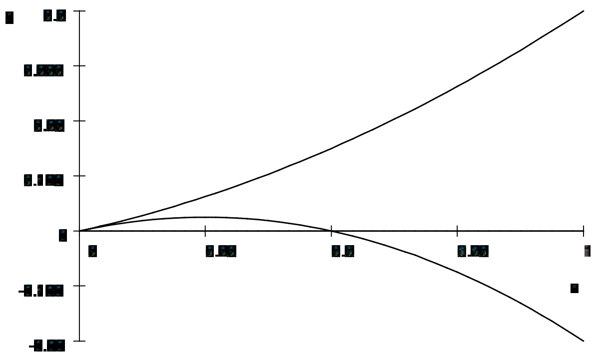

FIGURE 2.

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REFERENCES

[1] Z. Bai, *The method of lower and upper solutions for a bending of an elastic beam equation*, J. Math. Anal. Appl., **248** (2000), 195–202.

[2] A. Cabada, *The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems.* J. Math. Anal. Appl., **185**, 2, (1994), 302–320.

[3] C. de Coster and P. Habets, *Upper and lower solutions in the theory of ODE boundary value problems: Classical and Recent Results*, Institut de Mathématique Pure et Appliquée, Université Catholique de Louvain, Recherches de Mathématique, **52** (April 1996).

[4] C. de Coster and L. Sanchez, *Upper and lower solutions, Ambrosetti-Prodi problem and positive solutions for fourth-order O.D.E.*, Riv. Mat. Pura Appl., **14** (1994), 57-82.

[5] J. Ehme and P. Eloe and J. Henderson, *Upper and lower solutions method for fully nonlinear boundary value problems*, J. Diff. Eq., **180** (2002), 51-64.

[6] D. Franco, D. O'Regan, J. Perán, *Fourth-order problems with nonlinear boundary conditions*, J. Comput. Appl. Math., **174** (2005), 315-327.

[7] M. R. Grossinho and F. Minhós, *Existence result for some third order separated boundary value problems*, Nonlinear Anal. Series A. Theory and Methods, **47** (2001), 2407-2418.

[8] M. R. Grossinho and F. Minhós, *Upper and Lower Solutions for Higher Order Boundary Value Problems* (to appear in Nonlinear Studies).

[9] M. R. Grossinho and S. Tersian, *The dual variational principle and equilibria for a beam resting on a discontinuous nonlinear elastic foundation*, Nonlinear Anal., **41** (2000), 417-431.

[10] C. P. Gupta, *Existence and uniqueness theorems for the bending of an elastic beam equation*, Appl. Anal., **26** (1988), 289-304.

[11] C. P. Gupta, *Existence and uniqueness theorems for a fourth order boundary value problem of Sturm-Liouville type*, Differential and Integral Equations, **4**, 2, (March 1991), 397-410.

[12] T. Gyulov and S. Tersian, *Existence of Trivial and Nontrivial Solutions of a Fourth-Order Ordinary Differential Equation*, Electron. J. Diff. Eqns., **41** (2004), 1-14.

[13] J. Mawhin, *Topological degree methods in nonlinear boundary value problems.* Regional Conference Series in Mathematics, **40** (American Mathematical Society, Providence, Rhode Island, 1979).

[14] M. Ruyun and Z. Jihui and F. Shengmao, *The method of lower and upper solutions for fourth-order boundary value problems*, J. Math. Anal. Appl., **215** (1997), 415-422.

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E-mail address: fminhos@uevora.pt(F. Minhós)
E-mail address: tgulov@ru.acad.bg (T. Gyulov)
E-mail address: aims@uevora.pt (A.I.Santos)