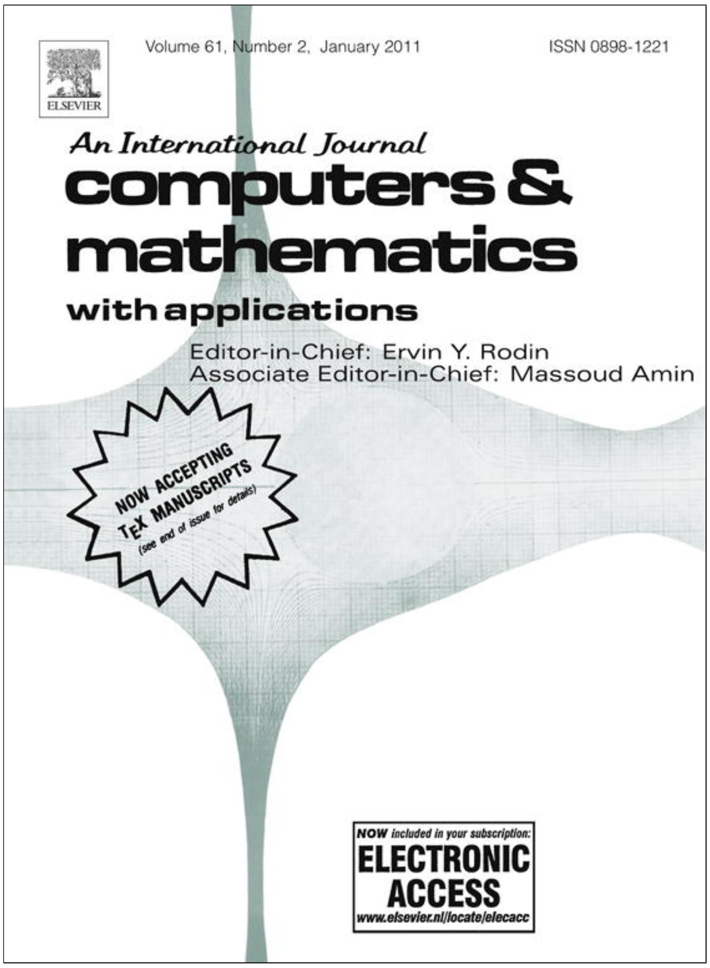
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Higher order boundary value problems with ϕ-Laplacian and functional boundary conditions

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ABSTRACT

We study the existence of solutions of the boundary value problem

$$\begin{aligned}
&\left(\phi(u^{(n-1)}(t))\right)' + f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right) = 0, \quad t \in (0, 1),\\
&g_i\left(u, u', \ldots, u^{(n-1)}, u^{(i)}(0)\right) = 0, \quad i = 0, \ldots, n-2,\\
&g_{n-1}\left(u, u', \ldots, u^{(n-1)}, u^{(n-2)}(1)\right) = 0,
\end{aligned}$$

where $n \geq 2$, ϕ and g_i, $i = 0, \ldots, n-1$, are continuous, and f is a Carathéodory function. We obtain an existence criterion based on the existence of a pair of coupled lower and upper solutions. We also apply our existence theorem to derive some explicit conditions for the existence of a solution of a special case of the above problem. In our problem, both the differential equation and the boundary conditions may have dependence on all lower order derivatives of the unknown function, and many boundary value problems with various boundary conditions, studied extensively in the literature, are special cases of our problem. Consequently, our results improve and cover a number of known results in the literature. Examples are given to illustrate the applicability of our theorems.

1. Introduction

In this paper, we are concerned with the existence of solutions of the higher order boundary value problem (BVP) consisting of the ϕ-Laplacian type differential equation

$$\left(\phi(u^{(n-1)}(t))\right)' + f\left(t, u(t), u'(t), \ldots, u^{(n-1)}(t)\right) = 0, \quad t \in (0, 1), \tag{1.1}$$

and the functional boundary condition (BC)

$$\begin{cases} g_i\left(u, u', \ldots, u^{(n-1)}, u^{(i)}(0)\right) = 0, & i = 0, \ldots, n-2, \\ g_{n-1}\left(u, u', \ldots, u^{(n-1)}, u^{(n-2)}(1)\right) = 0, \end{cases} \tag{1.2}$$

where $n \geq 2$ is an integer, ϕ is an increasing homeomorphism, $g_i : (C[0, 1])^n \times \mathbb{R} \to \mathbb{R}$, $i = 0, \ldots, n-1$, are continuous functions, and $f : (0, 1) \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, that is, (i) for any $(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, $f(t, x_0, \ldots, x_{n-1})$ is

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measurable on $(0, 1)$, (ii) for a.e. $t \in (0, 1), f(t, \cdot, \ldots, \cdot)$ is continuous on $\mathbb{R}^n$, and (iii) for every compact set $K \in \mathbb{R}^n$, there exists a nonnegative function $r_K \in L^1(0, 1)$ such that

$$|f(t, x_0, \ldots, x_{n-1})| \le r_K(t) \quad \text{for } (t, x_0, \ldots, x_{n-1}) \in (0, 1) \times K.$$

By a solution of BVP (1.1), (1.2), we mean a function $u(t) \in C^{n-1}[0, 1]$ such that $\phi(u^{(n-1)}(t))$ is absolutely continuous on $(0, 1)$, $u(t)$ satisfies Eq. (1.1) a.e. on $(0, 1)$, and $u(t)$ satisfies BC (1.2).

We comment that the functional BC (1.2) is very general in nature. It not only covers many classical BCs, such as various linear two-point, multi-point, integral, and delay or advanced BCs studied by many authors, but it may also include many new BCs not studied so far in the literature. In recent years, BVPs with linear and nonlinear BCs have been extensively investigated by numerous researchers. For a small sample of such work, we refer the reader to [1–11] for results on BVPs with linear BCs, [12–19] on BVPs with nonlinear BCs and [20–23] for functional BVPs. In these cited papers, a variety of methods and tools, such as the lower and upper solutions method and various fixed point theorems, have been successfully used to prove the existence of solutions of BVPs. As is well known, the lower and upper solutions method has been very useful in the study of BVPs for differential equations, and the analysis in this paper is mainly based on this technique. For the readers' information, we mention some works that used the lower and upper solutions method. Cabada and Pouso [12] studied the BVP consisting of Eq. (1.1) with $n = 2$ and the BC

$$g(u(0), u'(0), u'(1)) = 0, \qquad u(1) = h(u(0)).$$

By introducing the concepts of lower and upper solutions, they found sufficient conditions for the existence of solutions; Ehme et al. [13] considered the BVP consisting of Eq. (1.1), with $\phi(x) = x$ and $n = 4$, and the BC

$$g_i\left(u(0), u(1), u'(0), u'(1), u''(0), u''(1)\right) = 0, \quad i = 0, 1, 2, 3.$$

Existence results were obtained for the above BVP under the assumption that there exists a so-called strong upper–lower solutions pair. Franco et al. [14] extended these results to the BVP consisting of the same equation and a more general BC. Kong and Kong [19] studied the even order BVP consisting of the equation

$$u^{(2n)} + \lambda f\left(t, u, u'', \ldots, u^{(2n-2)}\right) = 0, \quad t \in (0, 1),$$

and the BC

$$\begin{cases} u^{(2i)}(0) = g_i(u^{(2i)}(t_1) \ldots, u^{(2i)}(t_m)), \\ u^{(2i)}(1) = h_i(u^{(2i)}(t_1) \ldots, u^{(2i)}(t_m)), \end{cases} \quad i = 0, \ldots, n-1,$$

and obtained conditions for the existence and nonexistence of positive solutions for different values of λ using the lower and upper solutions method. Graef and Kong [15] consider the BVP consisting of Eq. (1.1) with $n = 1$ and the BC

$$\begin{cases} u(0) = g\big(u(t_1), u'(t_1), u(t_2), u'(t_2), \ldots, u(t_m), u'(t_m)\big), \\ u(1) = h\big(u(t_1), u'(t_1), u(t_2), u'(t_2), \ldots, u(t_m), u'(t_m)\big). \end{cases}$$

Existence criteria were derived assuming the existence of a pair of so-called coupled lower and upper solutions.

In the context of lower and upper solutions, this paper is more related to the work in [13–15]. Motivated by these papers, we first introduce a definition for the coupled lower and upper solutions of BVP (1.1), (1.2) (see Definition 2.2). Then, in Theorem 2.1, we establish the existence of solutions of the problem based on the assumption that a pair of coupled lower and upper solutions exists. Appropriate Nagumo conditions are used here (see Definition 2.1). Moreover, as an application of Theorem 2.1, we find explicit conditions (see Theorem 4.1) for the existence of solutions of the BVP consisting of Eq. (1.1) and the BC

$$\begin{cases} u^{(i)}(0) = h_i\left(u, u', \ldots, u^{(n-1)}\right), \quad i = 0, \ldots, n-2, \\ u^{(n-2)}(1) = h_{n-1}\left(u, u', \ldots, u^{(n-1)}\right), \end{cases} \tag{1.3}$$

where $h_i : (C[0, 1])^n \to \mathbb{R}$, $i = 0, \ldots, n-1$, are continuous functions. In doing so, the conditions we provide (see (A1)–(A6) in Section 4) guarantee that there exists a pair of coupled lower and upper solutions for BVP (1.1), (1.2) in such a way that we may apply Theorem 2.1 to obtain the existence of solutions.

Throughout this paper, let $X = C^{n-1}[0, 1]$, and for any $u \in C[0, 1]$, define $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Let

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \ldots, \|u^{(n-1)}\|_\infty\}$$

and

$$\|u\|_p = \begin{cases} \left(\int_0^1 |u(t)|^p \mathrm{d}t\right)^{1/p}, & 1 \le p < \infty, \\ \inf\{M : \text{meas}\,\{t : |u(t)| > M\} = 0\}, & p = \infty, \end{cases}$$

stand for the norms in X and $L^p(0, 1)$, respectively, where meas $\{\cdot\}$ denotes the Lebesgue measure of a set.

The rest of this paper is organized as it follows: Section 2 contains some results concerning coupled lower and upper solutions and two illustrative examples; Section 3 contains the proof of Theorem 2.1 stated in Section 2; and an application of Theorem 2.1 and a related example are presented in Section 4.

2. Coupled lower and upper solutions

We first present two definitions.

Definition 2.1. Let $f : (0, 1) \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function and let $\alpha, \beta \in X$ satisfy

$$\alpha^{(i)}(t) \le \beta^{(i)}(t) \quad \text{for } t \in [0, 1] \text{ and } i = 0, \ldots, n-2. \tag{2.1}$$

We say that f satisfies a Nagumo condition with respect to α and β if for

$$\xi = \max\left\{\beta^{(n-2)}(1) - \alpha^{(n-2)}(0),\ \beta^{(n-2)}(0) - \alpha^{(n-2)}(1)\right\}, \tag{2.2}$$

there exist a constant $C = C(\alpha, \beta)$ with

$$C > \max\left\{\xi, \|\alpha^{(n-1)}\|_\infty, \|\beta^{(n-1)}\|_\infty\right\} \tag{2.3}$$

and functions $\psi \in C[0, \infty)$ and $w \in L^p(0, 1)$, $1 \le p \le \infty$, such that $\psi > 0$ on $[0, \infty)$,

$$|f(t, x_0, \ldots, x_{n-1})| \le w(t)\psi(|x_{n-1}|) \quad \text{on } (0, 1) \times \mathbb{D}_\alpha^\beta \times \mathbb{R}, \tag{2.4}$$

and

$$\int_{\phi(\xi)}^{\phi(C)} \frac{(\phi^{-1}(x))^{(p-1)/p}}{\psi(\phi^{-1}(x))} \mathrm{d}x > \|w\|_p \eta^{(p-1)/p}, \tag{2.5}$$

where $(p-1)/p \equiv 1$ for $p = \infty$,

$$\mathbb{D}_\alpha^\beta = [\alpha(t), \beta(t)] \times [\alpha'(t), \beta'(t)] \times \cdots \times [\alpha^{(n-2)}(t), \beta^{(n-2)}(t)], \tag{2.6}$$

and

$$\eta = \max_{t\in[0,1]} \beta^{(n-2)}(t) - \min_{t\in[0,1]} \alpha^{(n-2)}(t). \tag{2.7}$$

Remark 2.1. Let $\alpha, \beta \in X$ satisfy (2.1). Assume that $\lim_{x\to\infty} \phi(x) = \infty$ and there exist $w \in L^p(0, 1)$, $1 \le p \le \infty$, and $\sigma \in [0, \infty)$ such that

$$|f(t, x_0, \ldots, x_{n-1})| \le w(t)(1 + |\phi(x_{n-1})|^\sigma) \quad \text{on } (0, 1) \times \mathbb{D}_\alpha^\beta \times \mathbb{R} \tag{2.8}$$

and

$$\int_1^\infty \frac{(\phi^{-1}(x))^{(p-1)/p}}{1 + x^\sigma} \mathrm{d}x = \infty. \tag{2.9}$$

Then it is easy to see that f satisfies a Nagumo condition with respect to α and β.

Definition 2.2. Assume that $\alpha, \beta \in X$ satisfy (2.1), $\phi(\alpha^{(n-1)})$ and $\phi(\beta^{(n-1)})$ are absolutely continuous on $(0, 1)$, and let C be the constant introduced in Definition 2.1. Then α and β are said to be coupled lower and upper solutions of BVP (1.1), (1.2) if

$$\left(\phi(\alpha^{(n-1)}(t))\right)' + f\left(t, \alpha(t), \alpha'(t), \ldots, \alpha^{(n-1)}(t)\right) \ge 0 \quad \text{a.e. on } (0, 1), \tag{2.10}$$

$$\left(\phi(\beta^{(n-1)}(t))\right)' + f\left(t, \beta(t), \beta'(t), \ldots, \beta^{(n-1)}(t)\right) \le 0 \quad \text{a.e. on } (0, 1), \tag{2.11}$$

$$\begin{cases} \min\limits_{\|z\|_\infty \le C} g_i\left(\alpha, \ldots, \alpha^{(n-2)}, z, \alpha^{(i)}(0)\right) \ge 0, \quad i = 0, \ldots, n-2, \\ \min\limits_{\|z\|_\infty \le C} g_{n-1}\left(\alpha, \ldots, \alpha^{(n-2)}, z, \alpha^{(n-2)}(1)\right) \ge 0, \end{cases} \tag{2.12}$$

and

$$\begin{cases} \max\limits_{\|z\|_\infty \le C} g_i\left(\beta, \ldots, \beta^{(n-2)}, z, \beta^{(i)}(0)\right) \le 0, \quad i = 0, \ldots, n-2, \\ \max\limits_{\|z\|_\infty \le C} g_{n-1}\left(\beta, \ldots, \beta^{(n-2)}, z, \beta^{(n-2)}(1)\right) \le 0. \end{cases} \tag{2.13}$$

In what follows, a functional $\chi : C[0, 1] \to \mathbb{R}$ is said to be nondecreasing if $\chi(u_1) \ge \chi(u_2)$ for any $u_1, u_2 \in C[0, 1]$ with $u_1(t) \ge u_2(t)$ on $[0, 1]$. A similar definition holds for χ to be nondecreasing.

Remark 2.2. We wish to make the following comments on Definition 2.2.

(a) In BC (1.2), if $g_i, i = 0, \dots, n-1$, do not depend on $u^{(n-1)}$, then we may define the lower and upper solutions independently. For instance, instead of BC (1.2), if we consider the BC

$$\begin{cases} g_i^*\left(u, \dots, u^{(n-2)}, u^{(i)}(0)\right) = 0, \quad i = 0, \dots, n-2, \\ g_{n-1}^*\left(u, \dots, u^{(n-2)}, u^{(n-2)}(1)\right) = 0, \end{cases} \tag{2.14}$$

where $g_i^* : (C[0,1])^{n-1} \times \mathbb{R} \to \mathbb{R}, i = 0, \dots, n-1$, are continuous, then (2.12) and (2.13) become

$$\begin{cases} g_i^*\left(\alpha, \dots, \alpha^{(n-2)}, \alpha^{(i)}(0)\right) \ge 0, \quad i = 0, \dots, n-2, \\ g_{n-1}^*\left(\alpha, \dots, \alpha^{(n-2)}, \alpha^{(n-2)}(1)\right) \ge 0, \end{cases} \tag{2.15}$$

and

$$\begin{cases} g_i^*\left(\beta, \dots, \beta^{(n-2)}, \beta^{(i)}(0)\right) \le 0, \quad i = 0, \dots, n-2, \\ g_{n-1}^*\left(\beta, \dots, \beta^{(n-2)}, \beta^{(n-2)}(1)\right) \le 0, \end{cases} \tag{2.16}$$

respectively. In this case, we say that α is a lower solution of BVP (1.1), (2.14) if α satisfies (2.10) and (2.15), and that β is an upper solution of BVP (1.1), (2.14) if β satisfies (2.11) and (2.16).

(b) For $i = 0, \dots, n-1$ and $(y_0, \dots, y_{n-1}, x) \in (C[0,1])^n \times \mathbb{R}$, if $g_i(y_0, \dots, y_{n-1}, x)$ is monotone in y_{n-1}, say, for example, nondecreasing, then (2.12) and (2.13) become

$$\begin{cases} g_i\left(\alpha, \dots, \alpha^{(n-2)}, -C, \alpha^{(i)}(0)\right) \ge 0, \quad i = 0, \dots, n-2, \\ g_{n-1}\left(\alpha, \dots, \alpha^{(n-2)}, -C, \alpha^{(n-2)}(1)\right) \ge 0 \end{cases}$$

and

$$\begin{cases} g_i\left(\beta, \dots, \beta^{(n-2)}, C, \beta^{(i)}(0)\right) \le 0, \quad i = 0, \dots, n-2, \\ g_{n-1}\left(\beta, \dots, \beta^{(n-2)}, C, \beta^{(n-2)}(1)\right) \le 0, \end{cases}$$

respectively.

We now state the main theorem of this section.

Theorem 2.1. *Assume that the following conditions hold:*

(H1) *$\phi(x)$ is increasing on $\mathbb{R}$;*
(H2) *BVP (1.1), (1.2) has a pair of coupled lower and upper solutions α and β satisfying (2.1);*
(H3) *for $(t, x_0, \dots, x_{n-1}) \in (0,1) \times \mathbb{R}^n$ with $\alpha^{(i)}(t) \le x_i \le \beta^{(i)}(t),\ i = 0, \dots, n-3$, we have*

$$\begin{aligned} f(\cdot, \alpha(t), \dots, \alpha^{(n-3)}(t), x_{n-2}, x_{n-1}) &\le f(\cdot, x_0, \dots, x_{n-1}) \\ &\le f(\cdot, \beta(t), \dots, \beta^{(n-3)}(t), x_{n-2}, x_{n-1}); \end{aligned}$$

(H4) *f satisfies a Nagumo condition with respect to α and β;*
(H5) *for $i = 0, \dots, n-1$ and $(y_0, \dots, y_{n-1}, x) \in (C[0,1])^n \times \mathbb{R}$, $g_i(y_0, \dots, y_{n-1}, x)$ are nondecreasing in each of the arguments $y_0, \dots, y_{n-2}$.*

Then BVP (1.1), (1.2) has at least one solution $u(t)$ satisfying

$$\alpha^{(i)}(t) \le u^{(i)}(t) \le \beta^{(i)}(t) \quad \text{for } t \in [0,1] \text{ and } i = 0, \dots, n-2, \tag{2.17}$$

and

$$|u^{(n-1)}(t)| \le C \quad \text{for } t \in [0,1], \tag{2.18}$$

where $C = C(\alpha, \beta)$ is the constant introduced in Definition 2.1.

Remark 2.3. Theorem 2.1 improves and covers a number of results in the literature, such as those in [8,12–16,20]. Specifically, in our theorem, not only are the differential equation and the BC much more general, but also the usual assumption that $\phi(\mathbb{R}) = \mathbb{R}$ is not required, as illustrated by Example 2.2.

In the remainder of this section, we give two examples.

Example 2.1. The BVP consisting of the equation

$$\left((u''(t))^3\right)' + t^{-1/2}\left(u(t) - 2(u''(t))^3\right) = 0, \quad t \in (0,1), \tag{2.19}$$

and the BC

$$\begin{cases} \max\limits_{s\in[0,1]} u'(s) - 5u(0) + 1 = 0, \\ \displaystyle\int_0^1 u(s)\mathrm{d}s - 4\mathrm{e}^2 u'(0) = 0, \\ u''(1/2) - 2\mathrm{e}^2 u'(1) = 0, \end{cases} \tag{2.20}$$

has at least one nontrivial solution $u(t)$ satisfying

$$-(t+1)^2 \le u(t) \le (t+1)^2, \tag{2.21}$$

$$-2(t+1) \le u'(t) \le 2(t+1), \tag{2.22}$$

and

$$-12\mathrm{e}^2 \le u''(t) \le 12\mathrm{e}^2 \tag{2.23}$$

for $t \in [0, 1]$.

In fact, if we let $n = 3$, $\phi(x) = x^3$,

$$f(t, x_0, x_1, x_2) = t^{-1/2}\left(x_0 - 2x_2^3\right)$$

for $(x_0, x_1, x_2) \in \mathbb{R}^3$, and

$$g_0(y_0, y_1, y_2, x) = \max_{s\in[0,1]} y_1(s) - 5x + 1,$$

$$g_1(y_0, y_1, y_2, x) = \int_0^1 y_0(s)\mathrm{d}s - 4\mathrm{e}^2 x,$$

$$g_2(y_0, y_1, y_2, x) = y_2(1/2) - 2\mathrm{e}^2 x$$

for $(y_0, y_1, y_2, x) \in (C[0, 1])^3 \times \mathbb{R}$, then we see that BVP (2.19), (2.20) is of the form of BVP (1.1), (1.2). Moreover, it is clear that (H1) and (H5) hold.

Let $\alpha(t) = -(t+1)^2$ and $\beta(t) = (t+1)^2$. Obviously, $\alpha(t)$ and $\beta(t)$ satisfy (2.1). Define $w(t) = 4t^{-1/2}$ and $\psi(x) = 1+x^3$. Then, $w \in L^1(0, 1)$ with $\|w\|_1 = 8$, $\psi(x) > 0$ on $[0, \infty)$, and

$$|f(t, x_0, x_1, x_2)| = |t^{-1/2}(x_0 - 2x_2^3)| \le 4t^{-1/2}\left(1 + |x_2|^3\right) = w(t)\psi(|x_2|)$$

on $(0, 1) \times \mathbb{D}_\alpha^\beta \times \mathbb{R}$, where $\mathbb{D}_\alpha^\beta$ is given by (2.6). Thus, (2.4) holds. For ξ defined by (2.2), we have $\xi = 6$. With $C = 12\mathrm{e}^2$ and $p = 1$, it is easy to check that (2.3) and (2.5) hold. Hence, f satisfies a Nagumo condition with respect to α and β, i.e., (H4) holds.

By a simple computation, we see that $\alpha(t)$ and $\beta(t)$ satisfy (2.10)–(2.13), i.e., $\alpha(t)$ and $\beta(t)$ are coupled lower and upper solutions of BVP (2.19), (2.20). Then, (H2) holds.

Finally, (H3) obviously holds.

Therefore, by Theorem 2.1, BVP (2.19), (2.20) has a solution $u(t)$ satisfying (2.21)–(2.23). From the first equation in BC (2.20) we see that $u(t)$ is clearly nontrivial.

Example 2.2. Let

$$\phi(x) = \begin{cases} \arctan(x+5) - 125, & x < -5, \\ x^3, & -5 \le x \le 5, \\ \arctan(x-5) + 125, & x > 5. \end{cases}$$

Then, the BVP consisting of the equation

$$\left(\phi(u'''(t))\right)' + u^{1/3}(t) - 2u''(t) = 0, \quad t \in (0,1), \tag{2.24}$$

and the BC

$$\begin{cases} u'(1/2) - 2u(0) = 0, \\ u(0) - 2u'(0) - 1 = 0, \\ u'''(1/2) - 3u''(0) = 0, \\ \displaystyle\int_0^1 u(s)\mathrm{d}s + u'''(1/4) + u'''(3/4) - 10u''(1) = 0, \end{cases} \tag{2.25}$$

has at least one nontrivial solution $u(t)$ satisfying

$$-(t^2+t+1) \le u(t) \le t^2+t+1, \tag{2.26}$$
$$-(2t+1) \le u'(t) \le 2t+1, \tag{2.27}$$
$$-2 \le u''(t) \le 2, \tag{2.28}$$

and

$$-5 \le u'''(t) \le 5 \tag{2.29}$$

for $t \in [0, 1]$.

In fact, if we let $n = 4$, $f(t, x_0, x_1, x_2, x_3) = x_0^{1/3} - 2x_2$ for $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, and

$$g_0(y_0, y_1, y_2, y_3, x) = y_1(1/2) - 2x,$$
$$g_1(y_0, y_1, y_2, y_3, x) = y_0(0) - 2x - 1,$$
$$g_2(y_0, y_1, y_2, y_3, x) = y_3(1/2) - 3x,$$
$$g_3(y_0, y_1, y_2, y_3, x) = \int_0^1 y_0(s)\mathrm{d}s + y_3(1/4) + y_3(3/4) - 10x$$

for $(y_0, y_1, y_2, y_3, x) \in (C[0, 1])^4 \times \mathbb{R}$, then we see that BVP (2.24), (2.25) is of the form of BVP (1.1), (1.2). Moreover, it is clear that (H1) and (H5) hold.

Let $\alpha(t) = -(t^2+t+1)$ and $\beta(t) = t^2+t+1$. Obviously, $\alpha(t)$ and $\beta(t)$ satisfy (2.1). Since f does not depend on x_3, f evidently satisfies a Nagumo condition with respect to α and β. In fact, from (2.2), $\xi = 4$. Then, with $C = 5$, $w(t) = 1$, $\psi(x_{n-3}) = 6$, and $p = 1$, it is easy to check that (2.3)–(2.5) hold. Thus, (H4) holds. Moreover, (H3) obviously holds.

By a simple computation, we see that $\alpha(t)$ and $\beta(t)$ satisfy (2.10)–(2.13), i.e., $\alpha(t)$ and $\beta(t)$ are coupled lower and upper solutions of BVP (2.19), (2.20). Then, (H2) holds.

Therefore, by Theorem 2.1, BVP (2.24), (2.25) has a solution $u(t)$ satisfying (2.26)–(2.29). That $u(t)$ is nontrivial can be seen from the second equation in BC (2.25).

Remark 2.4. To the best of our knowledge, no known criteria can be applied to the above two examples. In particular, we want to point out again that, in Example 2.2, the condition $\phi(\mathbb{R}) = \mathbb{R}$, as is usually required in the literature, is not satisfied here.

3. Proof of Theorem 2.1

We assume that conditions (H1)–(H5) hold throughout this section. Let α and β be given in (H2). For $u \in C^{n-2}[0, 1]$ and $i = 0, \dots, n-2$, define

$$\tilde{u}^{[i]}(t) = \max\left\{\alpha^{(i)}(t), \min\left\{u^{(i)}(t), \beta^{(i)}(t)\right\}\right\}. \tag{3.1}$$

Then, for $i = 0, \dots, n-2$, $\tilde{u}^{[i]}(t)$ is continuous on [0],

$$\tilde{\alpha}^{[i]}(t) = \alpha^{(i)}(t), \qquad \tilde{\beta}^{[i]}(t) = \beta^{(i)}(t), \quad \text{and} \quad \alpha^{(i)}(t) \le \tilde{u}^{[i]}(t) \le \beta^{(i)}(t) \tag{3.2}$$

for $t \in [0, 1]$ and $i = 0, \dots, n-2$. Let $C = C(\alpha, \beta)$ be the constant introduced in Definition 2.1. Define

$$\varphi(x) = \begin{cases} \phi(x), & |x| \le C, \\ \dfrac{\phi(C) - \phi(-C)}{2C}x + \dfrac{\phi(C) + \phi(-C)}{2}, & |x| > C, \end{cases} \tag{3.3}$$

$$\hat{u}^{[n-1]}(t) = \max\left\{-C, \min\left\{u^{(n-1)}(t), C\right\}\right\}, \quad u \in X, \tag{3.4}$$

and a functional $F : (0, 1) \times X \to \mathbb{R}$ by

$$F(t, u(\cdot)) = f\left(t, \tilde{u}^{[0]}(t), \tilde{u}^{[1]}(t), \dots, \tilde{u}^{[n-2]}(t), \hat{u}^{[n-1]}(t)\right) + \frac{\tilde{u}^{[n-2]}(t) - u^{(n-2)}(t)}{1 + (u^{(n-2)}(t))^2}. \tag{3.5}$$

Then, in view of (H1), $\varphi : \mathbb{R} \to \mathbb{R}$ is increasing and continuous (hence φ^{-1} exists), and

$$\lim_{x\to-\infty} \varphi(x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} \varphi(x) = \infty. \tag{3.6}$$

Moreover, for $u \in X$ and $t \in (0, 1)$, $F(t, u(\cdot))$ is continuous in u, and from (2.4) and (3.2), we see that

$$|F(t, u(\cdot))| \le w(t) \max_{x\in[0,C]} \psi(x) + \|\alpha\| + \|\beta\| + 1. \tag{3.7}$$

Consider the BVP consisting of the equation

$$\left(\varphi(u^{(n-1)}(t))\right)' + F(t, u(\cdot)) = 0, \quad t \in (0, 1), \tag{3.8}$$

and the BC

$$\begin{cases} u^{(i)}(0) = g_i\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[i]}(0)\right) + \tilde{u}^{[i]}(0), \quad i = 0, \dots, n-2, \\ u^{(n-2)}(1) = g_{n-1}\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[n-2]}(1)\right) + \tilde{u}^{[n-2]}(1). \end{cases} \tag{3.9}$$

Lemma 3.1. *For any fixed $u \in X$, define $l(\cdot; u) : \mathbb{R} \to \mathbb{R}$ by*

$$l(x; u) = \int_0^1 \varphi^{-1}\left(x - \int_0^\tau F(s, u(\cdot))\mathrm{d}s\right)\mathrm{d}\tau + g_u, \tag{3.10}$$

where

$$\begin{aligned} g_u = {} & g_{n-2}\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[n-2]}(0)\right) + \tilde{u}^{[n-2]}(0) \\ & - g_{n-1}\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[n-2]}(1)\right) - \tilde{u}^{[n-2]}(1). \end{aligned} \tag{3.11}$$

Then the equation

$$l(x; u) = 0 \tag{3.12}$$

has a unique solution.

Proof. We first note that $l(\cdot; u)$ is continuous and increasing on $\mathbb{R}$. From (3.6), we have

$$\lim_{x\to-\infty} l(x; u) = -\infty \quad \text{and} \quad \lim_{x\to\infty} l(x; u) = \infty.$$

Then, the existence and uniqueness of a solution of Eq. (3.12) follows from the fact that $l(\cdot; u)$ is continuous and increasing on $\mathbb{R}$. □

Lemma 3.2. *For $u \in X$, let*

$$A_n u(t) = \int_0^t \varphi^{-1}\left(x_u - \int_0^\tau F(s, u(\cdot))\mathrm{d}s\right)\mathrm{d}\tau + g_{n-2}\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[n-2]}(0)\right) + \tilde{u}^{[n-2]}(0)$$

with x_u being the unique solution of Eq. (3.12). Then $u(t)$ is a solution of BVP (3.8), (3.9) if and only if $u(t)$ is a solution of the equation

$$u(t) = \begin{cases} A_2 u(t), & n = 2, \\ \dfrac{1}{(n-3)!}\displaystyle\int_0^t (t-s)^{n-3} A_n u(s)\mathrm{d}s + \sum_{i=0}^{n-3}\left(g_i\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[i]}(0)\right) + \tilde{u}^{[i]}(0)\right)\dfrac{t^i}{i!}, & n \geq 3, \end{cases}$$

where we take $0^0 = 1$.

Proof. This can be verified by direct computations. □

Lemma 3.3. *BVP (3.8), (3.9) has at least one solution.*

Proof. For any $u \in X$, define an operator $T : X \to X$ by

$$Tu(t) = \begin{cases} A_2 u(t), & n = 2, \\ \dfrac{1}{(n-3)!}\displaystyle\int_0^t (t-s)^{n-3} A_n u(s)\mathrm{d}s + \sum_{i=0}^{n-3}\left(g_i\left(\tilde{u}^{[0]}, \dots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[i]}(0)\right) + \tilde{u}^{[i]}(0)\right)\dfrac{t^i}{i!}, & n \geq 3. \end{cases}$$

Then, by Lemma 3.2, $u(t)$ is a solution of BVP (3.8), (3.9) if and only if $u(t)$ is a fixed point of T.

Let $\{u_k\}_{k=1}^{\infty} \subseteq X$ with $u_k \to u_0$ in X. We want to show that $Tu_k \to Tu_0$ in X. Let x_k be the unique solution of $l(x; u_k) = 0$, where l is given by (3.10). In view of (3.7), there exists $r \in L^1(0, 1)$ such that

$$|F(t, u_k(\cdot))| \leq r(t) \quad \text{on } (0, 1). \tag{3.13}$$

From the continuity of g_{n-2} and g_{n-1}, (3.2), (3.4) and (3.11), we see that g_{u_k} is bounded. Thus, $\{x_k\}$ is bounded. If $\{x_k\}$ is not convergent, then there exist two convergent subsequences $\{x_{i_k}\}$ and $\{x_{j_k}\}$ such that $\lim_{k\to\infty} x_{i_k} = c_1$, $\lim_{k\to\infty} x_{j_k} = c_2$, and $c_1 \neq c_2$. Then, by the Lebesgue dominated convergence theorem, we have

$$0 = \lim_{k\to\infty} l(x_{i_k}; u_{i_k}) = l(c_1; u_0)$$

and

$$0=\lim_{k\to\infty} l(x_{j_k};u_{j_k})=l(c_2;u_0),$$

which contradicts the fact that $l(c_1;u_0)\neq l(c_2;u_0)$. Hence, $\{x_k\}$ is convergent, say $\lim_{k\to\infty}x_k=x_0$. Thus, $l(x_0;u_0)=0$ and $\lim_{k\to\infty}A_nu_k(t)=A_nu_0(t)$. As a consequence, we also have

$$\lim_{k\to\infty}(Tu_k)^{(i)}(t)=(Tu)^{(i)}(t),\quad i=0,\dots,n-1.$$

This shows that $T:X\to X$ is continuous. From (3.13) and the fact that g_u is bounded for $u\in X$, a standard argument shows that $T(X)$ is compact. By the Schauder fixed point theorem, T has at least one fixed point $u\in X$, which is a solution of BVP (3.8), (3.9). This completes the proof of the lemma. □

Lemma 3.4. *If $u(t)$ is a solution of BVP (3.8), (3.9), then $u(t)$ satisfies (2.17).*

Proof. We first show that $u^{(n-2)}(t)\le\beta^{(n-2)}(t)$ on $[0,1]$. If $u^{(n-2)}(0)>\beta^{(n-2)}(0)$, then from (2.13), (H5), (3.1), (3.2), (3.4) and (3.9), we see that

$$\begin{aligned}u^{(n-2)}(0)&=g_{n-2}\left(\tilde u^{[0]},\dots,\tilde u^{[n-2]},\hat u^{[n-1]},\tilde u^{[n-2]}(0)\right)+\tilde u^{[n-2]}(0)\\&=g_{n-2}\left(\tilde u^{[0]},\dots,\tilde u^{[n-2]},\hat u^{[n-1]},\beta^{(n-2)}(0)\right)+\beta^{(n-2)}(0)\\&\le g_{n-2}\left(\beta,\dots,\beta^{(n-2)},\hat u^{[n-1]},\beta^{(n-2)}(0)\right)+\beta^{(n-2)}(0)\\&\le\max_{\|z\|_\infty\le C}g_{n-2}\left(\beta,\dots,\beta^{(n-2)},z,\beta^{(n-2)}(0)\right)+\beta^{(n-2)}(0)\\&\le\beta^{(n-2)}(0),\end{aligned}$$

which is a contradiction. Similarly, if $u^{(n-2)}(1)>\beta^{(n-2)}(1)$, we have

$$\begin{aligned}u^{(n-2)}(1)&=g_{n-2}\left(\tilde u^{[0]},\dots,\tilde u^{[n-2]},\hat u^{[n-1]},\tilde u^{[n-2]}(1)\right)+\tilde u^{[n-2]}(1)\\&=g_{n-2}\left(\tilde u^{[0]},\dots,\tilde u^{[n-2]},\hat u^{[n-1]},\beta^{(n-2)}(1)\right)+\beta^{(n-2)}(1)\\&\le g_{n-2}\left(\beta,\dots,\beta^{(n-2)},\hat u^{[n-1]},\beta^{(n-2)}(1)\right)+\beta^{(n-2)}(1)\\&\le\max_{\|z\|_\infty\le C}g_{n-2}\left(\beta,\dots,\beta^{(n-2)},z,\beta^{(n-2)}(1)\right)+\beta^{(n-2)}(1)\\&\le\beta^{(n-2)}(1).\end{aligned}$$

We again obtain a contradiction. Thus, $u^{(n-2)}(0)\le\beta^{(n-2)}(0)$ and $u^{(n-2)}(1)\le\beta^{(n-2)}(1)$. Now assume, to the contrary, that there exists $t^*\in(0,1)$ such that $u^{(n-2)}(t^*)>\beta^{(n-2)}(t^*)$. Then $u^{(n-2)}(t^*)-\beta^{(n-2)}(t^*)>0$. Without loss of generality, we may assume that $u^{(n-2)}(t)-\beta^{(n-2)}(t)$ is maximized at t^*. Then, $u^{(n-1)}(t^*)=\beta^{(n-1)}(t^*)$ and there exists a small right neighborhood $\mathcal N$ of t^* such that $u^{(n-2)}(t)-\beta^{(n-2)}(t)>0$ and $u^{(n-1)}(t)\le\beta^{(n-1)}(t)$ for all $t\in\mathcal N$. We claim that there exists $\bar t\in\mathcal N$ such that

$$\left(\varphi(u^{(n-1)}(\bar t))\right)'-\left(\varphi(\beta^{(n-1)}(\bar t))\right)'\le 0.\tag{3.14}$$

If this is not true, then $u^{(n-1)}(t)-\beta^{(n-1)}(t)$ is strictly increasing in $\mathcal N$. Hence, $u^{(n-1)}(t)-\beta^{(n-1)}(t)>0$ in $\mathcal N$. This contradicts the assumption that $u^{(n-2)}(t)-\beta^{(n-2)}(t)$ is maximized at t^*. Thus, (3.14) holds. From (2.11), we have

$$\left(\varphi(\beta^{(n-1)}(\bar t))\right)'+f\left(\bar t,\beta(\bar t),\beta'(\bar t),\dots,\beta^{(n-1)}(\bar t)\right)\le 0$$

and, by (H3), (3.5) and (3.8),

$$\begin{aligned}\left(\varphi(u^{(n-1)}(\bar t))\right)'-\left(\varphi(\beta^{(n-1)}(\bar t))\right)'&\ge -f\left(\bar t,\tilde u^{[0]}(\bar t),\tilde u^{[1]}(\bar t),\dots,\tilde u^{[n-2]}(\bar t),\hat u^{[n-1]}(\bar t)\right)-\frac{\beta^{(n-2)}(\bar t)-u^{(n-2)}(\bar t)}{1+(u^{(n-2)}(\bar t))^2}\\&\quad+f\left(\bar t,\beta(\bar t),\beta'(\bar t),\dots,\beta^{(n-1)}(\bar t)\right)\\&\ge\frac{u^{(n-2)}(\bar t)-\beta^{(n-2)}(\bar t)}{1+(u^{(n-2)}(\bar t))^2}>0,\end{aligned}$$

which is a contradiction with (3.14). Thus, $u^{(n-2)}(t)\le\beta^{(n-2)}(t)$ on $[0,1]$. Similarly. we can show that $u^{(n-2)}(t)\ge\alpha^{(n-2)}(t)$ on $[0,1]$. Hence,

$$\alpha^{(n-2)}(t)\le u^{(n-2)}(t)\le\beta^{(n-2)}(t)\quad\text{for } t\in[0,1].\tag{3.15}$$

Now, we prove that

$$\alpha^{(i)}(0)\le u^{(i)}(0)\le\beta^{(i)}(0),\quad i=0,\dots,n-3.\tag{3.16}$$

In fact, if there exists $i_0 \in \{0, \ldots, n-3\}$ such that $u^{(i_0)}(0) < \alpha^{(i_0)}(0)$, then, in view of (3.1), $\tilde{u}^{[i_0]}(t) = \alpha^{(i_0)}(t)$. Hence, from (2.12), (H5), (3.2), (3.4) and (3.9),

$$\begin{aligned}\alpha^{(i_0)}(0) &\le \min_{\|z\|_\infty \le C} g_i\left(\alpha, \ldots, \alpha^{(n-2)}, z, \alpha^{(i_0)}(0)\right) + \alpha^{(i_0)}(0)\\ &= \min_{\|z\|_\infty \le C} g_i\left(\alpha, \ldots, \alpha^{(n-2)}, z, \tilde{u}^{[i_0]}(0)\right) + \tilde{u}^{[i_0]}(0)\\ &\le g_i\left(\tilde{u}^{[0]}, \ldots, \tilde{u}^{[n-2]}, \hat{u}^{[n-1]}, \tilde{u}^{[i_0]}(0)\right) + \tilde{u}^{[i_0]}(0)\\ &= u^{(i_0)}(0).\end{aligned}$$

This is a contradiction. Thus, $u^{(i)}(0) \ge \alpha^{(i)}(0)$ for $i = 0, \ldots, n-3$. By a similar argument, we see that $u^{(i)}(0) \le \beta^{(i)}(0)$, $i = 0, \ldots, n-3$. Then, (3.16) holds.

Finally, integrating (3.15) and using (3.16), we see that $u(t)$ satisfies (2.17), hence completing the proof of the lemma. □

Lemma 3.5. *If $u(t)$ is a solution of BVP* (3.8), (3.9), *then $u^{(n-1)}(t)$ satisfies* (2.18).

Proof. By Lemma 3.4, $u(t)$ satisfies (2.17). If (2.18) does not hold, then there exists $\tilde{t} \in [0, 1]$ such that $u^{(n-1)}(\tilde{t}) > C$ or $u^{(n-1)}(\tilde{t}) < -C$. By the mean value theorem, there exists $\hat{t} \in [0, 1]$ such that $u^{(n-1)}(\hat{t}) = u^{(n-2)}(1) - u^{(n-2)}(0)$. Then, from (2.2), (2.3) and (2.17), we have

$$-C < -\xi \le \alpha^{(n-2)}(1) - \beta^{(n-2)}(0) \le u^{(n-1)}(\hat{t}) \le \beta^{(n-2)}(1) - \alpha^{n-2}(0) \le \xi < C.$$

If $u^{(n-1)}(\tilde{t}) > C$, there exist $s_1, s_2 \in [0, 1]$ such that $u^{(n-1)}(s_1) = \xi$, $u^{(n-1)}(s_2) = C$, and

$$\xi = u^{(n-1)}(s_1) \le u^{(n-1)}(t) \le u^{(n-1)}(s_2) = C \quad \text{for } t \in I, \tag{3.17}$$

where $I = [s_1, s_2]$ or $I = [s_2, s_1]$. In what follows, we only consider the case $I = [s_1, s_2]$ since the other case can be treated similarly. From (3.4) and (3.17), $\hat{u}^{[n-1]}(t) = u^{(n-1)}(t)$ on I, and in view of (2.17) and (3.1), we have $\tilde{u}^{[i]}(t) = u^{(i)}(t)$ for $t \in [0, 1]$ and $i = 0, \ldots, n-2$. Thus, from (3.5), $F(t, u(\cdot)) \equiv f(t, u(t), u'(t), \ldots, u^{(n-1)}(t))$ on I. Then, by a change of variables and from (2.4) and (3.8), we obtain

$$\begin{aligned}\int_{\phi(\xi)}^{\phi(C)} \frac{(\phi^{-1}(x))^{(p-1)/p}}{\psi(\phi^{-1}(x))}\mathrm{d}x &= \int_{\phi(u^{(n-1)}(s_1))}^{\phi(u^{(n-1)}(s_2))} \frac{(\phi^{-1}(x))^{(p-1)/p}}{\psi(\phi^{-1}(x))}\mathrm{d}x\\ &= \int_{s_1}^{s_2} \frac{\phi(u^{(n-1)}(s))'}{\psi(u^{(n-1)}(s))}(u^{(n-1)}(s))^{(p-1)/p}\mathrm{d}s\\ &= \int_{s_1}^{s_2} \frac{\varphi(u^{(n-1)}(s))'}{\psi(u^{(n-1)}(s))}(u^{(n-1)}(s))^{(p-1)/p}\mathrm{d}s\\ &= \int_{s_1}^{s_2} \frac{-f\left(s, u(s), u'(s), \ldots, u^{(n-1)}(s)\right)}{\psi(u^{(n-1)}(s))}(u^{(n-1)}(s))^{(p-1)/p}\mathrm{d}s\\ &\le \int_{s_1}^{s_2} w(s)(u^{(n-1)}(s))^{(p-1)/p}\mathrm{d}s.\end{aligned}$$

Hölder's inequality then implies

$$\int_{\phi(\xi)}^{\phi(C)} \frac{(\phi^{-1}(x))^{(p-1)/p}}{\psi(\phi^{-1}(x))}\mathrm{d}x \le \|w\|_p \left(\int_{s_1}^{s_2} u^{(n-1)}(s)\mathrm{d}s\right)^{(p-1)/p} \le \|w\|_p \eta^{(p-1)/p},$$

where η is defined by (2.7). But this contradicts (2.5). Therefore, $u^{(n-1)}(t)$ satisfies (2.18). If $u^{(n-1)}(\tilde{t}) < -C$, by a similar argument as above, we can still show that (2.18) holds. The proof is complete. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Note that any solution $u(t)$ of BVP (3.8), (3.9) satisfying (2.17) and (2.18) is a solution of BVP (1.1), (1.2). The conclusion readily follows from Lemmas 3.3–3.5. □

4. An application of Theorem 2.1

In this section, we apply Theorem 2.1 to derive some explicit conditions for the existence of solutions of BVP (1.1), (1.3). To do so, we need the following assumptions.

(A1) $\phi(x)$ is increasing on $\mathbb{R}$, $\lim_{x\to\infty} \phi(x) = \infty$, and $\phi(0) = 0$.
(A2) For $(t, x_0, \ldots, x_{n-1}) \in (0, 1) \times \mathbb{R}^n$, $f(t, x_0, \ldots, x_{n-1})$ is nondecreasing in each of $x_0, \ldots, x_{n-3}$.

(A3) There exist $1 \le p \le \infty$ and $\sigma \in \mathbb{R}$ such that for any $r > 0$, there exist $\mu_r \in L^p(0,1)$ such that (2.9) holds and

$$|f(t, x_0, \dots, x_{n-1})| \le \mu_r(t)(1 + |\phi(x_{n-1})|^\sigma) \quad \text{on } (0,1) \times \mathbb{D}_r \times \mathbb{R}, \tag{4.1}$$

where

$$\mathbb{D}_r = \underbrace{[-r, r] \times [-r, r] \times \cdots \times [-r, r]}_{n-1}. \tag{4.2}$$

(A4) There exist $\delta > 0$, $\vartheta \in L^1(0,1)$, and $\zeta \in C[0,\infty)$ such that $\vartheta > 0$ on $(0,1)$, $\zeta > 0$ on $[0,\infty)$, $\zeta(\phi^{-1}(\cdot))$ is locally Lipschitz on $[0,\infty)$,

$$x_0 f(t, x_0, \dots, x_{n-1}) \le \vartheta(t)|x_0|\zeta(|x_{n-1}|) \quad \text{on } \mathbb{E}_\delta, \tag{4.3}$$

and

$$\int_0^\infty \frac{\mathrm{d}x}{\zeta(\phi^{-1}(x))} > \int_0^1 \vartheta(s)\mathrm{d}s, \tag{4.4}$$

where

$$\mathbb{E}_\delta = \{(t, x_0, \dots, x_{n-1}) \in (0,1) \times \mathbb{R}^n \mid x_i \ge \delta, i = 0, \dots, n-2, x_{n-1} \le 0\}$$
$$\cup \{(t, x_0, \dots, x_{n-1}) \in (0,1) \times \mathbb{R}^n \mid x_i \le -\delta, i = 0, \dots, n-2, x_{n-1} \ge 0\}.$$

(A5) For $i = 0, \dots, n-1$ and $(y_0, \dots, y_{n-1}) \in (C[0,1])^n$, $h_i(y_0, \dots, y_{n-1})$ is nondecreasing in each of its arguments $y_0, \dots, y_{n-2}$.

(A6) For $i = 0, \dots, n-1$ and any $u, v \in C^{n-2}[0,1]$, we have

$$-\infty < \inf_{z \in C[0,1]} |h_i(u, \dots, u^{(n-2)}, z)| \le \sup_{z \in C[0,1]} |h_i(u, \dots, u^{(n-2)}, z)| < \infty, \tag{4.5}$$

and there exists $c_i \ge 0$ such that

$$\left| \sup_{z \in C[0,1]} \left|h_i(u, \dots, u^{(n-2)}, z)\right| - \sup_{z \in C[0,1]} \left|h_i(v, \dots, v^{(n-2)}, z)\right| \right| \le c_i \|u - v\|, \tag{4.6}$$

$$\left| \inf_{z \in C[0,1]} \left|h_i(u, \dots, u^{(n-2)}, z)\right| - \inf_{z \in C[0,1]} \left|h_i(v, \dots, v^{(n-2)}, z)\right| \right| \le c_i \|u - v\|, \tag{4.7}$$

and

$$\Lambda := \sum_{i=0}^{n-1} c_i < 1. \tag{4.8}$$

Theorem 4.1. *Assume that* (A1)–(A6) *hold. Then BVP* (1.1), (1.3) *has at least one solution* $u(t)$. *Moreover,* $u(t)$ *is nontrivial if one of the following conditions hold:*

(B1) *there exists a subset* S *of* $(0,1)$ *with positive measure such that* $f(t, 0, \dots, 0) \not\equiv 0$ *for* $t \in S$;

(B2) *there exists* $i_0 \in \{0, \dots, n-1\}$ *such that* $h_{i_0}(0, \dots, 0) \ne 0$.

Remark 4.1. Theorem 4.1 includes [15, Theorem 2.2] and [16, Theorem 2.6] as special cases.

Before proving Theorem 4.1, we give an example that cannot be handled using existing results in the literature.

Example 4.1. Consider the BVP consisting of the equation

$$\left((u^{(''')}(t))^3\right)' - \exp\left(u''(t)\right)\left(\pi - \arctan u(t) + (u'''(t))^4\right) = 0, \quad t \in (0,1), \tag{4.9}$$

and the BC

$$\begin{cases} u(0) = (1/4)u''(1/2) + a\sin(u'''(2/3)) + \lambda_0, \\ u'(0) = (1/8)\displaystyle\int_0^1 u'(s)\mathrm{d}s + b\arctan(u'''(1/3)) + \lambda_1, \\ u''(0) = (1/8)u(3/4) + c\cos\left(\displaystyle\max_{s \in [0,1]} u'''(s)\right) + \lambda_2, \\ u''(1) = (1/4)\displaystyle\int_0^1 u(s)\mathrm{d}s + d/\left(1 + (u'''(1/4))^2\right) + \lambda_3, \end{cases} \tag{4.10}$$

where $a, b, c, d, \lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

We claim that BVP (4.9), (4.10) has at least one nontrivial solution.

In fact, if we let $n = 4$, $\phi(x) = x^3$,

$$f(t, x_0, x_1, x_2, x_3) = -e^{x_2}\left(\pi - \arctan x_0 + x_3^4\right)$$

for $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, and

$$h_0(y_0, y_1, y_2, y_3) = (1/4)y_2(1/2) + a\sin(y_3(2/3)) + \lambda_0, \tag{4.11}$$

$$h_1(y_0, y_1, y_2, y_3) = (1/8)\int_0^1 y_1(s)\mathrm{d}s + b\arctan(y_3(1/3)) + \lambda_1, \tag{4.12}$$

$$h_2(y_0, y_1, y_2, y_3) = (1/8)y_0(3/4) + c\cos\left(\max_{s\in[0,1]} y_3(s)\right) + \lambda_2, \tag{4.13}$$

$$h_2(y_0, y_1, y_2, y_3) = (1/4)\int_0^1 y_0(s)\mathrm{d}s + d/\left(1 + (y_3(1/4))^2\right) + \lambda_3, \tag{4.14}$$

for $(y_0, y_1, y_2, y_3) \in (C[0, 1])^4$, then it is easy to see that BVP (4.9), (4.10) is of the form of BVP (1.1), (1.3). Clearly, (A1), (A2), (A5), and (B1) hold.

Let $p = \infty$ and $\sigma = 4/3$, and for any $r > 0$, let $\mu_r(t) = 3\pi e^r/2$. Then, $\mu_r \in L^p(0, 1)$ and

$$\begin{aligned}|f(t, x_0, x_1, x_2, x_3)| &= e^{x_2}\left(\pi - \arctan x_0 + (x_3)^4\right)\\ &\le e^r\left(3\pi/2 + x_3^4\right) \le (3\pi e^r/2)\left(1 + x_3^4\right)\\ &= \mu_r(t)\left(1 + |\phi(x_3)|^\sigma\right) \quad \text{on } (0, 1)\times\mathbb{D}_r\times\mathbb{R},\end{aligned}$$

where $\mathbb{D}_r$ is defined by (4.2), i.e., (4.1) holds. Moreover, we have

$$\int_1^\infty \frac{(\phi^{-1}(x))^{(p-1)/p}}{1 + x^\sigma}\mathrm{d}x = \int_1^\infty \frac{x^{1/3}}{1 + x^{4/3}}\mathrm{d}x = \infty,$$

so (2.9) holds. Thus, (A3) holds.

For any $\delta > 0$, if $x_i \ge \delta$, $i = 0, 1, 2$, then we have

$$f(t, x_0, x_1, x_2, x_3) \le -e^\delta\left(\pi/2 + x_3^4\right) < 0,$$

and if $x_i \le -\delta$, $i = 0, 1, 2$, then we have

$$f(t, x_0, x_1, x_2, x_3) \ge -e^{-\delta}\left(3\pi/2 + x_3^4\right).$$

Let $\vartheta(t) = 1$, $\zeta(x_3) = e^{-\delta}\left(3\pi/2 + x_3^4\right)$. Clearly, $\vartheta \in L^1(0, 1)$ and $\zeta \in C[0, \infty)$ satisfy $\vartheta > 0$ on $(0, 1)$, $\zeta > 0$ on $[0, \infty)$, $\zeta(\phi^{-1}(\cdot))$ is locally Lipschitz on $[0, \infty)$, and (4.3) holds. Note that

$$\int_0^1 \vartheta(s)\mathrm{d}s = 1$$

and

$$\int_0^\infty \frac{\mathrm{d}x}{\zeta(\phi^{-1}(x))} = e^\delta\int_0^\infty \frac{\mathrm{d}x}{3\pi/2 + x^{4/3}} > 1 \quad \text{if } \delta \text{ is large.}$$

Thus, (4.4) holds for a large δ, and so (A4) holds.

Let $c_0 = c_3 = 1/4$ and $c_1 = c_2 = 1/8$. Then, from (4.11)–(4.14), we see that (4.5)–(4.8) hold, i.e., (A6) holds.

Therefore, by Theorem 4.1, BVP (4.9), (4.10) has at least one nontrivial solution.

To prove Theorem 4.1, we need the following lemma which is taken from [15, Lemma 3.7].

Lemma 4.1. *Let ϑ and ζ be given in* (A4). *Then the initial value problem (IVP)*

$$z'(t) = -\vartheta(t)\zeta(\phi^{-1}(|z(t)|)), \quad z(0) = 0, \tag{4.15}$$

has a unique solution $\nu(t)$ satisfying $\nu(t) \le 0$ on $[0, 1]$.

Now, we prove Theorem 4.1.

Proof of Theorem 4.1. Let $\nu(t)$ be the unique solution of IVP (4.15). For any $k > 0$, we show that the BVP consisting of the equation

$$\phi(u^{(n-1)}(t)) = \nu(t), \quad t \in [0, 1], \tag{4.16}$$

and the BC

$$\begin{cases} u^{(i)}(0) = k + \sup\limits_{z \in C[0,1]} \left|h_i\left(u, \dots, u^{(n-2)}, z\right)\right|, \quad i = 0, \dots, n-3, \\ u^{(n-2)}(0) = k + \sup\limits_{z \in C[0,1]} \left|h_{n-2}\left(u, \dots, u^{(n-2)}, z\right)\right| + \sup\limits_{z \in C[0,1]} \left|h_{n-1}\left(u, \dots, u^{(n-2)}, z\right)\right|, \end{cases} \tag{4.17}$$

has a unique solution $\beta_k(t)$. In view of (4.5), BC (4.17) is well defined.

For any $k > 0$ and $u \in C^{n-2}[0, 1]$, define an operator $A_k : C^{n-2}[0, 1] \to C^{n-2}[0, 1]$ by

$$\begin{aligned} A_k u(t) = & \sum_{i=0}^{n-2} \left(k + \sup_{z \in C[0,1]} \left|h_i\left(u, \dots, u^{(n-2)}, z\right)\right|\right) \frac{t^i}{i!} + \sup_{z \in C[0,1]} \left|h_{n-1}\left(u, \dots, u^{(n-2)}, z\right)\right| \frac{t^{n-2}}{(n-2)!} \\ & + \frac{1}{(n-2)!} \int_0^t (t-s)^{n-2} \nu(s) \mathrm{d}s. \end{aligned} \tag{4.18}$$

Clearly, a solution of BVP (4.16), (4.17) is a fixed point of A_k. For any $u, v \in C^{n-2}[0, 1]$, $t \in [0, 1]$, and $l = 0, \dots, n-2$, from (4.6), (4.8) and (4.18), we see that

$$\begin{aligned} |(A_k u)^{(l)}(t) - (A_k v)^{(l)}(t)| = & \left| \sum_{i=l}^{n-2} \left(\sup_{z \in C[0,1]} \left|h_i\left(u, \dots, u^{(n-2)}, z\right)\right| - \sup_{z \in C[0,1]} \left|h_i\left(v, \dots, v^{(n-2)}, z\right)\right| \right) \frac{t^{i-l}}{(i-l)!} \right. \\ & \left. + \left(\sup_{z \in C[0,1]} \left|h_{n-1}\left(u, \dots, u^{(n-2)}, z\right)\right| - \sup_{z \in C[0,1]} \left|h_{n-1}\left(v, \dots, v^{(n-2)}, z\right)\right| \right) \frac{t^{n-2-l}}{(n-2-l)!} \right| \\ \le & \sum_{i=0}^{n-2} \left| \sup_{z \in C[0,1]} \left|h_i\left(u, \dots, u^{(n-2)}, z\right)\right| - \sup_{z \in C[0,1]} \left|h_i\left(v, \dots, v^{(n-2)}, z\right)\right| \right| \\ & + \left| \sup_{z \in C[0,1]} \left|h_{n-1}\left(u, \dots, u^{(n-2)}, z\right)\right| - \sup_{z \in C[0,1]} \left|h_{n-1}\left(v, \dots, v^{(n-2)}, z\right)\right| \right| \\ = & \sum_{i=0}^{n-1} \left| \sup_{z \in C[0,1]} \left|h_i\left(u, \dots, u^{(n-2)}, z\right)\right| - \sup_{z \in C[0,1]} \left|h_i\left(v, \dots, v^{(n-2)}, z\right)\right| \right| \\ \le & \sum_{i=0}^{n-1} c_i \|u - v\| = \Lambda \|u - v\|. \end{aligned}$$

Since $\Lambda < 1$, we see that A_k is a contraction mapping. Hence, for any $k > 0$, A_k has a unique fixed point β_k in $C^{n-2}[0, 1]$, and consequently, BVP (4.16), (4.17) has a unique solution $\beta_k(t)$. Choose k_1 large enough so that

$$k_1 + \int_0^1 \phi^{-1}(\nu(s)) \mathrm{d}s \ge 0 \tag{4.19}$$

and

$$\beta^{(i)}(t) := \beta_{k_1}^{(i)}(t) \ge \delta \quad \text{for } t \in [0, 1] \text{ and } i = 0, \dots, n-2, \tag{4.20}$$

where δ is given in (A4). From (4.15) and (4.16), it is clear that

$$\left(\phi(\beta^{(n-1)}(t))\right)' = \nu'(t) = -\vartheta(t)\zeta(\phi^{-1}(|\nu(t)|)) = -\vartheta(t)\zeta(|\beta^{(n-1)}(t)|). \tag{4.21}$$

In view of (A1), we have $\beta^{(n-1)}(t) = \phi^{-1}(\nu(t)) \le 0$ on $[0, 1]$. Then, noting (4.20) and from (4.3) and (4.21), it follows that

$$\left(\phi(\beta^{(n-1)}(t))\right)' + f(t, \beta(t), \beta'(t), \dots, \beta^{(n-1)}(t)) \le 0 \quad \text{for } t \in (0, 1),$$

i.e., $\beta(t)$ satisfies (2.11).

Now, for any $k > 0$, consider the BVP consisting of the equation

$$\phi(u^{(n-1)}(t)) = -\nu(t), \quad t \in [0, 1], \tag{4.22}$$

and the BC

$$\begin{cases} u^{(i)}(0) = -k - \inf_{z\in C[0,1]} \left|h_i\left(u, \ldots, u^{(n-2)}, z\right)\right|, \quad i = 0, \ldots, n-3, \\ u^{(n-2)}(0) = -k - \inf_{z\in C[0,1]} \left|h_{n-2}\left(u, \ldots, u^{(n-2)}, z\right)\right| - \sup_{z\in C[0,1]} \left|h_{n-1}\left(u, \ldots, u^{(n-2)}, z\right)\right|. \end{cases} \tag{4.23}$$

In view of (4.5), we see that BC (4.23) is well defined, and a solution of BVP (4.22), (4.23) is a fixed point of the operator B_k defined by

$$B_k u(t) = -\sum_{i=0}^{n-2}\left(k + \inf_{z\in C[0,1]}\left|h_i\left(u, \ldots, u^{(n-2)}, z\right)\right|\right)\frac{t^i}{i!} - \inf_{z\in C[0,1]}\left|h_{n-1}\left(u, \ldots, u^{(n-2)}, z\right)\right|\frac{t^{n-2}}{(n-2)!}$$
$$- \frac{1}{(n-2)!}\int_0^t (t-s)^{n-2}\nu(s)\mathrm{d}s. \tag{4.24}$$

Using (4.7) and an argument similar to the one above, we can show that there exists k_2 large enough so that

$$-k_2 - \int_0^1 \phi^{-1}(\nu(s))\mathrm{d}s \le 0 \tag{4.25}$$

and BVP (4.22), (4.23) with $k = k_2$ has a unique solution $\alpha(t)$ satisfying

$$\alpha^{(i)}(t) \le -\delta \quad \text{for } t \in [0,1] \text{ and } i = 0, \ldots, n-2. \tag{4.26}$$

From (4.15) and (4.22), we have

$$\left(\phi(\alpha^{(n-1)}(t))\right)' = \nu'(t) = \vartheta(t)\zeta(\phi^{-1}(|\nu(t)|)) = \vartheta(t)\zeta(|\alpha^{(n-1)}(t)|). \tag{4.27}$$

In view of (4.20) and the fact that $\alpha^{(n-1)}(t) = \phi^{-1}(-\nu(t)) \ge 0$ on $[0, 1]$, then, from (4.3) and (4.27), it follows that

$$\left(\phi(\alpha^{(n-1)}(t))\right)' + f(t, \alpha(t), \alpha'(t), \ldots, \alpha^{(n-1)}(t)) \ge 0 \quad \text{for } t \in (0,1),$$

i.e., $\alpha(t)$ satisfies (2.10).

From (4.20) and (4.26), $\alpha(t)$ and $\beta(t)$ satisfy (2.1). Let $r = \max\{\|\alpha\|, \|\beta\|\}$. Then (4.1) implies that (2.8) holds with $w(t) = \mu_r(t)$ and $\psi(|x_{n-1}|) = 1 + |\phi(|x_{n-1}|^\sigma)$. From (A1), (A3), and Remark 2.1, f satisfies a Nagumo condition with respect to α and β, i.e., (H4) holds. Let C be the constant given in Definition 2.1, for the pair $\alpha(t)$ and $\beta(t)$. Note that α and β are fixed points of B_{k_2} and A_{k_1}, respectively. From (4.18), (4.19), (4.24) and (4.25), it is easy to see that $\alpha(t)$ and $\beta(t)$ satisfy (2.12) and (2.13) with

$$g_i\left(u, u', \ldots, u^{n-1}, u^{(i)}(0)\right) = h_i\left(u, u', \ldots, u^{n-1}\right) - u^{(i)}(0), \quad i = 0, \ldots, n-2, \tag{4.28}$$

and

$$g_{n-1}\left(u, u', \ldots, u^{n-1}, u^{(i)}(1)\right) = h_{n-1}\left(u, u', \ldots, u^{n-2}\right) - u^{(n-1)}(1). \tag{4.29}$$

Thus, $\alpha(t)$ and $\beta(t)$ are coupled lower and upper solutions of BVP (1.1), (1.2), or equivalently BVP (1.1), (1.3). Then, (H2) holds. Moreover, under the assumptions (A1), (A2), and (A5), we have that (H1), (H3), and (H5) (with g_i, $i = 0, \ldots, n-1$ given in (4.28) and (4.29)) hold. Therefore, by Theorem 2.1, BVP (1.1), (1.3) has at least one solution. Finally, it is obvious that if either (B1) or (B2) holds, then $u(t)$ is nontrivial. This completes the proof of the theorem. □

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