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ORIGINAL RESEARCH

Higher Order ϕ-Laplacian BVP with Generalized Sturm–Liouville Boundary Conditions

John R. Graef · Lingju Kong · Feliz M. Minhós

Abstract This work presents an existence and location result for the higher order boundary value problem

$$-\left(\phi\left(u^{(n-1)}(x)\right)\right)' = f(x, u(x), \ldots, u^{(n-1)}(x)),$$
$$u^{(i)}(0) = A_i, \quad i = 0, \ldots, n-3,$$
$$g_1\left(u^{(n-2)}(0)\right) - g_2\left(u^{(n-1)}(0)\right) = B,$$
$$g_3\left(u^{(n-2)}(1)\right) + g_4\left(u^{(n-1)}(1)\right) = C,$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and continuous function such that $\phi(0) = 0$, $n \geq 2$ is an integer, $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, $A_i, B, C \in \mathbb{R}$, and $g_j : \mathbb{R} \to \mathbb{R}$ are continuous functions such that g_1, g_3 are increasing and g_2, g_4 are nondecreasing. In view of the assumptions on ϕ and f, this paper generalizes several problems due to the dependence on the $(n-1)$-st derivative not only in the differential equation but also in the boundary conditions.

Keywords Boundary value problem · Sturm–Liouville boundary conditions · ϕ-Laplacian · Higher order problems

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Introduction

In this paper, we study the n-th order boundary value problem (BVP) composed of the ϕ-Laplacian type differential equation

$$-\left(\phi\left(u^{(n-1)}(x)\right)\right)' = f\left(x, u(x), \ldots, u^{(n-1)}(x)\right), \quad x \in (0,1), \tag{1}$$

and the generalized Sturm–Liouville boundary conditions

$$\begin{aligned} &u^{(i)}(0) = A_i, \quad i = 0, \ldots, n-3, \\ &g_1\left(u^{(n-2)}(0)\right) - g_2\left(u^{(n-1)}(0)\right) = B, \\ &g_3\left(u^{(n-2)}(1)\right) + g_4\left(u^{(n-1)}(1)\right) = C, \end{aligned} \tag{2}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and continuous function with $\phi(0) = 0, n \geq 2$ is an integer, $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function, $A_i, B, C \in \mathbb{R}$, and $g_j : \mathbb{R} \to \mathbb{R}$ are continuous functions such that g_1, g_3 are increasing and g_2, g_4 are nondecreasing.

We remark that in case $n = 2$, Eq. 1 includes the classical ϕ-Laplacian. Moreover, in this case, the first set of boundary conditions in (2) do not exist.

Higher order two-point boundary value problems have been studied by several authors such as in [7,12] for n-th order differential equations, in [6,8] for even-order equations, and in [10,13] for multi-point boundary conditions. These papers do not consider a fully nonlinear differential equation in that the nonlinear part does not depend on all derivatives up to and including the one of order $n-1$. This dependence is overcome by assuming a Nagumo-type growth condition on the nonlinear part, as suggested in [5,9]. Taking the higher order ϕ-Laplacian fully nonlinear equation not only improves results in previous papers, but it also generalizes several works on second and higher order equations with two-point boundary conditions. We refer the reader to the references contained in the above mentioned papers for other works improved by the results here. We should also note that because we are using a truncation technique, we do not assume that $\phi(\mathbb{R}) = \mathbb{R}$ as is usually the case, nor do we place strong monotonicity conditions on f as many authors have done.

On the other hand, the general Sturm–Liouville boundary conditions considered here are not covered by several other multi-point boundary value problems of higher order such as those considered, for example, in [3,4] for n-th order problems, or in [2] for second order ones.

The arguments used below make use of fixed point theory, some techniques suggested by [1], and the lower and upper solution method. We obtain existence and location results, meaning that not only do we prove the existence of solutions, but some information about their location and that of some of their derivatives is also obtained. This type of information can be used to study some qualitative properties of the solution such as, for example, the sign of the solution. In fact, the existence of positive (negative) solutions can be seen as a particular case of the above results by considering a nonnegative lower solution or a nonpositive upper one. The last section of the paper presents an example where these data are useful.

Definitions and preliminary results

In this section, we present some definitions and results to be used in the remainder of the paper. Let

$$||u||_p = \begin{cases} \left(\int_0^1 |u(t)|^p dt\right)^{1/p}, & 1 \le p < \infty, \\ \sup\{|u(t)| \ : \ t \in [0,1]\}, & p = \infty, \end{cases}$$

denote the norm in $L^p([0,1])$. The function $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is said to be a *Carathéodory function* if it satisfies the following conditions:

(i) For each $y \in \mathbb{R}^n$, the function $f(\cdot, y)$ is measurable on $[0, 1]$;
(ii) For a. e. $x \in [0, 1]$, the function $f(x, \cdot)$ is continuous on $\mathbb{R}^n$;
(iii) For each compact set $K \subset \mathbb{R}^n$, there is a function $m_K \in L^1([0,1])$ such that $|f(x, y)| \le m_K(x)$ for a. e. $x \in [0, 1]$ and all $y \in K$.

The concept of a Nagumo-type condition, as defined next, and a priori estimates for the derivative $u^{(n-1)}$ play a key role in this paper.

Definition 1 Given a subset $E \subset [0,1] \times \mathbb{R}^n$, a function $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies a Nagumo-type condition in the set

$$E := \left\{(x, y_0, \dots, y_{n-1}) \in [0,1] \times \mathbb{R}^n : \gamma_j(x) \le y_j \le \Gamma_j(x), \ j = 0, \dots, n-2\right\},$$

with $\gamma_j, \Gamma_j \in C([0,1], \mathbb{R})$ and

$$\gamma_j(x) \le \Gamma_j(x) \quad \text{for all } x \in [0,1], \ j = 0, \dots, n-2,$$

if there exists $h_E \in L^1(0,1)$ and $R > r$, where

$$r := \max\{\Gamma_{n-2}(1) - \gamma_{n-2}(0), \Gamma_{n-2}(0) - \gamma_{n-2}(1)\}, \tag{3}$$

such that

$$|f(x, y_0, \dots, y_{n-1})| \le h_E(|y_{n-1}|) \quad \text{for all } (x, y_0, \dots, y_{n-1}) \in E \tag{4}$$

and

$$\int_{\phi(r)}^{\phi(R)} \frac{|\phi^{-1}(s)|}{h_E\left(|\phi^{-1}(s)|\right)} ds > \max_{x \in [0,1]} \Gamma_{n-2}(x) - \min_{x \in [0,1]} \gamma_{n-2}(x). \tag{5}$$

An a priori bound on the $(n-1)$-st derivative is given by next result.

Lemma 2 *Suppose $\gamma_j, \Gamma_j \in C([0,1], \mathbb{R})$ are such that*

$$\gamma_j(x) \le \Gamma_j(x) \quad \textit{for all } x \in [0,1], \ j = 0, \dots, n-2,$$

and let $f : E \to \mathbb{R}$ be a Carathéodory function satisfying a Nagumo-type condition in E. Then there exists $R > 0$ (depending only on γ_{n-2}, Γ_{n-2}, and h_E) such that every solution $u(x)$ of (1) with

$$\gamma_j(x) \le u^{(j)}(x) \le \Gamma_j(x) \quad \textit{for all } x \in [0,1], \ j = 0, \dots, n-2, \tag{6}$$

satisfies $\left\|u^{(n-1)}\right\|_\infty < R$.

Proof Let u be a solution of the differential equation (1) satisfying (6). By the Mean Value Theorem, there is an $x_0 \in (0, 1)$ such that

$$u^{(n-1)}(x_0) = u^{(n-2)}(1) - u^{(n-2)}(0)$$

and

$$-R < -r \le \gamma_{n-2}(1) - \Gamma_{n-2}(0) \le u^{(n-1)}(x_0) \le \Gamma_{n-2}(1) - \gamma_{n-2}(0) \le r < R.$$

If $\left|u^{(n-1)}(x)\right| \le R$, for every $x \in [0, 1]$, then the proof would be finished. If this is not the case, choose $x_1 \in [0, 1]$ such that $u^{(n-1)}(x_1) > R$ or $u^{(n-1)}(x_1) < -R$. In the first case, consider an interval $[x_2, x_3]$ such that

$$u^{(n-1)}(x_2) = \max\left\{0, u^{(n-1)}(x_0)\right\}$$

and

$$u^{(n-1)}(x_2) \le u^{(n-1)}(x) \le u^{(n-1)}(x_3) = R \quad \text{for all} \quad x \in [x_2, x_3].$$

Then, the fact that f satisfies (4) and a convenient change of variables yield

$$\begin{aligned}\int_{\phi(r)}^{\phi(R)} \frac{|\phi^{-1}(s)|}{h_E(|\phi^{-1}(s)|)} ds &\le \int_{\phi(u^{(n-1)}(x_2))}^{\phi(u^{(n-1)}(x_3))} \frac{|\phi^{-1}(s)|}{h_E(|\phi^{-1}(s)|)} ds \\ &= \int_{x_2}^{x_3} \frac{|u^{(n-1)}(s)|}{h_E(u^{(n-1)}(s))} \left(\phi\left(u^{(n-1)}(x)\right)\right)' dx \\ &\le \int_{x_2}^{x_3} \frac{|f(x, u(x), u'(x), \ldots, u^{(n-1)}(x))|}{h_E(u^{(n-1)}(s))} \left|u^{(n-1)}(x)\right| dx \\ &\le \int_{x_2}^{x_3} u^{(n-1)}(x)\, dx = u^{(n-2)}(x_3) - u^{(n-2)}(x_2) \\ &\le \max_{x\in[0,1]} \Gamma_{n-2}(x) - \min_{x\in[0,1]} \gamma_{n-2}(x)\end{aligned}$$

which contradicts (5). Therefore, $u^{(n-1)}(x) < R$ for every $x \in [0, 1]$. In a similar way, we can show that $u^{(n-1)}(x) > -R$ for every $x \in [0, 1]$. □

Our next lemma proves an existence and uniqueness result for a problem related to (1–2). For convenience, we take $0^0 = 1$.

Lemma 3 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism such that $\varphi(0) = 0$ and $\varphi(\mathbb{R}) = \mathbb{R}$, let $p : [0, 1] \to \mathbb{R}$ satisfy $p \in L^1([0, 1])$, and let A_i, $i = 0, \ldots, n-3$, $B_1, C_1 \in \mathbb{R}$. Then the problem*

$$\begin{cases} -\left(\varphi\left(u^{(n-1)}(x)\right)\right)' = p(x), & \text{for a. e. } x \in [0,1], \\ u^{(i)}(0) = A_i, & i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = B_1, \\ u^{(n-2)}(1) = C_1, \end{cases} \tag{7}$$

has a unique solution given by

$$u(x) = B_1 + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\,dr\right) ds$$

if $n = 2$*, and*

$$u(x) = \sum_{k=0}^{n-3} A_k \frac{x^k}{k!} + \int_0^x \frac{(x-r)^{n-3}}{(n-3)!} v(r)dr \tag{8}$$

if $n \geq 3$*, where*

$$v(x) := B_1 + \int_0^x \varphi^{-1}\left(\tau_v - \int_0^s p(r)\,dr\right) ds$$

and $\tau_v \in \mathbb{R}$ *is the unique solution of the equation*

$$C_1 - B_1 = \int_0^1 \varphi^{-1}\left(\tau_v - \int_0^s p(r)\,dr\right) ds. \tag{9}$$

Proof Defining $v(x) := u^{(n-2)}(x)$, from (7) we obtain the Dirichlet problem

$$-\left(\varphi\left(v'(x)\right)\right)' = p(x), \quad \text{for a. e. } x \in [0,1], \tag{10}$$

$$v(0) = B_1, \quad v(1) = C_1. \tag{11}$$

Therefore, for some $\tau \in \mathbb{R}$,

$$v'(x) = \varphi^{-1}\left(\tau - \int_0^x p(r)\,dr\right)$$

and

$$v(x) = B_1 + \int_0^x \varphi^{-1}\left(\tau - \int_0^s p(r)\,dr\right) ds. \tag{12}$$

Since φ^{-1} is increasing, we have

$$\begin{aligned} v_*(\tau) &:= B_1 + \varphi^{-1}(\tau - \|p\|_1) \leq v(1) \\ &\leq B_1 + \varphi^{-1}(\tau + \|p\|_1) := v^*(\tau) \end{aligned} \tag{13}$$

for each $\tau \in \mathbb{R}$. Now $\varphi^{-1}(\mathbb{R}) = \mathbb{R}$ and the functions v_* and v^* are continuous and increasing, so $v_*(\mathbb{R}) = v^*(\mathbb{R}) = \mathbb{R}$. Thus, there is a unique τ_v satisfying (9). If $n = 2$, we are done. If $n \geq 3$, then repeatedly integrating (12) and applying the boundary conditions, we obtain (8). This completes the proof of the lemma. □

Some properties of truncated functions that we will use in the remainder of the paper are given in the next result (see [11, Lemma 2]).

Lemma 4 *For $z, w \in C([0,1])$ with $z(x) \le w(x)$ for every $x \in [0,1]$, define*

$$\mu(x,u) = max\{z, min\{u, w\}\}.$$

Then, for each $u \in C^1([0,1])$, the following properties hold:

(a) $\frac{d}{dx}\mu(x,u(x))$ *exists for a.e.* $x \in [0,1]$;

(b) *If $u, u_m \in C^1([0,1])$ and $u_m \to u$ in $C^1([0,1])$, then*

$$\frac{d}{dx}\mu(x,u_m(x)) \to \frac{d}{dx}\mu(x,u(x)) \quad \text{for a.e. } x \in [0,1].$$

The notions of lower and upper solutions are defined as follows.

Definition 5 Let $n \ge 2$ and $A_i, B, C \in \mathbb{R}$ for $i = 0, \ldots, n-3$. A function $\alpha \in C^{n-1}([0,1])$ such that $\phi\left(\alpha^{(n-1)}(x)\right) \in AC([0,1])$ is a lower solution of the problem (1–2) if

$$-\left(\phi\left(\alpha^{(n-1)}(x)\right)\right)' \le f\left(x, \alpha(x), \alpha'(x), \ldots, \alpha^{(n-1)}(x)\right) \tag{14}$$

for $x \in [0,1]$, and

$$\begin{aligned} &\alpha^{(i)}(0) \le A_i, \quad i = 0, \ldots, n-3, \\ &g_1\left(\alpha^{(n-2)}(0)\right) - g_2\left(\alpha^{(n-1)}(0)\right) \le B, \\ &g_3\left(\alpha^{(n-2)}(1)\right) + g_4\left(\alpha^{(n-1)}(1)\right) \le C. \end{aligned} \tag{15}$$

A function $\beta \in C^{n-1}([0,1])$ such that $\phi\left(\beta^{(n-1)}(x)\right) \in AC([0,1])$ is an upper solution of problem (1–2), if the reverse inequalities hold.

Main theorem

Our main result is an existence and location theorem, as is usual in applying the lower and upper solutions technique. In this case, some data on the location of the derivatives up to and including the one of $(n-2)$-nd order are also given.

Theorem 6 *Let $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a L^1-Carathéodory function. Assume that α and β are lower and upper solutions of the problem (1–2), respectively, such that*

$$\alpha^{(n-2)}(x) \le \beta^{(n-2)}(x) \quad \text{for all } x \in [0,1], \tag{16}$$

f satisfies a Nagumo-type condition (4) in the set

$$E_* = \left\{(x, y_0, \ldots, y_{n-1}) \in [0,1] \times \mathbb{R}^n : \alpha^{(i)}(x) \le y_i \le \beta^{(i)}(x),\ i = 0, \ldots, n-2\right\},$$

and

$$\begin{aligned} f\left(x, \alpha(x), \ldots, \alpha^{(n-3)}(x), y_{n-2}, y_{n-1}\right) &\le f(x, y_0, \ldots, y_{n-1}) \\ &\le f\left(x, \beta(x), \ldots, \beta^{(n-3)}(x), y_{n-2}, y_{n-1}\right), \end{aligned} \tag{17}$$

for fixed x, y_{n-2}, y_{n-1} and $\alpha^{(i)}(x) \le y_i \le \beta^{(i)}(x), i = 0, \ldots, n-3$, for all $x \in [0,1]$. If the functions $g_j : \mathbb{R} \to \mathbb{R}$ are continuous with g_1 and g_3 increasing and g_2 and g_4 nondecreasing, then the problem (1–2) has at least one solution u such that

$$\alpha^{(i)}(x) \le u^{(i)}(x) \le \beta^{(i)}(x)$$

for every $x \in [0,1]$ and $i = 0, \ldots, n-2$.

Remark 7 The relations $\alpha^{(j)}(x) \leq \beta^{(j)}(x)$, $j = 0, \ldots, n-3$ are obtained from (16) by successive integrations and applying the boundary conditions given in Definition 5.

Proof For $i = 0, \ldots, n-2$, define the continuous truncations

$$\delta_i(x, y_i) = \begin{cases} \beta^{(i)}(x) & \text{if } y_i > \beta^{(i)}(x), \\ y_i & \text{if } \alpha^{(i)}(x) \leq y_i \leq \beta^{(i)}(x), \\ \alpha^{(i)}(x) & \text{if } y_i < \alpha^{(i)}(x). \end{cases} \tag{18}$$

Let

$$R > \max\left\{r, \left\|\alpha^{(n-1)}\right\|_\infty, \left\|\beta^{(n-1)}\right\|_\infty\right\}, \tag{19}$$

where r satisfies (3) with $\gamma_{n-2}(x) = \alpha^{(n-2)}(x)$ and $\Gamma_{n-2}(x) = \beta^{(n-2)}(x)$. Consider the functions

$$q(y) = \max\{-R, \min\{y, R\}\} \tag{20}$$

and $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} \phi(y), & \text{if } |y| \leq R, \\ \frac{\phi(R)-\phi(-R)}{2R}\, y + \frac{\phi(R)+\phi(-R)}{2}, & \text{if } |y| > R. \end{cases}$$

Define the modified problem composed of the differential equation

$$\begin{aligned} &-\left(\varphi\left(u^{(n-1)}(x)\right)\right)' \\ &\quad = f\left(x, \delta_0(x, u), \ldots, \delta_{n-2}\left(x, u^{(n-2)}\right), q\left(\frac{d}{dx}\delta_{n-2}(x, u^{(n-2)})\right)\right) \equiv F_u(x) \end{aligned} \tag{21}$$

and the boundary conditions

$$\begin{aligned} u^{(i)}(0) &= A_i, \quad i = 0, \ldots, n-3, \\ u^{(n-2)}(0) &= \delta_{n-2}\left(0, g_1^{-1}\left(B + g_2\left(u^{(n-1)}(0)\right)\right)\right), \\ u^{(n-2)}(1) &= \delta_{n-2}\left(1, g_3^{-1}\left(C - g_4\left(u^{(n-1)}(1)\right)\right)\right). \end{aligned} \tag{22}$$

A function $u \in C^{n-1}([0, 1])$ such that $\phi \circ u^{(n-1)} \in AC([0, 1])$ is a solution of problem (21–22) if it satisfies the above equalities.

Step 1: Every solution u of problem (21–22) satisfies

$$\alpha^{(i)}(x) \leq u^{(i)}(x) \leq \beta^{(i)}(x) \quad , for \ \ i = 0, \ldots, n-2, \tag{23}$$

$$-R \leq u^{(n-1)}(x) \leq R, \tag{24}$$

for $x \in [0, 1]$.

For a solution u of (21–22), assume, for the sake of a contradiction, that the right hand inequality in (23) does not hold for $i = n-2$ and define

$$\max_{x\in[0,1]} (u - \beta)^{(n-2)}(x) := (u - \beta)^{(n-2)}(x_0) > 0.$$

By (22), $u^{(n-2)}(0) \leq \beta^{(n-2)}(0)$ and $u^{(n-2)}(1) \leq \beta^{(n-2)}(1)$. So, $x_0 \in (0, 1)$, $u^{(n-1)}(x_0) = \beta^{(n-1)}(x_0)$, and there is $\delta > 0$ such that

$$u^{(n-2)}(x_0 + \delta) = \beta^{(n-2)}(x_0 + \delta)$$

and $u^{(n-2)}(x) > \beta^{(n-2)}(x)$ on $[x_0, x_0+\delta)$.

On $(x_0, x_0+\delta)$, by Definition 5, (17), (18), (20), and (19), we have

$$\begin{aligned}
-\left(\varphi\left(u^{(n-1)}(x)\right)\right)' &= f\left(x, \delta_0(x,u), \ldots, \delta_{n-2}(x, u^{(n-2)}), q\left(\frac{d}{dx}(\delta_{n-2}(x, u^{(n-2)})\right)\right) \\
&= f\left(x, \delta_0\left(x,u\right), \ldots, \delta_{n-3}\left(x, u^{(n-3)}\right), \beta^{(n-2)}\left(x\right), \beta^{(n-1)}\left(x\right)\right) \\
&\leq f\left(x, \beta\left(x\right), \ldots, \beta^{(n-3)}, \beta^{(n-2)}\left(x\right), \beta^{(n-1)}\left(x\right)\right) \\
&\leq -\left(\phi\left(\beta^{(n-1)}(x)\right)\right)' = -\left(\varphi\left(\beta^{(n-1)}(x)\right)\right)'.
\end{aligned}$$

Therefore, $u^{(n-1)}(x) \geq \beta^{(n-1)}(x)$ on $(x_0, x_0+\delta)$, which is a contradiction to the definition of $[x_0, x_0+\delta)$. Hence, $u^{(n-2)}\left(x\right) \leq \beta^{(n-2)}\left(x\right)$ for every $x \in [0,1]$. By analogous arguments, it can be shown that $\alpha^{(n-2)}\left(x\right) \leq u^{(n-2)}\left(x\right)$ in $[0,1]$.

Integrating the inequalities

$$\alpha^{(n-2)}\left(x\right) \leq u^{(n-2)}\left(x\right) \leq \beta^{(n-2)}\left(x\right)$$

in $[0,x]$ and applying the boundary conditions (22) for lower and upper solutions, we see that (23) holds for $i = 0, \ldots, n-2$.

From Lemma 4 and the definition of q, the right hand side of (21) is an L^1–function. Therefore, Lemma 2 can be applied with $\gamma_j(x) = \alpha^{(j)}(x)$ and $\Gamma_j(x) = \beta^{(j)}(x)$ for $j = 0, \ldots, n-2$, that is, condition (24) holds.

Step 2: Problem (21–22) has a solution $u_1(x)$.

Let $u \in C^{n-1}([0,1])$ be fixed. If $n \geq 3$, by Lemma 3, solutions of problem (21–22) are the fixed points of the operator

$$\mathcal{T}u(x) = \sum_{k=0}^{n-3} A_k \frac{x^k}{k!} + \int_0^x \frac{(x-r)^{n-3}}{(n-3)!} v_u(r)dr$$

with

$$v_u\left(x\right) := \delta_{n-2}\left(0, g_1^{-1}\left(B + g_2\left(u^{(n-1)}(0)\right)\right)\right) + \int_0^x \varphi^{-1}\left(\tau_u - \int_0^s F_u\left(r\right)\, dr\right) ds$$

and $\tau_u \in \mathbb{R}$ is the unique solution of the equation

$$\begin{aligned}
&\delta_{n-2}\left(1, g_3^{-1}\left(C - g_4\left(u^{(n-1)}(1)\right)\right)\right) - \delta_{n-2}\left(0, g_1^{-1}\left(B + g_2\left(u^{(n-1)}(0)\right)\right)\right) \\
&= \int_0^1 \varphi^{-1}\left(\tau_u - \int_0^s F_u\left(r\right)\, dr\right) ds. \qquad (25)
\end{aligned}$$

It is easy to verify that $T : C^{n-1}([0,1]) \to C^{n-1}([0,1])$ and is continuous. By (21), there is a function $\omega \in L^1\left([0,1]\right)$ such that

$$\mid F_u(s) \mid \leq \omega(s) \quad \text{for a. e. } s \in [0,1] \text{ and for all } u \in C^{n-1}([0,1]).$$

In view of (25), there exists $L > 0$ such that

$$\mid \tau_u \mid \leq L \quad \text{for all } u \in C^{n-1}([0,1]).$$

So, we conclude that the operator $\mathcal{T}(C^{n-1}([0,1]))$ is bounded in $C^{n-1}([0,1])$ and, by Schauder's fixed point theorem, the operator $\mathcal{T}$ has a fixed point u_1. If $n = 2$, the proof is similar.

Step 3: *$u_1(x)$ is a solution of the problem (1–2).*

To show that this function $u_1(x)$ is a solution of the original problem (1–2), it suffices to prove that

$$\alpha^{(n-2)}(0) \le g_1^{-1}\left(B + g_2\left(u_1^{(n-1)}(0)\right)\right) \le \beta^{(n-2)}(0) \tag{26}$$

and

$$\alpha^{(n-2)}(1) \le g_3^{-1}\left(C - g_4\left(u_1^{(n-1)}(1)\right)\right) \le \beta^{(n-2)}(1). \tag{27}$$

For the first part, assume for the sake of a contradiction, that

$$g_1^{-1}\left(B + g_2\left(u_1^{(n-1)}(0)\right)\right) < \alpha^{(n-2)}(0).$$

Therefore, by (22), $u_1^{(n-2)}(0) = \alpha^{(n-2)}(0)$, and by (23), $u_1^{(n-1)}(0) \ge \alpha^{(n-1)}(0)$. By the monotonicity assumptions on the functions g_1 and g_2, we have

$$g_1\left(\alpha^{(n-2)}(0)\right) > B + g_2\left(u_1^{(n-1)}(0)\right) \ge B + g_2\left(\alpha^{(n-1)}(0)\right),$$

which contradicts (16). Thus, $\alpha^{(n-2)}(0) \le g_1^{-1}\left(B + g_2\left(u_1^{(n-1)}(0)\right)\right)$. By the same method, we can show that $g_1^{-1}\left(B + g_2\left(u_1^{(n-1)}(0)\right)\right) \le \beta^{(n-2)}(0)$.

Similar arguments can be used to prove (27). □

Example

Let

$$\phi(x) = \begin{cases} \arctan(x-5) + 125, & x > 5, \\ x^3, & -5 \le x \le 5, \\ \arctan(x+5) - 125, & x < -5, \end{cases}$$

and consider the boundary value problem consisting of the differential equation

$$-\left(\phi(u^{(n-1)}(x))\right)' = (u(x))^3 - 2\left(u^{(n-2)}(x)\right)^{2m-1} + \left|u^{(n-1)}(x)\right|^{\theta}, \tag{28}$$

with $n \ge 3$, $m \in \mathbb{N}$, $0 \le \theta \le 2$, and the boundary conditions

$$\begin{aligned} &u^{(i)}(0) = 0, \quad i = 0, \ldots, n-3, \\ &u^{(n-2)}(0) - \left(u^{(n-1)}(0)\right)^3 = B, \\ &\left(u^{(n-2)}(1)\right)^{2k+1} + e^{u^{(n-1)}(1)} = C, \end{aligned} \tag{29}$$

where $k \in \mathbb{N}$, $-1 \le B \le 1$, $0 \le C \le 2$, and either $B \ne 0$ or $C \ne 1$.

Clearly, (28–29) is a particular case of problem (1–2) with

$$f(x, y_0, \dots, y_{n-1}) = (y_0)^3 - 2(y_{n-2})^{2m-1} + |y_{n-1}|^\theta,$$
$$A_i = 0, \quad i = 0, \dots, n-3,$$
$$g_1(z) = z, \quad g_2(z) = z^3, \quad g_3(z) = z^{2k+1}, \quad g_4(z) = e^z.$$

We note that the existence results for the ϕ-Laplacian with the assumption $\phi(\mathbb{R}) = \mathbb{R}$, are not applicable to Eq. 28. The polynomial functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ given by

$$\alpha(x) = -\frac{x^{n-2}}{(n-2)!} \quad \textit{and} \quad \beta(x) = \frac{x^{n-2}}{(n-2)!}$$

are, respectively, lower and upper solutions of (28–29). We see that f is a L^1-Carathéodory function and satisfies the Nagumo-type growth condition in the set

$$E_* := \left\{ \begin{array}{c} (x, y_0, \dots, y_{n-1}) \in [0,1] \times \mathbb{R}^n : -\frac{x^{n-2}}{(n-2)!} \le y_0 \le \frac{x^{n-2}}{(n-2)!}, \\ -x \le y_{n-3} \le x, \quad -1 \le y_{n-2} \le 1 \end{array} \right\},$$

with $h_E(|y_{n-1}|) := 3 + |y_{n-1}|^\theta$. In fact, it is easy to see that $r = 2$. If we choose $R = 3$, then

$$\int_{\phi(r)}^{\phi(R)} \frac{|\phi^{-1}(s)|}{h_E(|\phi^{-1}(s)|)} ds = \int_8^{27} \frac{s^{\frac{1}{3}}}{3 + s^{\frac{\theta}{3}}} ds \ge \int_8^{27} \frac{2}{3 + 3^\theta} ds > 2$$

since $0 \le \theta \le 2$. Therefore, by Theorem 6, there is a nontrivial solution u of the problem (28–29) such that

$$\alpha^{(i)}(x) \le u^{(i)}(x) \le \beta^{(i)}(x)$$

for every $x \in [0, 1]$ and $i = 0, \dots, n-2$. Moreover, by the location part, this solution u can not be a polynomial of order greater than $n-3$. For example, if $n = 4$, this solution can not be a parabola.

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References

1. Cabada, A., Minhós, F.: Fully nonlinear fourth order equations with functional boundary conditions. J. Math. Anal. Appl. **340**, 239–251 (2008)
2. Graef, J.R., Kong, L.: Existence solutions for nonlinear boundary value problems. Commun. Appl. Nonlinear Anal. **14**, 39–60 (2007)
3. Graef, J.R., Kong, L., Yang, B.: Existence for a higher order multi-point boundary value problem. Results Math. **53**, 77–101 (2009)
4. Graef, J.R., Kong, L., Kong, Q.: Higher order multi-point boundary value problems. Math. Nachr. (to appear)
5. Grossinho, M.R., Minhós, F.: Upper and lower solutions for higher order boundary value problems. Nonlinear Stud. **12**, 165–176 (2005)
6. Liu, Y.: Solutions of two-point boundary value problems for even-order differential equations. J. Math. Anal. Appl. **323**, 721–740 (2006)
7. Liu, Y.: An existence result for solutions of nonlinear Sturm-Liouville boundary value problems for higher order p-Laplacian differential equations. Rocky Mountain J. Math. **39**, 147–163 (2009)

8. Ma, Y.: Existence of positive solutions of Lidstone boundary value problems. J. Math. Anal. Appl. **314**, 97–108 (2006)
9. Minhós, F., Santos, A.I.: Higher order two-point boundary value problems with asymmetric growth. Discret. Continuous Dynam. Syst. Ser. S **1**, 127–137 (2008)
10. Pang, C., Dong, W., Wei, Z.: Green's function and positive solutions of nth order m-point boundary value problem. Appl. Math. Comput. **182**, 1231–1239 (2006)
11. Wang, M., Cabada, A., Nieto, J.J.: Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions. Ann. Polon. Math. **58**, 221–235 (1993)
12. Wong, F.H.: An application of Schauder's fixed point theorem with respect to higher order BVPs. Proc. Am. Math. Soc. **126**, 2389–2397 (1998)
13. Xu, F., Liu, L., Wu, Y.: Multiple positive solutions of four-point nonlinear boundary value problems for a higher-order p-Laplacian operator of all derivatives. Nonlinear Anal. **71**, 4309–4319 (2009)