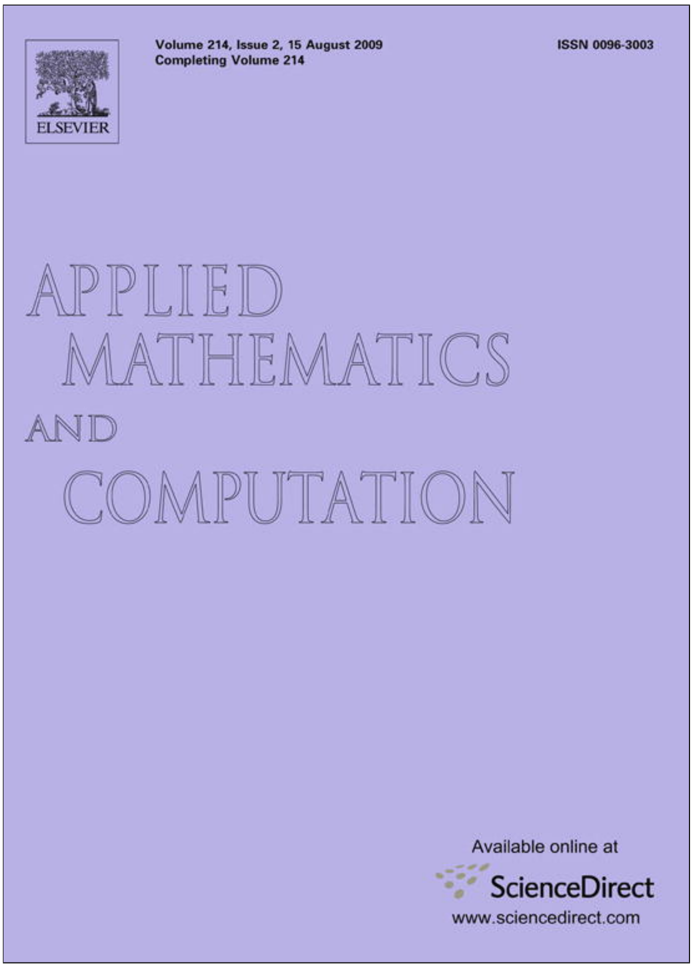
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A discrete fourth-order Lidstone problem with parameters

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ABSTRACT

Various existence, multiplicity, and nonexistence results for nontrivial solutions to a nonlinear discrete fourth-order Lidstone boundary value problem with dependence on two parameters are given, using a symmetric Green's function approach. An existence result is also given for a related semipositone problem, thus relaxing the usual assumption of nonnegativity on the nonlinear term.

1. Introduction to the fourth-order discrete problem

Recently there has been a large amount of attention paid to fourth-order differential equations that arise from various beam problems [4,6,11,16–19,21]. Similarly there has been a parallel interest in results for the analogous discrete fourth-order problem, for example [5,20], and in particular the discrete problem with Lidstone boundary conditions [1,12–15]. In what follows we seek to enrich the discussion found in the above cited literature by exploring two additional aspects of the discrete fourth-order Lidstone problem heretofore not considered, namely explicit dependence on two parameters and a semipositone result (relaxing the nonnegative assumption on the nonlinearity).

With this goal in mind, we introduce the nonlinear discrete fourth-order Lidstone boundary value problem with explicit parameters β and λ given by

$$\begin{cases} \Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = \lambda f(t, y(t)), & t \in [a+1, b-1]_{\mathbb{Z}}, \\ y(a) = 0 = \Delta^2 y(a-1), & y(b) = 0 = \Delta^2 y(b-1), \end{cases} \tag{1.1}$$

where Δ is the usual forward difference operator given by $\Delta y(t) = y(t+1) - y(t)$, $\Delta^n y(t) = \Delta^{n-1}(\Delta y(t))$, $[c,d]_{\mathbb{Z}} := \{c, c+1, \ldots, d-1, d\}$, and $\beta > 0$ and $\lambda > 0$ are real parameters; specific assumptions on the nature of the nonlinearity f will be made clear in the sequel. Boundary value problem (1.1) can be viewed as a discretization of the differential equations case studied in the papers cited previously and the references therein. Indeed, over the real unit interval $[0,1]$, the boundary value problem (1.1) becomes

$$\begin{cases} y^{(4)} - \beta y'' = \lambda f(t, y), & 0 \leqslant t \leqslant 1, \\ y(0) = 0 = y''(0), & y(1) = 0 = y''(1). \end{cases}$$

In our discrete version we will employ a symmetric Green's function approach, and apply fixed point theorems due to Krasnosel'skiĭ, and Leggett and Williams.

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The paper will proceed as follows: In Section 2 we construct the necessary Green's functions. Section 3 gives existence results for at least one, two, or no solutions of (1.1) in terms of λ. The existence of at least three solutions is discussed in Section 4, followed by an existence result for a related semipositone problem in Section 5.

2. Preliminary results

In this section we will find symmetric expressions for the corresponding Green's functions for a factored form of the difference equation in the first line of (1.1) with boundary conditions in the second line of (1.1) in such a way that we can find bounds on it for later use. The kernel of the summation operator will have explicit dependence on the parameter β, a first for this type of discussion.

Lemma 2.1. *Let $h : [a+1, b-1]_{\mathbb{Z}} \to \mathbb{R}$ be a function. Then the linear discrete fourth-order Lidstone boundary value problem*

$$\begin{cases} \Delta^4 y(t-2) - \beta\Delta^2 y(t-1) = h(t), & t \in [a+1, b-1]_{\mathbb{Z}}, \\ y(a) = 0 = \Delta^2 y(a-1), \quad y(b) = 0 = \Delta^2 y(b-1), \end{cases} \tag{2.1}$$

has solution

$$y(t) = \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) h(z), \quad t \in [a-1, b+1]_{\mathbb{Z}}, \tag{2.2}$$

where $G_2(t,s)$ given by

$$G_2(t,s) = \frac{1}{\ell(1,0)\ell(b,a)} \begin{cases} \ell(t,a)\ell(b,s): & t \leqslant s, \\ \ell(s,a)\ell(b,t): & s \leqslant t, \end{cases} \quad (t,s) \in [a-1, b+1]_{\mathbb{Z}} \times [a,b]_{\mathbb{Z}} \tag{2.3}$$

with $\ell(t,s) = \mu^{t-s} - \mu^{s-t}$ for $\mu = \frac{\beta+2+\sqrt{\beta(\beta+4)}}{2}$ is the Green's function for the second-order discrete boundary value problem

$$\begin{cases} -\left(\Delta^2 y(t-1) - \beta y(t)\right) = 0, & t \in [a,b]_{\mathbb{Z}}, \\ y(a) = 0 = y(b), \end{cases} \tag{2.4}$$

and $G_1(s,z)$ given by

$$G_1(s,z) = \frac{1}{b-a} \begin{cases} (s-a)(b-z): & s \leqslant z, \\ (z-a)(b-s): & z \leqslant s, \end{cases} \quad (s,z) \in [a,b]_{\mathbb{Z}} \times [a+1, b-1]_{\mathbb{Z}} \tag{2.5}$$

is the Green's function for the second-order discrete boundary value problem

$$\begin{cases} -\Delta^2 u(s-1) = 0, & s \in [a+1, b-1]_{\mathbb{Z}}, \\ u(a) = 0 = u(b). \end{cases} \tag{2.6}$$

If h is symmetric on $[a+1, b-1]_{\mathbb{Z}}$, then the solution (2.2) is likewise symmetric on $[a-1, b+1]_{\mathbb{Z}}$.

Proof. Let y be a solution of boundary value problem (2.1). Note that the difference equation in the top line of (2.1) can be written as

$$\Delta^2\left(\Delta^2 y(t-2) - \beta y(t-1)\right) = h(t), \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

If we let $u(t) = -(\Delta^2 y(t-1) - \beta y(t))$, then $\Delta^2 u(t-1) = -h(t)$, with $u(a) = 0$ and $u(b) = 0$ and

$$-\left(\Delta^2 y(t-1) - \beta y(t)\right) = u(t), \quad y(a) = 0 = y(b).$$

By [7, Example 6.12], the Green's function for (2.6) is given by (2.5), so that

$$u(t) = \sum_{z=a+1}^{b-1} G_1(t,z) h(z), \quad t \in [a,b]_{\mathbb{Z}}.$$

Now let

$$\ell(t,s) = \mu^{t-s} - \mu^{s-t} \quad \text{for } \mu = \frac{\beta + 2 + \sqrt{\beta(\beta+4)}}{2}, \tag{2.7}$$

and consider

$$y(t) = \sum_{s=a}^{b} G_2(t,s) u(s) = \frac{\ell(b,t)}{L} \sum_{s=a}^{t-1} \ell(s,a) u(s) + \frac{\ell(t,a)}{L} \sum_{s=t}^{b} \ell(b,s) u(s),$$

where $L = \ell(1,0)\ell(b,a)$. Using the product rule for differences, we have

$$\Delta y(t) = \frac{\Delta\ell(b,t)}{L}\sum_{s=a}^{t}\ell(s,a)u(s) + \frac{\Delta\ell(t,a)}{L}\sum_{s=t+1}^{b}\ell(b,s)u(s),$$

from which it follows that

$$\Delta^2 y(t-1) - \beta y(t) = \frac{u(t)}{L}[\ell(t,a)\Delta\ell(b,t) - \ell(b,t)\Delta\ell(t,a)] + \frac{1}{L}\left[\Delta^2\ell(b,t-1) - \beta\ell(b,t)\right]\sum_{s=a}^{t-1}\ell(s,a)u(s)$$
$$+\frac{1}{L}\left[\Delta^2\ell(t-1,a) - \beta\ell(t,a)\right]\sum_{s=t}^{b}\ell(b,s)u(s).$$

By the definition of ℓ in terms of β this simplifies to

$$\Delta^2 y(t-1) - \beta y(t) = \frac{u(t)}{L}[\ell(t,a)\ell(b,t+1) - \ell(b,t)\ell(t+1,a)] = \frac{-Lu(t)}{L} = -u(t).$$

The result follows. □

Lemma 2.2. *Let $h : [a+1,b-1]_{\mathbb{Z}} \to \mathbb{R}$ be a function, and let y be the solution of (2.1). Then*

$$y(t) \geqslant \sigma\|y\| \quad \text{for } t \in [a+1,b-1]_{\mathbb{Z}},$$

where $\|y\| = \max_{t\in[a-1,b+1]_{\mathbb{Z}}}|y(t)|$ and

$$\sigma := \frac{4\ell^2(1,0)\ell(b,a+1)}{(b-a)^2\ell(b,a)\ell^2(b/2,a/2)}, \tag{2.8}$$

where ℓ is given in (2.7) in terms of β.

Proof. Let $h : [a+1,b-1]_{\mathbb{Z}} \to \mathbb{R}$ be a function, and let y be the solution of (2.1). By Lemma 2.1 we have that

$$y(t) = \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1}G_2(t,s)G_1(s,z)h(z), \quad t \in [a-1,b+1]_{\mathbb{Z}},$$

since

$$G_2(t,a)G_1(a,z) = 0 = G_2(t,b)G_1(b,z) \quad \text{for all } (t,z) \in [a-1,b+1]_{\mathbb{Z}} \times [a+1,b-1]_{\mathbb{Z}}.$$

Since $y(a) = y(b) = 0$, $y(a-1) = -y(a+1)$ and $y(b+1) = -y(b-1)$, the maximum of y occurs on $[a+1,b-1]_{\mathbb{Z}}$. Thus for $(t,s,z) \in [a+1,b-1]^3_{\mathbb{Z}}$ we have that

$$G_2(t,s)G_1(s,z) \leqslant G_2(s,s)G_1(z,z) \leqslant \frac{\ell^2(b/2,a/2)}{\ell(1,0)\ell(b,a)}\cdot\frac{b-a}{4},$$

where we are allowing ℓ to be evaluated as a function over the real line, not just over the integers. Likewise

$$G_2(t,s)G_1(s,z) \geqslant \frac{\min\{\ell(t,a),\ell(b,t)\}}{\ell(b,a)}G_2(s,s)\cdot\frac{\min\{s-a,b-s\}}{b-a-1}G_1(z,z) \geqslant \frac{\ell(1,0)}{\ell(b,a)}G_2(s,s)\cdot\frac{1}{b-a-1}G_1(z,z)$$
$$\geqslant \frac{\ell(1,0)\ell(b,a+1)}{\ell^2(b,a)}\cdot\frac{1}{b-a}.$$

Thus, if we define

$$m := \frac{\ell(1,0)\ell(b,a+1)}{(b-a)\ell^2(b,a)} \tag{2.9}$$

and

$$M := \frac{(b-a)\ell^2(b/2,a/2)}{4\ell(1,0)\ell(b,a)}, \tag{2.10}$$

then

$$y(t) \geqslant \frac{m}{M}\|y\| = \sigma\|y\|, \quad t \in [a+1,b-1]_{\mathbb{Z}},$$

and the proof is complete. □

3. Existence of one or two solutions

Let $\mathscr{S}$ denote the Banach space of real-valued functions on $[a-1,b+1]_{\mathbb{Z}}$, with the maximum norm $\|y\| = \max_{t\in[a-1,b+1]_{\mathbb{Z}}} | y(t) |$. For σ as in (2.8), define the cone $\mathscr{P} \subset \mathscr{S}$ via

$$\mathscr{P} := \{y \in \mathscr{S} : y(a) = 0 = y(b), y \not\equiv 0, y(t) \geqslant \sigma\|y\|, t \in [a+1,b-1]_{\mathbb{Z}}\}. \tag{3.1}$$

Define for $t \in [a-1,b+1]_{\mathbb{Z}}$ the functional operator A_λ by

$$A_\lambda y(t) := \lambda \sum_{s=a}^{b} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)f(z,y(z)),$$

where $G_2(t,s)$ and $G_1(s,z)$ are the Green's functions given in (2.3) and (2.5), respectively. Since

$$G_2(t,a)G_1(a,z) = 0 = G_2(t,b)G_1(b,z) \quad \text{for all } (t,z) \in [a-1,b+1]_{\mathbb{Z}} \times [a+1,b-1]_{\mathbb{Z}},$$

we have

$$A_\lambda y(t) = \lambda \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)f(z,y(z)), \tag{3.2}$$

and by Lemma 2.1 the fixed points of A_λ are solutions of boundary value problem (1.1). We first employ below the following theorem, due to Krasnosel'skiĭ [8].

Theorem 3.1. *Let $\mathscr{S}$ be a Banach space, $\mathscr{P} \subseteq \mathscr{S}$ be a cone, and suppose that Ω_1, Ω_2 are bounded open balls of $\mathscr{S}$ centered at the origin, with $\overline{B}_1 \subset \Omega_2$. Suppose further that $\mathscr{A} : \mathscr{P} \cap (\overline{B}_2 \setminus \Omega_1) \to \mathscr{P}$ is a completely continuous operator such that either*

$$\|\mathscr{A}u\| \leqslant \|u\|, u \in \mathscr{P} \cap \partial\Omega_1 \quad \text{and} \quad \|\mathscr{A}u\| \geqslant \|u\|, u \in \mathscr{P} \cap \partial\Omega_2,$$

or

$$\|\mathscr{A}u\| \geqslant \|u\|, u \in \mathscr{P} \cap \partial\Omega_1 \quad \text{and} \quad \|\mathscr{A}u\| \leqslant \|u\|, u \in \mathscr{P} \cap \partial\Omega_2$$

holds. Then $\mathscr{A}$ has a fixed point in $\mathscr{P} \cap (\overline{B}_2 \setminus \Omega_1)$.

Before we proceed to our first results we must mention some of the conditions we will impose on the nonlinearity f in (1.1). We note here that in the remainder of this section we assume some combination of

(H_1) $f : [a+1,b-1]_{\mathbb{Z}} \times [0,\infty) \to [0,\infty)$ is continuous with $f(\cdot,y) > 0$ for $y > 0$;
(H_2) $f(t,y) = g(t)w(y)$, where $g : [a+1,b-1]_{\mathbb{Z}} \to [0,\infty)$ with $\sum_{z=a+1}^{b-1} g(z) > 0$, and $w : [0,\infty) \to [0,\infty)$ is continuous with $w(y) > 0$ for $y > 0$;
(H_3) $f(t,y) = g(t)w(y)$, where g as in (H_2), $w : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing, and there exists $\theta \in (0,1)$ such that $w(\kappa y) \geqslant \kappa^\theta w(y)$ for $\kappa \in (0,1)$ and $y \in [0,\infty)$.

Theorem 3.2. *Assume* (H_1). *Suppose further that there exist positive numbers $0 < r < R < \infty$ such that for all $t \in [a+1,b-1]_{\mathbb{Z}}$, the nonlinearity f satisfies*

(H_5) $f(t,y) \leqslant \frac{y}{\lambda M(b-a-1)^2}$ *for* $y \in [0,r]$, *and* $f(t,y) \geqslant \frac{y}{\lambda\sigma m(b-a-1)^2}$ *for* $y \in [R,\infty)$.

Then (1.1) has a nontrivial solution y such that, for σ as in (2.8),

$$\sigma r \leqslant y(t) \leqslant R/\sigma, \quad t \in [a+1,b-1]_{\mathbb{Z}}.$$

Proof. If $y \in \mathscr{P}$, then $A_\lambda y(a) = 0 = A_\lambda y(b)$ by Lemma 2.1, and $A_\lambda y(t) \geqslant \sigma | A_\lambda y |$ for $t \in [a+1,b-1]_{\mathbb{Z}}$ by Lemma 2.2. Consequently, $A_\lambda(\mathscr{P}) \subset \mathscr{P}$. Moreover, A_λ is completely continuous using standard arguments. Define bounded open balls centered at the origin by

$$\Omega_1 = \{y \in \mathscr{S} : \|y\| < r\}, \qquad \Omega_2 = \{y \in \mathscr{S} : \|y\| < R'\},$$

where $R' := MR/m$. Then $0 \in \Omega_1 \subset \Omega_2$. For $y \in \mathscr{P} \cap \partial\Omega_1$ so that $| y |= r$, we have

$$A_\lambda y(t) = \lambda \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)f(z,y(z)) \overset{(2.10)}{\leqslant} \lambda M \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} f(z,y(z)) \leqslant \frac{1}{(b-a-1)^2} \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} y(z) \leqslant \|y\|$$

for $t \in [a+1,b-1]_{\mathbb{Z}}$. Thus, $| A_\lambda y |\leqslant| y |$ for $y \in \mathscr{P} \cap \partial\Omega_1$. Similarly, let $y \in \mathscr{P} \cap \partial\Omega_2$, so that $| y |= R'$. Then

$$y(z) \geqslant \sigma\|y\| = \frac{m}{M}R' = R, \quad z \in [a+1,b-1]_{\mathbb{Z}},$$

and

$$A_\lambda y(t) \overset{(2.9)}{\geqslant} \lambda m \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} f(z,y(z)) \geqslant \frac{1}{\sigma(b-a-1)^2}\sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} y(z) \geqslant \|y\|.$$

Consequently, $\|A_\lambda y\| \geqslant \|y\|$ for $y \in \mathscr{P} \cap \partial\Omega_2$. By Theorem 3.1, A_λ has a fixed point $y \in \mathscr{P} \cap (\overline{B}_{R'} \setminus \Omega_1)$, which is a nontrivial solution of (1.1), such that $r \leqslant \|y\| \leqslant R'$. Using the fact that $y \in \mathscr{P}$ and the definition of σ in (2.8), the bounds on y follow. □

The proof of the next theorem is similar to that just completed.

Theorem 3.3. *Assume* (H_1). *In addition, suppose that there exist positive numbers* $0 < r < R < \infty$ *such that for all* $t \in [a+1, b-1]_{\mathbb{Z}}$, *the nonlinearity f satisfies*

(H_6) $f(t,y) \leqslant \frac{y}{\lambda M(b-a-1)^2}$ *for* $y \in [R,\infty)$, *and* $f(t,y) \geqslant \frac{y}{\lambda\sigma m(b-a-1)^2}$ *for* $y \in [0,r]$.

Then (1.1) has a nontrivial solution y such that

$$\sigma r \leqslant y(t) \leqslant R/\sigma, \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

With an additional assumption one can prove the existence of at least two nontrivial solutions to (1.1). The proofs are modifications of the proof in Theorem 3.2and are omitted.

Theorem 3.4. *Assume* (H_1). *In addition, suppose that there exist positive numbers* $0 < r < N < R < \infty$ *such that for all* $t \in [a+1, b-1]_{\mathbb{Z}}$, *the nonlinearity f satisfies*

(H_7) $f(t,y) < \frac{N}{\lambda M(b-a-1)^2}$ *for* $y \in [\sigma N, N]$, *and* $f(t,y) \geqslant \frac{y}{\lambda\sigma m(b-a-1)^2}$ *for* $y \in [0,r] \cup [R,\infty)$.

Then (1.1) has at least two nontrivial solutions y_1, y_2 *such that* $\|y_1\| < N < \|y_2\|$, *and*

$$\sigma r \leqslant y_1(t) < N, \quad \sigma N < y_2(t) \leqslant R/\sigma, \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

Theorem 3.5. *Assume* (H_1). *In addition, suppose that there exist positive numbers* $0 < r < N < R < \infty$ *such that for* $t \in [a+1, b-1]_{\mathbb{Z}}$, *the nonlinearity f satisfies*

(H_8) $f(t,y) > \frac{N}{\lambda\sigma m(b-a-1)^2}$ *for* $y \in [\sigma N, N]$, *and* $f(t,y) \leqslant \frac{y}{\lambda M(b-a-1)^2}$ *for* $y \in [0,r] \cup [R,\infty)$.

Then (1.1) has at least two nontrivial solutions y_1, y_2 *such that* $\|y_1\| < N < \|y_2\|$, *and*

$$\sigma r \leqslant y_1(t) < N, \quad \sigma N < y_2(t) \leqslant R/\sigma, \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

The next theorem allows us to summarize the above results thus far in terms of λ.

Theorem 3.6. *Assume* (H_1). *For* $t \in [a+1, b-1]_{\mathbb{Z}}$, *define*

$$f_0(t) := \lim_{y\to 0^+} \frac{f(t,y)}{y}, \qquad f_\infty(t) := \lim_{y\to\infty} \frac{f(t,y)}{y}. \tag{3.3}$$

Then we have the following statements for $t \in [a+1, b-1]_{\mathbb{Z}}$.

(a) *If* $f_0(t) = 0$ *and* $f_\infty(t) = \infty$, *then (1.1) has a nontrivial solution for all* $\lambda \in (0,\infty)$.
(b) *If* $f_0(t) = \infty$ *and* $f_\infty(t) = 0$, *then (1.1) has a nontrivial solution for all* $\lambda \in (0,\infty)$.
(c) *If* $f_0(t) = f_\infty(t) = \infty$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has at least two nontrivial solutions for* $0 < \lambda < \lambda_0$.
(d) *If* $f_0(t) = f_\infty(t) = 0$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has at least two nontrivial solutions for* $\lambda > \lambda_0$.
(e) *If* $f_0(t), f_\infty(t) < \infty$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has no nontrivial solutions for* $0 < \lambda < \lambda_0$.
(f) *If* $0 < f_0(t), f_\infty(t)$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has no nontrivial solutions for* $\lambda > \lambda_0$.

Proof. If $f_0(t) = 0$ and $f_\infty(t) = \infty$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$, then (H_5) is satisfied for sufficiently small $r > 0$ and sufficiently large $R > 0$. If $f_0(t) = \infty$ and $f_\infty(t) = 0$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$, then (H_6) is satisfied. Likewise if $f_0(t) = f_\infty(t) = \infty$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$, then (H_7) holds for $\lambda > 0$ sufficiently small, and if $f_0(t) = f_\infty(t) = 0$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$, then (H_8) holds if λ is sufficiently large. To see (e), since $f_0(t)$ and $f_\infty(t) < \infty$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$, there exist positive constants η_1, η_2, r, and R such that $r < R$ and

$$f(t,y) \leqslant \eta_1 y \quad \text{for } y \in [0,r],$$
$$f(t,y) \leqslant \eta_2 y \quad \text{for } y \in [R,\infty)$$

for all $t \in [a+1, b-1]_{\mathbb{Z}}$. Let $\eta > 0$ be given by

$$\eta = \max\left\{\eta_1, \eta_2, \max_{y\in[r,R], t\in[a+1,b-1]_{\mathbb{Z}}} \frac{f(t,y)}{y}\right\}.$$

Then $f(t,y) \leqslant \eta y$ for all $y \in (0,\infty)$ and all $t \in [a+1, b-1]_{\mathbb{Z}}$. If x is a nontrivial solution of (1.1), then since $A_\lambda x = x$, we have

$$\|x\| = \|A_\lambda x\| \leqslant \lambda\eta M \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} x(z) \leqslant \lambda\eta M(b-a-1)^2\|x\| < \|x\|$$

for $0 < \lambda < 1/(\eta M(b-a-1)^2)$, a contradiction. The proof of part (f) is similar and thus omitted. □

The final two theorems in this section allow us to substitute either hypothesis (H_2) or (H_3) for (H_1).

Theorem 3.7. *Assume* (H_2). *For* $t \in [a+1, b-1]_{\mathbb{Z}}$, *define*

$$w_0 := \lim_{y\to 0^+} \frac{w(y)}{y}, \qquad w_\infty := \lim_{y\to\infty} \frac{w(y)}{y}. \tag{3.4}$$

Then we have the following statements.

(a) *If* $w_0 = 0$ *or* $w_\infty = 0$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has a nontrivial solution for* $\lambda > \lambda_0$.
(b) *If* $w_0 = \infty$ *or* $w_\infty = \infty$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has a nontrivial solution for* $0 < \lambda < \lambda_0$.
(c) *If* $w_0 = w_\infty = 0$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has at least two nontrivial solutions for* $\lambda > \lambda_0$.
(d) *If* $w_0 = w_\infty = \infty$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has at least two nontrivial solutions for* $0 < \lambda < \lambda_0$.
(e) *If* $w_0, w_\infty < \infty$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has no nontrivial solutions for* $0 < \lambda < \lambda_0$.
(f) *If* $w_0, w_\infty > 0$, *then there exists* $\lambda_0 > 0$ *such that (1.1) has no nontrivial solutions for* $\lambda > \lambda_0$.

Theorem 3.8. *Assume* (H_3). *Then, for any* $\lambda \in (0,\infty)$, *(1.1) has a unique solution* y_λ. *Furthermore, such a solution* y_λ *satisfies the following properties:*

(i) y_λ *is nondecreasing in* λ;
(ii) $\lim_{\lambda\to 0^+}\|y_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty}\|y_\lambda\| = \infty$;
(iii) y_λ *is continuous in* λ, *that is, if* $\lambda \to \lambda_0$, *then* $\|y_\lambda - y_{\lambda_0}\| \to 0$.

Proof. This proof is modelled after [3, Theorem 2.2]. We first show that (1.1) has a solution for any fixed $\lambda \in (0,\infty)$. From (H_3) we see that A_λ is nondecreasing, and for $t \in [a,b]_{\mathbb{Z}}$ satisfies

$$A_\lambda(\kappa y(t)) = \lambda \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)g(z)w(\kappa y(z)) \geqslant \kappa^\theta\lambda \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)g(z)w(y(z)) = \kappa^\theta A_\lambda y(t) \tag{3.5}$$

for $y \in \mathscr{P}$. Let

$$L_\lambda = \lambda(b-a-1)\sum_{z=a+1}^{b-1} g(z), \tag{3.6}$$

and let

$$\bar{y}(t) = \begin{cases} -L_\lambda: & t = a-1, b-1, \\ 0: & t = a, b, \\ L_\lambda: & t \in [a+1, b-1]_{\mathbb{Z}}. \end{cases}$$

Then $\bar{y} \in \mathscr{P}$ and $\bar{y}(t) > 0$ for $t \in [a+1, b-1]_{\mathbb{Z}}$. By (H_3) we have $w(0) > 0$ with w nondecreasing. Thus for $t \in [a+1, b-1]_{\mathbb{Z}}$,

$$A_\lambda\bar{y}(t) \geqslant \lambda m w(0) \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} g(z) = m w(0) L_\lambda$$

for m in (2.9) and L_λ in (3.6), and

$$A_\lambda\bar{y}(t) \leqslant \lambda M w(\bar{y}) \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} g(z) = M w(\bar{y}) L_\lambda$$

for M in (2.10). Thus

$$m w(0) L_\lambda \leqslant A_\lambda\bar{y}(t) \leqslant M w(L_\lambda) L_\lambda, \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

Define c^* and d_* via

$$c^* = \sup\{x : xL_\lambda \leqslant A_\lambda \bar{y}(t)\} \quad \text{and} \quad d_* = \inf\{x : A_\lambda \bar{y}(t) \leqslant xL_\lambda\}.$$

Clearly $c^* \geqslant mw(0)$ and $d_* \leqslant Mw(L_\lambda)$. Choose c and d such that

$$0 < c < \min\left\{1, (c^*)^{\frac{1}{1-\theta}}\right\} \quad \text{and} \quad \max\left\{1, (d_*)^{\frac{1}{1-\theta}}\right\} < d < \infty.$$

Define two sequences $\{u_k(t)\}_{k=1}^{\infty}$ and $\{v_k(t)\}_{k=1}^{\infty}$ via

$$u_1(t) = \begin{cases} -cL_\lambda : & t = a-1, b-1, \\ 0 : & t = a, b, \\ cL_\lambda : & t \in [a+1, b-1]_{\mathbb{Z}}, \end{cases} \qquad u_{k+1}(t) = A_\lambda u_k(t), \quad t \in [a-1, b+1]_{\mathbb{Z}}, k \in \mathbb{N},$$

and

$$v_1(t) = \begin{cases} -dL_\lambda : & t = a-1, b-1, \\ 0 : & t = a, b, \\ dL_\lambda : & t \in [a+1, b-1]_{\mathbb{Z}}, \end{cases} \qquad v_{k+1}(t) = A_\lambda v_k(t), \quad t \in [a-1, b+1]_{\mathbb{Z}}, k \in \mathbb{N}.$$

From the monotonicity of A_λ and (3.5) we see that on $[a+1, b-1]_{\mathbb{Z}}$ we have

$$cL_\lambda = u_1 \leqslant u_2 \leqslant \cdots \leqslant u_k \leqslant \cdots \leqslant v_k \leqslant \cdots \leqslant v_2 \leqslant v_1 = dL_\lambda. \tag{3.7}$$

Let $\delta = c/d \in (0,1)$. We claim that

$$u_k(t) \geqslant \delta^{\theta^k} v_k(t), \quad t \in [a, b]_{\mathbb{Z}}. \tag{3.8}$$

Clearly $u_1 = \delta v_1$ on $[a-1, b+1]_{\mathbb{Z}}$. Assume (3.8) holds for $k = n$; then, from the monotonicity of A_λ and (3.5) we obtain

$$u_{n+1}(t) \geqslant A_\lambda u_n(t) \geqslant A_\lambda\left(\delta^{\theta^n} v_n(t)\right) \geqslant \left(\delta^{\theta^n}\right)^{\theta} A_\lambda v_n(t) = \delta^{\theta^{n+1}} v_{n+1}(t)$$

for $t \in [a, b]_{\mathbb{Z}}$. It follows from mathematical induction that (3.8) holds. From (3.7) and (3.8) we have

$$0 \leqslant u_{k+l}(t) - u_k(t) \leqslant v_k(t) - u_k(t) \leqslant \left(1 - \delta^{\theta^k}\right) v_1(t) = \left(1 - \delta^{\theta^k}\right) dL_\lambda$$

for $t \in [a, b]_{\mathbb{Z}}$, where l is a nonnegative integer. Hence

$$\|u_{k+l} - u_k\| \leqslant \|v_k - u_k\| \leqslant \left(1 - \delta^{\theta^k}\right) dL_\lambda.$$

As a result, there exists a function $y \in \mathscr{P}$ such that

$$\lim_{k\to\infty} u_k(t) = \lim_{k\to\infty} v_k(t) = y(t), \quad t \in [a-1, b+1]_{\mathbb{Z}},$$

and y is a nontrivial solution of (1.1). If there exist two nontrivial solutions y_1 and y_2 of (1.1), then $A_\lambda y_1(t) = y_1(t)$ and $A_\lambda y_2(t) = y_2(t)$ for $t \in [a-1, b+1]_{\mathbb{Z}}$. Then there exists an $\alpha > 0$ such that $y_1 \geqslant \alpha y_2$ on $[a, b]_{\mathbb{Z}}$; set $\alpha_0 = \sup\{\alpha : y_1(t) \geqslant \alpha y_2(t)\}$. Then $\alpha_0 \in (0, \infty)$, and $y_1(t) \geqslant \alpha_0 y_2(t)$ for $t \in [a, b]_{\mathbb{Z}}$. If $\alpha_0 < 1$, then, from (H_3), $w(\alpha_0 y_2(t)) > \alpha_0 w(y_2(t))$ on $[a, b]_{\mathbb{Z}}$. This, together with the monotonicity of w, implies that

$$y_1(t) = A_\lambda y_1(t) \geqslant A_\lambda(\alpha_0 y_2(t)) > \alpha_0 A_\lambda y_2(t) = \alpha_0 y_2(t), \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

Thus, we can find $\tau > 0$ such that $y_1(t) \geqslant (\alpha_0 + \tau) y_2(t)$ on $[a, b]_{\mathbb{Z}}$, which contradicts the definition of α_0. Hence, $y_1(t) \geqslant y_2(t)$ for $t \in [a, b]_{\mathbb{Z}}$. Similarly, we can show that $y_2(t) \geqslant y_1(t)$ for $t \in [a, b]_{\mathbb{Z}}$. Therefore, (1.1) has a unique solution.

Using exactly the same argument as in the second part of the proof of [10, Theorem 6], we can show that (i), (ii), and (iii) hold. The details are omitted here. This completes the proof of the theorem. □

4. Existence of three solutions

In this section we employ the Leggett–Williams Theorem [9] to establish the existence of at least three nontrivial solutions to (1.1). Before proceeding to the theorem, however, we first introduce some notation.

A map ψ is a nonnegative continuous concave functional on a cone $\mathscr{P}$ if it satisfies the following conditions:

(i) $\psi : \mathscr{P} \to [0, \infty)$ is continuous;
(ii) $\psi(\zeta x + (1-\zeta)y) \geqslant \zeta\psi(x) + (1-\zeta)\psi(y)$ for all $x, y \in \mathscr{P}$ and $0 \leqslant \zeta \leqslant 1$.

Take the same cone $\mathscr{P}$ in (3.1) as before, and let

$$\mathscr{P}_c := \{y \in \mathscr{P} : \|y\| < c\}$$

and

$$\mathscr{P}(\psi, q, d) := \{y \in \mathscr{P} : q \leqslant \psi(y),\ \|y\| \leqslant d\}.$$

The following theorem is due to Leggett and Williams [9].

Theorem 4.1. *Let $\mathscr{P}$ be a cone in the real Banach space $\mathscr{S}$, $A : \overline{\mathscr{P}_c} \to \overline{\mathscr{P}_c}$ be completely continuous and ψ be a nonnegative continuous concave functional on $\mathscr{P}$ with $\psi(y) \leqslant \|y\|$ for all $y \in \overline{\mathscr{P}_c}$. Suppose there exists $0 < p < q < d \leqslant c$ such that the following conditions hold:*

(i) $\{y \in \mathscr{P}(\psi, q, d) : \psi(y) > q\} \neq \emptyset$ and $\psi(Ay) > q$ for all $y \in \mathscr{P}(\psi, q, d)$;
(ii) $\|Ay\| < p$ for $\|y\| \leqslant p$;
(iii) $\psi(Ay) > q$ for $y \in \mathscr{P}(\psi, q, c)$ with $\|Ay\| > d$.

Then A has at least three fixed points y_1, y_2, and y_3 in $\overline{\mathscr{P}_c}$ satisfying:

$$\|y_1\| < p, \quad \psi(y_2) > q, \quad p < \|y_3\| \quad \text{with } \psi(y_3) < q.$$

Let the nonnegative continuous concave functional $\psi : \mathscr{P} \to [0, \infty)$ by defined by

$$\psi(y) = \min_{t \in [a+1, b-1]_{\mathbb{Z}}} y(t), \quad y \in \mathscr{P}; \tag{4.1}$$

note that for $y \in \mathscr{P}$, $0 < \psi(y) \leqslant \|y\|$ by the choice of cone $\mathscr{P}$ in (3.1).

Theorem 4.2. *Assume (H_1). Suppose that there exist constants $0 < p < q < q/\sigma \leqslant c$ such that, for $t \in [a+1, b-1]_{\mathbb{Z}}$,*

(H_9) $f(t, y) < \frac{p}{\lambda M(b-a-1)^2}$ *if* $y \in [0, p]$,
(H_{10}) $f(t, y) > \frac{q}{\lambda m(b-a-1)^2}$ *if* $y \in [q, q/\sigma]$,
(H_{11}) $f(t, y) \leqslant \frac{c}{\lambda M(b-a-1)^2}$ *if* $y \in [0, c]$,

where m and M are as defined in (2.9) and (2.10), respectively, and $\sigma = m/M$ as in (2.8). Then the boundary value problem (1.1) has at least three nontrivial solutions y_1, y_2, y_3 satisfying

$$\|y_1\| < p, \qquad q < \psi(y_2), \qquad \|y_3\| > p \quad \text{with } \psi(y_3) < q,$$

where ψ is given in (4.1).

Proof. Define the operator $A_\lambda : \mathscr{P} \to \mathscr{S}$ as in (3.2). As mentioned in the proof to Theorem 3.2, $A_\lambda : \mathscr{P} \to \mathscr{P}$ and A_λ is completely continuous. We now show that all of the conditions of Theorem 4.1 are satisfied. For all $y \in \mathscr{P}$ we have $\psi(y) \leqslant \|y\|$. If $y \in \overline{\mathscr{P}_c}$, then $\|y\| \leqslant c$ and assumption (H_{11}) implies $f(z, y(z)) \leqslant c/(\lambda M(b-a-1)^2)$ for $z \in [a+1, b-1]_{\mathbb{Z}}$. As a result,

$$\|A_\lambda y\| = \max_{t \in [a+1, b-1]_{\mathbb{Z}}} \lambda \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) f(z, y(z)) \leqslant \frac{c}{M(b-a-1)^2} \max_{t \in [a+1, b-1]_{\mathbb{Z}}} \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z)$$
$$\leqslant \frac{cM(b-a-1)^2}{M(b-a-1)^2} = c.$$

Therefore $A_\lambda : \overline{\mathscr{P}_c} \to \overline{\mathscr{P}_c}$. In the same way, if $y \in \overline{\mathscr{P}_p}$, then assumption (H_9) yields $f(t, y(t)) < \frac{p}{\lambda M(b-a-1)^2}$ for $t \in [a+1, b-1]_{\mathbb{Z}}$; as in the argument above, it follows that $A_\lambda : \overline{\mathscr{P}_p} \to \mathscr{P}_p$. Hence, condition (ii) of Theorem 4.1 is satisfied. To check condition (i) of Theorem 4.1, choose $y_{\mathscr{P}}(t) \equiv q/\sigma$ for $t \in [a+1, b-1]_{\mathbb{Z}}$. Then $y_{\mathscr{P}} \in \mathscr{P}(\psi, q, q/\sigma)$ and $\psi(y_{\mathscr{P}}) = \psi(q/\sigma) > q$, so that $\{y \in \mathscr{P}(\psi, q, q/\sigma) : \psi(y) > q\} \neq \emptyset$. Consequently, if $y \in \mathscr{P}(\psi, q, q/\sigma)$, then $q \leqslant y(z) \leqslant q/\sigma$ for $z \in [a+1, b-1]_{\mathbb{Z}}$. From assumption (H_{10}) we have that

$$f(z, y(z)) > \frac{q}{\lambda m(b-a-1)^2}$$

for all $z \in [a+1, b-1]_{\mathbb{Z}}$; we see that

$$\psi(A_\lambda y) = \min_{t \in [a+1, b-1]_{\mathbb{Z}}} \lambda \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) f(z, y(z)) > \frac{\lambda m q(b-a-1)^2}{\lambda m(b-a-1)^2} = q.$$

Thus we have

$$\psi(A_\lambda y) > q, \quad y \in \mathscr{P}(\psi, q, q/\sigma),$$

so that condition (*i*) of Theorem 4.1 holds. Lastly we consider Theorem 4.1 (*iii*). Suppose $y \in \mathscr{P}(\psi, q, c)$ with $\|A_\lambda y\| > q/\sigma$. By the definitions of ψ and the cone $\mathscr{P}$,

$$\psi(A_\lambda y) = \min_{t \in [a+1, b-1]_{\mathbb{Z}}} A_\lambda y(t) \geqslant \sigma \|A_\lambda y\| > \sigma q/\sigma = q.$$

An application of Theorem 4.1 yields the conclusion. □

5. Semipositone result

In this section we establish the existence of at least one nontrivial solution for the boundary value problem (1.1), with modified conditions on the nonlinearity f given as follows:

(H_{12}) $f : [a+1, b-1]_{\mathbb{Z}} \times [0, \infty) \to [0, \infty)$ is continuous, and there exist $t_1, t_2 \in (a+1, b-1)_{\mathbb{Z}}$ such that

$$\lim_{y \to \infty} \frac{f(t,y)}{y} = \infty$$

uniformly on $[t_1, t_2]_{\mathbb{Z}}$; (H_{13}) there exists $B > 0$ such that $f(t,y) \geqslant -B$ for all $t \in [a+1, b-1]_{\mathbb{Z}}$ and all $y \geqslant 0$.

We remark that (H_{12}) is a superlinear type of condition, whereas (H_{13}) allows $f(t,y)$ to be semipositone. The next lemma is needed in the derivation of the main result of this section. These techniques are modeled after Bai and Xu [2].

Lemma 5.1. *Let y_1 be the unique nontrivial solution of the linear boundary value problem*

$$\begin{cases} \Delta^4 y(t-2) - \beta \Delta^2 y(t-1) = 1, & t \in [a+1, b-1]_{\mathbb{Z}}, \\ y(a) = 0 = \Delta^2 y(a-1), \quad y(b) = 0 = \Delta^2 y(b-1). \end{cases} \tag{5.1}$$

Then,

$$y_1(t) \leqslant M^2 (b-a-1)^2 \sigma / m, \quad t \in [a+1, b-1]_{\mathbb{Z}}, \tag{5.2}$$

where σ is given in (2.8) , m is given in (2.9), and M is given in (2.10).

Proof. The conclusion is immediate from Lemma 2.2. □

We will now apply Theorem 3.1 to obtain our main result in this section. Again let the Banach space be denoted $\mathscr{S}$ and the cone $\mathscr{P}$ as in (3.1).

Theorem 5.2. *Assume (H_{12}) and (H_{13}). Let $r > 0$, and take y_1 as in Lemma 5.1. If*

$$0 < \lambda \leqslant \min\left\{ \frac{r}{k_r \|y_1\|}, \frac{rm}{M^2 (b-a-1)^2 B} \right\}, \tag{5.2}$$

where

$$k_r = \sup_{t \in [a+1, b-1]_{\mathbb{Z}},\ 0 \leqslant y \leqslant r} f(t,y) + B, \tag{5.3}$$

then the boundary value problem (1.1) has a nontrivial solution y^*.

Proof. Let $x(t) = \lambda B y_1(t)$. We will show that the following boundary value problem

$$\begin{cases} \Delta^4 y(t-2) - \beta \Delta^2 y(t-1) = \lambda F(t, y(t) - x(t)), & t \in [a+1, b-1]_{\mathbb{Z}}, \\ y(a) = 0 = \Delta^2 y(a-1), \quad y(b) = 0 = \Delta^2 y(b-1), \end{cases} \tag{5.4}$$

where

$$F(t,w) = \begin{cases} f(t,w) + B : & w \geqslant 0, \\ f(t,0) + B : & w \leqslant 0, \end{cases}$$

has a nontrivial solution. Thereafter we will obtain a nontrivial solution for the boundary value problem 1.1. The problem (5.4) is equivalent to the fixed point equation $A_\lambda y(t) = y(t)$ for A_λ given in (3.2). We will prove, by Theorem 3.1, that A_λ has a fixed point in $\mathscr{P}$.

We proceed as in the proof of Theorem 3.2. Let $\Omega_1 = \{y \in \mathscr{S} : \|y\| < r\}$; for $y \in \mathscr{P} \cap \partial\Omega_1$, we have $\|y\| = r$ and

$$A_\lambda y(t) = \lambda \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) F(z, y(z) - x(z)) \overset{(5.3)}{\leqslant} \lambda k_r \sum_{s=a+1}^{b-1} \sum_{z=a+1}^{b-1} G_2(t,s) G_1(s,z) = \lambda k_r y_1(t) \leqslant \lambda k_r \|y_1\| \overset{(5.2)}{\leqslant} r = \|y\|.$$

Thus, $\|A_\lambda y\| \leqslant \|y\|$ for $y \in \mathscr{P} \cap \partial\Omega_1$. Now let K be a positive real number such that

$$\frac{1}{2}\lambda\sigma K\left(\max_{t\in[a+1,b-1]_{\mathbb{Z}}}\sum_{s=a+1} b-1\sum_{z=t_1}^{t_2} G_2(t,s)G_1(s,z)\right) > 1. \tag{5.5}$$

In view of (H_{12}), there exists $J > 0$ such that for all $w \geqslant J$ and $t \in [t_1, t_2]_{\mathbb{Z}}$,

$$F(t,w) = f(t,w) + B \geqslant Kw. \tag{5.6}$$

Set $\Omega_2 = \{y \in \mathscr{S} : \|y\| < R\}$, where

$$R = r + \max\left\{2\lambda M^2(b-a-1)^2B/m, 2J/\sigma\right\}. \tag{5.7}$$

For $y \in \mathscr{P} \cap \partial\Omega_2$, we have $\|y\| = R$ and

$$x(t) = \lambda B y_1(t) \leqslant \lambda M^2(b-a-1)^2 B\sigma/m \leqslant \lambda M^2(b-a-1)^2 B \cdot \frac{y(t)}{mR}.$$

This implies for $t \in [a+1, b-1]_{\mathbb{Z}}$ that

$$y(t) - x(t) \geqslant \left(1 - \lambda M^2(b-a-1)^2 B\frac{1}{mR}\right)y(t) \geqslant \left(1 - \lambda M^2(b-a-1)^2 B\frac{1}{mR}\right)\sigma R;$$

by (5.7) it follows for $t \in [t_1, t_2]_{\mathbb{Z}}$ that

$$y(t) - x(t) \geqslant \left(1 - \lambda M^2(b-a-1)^2 B\frac{1}{mR}\right)\sigma R \geqslant \frac{1}{2}\sigma R \geqslant J. \tag{5.8}$$

Hence, by (5.6) and (5.8), we see that for $z \in [t_1, t_2]_{\mathbb{Z}}$,

$$F(z, y(z) - x(z)) \geqslant K(y(z) - x(z)) \geqslant \frac{1}{2}\sigma K R. \tag{5.9}$$

Applying (5.5) and (5.9), we find

$$\|A_\lambda y\| = \lambda \max_{t\in[a+1,b-1]_{\mathbb{Z}}} \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)F(z, y(z) - x(z)) \geqslant \lambda \max_{t\in[a+1,b-1]_{\mathbb{Z}}} \sum_{s=a+1}^{b-1}\sum_{z=t_1}^{t_2} G_2(t,s)G_1(s,z)F(z, y(z) - x(z))$$
$$\geqslant \frac{1}{2}\lambda\sigma K R \max_{t\in[a+1,b-1]_{\mathbb{Z}}} \sum_{s=a+1}^{b-1}\sum_{z=t_1}^{t_2} G_2(t,s)G_1(s,z) \geqslant R.$$

This shows that $\|A_\lambda y\| \geqslant R = \|y\|$ for $y \in \mathscr{P} \cap \partial\Omega_2$. It now follows from Theorem 3.1 that A_λ has a fixed point $u^* \in \mathscr{P}$ with $r \leqslant \|u^*\| \leqslant R$. Further, using (5.2) and Lemma 2.2, we get for $t \in [a+1, b-1]_{\mathbb{Z}}$ that

$$u^*(t) \geqslant \sigma\|u^*\| \geqslant r\sigma \geqslant \lambda M^2(b-a-1)^2 B\sigma/m \geqslant \lambda B y_1(t) = x(t).$$

Therefore, let us define

$$y^*(t) := u^*(t) - x(t) \geqslant 0, \quad t \in [a+1, b-1]_{\mathbb{Z}}.$$

We will prove that y^* is in fact a nontrivial solution of the boundary value problem (1.1). To see this, we have for $t \in [a+1, b-1]_{\mathbb{Z}}$ that $A_\lambda u^*(t) = u^*(t)$, or

$$\lambda \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)[f(z, u^*(z) - x(z)) + B] = u^*(t),$$

which, noting that $y_1(t) = \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)$, gives

$$\lambda \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)f(z, u^*(z) - x(z)) + \lambda B y_1(t) = u^*(t),$$

or equivalently

$$\lambda \sum_{s=a+1}^{b-1}\sum_{z=a+1}^{b-1} G_2(t,s)G_1(s,z)f(z, u^*(z) - x(z)) = u^*(t) - x(t) = y^*(t).$$

The proof is complete. □

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