RELAXATION IN $\mathbf{SBV}_p\left(\Omega;S^{d-1}\right)$

GRAÇA CARITA, IRENE FONSECA, AND GIOVANNI LEONI

Abstract. An integral representation formula is obtained for the relaxation of a class of energy functionals defined in the class of SBV_p functions that are constrained to have values on the sphere S^{d-1}.

1. Introduction

Equilibrium problems for liquid crystals and magnetostrictive materials lead naturally to the study of variational problems in which the underlying function space is a subset of Borel functions with values on the sphere (see [26], [43]). More generally, for bulk energies there is a large literature on lower semicontinuity, relaxation, and regularity for functionals of the type

$$E(u) := \int_\Omega f(x,u,\nabla u)\ dx, \qquad u \in W^{1,p}(\Omega;\mathcal{M}),$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $1 \le p < \infty$, and $\mathcal{M} \subset \mathbb{R}^d$ is a regular m-dimensional manifold, $m \in \mathbb{N}$, (see, e.g., [16], [24], [33], [34], [35]). If $f(x,u,\cdot)$ is nonconvex, usually $u \mapsto E(u)$ fails to be lower semicontinuous, and thus we must consider the relaxed energy

$$\mathcal{E}(u) := \inf\left\{\liminf_{n\to\infty} E(u_n) :\ u_n \in W^{1,p}(\Omega;\mathcal{M}),\ u_n \to u \text{ in } L^1(\Omega;\mathcal{M})\right\}.$$

One of the main objectives of relaxation theory is to find an integral representation for $\mathcal{E}(u)$. If $p > 1$ and the integrand f satisfies a coercivity hypothesis of the type

$$f(x,u,\xi) \ge \frac{1}{C}|\xi|^p$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $u \in \mathcal{M}$ and $\xi \in \mathbb{R}^{d\times N}$ and for some $C > 0$, then the domain of $\mathcal{E}$ remains in the Sobolev space $W^{1,p}(\Omega;\mathcal{M})$. On the other hand, if $p = 1$, then it may happen that discontinuous fields are approached by sequences of smooth maps with bounded energy, in which case the domain of $\mathcal{E}$ may escape $W^{1,1}(\Omega;\mathcal{M})$ and include bounded variation type fields. In this context the relaxed energy $\mathcal{E}$ has been studied by Alicandro, Corbo Esposito, and Leone [1] when $\mathcal{M} = S^{d-1}$, the unit sphere in $\mathbb{R}^d$, and f has linear growth. This result was later extended by Mucci [41] to general manifolds and for a restricted class of integrands satisfying an isotropy condition, and subsequently by Babadjian and Millot [8], who removed this restriction. Note that the integrands treated and the arguments used in [1] and [8] fall within the general theory developed for the unconstrained case in [4], [31], and [13].

The key arguments in [1], [41], and [8] are the density of smooth functions in $W^{1,p}(\Omega;\mathcal{M})$ (see [10], [11], and [37] for the precise statement) and a projection technique introduced in [39], [38].

In this paper we address a constrained variational problem that seems to fall outside the scope of these techniques. Precisely, we consider the functional

$$F(u) := \int_\Omega f(x,u,\nabla u)\ dx + \int_{S(u)} g\left(x,u^+,u^-,\nu_u\right)\ d\mathcal{H}^{N-1} \qquad u \in SBV_p\left(\Omega;S^{d-1}\right),$$

where $p > 1$, the functions $f : \Omega \times S^{d-1} \times \mathbb{R}^{d\times N} \to [0,\infty)$ and $g : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0,\infty)$ satisfy the hypotheses:

(F_1) f is Carathéodory,
(F_2) there exists $C > 0$ such that

$$\frac{1}{C} |\xi|^p \leq f(x, u, \xi) \leq C(1 + |\xi|^p)$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $u \in S^{d-1}$ and $\xi \in \mathbb{R}^{d \times N}$.
(G_1) g is continuous,
(G_2) there exists $C > 0$ such that

$$\frac{1}{C} \leq g(x, \lambda, \theta, \nu) \leq C$$

for all $x \in \Omega$, λ, $\theta \in S^{d-1}$ and $\nu \in S^{N-1}$,
(G_3) $g(x, \lambda, \theta, -\nu) = g(x, \theta, \lambda, \nu)$ for all $x \in \Omega$, λ, $\theta \in S^{d-1}$ and $\nu \in S^{N-1}$.

Here it is important to observe that functions in $SBV_p(\Omega; S^{d-1})$ cannot be approximated by smooth functions. Instead, we adapt to the constrained case an approximation result due to Braides and Chiadò-Piat (see Lemma 5.2 in [15]) using regularity results developed by Carriero and Leaci [18] (see also [22]) for a constrained Mumford-Shah type functional, which allows us to replace the projection argument in [39], [38], with the one due to Carriero and Leaci (see [18], Lemma 3.5).

The purpose of this paper is to obtain an integral representation for the localized relaxed energy

$$\mathcal{F}(u; A) := \inf \left\{ \liminf_{n \to \infty} F(u_n; A) : \, u_n \in SBV_p\left(A; S^{d-1}\right), \, u_n \to u \text{ in } L^1\left(A; \mathbb{R}^d\right) \right\}, \tag{1.1}$$

with $A \in \mathcal{A}(\Omega)$ and $u \in SBV_p(\Omega; S^{d-1})$, where $\mathcal{A}(\Omega)$ denotes the family of all open subsets of Ω.

Precisely,

Theorem 1.1. *Assume that*

$$p = 2$$

and that f and g satisfy (F_1), (F_2) and (G_1), (G_2), (G_3), respectively. Then for every $u \in SBV_2(\Omega; S^{d-1})$ and $A \in \mathcal{A}(\Omega)$,

$$\mathcal{F}(u; A) = \int_A Q_T f(x, u, \nabla u) \, dx + \int_{S(u) \cap A} Rg\left(x, u^+, u^-, \nu_u\right) \, d\mathcal{H}^{N-1},$$

where $Q_T f(x, u, \cdot)$ and $Rg(x, \cdot, \cdot, \cdot)$ denote, respectively, the tangential quasiconvex envelope of $f(x, u, \cdot)$, and the BV-elliptic envelope of $g(x, \cdot, \cdot, \cdot)$.

The treatment of the unconstrained case may be found in [5], [14] and [15].

This paper is organized as follows. In Section 2 we give a brief overview of preliminary results, and in Section 3 we establish the lower bound for the relaxed energy $\mathcal{F}$. To obtain the upper bound for $\mathcal{F}$, in Section 4, we show that $\mathcal{F}$ is a variational functional (see Definition 4.1 in [6]), that is,

(H_1) $\mathcal{F}(u; \cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure;
(H_2) $\mathcal{F}$ is local, i.e., $\mathcal{F}(u; A) = \mathcal{F}(v; A)$ whenever $u = v$ $\mathcal{L}^N$-a.e. in $A \in \mathcal{A}(\Omega)$;
(H_3) $\mathcal{F}(\cdot; A)$ is $L^1(\Omega; \mathbb{R}^d)$ sequentially lower semicontinuous, that is,

$$\mathcal{F}(u; A) \leq \liminf_{n \to \infty} \mathcal{F}(u_n; A)$$

whenever $A \in \mathcal{A}(\Omega)$ and $u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$.

Note that property (H_2) follows from (1.1), while property (H_3) from (1.1) and a diagonal argument. The main difficulty is to prove that $\mathcal{F}(u; \cdot)$ satisfies (H_1). We will show that $\mathcal{F}(u; \cdot)$ satisfies (H_1) for a special class of functions u such that

$$u \in C\left(\Omega \setminus K; S^{d-1}\right) \cap SBV_p\left(\Omega; S^{d-1}\right), \tag{1.2}$$

where the compact set $K \subset \Omega$ satisfies suitable conditions (see (4.2) below). The key result is Lemma 4.3, in which we show that every admissible sequence for $\mathcal{F}(u; A)$ can be modified to match u in a neighborhood of ∂A, without increasing the energy. The proof of this lemma relies strongly on the regularity of u away from K, together with a projection argument introduced by Carriero and Leaci (see [18], Lemma 3.5).

Once (H_1) is established, using blow up techniques developed in [31], [32], we obtain the integral representation for all u as in (1.2). To remove the additional smoothness of u, we use the regularity results of Carriero and Leaci [18] and of Schoen and Uhlenbeck (see Theorem 2.2.4 in [40]) for sphere-valued minimizers of the Mumford-Shah functional, in order to approximate any u in $SBV_2\left(\Omega; S^{d-1}\right)$ in a strong sense by a sequence $\{u_n\}$ of the type (1.2) (see Lemma 4.4). The fact that $p = 2$ is only used to ensure C^∞ regularity outside the set K. Indeed, for $p > 1$, $p \neq 2$, it is known that p-harmonic functions are only $C^{1,\alpha}$, and this prevents the use of Sard's theorem (see the proof of Case 4 of Substep 1b in Theorem 4.1). We remark that all the other preparatory results do not need this restriction, and thus we present them for arbitrary $p > 1$, although a different argument is needed to treat Case 4 of Substep 1b in Theorem 4.1 for $p \neq 2$, and this is ongoing work.

2. Preliminaries

In the following $\Omega \subset \mathbb{R}^N$ is an open bounded set and we denote by $\mathcal{A}(\Omega)$ and $\mathcal{B}(\Omega)$ the families of open and Borel subsets of Ω, respectively. The Lebesgue N-dimensional measure is denoted by $\mathcal{L}^N$, while $\mathcal{H}^{N-1}$ stands for the $(N-1)$-dimensional Hausdorff measure. The unit cube in $\mathbb{R}^N$, $\left(-\frac{1}{2}, \frac{1}{2}\right)^N$, is denoted by Q and we set $Q(x_0, \varepsilon) := x_0 + \varepsilon Q$ for $\varepsilon > 0$. We define $Q_\nu := R_\nu(Q)$, where R_ν is a rotation such that $R_\nu(e_N) = \nu$. The constant C may vary from line to line.

Definition 2.1. *A function $u \in L^1\left(\Omega; \mathbb{R}^d\right)$ is said to be of* bounded variation*, and we write $u \in BV\left(\Omega; \mathbb{R}^d\right)$, if for all $i = 1, \dots, d$, and $j = 1, \dots, N$, there exists a Radon measure μ_{ij} such that*

$$\int_\Omega u^i(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = -\int_\Omega \varphi \, d\mu_{ij}$$

for every $\varphi \in C_c^1(\Omega; \mathbb{R})$.

The distributional derivative Du is a $d \times N$ matrix-valued measure with components μ_{ij}. The *total variation* of the measure Du is given by

$$|Du|(\Omega) := \sup\left\{\sum_{i=1}^d \int_\Omega u^i \operatorname{div} \varphi_i \, dx : \varphi \in C_c^1\left(\Omega; \mathbb{R}^{d\times N}\right), \ \|\varphi\|_\infty \leq 1\right\}.$$

We briefly recall some facts about functions of bounded variation. For more details we refer the reader to [5], [27], [36], and [44].

Definition 2.2. *Given $u \in BV\left(\Omega; \mathbb{R}^d\right)$ the* approximate upper limit *and the* approximate lower limit *of each component u^i, $i = 1, \dots, d$, are defined by*

$$\left(u^i\right)^+(x) := \inf\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}^N\left(\left\{y \in \Omega \cap Q(x, \varepsilon) : u^i(y) > t\right\}\right)}{\varepsilon^N} = 0\right\}$$

and

$$\left(u^i\right)^-(x) := \sup\left\{t \in \mathbb{R} : \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}^N\left(\left\{y \in \Omega \cap Q(x, \varepsilon) : u^i(y) < t\right\}\right)}{\varepsilon^N} = 0\right\},$$

respectively. The jump set *of u is defined by*

$$S(u) := \bigcup_{i=1}^d \left\{x \in \Omega : \left(u^i\right)^-(x) < \left(u^i\right)^+(x)\right\}.$$

It can be shown that $S(u)$ and the complement of the set of Lebesgue points of u differ, at most, by a set of $\mathcal{H}^{N-1}$ measure zero. Moreover, $S(u)$ is $(N-1)$-rectifiable, i.e., there are C^1 hypersurfaces Γ_i such that

$$\mathcal{H}^{N-1}\left(S(u)\setminus\cup_{i=1}^{\infty}\Gamma_i\right)=0.$$

In addition, for $\mathcal{H}^{N-1}$-a.e. $x\in S(u)$ it is possible to find a, $b\in\mathbb{R}^d$ and $\nu\in S^{N-1}$ such that

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q_\nu^+(x,\varepsilon)}|u(y)-a|\,dy=0,\qquad \lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^N}\int_{Q_\nu^-(x,\varepsilon)}|u(y)-b|\,dy=0,$$

where $Q_v^+(x,\varepsilon):=\{y\in Q_\nu(x,\varepsilon):\langle y-x,\nu\rangle>0\}$ and $Q_\nu^-(x,\varepsilon):=\{y\in Q_\nu(x,\varepsilon):\langle y-x,\nu\rangle<0\}$. The triplet (a,b,ν) is uniquely determined up to a change of sign of ν and an interchange between a and b and it will be denoted by $(u^+(x),u^-(x),\nu_u(x))$. In the sequel, we write that

$$(a,b,\nu)\backsim(a',b',\nu') \tag{2.1}$$

if $(a,b,\nu)=(a',b',\nu')$ or $(a,b,\nu)=(b',a',-\nu')$.

Choosing a normal $\nu_u(x)$ to $S(u)$ at x, we denote the *jump* of u across $S(u)$ by $[u]:=u^+-u^-$. The distributional derivative of $u\in BV(\Omega;\mathbb{R}^d)$ admits the decomposition

$$Du=\nabla u\mathcal{L}^N\lfloor\Omega+([u]\otimes\nu_u)\mathcal{H}^{N-1}\lfloor S(u)+C(u),$$

where ∇u represents the density of the absolutely continuous part of the Radon measure Du with respect to the Lebesgue measure. The *Hausdorff*, or *jump*, *part* of Du is represented by $([u]\otimes\nu_u)\mathcal{H}^{N-1}\lfloor(S(u)\cap\Omega)$ and $C(u)$ is the *Cantor part* of Du. The measure $C(u)$ is singular with respect to the Lebesgue measure and is diffuse, i.e., every Borel set $E\subset\Omega$ with $\mathcal{H}^{N-1}(E)<\infty$ has Cantor measure zero.

We say that a set $E\subset\mathbb{R}^N$ is a set of *finite perimeter* in Ω if $\chi_E\in BV(\Omega)$, that is,

$$\sup\left\{\int_E\operatorname{div}\varphi\,dx:\ \varphi\in C_0^1\left(\Omega;\mathbb{R}^d\right),\ \|\varphi\|_\infty\le1\right\}<\infty.$$

The *perimeter* of E in Ω is the total variation of the characteristic function χ_E in Ω and it is denoted by $\operatorname{Per}(E;\Omega)$.

The relation between functions in $BV(\Omega)$ and sets of finite perimeter is given by the *Fleming-Rishel coarea formula*

$$|Du|(\Omega)=\int_{-\infty}^{\infty}\operatorname{Per}(\{x\in\Omega:\ u(x)>t\};\Omega)\,dt. \tag{2.2}$$

For every set E of finite perimeter in Ω, we have

$$\operatorname{Per}(E;\Omega)=\mathcal{H}^{N-1}(\partial^*E),$$

where ∂^*E represents the *reduced boundary* of E in Ω, i.e., $\partial^*E\cap\Omega=S(\chi_E)\cap\Omega$.

Special functions of bounded variation were introduced by De Giorgi and Ambrosio [21] in the study of image segmentation in computer vision.

Definition 2.3. *The space of* special functions of bounded variation, $SBV\left(\Omega;\mathbb{R}^d\right)$, *is the space of all functions u in $BV\left(\Omega;\mathbb{R}^d\right)$ such that $C(u)=0$.*

We say that a function $u\in SBV\left(\Omega;\mathbb{R}^d\right)$ belongs to $SBV_p\left(\Omega;\mathbb{R}^d\right)$, $p>1$, if

$$\nabla u\in L^p\left(\Omega;\mathbb{R}^d\right)\text{ and }\mathcal{H}^{N-1}(S(u))<\infty.$$

A sequence $\{u_n\}\subset SBV_p\left(\Omega;\mathbb{R}^d\right)$ *converges strongly to u in SBV_p* if

$$\begin{aligned}&u_n\to u\text{ in }L^1\left(\Omega;\mathbb{R}^d\right),\qquad \nabla u_n\to\nabla u\text{ in }L^p\left(\Omega;\mathbb{R}^{d\times N}\right),\\&\mathcal{H}^{N-1}(S(u_n)\triangle S(u))\to0,\qquad \int_{S(u_n)\cup S(u)}\left(|u_n^+-u^+|+|u_n^--u^-|\right)\,d\mathcal{H}^{N-1}\to0.\end{aligned} \tag{2.3}$$

Here we choose the orientation

$$\nu_{u_n} = \nu_u \quad \mathcal{H}^{N-1}\text{-a.e. on } S(u_n) \cap S(u) . \tag{2.4}$$

The space $SBV_0(\Omega; \mathbb{R}^d)$ is defined by

$$SBV_0\left(\Omega; \mathbb{R}^d\right) := \left\{u \in SBV\left(\Omega; \mathbb{R}^d\right) : \nabla u = 0 \ \mathcal{L}^N\text{-a.e. in } \Omega \text{ and } \mathcal{H}^{N-1}(S(u)) < \infty\right\}.$$

We recall the definition of quasiconvexity.

Definition 2.4. *A Borel function $f : \mathbb{R}^{d\times N} \to [-\infty, \infty]$ is said to be* quasiconvex *if*

$$f(\xi) \le \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega f(\xi + \nabla\varphi(y))\, dy \tag{2.5}$$

for every open bounded $\Omega \subset \mathbb{R}^N$ with $\mathcal{L}^N(\partial\Omega) = 0$, for every $\xi \in \mathbb{R}^{d\times N}$ and for every $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ whenever the right hand side of (2.5) exists as a Lebesgue integral.

Here, and in what follows, the space $W_0^{1,\infty}(\Omega; \mathbb{R}^d)$ denotes the $W^{1,\infty}$ weak * closure of $C_c^\infty(\Omega; \mathbb{R}^d)$.

Given $f : \mathbb{R}^{d\times N} \to (-\infty, \infty]$, the *quasiconvex envelope* $Qf : \mathbb{R}^{d\times N} \to (-\infty, \infty]$ of f is defined by

$$Qf(\xi) := \sup\left\{\overline{f}(\xi) : \overline{f} : \mathbb{R}^{d\times N} \to (-\infty, \infty] \text{ is quasiconvex, } \overline{f} \le f\right\},$$

where $\xi \in \mathbb{R}^{d\times N}$, where we use the convention that $\sup\emptyset = -\infty$.

If $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a Borel function locally bounded from below, then it can be shown that the quasiconvex envelope of f is given by

$$Qf(\xi) = \inf\left\{\int_Q f(\xi + \nabla\varphi(y))\, dy : \varphi \in W_0^{1,\infty}\left(Q; \mathbb{R}^d\right)\right\},$$

see [30].

For manifold-constrained fields the appropriate notion of quasiconvexity was introduced in [24], precisely,

Definition 2.5. *Let $\mathcal{M} \subset \mathbb{R}^d$ be an m-dimensional manifold of class C^1, with $1 \le m \le d$, and let $f : \mathbb{R}^{d\times N} \to \mathbb{R}$ be a Borel function locally bounded from below. The* tangential quasiconvex envelope*, $Q_T f$, of f is defined by*

$$Q_T f(z, \xi) := \inf\left\{\int_Q f(\xi + \nabla\varphi(x))\ dx : \varphi \in W_0^{1,\infty}(Q; T_z(\mathcal{M}))\right\},$$

$z \in \mathcal{M}$ and $\xi \in [T_z(\mathcal{M})]^N$, where $T_z(\mathcal{M})$ is the tangent space to $\mathcal{M}$ at z.

Setting $\overline{\mathcal{M}} := \left\{(z, \xi) \in \mathcal{M} \times \mathbb{R}^{d\times N} : \xi \in [T_z(\mathcal{M})]^N\right\}$, a Borel function $f : \overline{\mathcal{M}} \to \mathbb{R}$ is said to be *tangentially quasiconvex* if

$$f(z, \xi) = Q_T f(z, \xi) \qquad \text{for all } (z, \xi) \in \overline{\mathcal{M}}.$$

It was proved in [24] that under the conditions of Definition 2.5, one has

$$Q_T f(z, \xi) = Q\overline{f}(z, \xi) \tag{2.6}$$

for all $z \in \mathcal{M}$ and $\xi \in [T_z(\mathcal{M})]^N$, where $\overline{f} : \mathcal{M} \times \mathbb{R}^{d\times N} \to \mathbb{R}$ is the function defined by

$$\overline{f}(z, \xi) := f(P_z\xi), \tag{2.7}$$

$(z, \xi) \in \mathcal{M}\times\mathbb{R}^{d\times N}$. Here $P_z\xi := \left(P_z\xi^1, \ldots, P_z\xi^N\right)$, where ξ^i stands for the i^{th} column of the matrix $\xi \in \mathbb{R}^{d\times N}$, and P_z is the orthogonal projection of $\mathbb{R}^d$ onto the tangent space $T_z(\mathcal{M})$. In the special case in which $\mathcal{M}$ is the unit sphere S^{d-1}, then for $z \in \mathbb{R}^d \setminus \{0\}$ and $\xi \in \mathbb{R}^{d\times N}$,

$$P_z\xi = \left(\mathbb{I}_{d\times d} - \frac{z}{|z|} \otimes \frac{z}{|z|}\right)\xi \tag{2.8}$$

is the orthogonal projection of ξ onto the plane perpendicular to $\frac{z}{|z|}$, i.e., $P_z\xi \in T_{\frac{z}{|z|}}\left(S^{d-1}\right)$ and so the function (2.7) takes the simple form

$$\overline{f}(z,\xi) := f\left(\left(\mathbb{I}_{d\times d} - z\otimes z\right)\xi\right),$$

$(z,\xi)\in S^{d-1}\times\mathbb{R}^{d\times N}$. Note that we may extend $\overline{f}$ to $\mathbb{R}^d\setminus\{0\}\times\mathbb{R}^{d\times N}$ by

$$\overline{f}(z,\xi) := f\left(\left(\mathbb{I}_{d\times d} - \frac{z}{|z|}\otimes\frac{z}{|z|}\right)\xi\right).$$

Definition 2.6. *Given a Borel set $E\subset\mathbb{R}^d$, a Borel function $g: E\times E\times S^{N-1}\to[0,\infty]$ is said to be* BV-elliptic *if for every $(a,b,\nu)\in E\times E\times S^{N-1}$,*

$$\int_{S(u)} g\left(u^+,u^-,\nu_u\right)\, d\mathcal{H}^{N-1} \geq g(a,b,\nu)$$

for all functions $u\in SBV_0\left(Q_\nu;\mathbb{R}^d\right)\cap L^\infty\left(Q_\nu;\mathbb{R}^d\right)$ that take values in E and such that $u=u_{a,b,\nu}$ in a neighborhood of ∂Q_ν. Here

$$u_{a,b,\nu}(x) := \begin{cases} b & \text{if } x\cdot\nu\geq 0, \\ a & \text{if } x\cdot\nu<0. \end{cases} \tag{2.9}$$

If the set $E\subset\mathbb{R}^d$ is bounded, then it turns out that BV-ellipticity is a necessary and sufficient condition for sequential lower semicontinuity of functionals of the form

$$u\in SBV_0(\Omega;E)\longmapsto\int_{S(u)} g\left(u^+,u^-,\nu_u\right)\, d\mathcal{H}^{N-1}$$

under appropriate conditions on the integrand g. We refer to Theorem 5.14 in [5] for more details.

Definition 2.7. *Given a Borel set $E\subset\mathbb{R}^d$ and a Borel function $g: E\times E\times S^{N-1}\to[0,\infty)$, the* BV-elliptic envelope *$Rg: E\times E\times S^{N-1}\to[0,\infty]$ of g is defined by*

$$Rg(a,b,\nu) := \inf\left\{\int_{S(u)} g\left(u^+,u^-,\nu_u\right)\, d\mathcal{H}^{N-1} : u\in SBV_0\left(Q_\nu;\mathbb{R}^d\right)\cap L^\infty\left(Q_\nu;\mathbb{R}^d\right),\ u=u_{a,b,\nu} \text{ on } \partial Q_\nu\right\}, \tag{2.10}$$

$(a,b,\nu)\in E\times E\times S^{N-1}$.

3. Lower bound

Set

$$\overline{F}(u;A) := \int_A Q_T f(x,u,\nabla u)\, dx + \int_{S(u)\cap A} Rg\left(x,u^+,u^-,\nu_u\right)\, d\mathcal{H}^{N-1}, \tag{3.1}$$

where $u\in SBV_p\left(\Omega;S^{d-1}\right)$ and $A\in\mathcal{A}(\Omega)$.

The main result of this section is the following sequential lower semicontinuity theorem.

Theorem 3.1. *Let $\Omega\subset\mathbb{R}^N$ be open and bounded. Assume that $f:\Omega\times S^{d-1}\times\mathbb{R}^{d\times N}\to[0,\infty)$ and $g:\Omega\times S^{d-1}\times S^{d-1}\times S^{N-1}\to[0,\infty)$ satisfy hypotheses (F_1), (F_2) and (G_1), (G_2), (G_3), respectively. Then for every $u\in SBV_p\left(\Omega;S^{d-1}\right)$, $p>1$, $A\in\mathcal{A}(\Omega)$, and every sequence $\{u_n\}\subset SBV_p\left(A;S^{d-1}\right)$ converging to u in $L^1\left(A;\mathbb{R}^d\right)$,*

$$\overline{F}(u;A)\leq\liminf_{n\to\infty} F(u_n;A). \tag{3.2}$$

The proof of the previous theorem uses the next two lemmas.

Lemma 3.2. *Assume that* $g : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0, \infty)$ *satisfies conditions* (G_1) *and* (G_2), *let* $a,\, b \in S^{d-1}$, $\nu \in S^{N-1}$, $\{\omega_n\} \subset C\left(Q_\nu; \Omega\right)$, $\{u_n\} \subset SBV_p\left(Q_\nu; S^{d-1}\right)$ *be such that* $u_n \to u_{a,b,\nu}$ *in* $L^1\left(Q_\nu; \mathbb{R}^d\right)$ *and*

$$\liminf_{n\to\infty} \int_{S(u_n)} g\left(\omega_n(y), u_n^+(y), u_n^-(y), \nu_{u_n}(y)\right) \, d\mathcal{H}^{N-1}(y) < \infty, \qquad \lim_{n\to\infty} \int_{Q_\nu} |\nabla u_n(y)|^p \, dy = 0. \tag{3.3}$$

Then there exists a sequence $\{v_n\} \subset SBV_p(Q_\nu; S^{d-1})$ *such that* $v_n \to u_{a,b,\nu}$ *in* $L^1(Q_\nu; \mathbb{R}^d)$, $v_n = u_{a,b,\nu}$ *in a neighborhood of* ∂Q_ν,

$$\begin{aligned}\liminf_{n\to\infty} &\int_{S(v_n)} g\left(\omega_n(y), v_n^+(y), v_n^-(y), \nu_{v_n}(y)\right) \, d\mathcal{H}^{N-1}(y) \\ &\le \liminf_{n\to\infty} \int_{S(u_n)} g\left(\omega_n(y), u_n^+(y), u_n^-(y), \nu_{u_n}(y)\right) \, d\mathcal{H}^{N-1}(y),\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{Q_\nu} |\nabla v_n(y)|^p \, dy = 0.$$

Proof. Without loss of generality, we take $a = e_d$, $b = e_1$, $\nu = e_N$, we denote $u_{a,b,\nu}$ by u_0, i.e.,

$$u_0(y) := \begin{cases} e_1 & \text{if } y_N > 0, \\ e_d & \text{if } y_N \le 0, \end{cases}$$

and we write Q in place of Q_ν.

Extract a subsequence (not relabeled) such that

$$\liminf_{n\to\infty} \int_{S(u_n)} g\left(\omega_n, u_n^+, u_n^-, \nu_{u_n}\right) \, d\mathcal{H}^{N-1} = \lim_{n\to\infty} \int_{S(u_n)} g\left(\omega_n, u_n^+, u_n^-, \nu_{u_n}\right) \, d\mathcal{H}^{N-1} < \infty.$$

In view of (G_2) and since $\{u_n\}$ converges to u_0 in $L^1\left(Q; \mathbb{R}^d\right)$, we may also assume that the sequence of Radon measures $\nu_n := \mathcal{H}^{N-1}\lfloor(S(u_n) \cap Q)$ weakly star converges in the sense of measures to some nonnegative Radon measure ν and that $\{u_n\}$ converges to u_0 pointwise $\mathcal{L}^N$-a.e. in Q.

Using an argument of Carriero and Leaci (see [18], Lemma 3.5), we modify the sequence $\{u_n\}$ in such a way that its projection onto the sphere is Lipschitz. For each $z \in \mathbb{R}^d$ set $z' := \left(z^1, \ldots, z^{d-1}\right)$ and $z'' := \left(z^2, \ldots, z^d\right)$ so that $\left(z', z^d\right) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and $\left(z^1, z''\right) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Set

$$Q^+ := \{y \in Q : \, y_N \ge 0\}, \qquad Q^- := \{y \in Q : \, y_N < 0\},$$

and define

$$\widehat{u}_n(y) := \begin{cases} \left(\max\left(u_n^1(y), \frac{1}{2}\right), u_n''(y)\right) & \text{if } y \in Q^+, \\ \left(u_n'(y), \max\left(u_n^d(y), \frac{1}{2}\right)\right) & \text{if } y \in Q^-. \end{cases}$$

Since $|u_0| = 1$ in Q and $\{u_n\}$ converges to u_0 pointwise $\mathcal{L}^N$-a.e. in Q, we have that $\{\widehat{u}_n\}$ still converges to u_0 pointwise $\mathcal{L}^N$-a.e. in Q, and since $\frac{1}{2} \le |\widehat{u}_n| \le 2$, by the Lebesgue dominated convergence theorem, we have that $\{\widehat{u}_n\}$ converges to u_0 in $L^q\left(Q; \mathbb{R}^d\right)$ for every $1 \le q < \infty$.

Using the fact that the function $f_1(t) := \max\left(t, \frac{1}{2}\right)$, $t \in \mathbb{R}$, is Lipschitz, by Corollary 3.1 in [3] and Corollary 3.89 in [5] (with Ω replaced by Q^+), it follows that $\widehat{u}_n$ belongs to $SBV_p\left(Q; \mathbb{R}^d\right)$, with

$$\nabla \widehat{u}_n^1 = \begin{cases} 0 & \mathcal{L}^N\text{-a.e. in } Q^+ \cap \left\{u_n^1 \le \frac{1}{2}\right\}, \\ \nabla u_n^1 & \text{otherwise}, \end{cases} \tag{3.4}$$

$$\nabla \widehat{u}_n^d = \begin{cases} 0 & \mathcal{L}^N\text{-a.e. in } Q^- \cap \left\{u_n^d \le \frac{1}{2}\right\}, \\ \nabla u_n^d & \text{otherwise}, \end{cases} \tag{3.5}$$

$\nabla \widehat{u}_n^i = \nabla u_n^i$ for $i = 2, \ldots, d-1$, and

$$S(\widehat{u}_n) \subset S(u_n) \cup \{y \in Q : \, y_N = 0\}, \tag{3.6}$$

In what follows, for simplicity of notation we abbreviate $\{y \in Q : \widehat{u}_n(y) \neq u_n(y)\}$ as $\{\widehat{u}_n \neq u_n\}$ and $Q(0,s)$ as Q_s. Observe that

$$\mathcal{L}^N(\{\widehat{u}_n \neq u_n\}) = \mathcal{L}^N\left(Q^+ \cap \left\{u_n^1 < \frac{1}{2}\right\}\right) + \mathcal{L}^N\left(Q^- \cap \left\{u_n^d < \frac{1}{2}\right\}\right) \leq \mathcal{L}^N\left(Q \cap \left\{|u_n - u_0| > \frac{1}{2}\right\}\right),$$

therefore $\mathcal{L}^N(\{\widehat{u}_n \neq u_n\}) \to 0$ as $n \to \infty$. By Fubini's theorem we deduce that

$$\int_0^1 \mathcal{H}^{N-1}(\{\widehat{u}_n \neq u_n\} \cap \partial Q_s)\, ds = \mathcal{L}^N(\{\widehat{u}_n \neq u_n\}) \to 0 \qquad \text{as } n \to \infty,$$

and so, up to a subsequence (not relabeled),

$$\mathcal{H}^{N-1}(\{\widehat{u}_n \neq u_n\} \cap \partial Q_s) \to 0 \qquad \text{as } n \to \infty \text{ for } \mathcal{L}^1\text{-a.e. } s \in (0,1). \tag{3.7}$$

Fix $\delta > 0$, and in view of (3.7) choose $s_\delta \in (1-\delta, 1)$ such that

$$\mathcal{H}^{N-1}(\{\widehat{u}_n \neq u_n\} \cap \partial Q_{s_\delta}) \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad \nu(\partial Q_{s_\delta}) = 0. \tag{3.8}$$

Consider $m \in \mathbb{N}$ so large that $\delta + \frac{1}{m} < 1$ and let $\{\varphi_m\}$ be a sequence of smooth cut-off functions such that $\varphi_m = 1$ in Q_{s_δ}, $\varphi_m = 0$ in $Q \setminus Q_{s_\delta + \frac{1}{m}}$, and $\|\nabla \varphi_m\|_{L^\infty(Q;\mathbb{R}^N)} = O(m)$. Define

$$u_{n,m,\delta} := \begin{cases} u_n & \text{in } Q_{s_\delta}, \\ P(\varphi_m \widehat{u}_n + (1-\varphi_m) u_0) & \text{in } Q \setminus \overline{Q}_{s_\delta}, \end{cases}$$

where P is the projection onto the sphere S^{d-1}. Note that $u_{n,m,\delta} = u_0$ on ∂Q because $Pu_0 = u_0$. Since

$$\widehat{u}_n^1 \geq \frac{1}{2} \quad \text{in } \left(Q \setminus \overline{Q}_{s_\delta}\right) \cap Q^+, \qquad \widehat{u}_n^d \geq \frac{1}{2} \quad \text{in } \left(Q \setminus \overline{Q}_{s_\delta}\right) \cap Q^-,$$

then

$$\varphi_m \widehat{u}_n^1 + (1-\varphi_m) u_0^1 \geq \frac{1}{2} \quad \text{in } \left(Q \setminus \overline{Q}_{s_\delta}\right) \cap Q^+, \qquad \varphi_m \widehat{u}_n^d + (1-\varphi_m) u_0^d \geq \frac{1}{2} \quad \text{in } \left(Q \setminus \overline{Q}_{s_\delta}\right) \cap Q^-.$$

Using the fact that the projection $P : \mathbb{R}^d \setminus B_d\left(0, \frac{1}{2}\right) \to S^{d-1}$ is Lipschitz, by Corollary 3.89 in [5],

$$\nabla u_{n,m,\delta} = \begin{cases} \nabla u_n & \text{in } Q_{s_\delta}, \\ \nabla P(\varphi_m \widehat{u}_n + (1-\varphi_m) u_0) \nabla(\varphi_m \widehat{u}_n + (1-\varphi_m) u_0) & \text{in } Q \setminus \overline{Q}_{s_\delta}, \end{cases} \tag{3.9}$$

and

$$S(u_{n,m,\delta}) \subset S(u_n) \cup ((Q \setminus Q_{s_\delta}) \cap \{y_N = 0\}) \cup (\partial Q_{s_\delta} \cap \{\operatorname{tr}(\widehat{u}_n) \neq \operatorname{tr}(u_n)\}), \tag{3.10}$$

where we have used (3.6). By (3.4), (3.5), and (3.9), we obtain

$$|\nabla u_{n,m,\delta}| \leq \operatorname{Lip}\left(P|_{\mathbb{R}^d \setminus B_d\left(0,\frac{1}{2}\right)}\right)(|\nabla \widehat{u}_n| + |\nabla \varphi_m| |\widehat{u}_n - u_0|) \leq C(|\nabla u_n| + m|\widehat{u}_n - u_0|)$$

in $Q_{s_\delta + \frac{1}{m}} \setminus \overline{Q}_{s_\delta}$, and so

$$\int_Q |\nabla u_{n,m,\delta}|^p \, dy \leq C\left(\int_Q |\nabla u_n|^p \, dy + m^p \int_Q |\widehat{u}_n - u_0|^p \, dy\right).$$

Since $\widehat{u}_n \to u_0$ in $L^p(Q;\mathbb{R}^d)$, also by (3.3), we get

$$\lim_{n\to\infty} \int_Q |\nabla u_{n,m,\delta}|^p \, dy = 0. \tag{3.11}$$

On the other hand, by (G_2), (3.6), and (3.10), we deduce that

$$\int_{S(u_{n,m,\delta})} g\left(\omega_n, u_{n,m,\delta}^+, u_{n,m,\delta}^-, \nu_{u_{n,m,\delta}}\right) d\mathcal{H}^{N-1} \leq \int_{S(u_n)} g(\omega_n, u_n^+, u_n^-, \nu_{u_n}) \, d\mathcal{H}^{N-1}$$
$$+ C\mathcal{H}^{N-1}(\{\widehat{u}_n \neq u_n\} \cap \partial Q_{s_\delta}) + C\mathcal{H}^{N-1}\left(S(u_n) \cap \left(Q_{s_\delta + \frac{1}{m}} \setminus Q_{s_\delta}\right)\right) + C\mathcal{H}^{N-1}(\{y_N = 0\} \cap (Q \setminus Q_{s_\delta})).$$

Letting $n \to \infty$ and using (3.8) and the fact that $\nu_n \overset{\star}{\rightharpoonup} \nu$ in the sense of measures, we obtain

$$\limsup_{n\to\infty} \int_{S(u_{n,m,\delta})} g\left(\omega_n, u^+_{n,m,\delta}, u^-_{n,m,\delta}, \nu_{u_{n,m,\delta}}\right)\, d\mathcal{H}^{N-1}$$
$$\leq \lim_{n\to\infty} \int_{S(u_n)} g\left(\omega_n, u^+_n, u^-_n, \nu_{u_n}\right)\, d\mathcal{H}^{N-1} + C\nu\left(\overline{Q_{s_\delta+\frac{1}{m}}} \setminus Q_{s_\delta}\right) + C\mathcal{H}^{N-1}\left(\{y_N = 0\} \cap (Q \setminus Q_{s_\delta})\right).$$

When $m \to \infty$ we have that

$$\nu\left(\overline{Q_{s_\delta+\frac{1}{m}}} \setminus Q_{s_\delta}\right) \to \nu\left(\partial Q_{s_\delta}\right) = 0$$

by (3.8). It suffices to let $\delta \to 0^+$ to conclude that

$$\limsup_{\delta\to0^+} \limsup_{m\to\infty} \limsup_{n\to\infty} \int_{S(u_{n,m,\delta})} g\left(\omega_n, u^+_{n,m,\delta}, u^-_{n,m,\delta}, \nu_{u_{n,m,\delta}}\right)\, d\mathcal{H}^{N-1}$$
$$\leq \lim_{n\to\infty} \int_{S(u_n)} g\left(\omega_n, u^+_n, u^-_n, \nu_{u_n}\right)\, d\mathcal{H}^{N-1}.$$

This, together with (3.11), and a simple diagonalization argument, yields the desired result. □

The next lemma allows us to work with sequences in $SBV_0\left(\Omega; S^{d-1}\right)$ in order to use the BV-ellipticity condition. A similar argument already appears in Theorem 3.3 of [2].

Lemma 3.3. *Let a, $b \in S^{d-1}$, $\nu \in S^{N-1}$, let $\{u_n\} \subset SBV_p\left(Q_\nu; S^{d-1}\right)$, $p > 1$, be a sequence of functions satisfying*

$$\sup_n \mathcal{H}^{N-1}\left(S\left(u_n\right)\right) < \infty, \qquad \lim_{n\to\infty} \int_{Q_\nu} |\nabla u_n|^p\, dy = 0, \tag{3.12}$$

and $u_n = u_{a,b,\nu}$ in a neighborhood of ∂Q_ν (depending on n).

Then there exists a sequence $\{\widetilde{v}_n\} \subset SBV_0\left(Q_\nu; S^{d-1}\right)$ such that

$$\lim_{n\to\infty} \|\widetilde{v}_n - u_n\|_{L^\infty(Q_\nu; S^{d-1})} = 0, \qquad \lim_{n\to\infty} \mathcal{H}^{N-1}\left(S\left(\widetilde{v}_n\right) \setminus S\left(u_n\right)\right) = 0,$$

and $\widetilde{v}_n = u_{a,b,\nu}$ in a neighborhood of ∂Q_ν.

Proof. Let k_n be the integer part of $\left(\int_{Q_\nu} |\nabla u_n|\, dy\right)^{-1/2}$, so that $k_n \to \infty$ and

$$k_n \int_{Q_\nu} |\nabla u_n|\, dy \to 0 \qquad \text{as } n \to \infty.$$

Take n so large that $1/k_n < |b^i - a^i|$, whenever $b^i \neq a^i$, $i = 1, \ldots, d$, where a^i and b^i are the components of the vectors a and b, respectively. For every $j = 0, \ldots, 3k_n - 1$ and every $i = 1, \ldots, d$, by the coarea formula (see (2.2)), we have

$$\int_{\{\alpha_j < u^i_n \leq \alpha_{j+1}\}} \left|\nabla u^i_n\right|\, dy = \left|Du^i_n\right|\left(\left(Q_\nu \setminus S\left(u^i_n\right)\right) \cap \left\{\alpha_j < u^i_n \leq \alpha_{j+1}\right\}\right)$$
$$= \int_{\alpha_j}^{\alpha_{j+1}} \mathcal{H}^{N-1}\left(\left(Q_\nu \setminus S\left(u^i_n\right)\right) \cap \partial^*\left(\left\{y \in Q_\nu :\, u^i_n(y) > t\right\}\right)\right)\, dt,$$

where $\alpha_0 := -2$, $\alpha_j := -2 + \frac{j}{k_n}$, $\alpha_{3k_n} := 1$, and so there exists $t^i_j \in (\alpha_j, \alpha_{j+1})$ such that

$$\int_{\{\alpha_j < u^i_n \leq \alpha_{j+1}\}} \left|\nabla u^i_n\right|\, dy \geq \frac{1}{k_n} \mathcal{H}^{N-1}\left(\left(Q_\nu \setminus S(u^i_n)\right) \cap \partial^*\{y \in Q_\nu :\, u^i(y) > t^i_j\}\right).$$

Summing over j yields

$$\int_{Q_\nu} \left|\nabla u_n^i\right| \, dy \geq \frac{1}{k_n} \sum_{j=0}^{3k_n-1} \mathcal{H}^{N-1}\left(\left(\overline{Q}_\nu \setminus S(u_n^i)\right) \cap \partial^* \{y \in \overline{Q}_\nu : \ u_n^i(y) > t_j^i\}\right) \, dt.$$

Let $t_{-1}^i := -2$, $t_{3k_n}^i := 1$, and for $j \in \{-1, \ldots, 3k_n - 1\}$ set

$$E_j^i := \left\{y \in Q_\nu : \ t_j^i < u_n^i(y) \leq t_{j+1}^i\right\},$$
$$v_n^i(y) := \begin{cases} a^i & \text{if } y \in E_j^i \text{ and } t_j^i < a^i \leq t_{j+1}^i, \\ b^i & \text{if } y \in E_j^i \text{ and } t_j^i < b^i \leq t_{j+1}^i, \\ t_j^i & \text{otherwise in } E_j^i. \end{cases}$$

Since $\frac{1}{k_n} < \left|b^i - a^i\right|$ whenever $b^i \neq a^i$, then either a^i or b^i is in $\left(t_j^i, t_{j+1}^i\right]$ but not both simultaneously, so v_n^i is well-defined and $\|v_n - u_n\|_{L^\infty(Q_\nu;\mathbb{R}^d)} \leq \frac{\sqrt{d}}{k_n}$.

Moreover, $v_n = u_{a,b,\nu}$ in a neighborhood of ∂Q_ν. Indeed, since $u_n = u_{a,b,\nu}$ in a neighborhood of ∂Q_ν, if $y_0 \in \partial Q_\nu$ and $y_0 \cdot \nu > 0$, then $u_n^i(y) = b^i$ for all $y \in Q_\nu$ near y_0. Using the fact that for every fixed $i \in \{1, \ldots, d\}$ the family $\left\{E_j^i\right\}_{j=-1}^{3k_n-1}$ is a partition of Q_ν, there is $j \in \{-1, \ldots, 3k_n - 1\}$ such that $y \in E_j^i$ for all y near y_0 with $t_j^i < b^i < u_n^i(y) \leq t_{j+1}^i$. Thus $v_n^i(y) = b^i$ for all such y, by the definition of v_n^i. In turn, $v_n^i(y_0) = b^i$. Similarly, $v_n(y) = a$ for all $y \in \partial Q_\nu$ with $y \cdot \nu < 0$.

We have

$$S(v_n) \subset \bigcup_{i=1}^{d} \bigcup_{j=0}^{3k_n-1} \partial^* E_j^i \subset \bigcup_{i=1}^{d} \bigcup_{j=0}^{3k_n-1} \partial^* \{y \in Q_\nu : \ u_n^i(y) > t_j^i\},$$

so that

$$\mathcal{H}^{N-1}(S(v_n) \backslash S(u_n)) \leq \sum_{i=1}^{d} \sum_{j=0}^{3k_n-1} \mathcal{H}^{N-1}\left(\left(Q_\nu \setminus S(u_n^i)\right) \cap \partial^* \{y \in Q_\nu : \ u_n^i > t_j^i\}\right) \leq C k_n \int_{Q_\nu} |\nabla u_n| \, dy \to 0,$$

where we have used (3.12). Moreover, for $y \in Q_\nu$ we have

$$1 = |u_n(y)| \leq |u_n(y) - v_n(y)| + |v_n(y)| \leq \frac{\sqrt{d}}{k_n} + |v_n(y)|.$$

Hence, $|v_n(y)| \geq 1 - \frac{\sqrt{d}}{k_n} \geq \frac{1}{2}$ for $\mathcal{L}^N$-a.e. $y \in Q_\nu$ and for all n sufficiently large. Define $\widetilde{v}_n := P(v_n)$. Then

$$\|\widetilde{v}_n - u_n\|_{L^\infty(Q_\nu;\mathbb{R}^d)} \leq C \|v_n - u_n\|_{L^\infty(Q_\nu;\mathbb{R}^d)} \leq \frac{C}{k_n} \to 0.$$

Since $P : \mathbb{R}^d \setminus B_d\left(0, \frac{1}{2}\right) \to S^{d-1}$ is Lipschitz, by Corollary 3.1 in [3], $\widetilde{v}_n \in SBV_0\left(Q_\nu; \mathbb{R}^d\right)$ and $S(\widetilde{v}_n) \subset S(v_n)$. Thus, $\mathcal{H}^{N-1}(S(\widetilde{v}_n) \setminus S(u_n)) \to 0$ and the proof is complete. $\square$

Remark 3.4. *Consider the function* $\psi : \mathbb{R}^d \to \mathbb{R}$ *defined by* $\psi(z) := |z|^2$, $z \in \mathbb{R}^d$. *Since* ψ *is locally Lipschitz, for any* $u \in SBV_p\left(\Omega; S^{d-1}\right)$ *we have* $\psi \circ u \in SBV_p(\Omega; \mathbb{R})$ *by Corollary 3.1 in* [3], *and* $0 = \nabla(\psi \circ u) = \nabla\psi(u) \nabla u = 2(\nabla u)^T u$ $\mathcal{L}^N$*-a.e. in* Ω. *Hence,*

$$(\nabla u(x))^T u(x) = 0 \qquad \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega. \tag{3.13}$$

Proof of Theorem 3.1. Without loss of generality, we may assume that

$$\liminf_{n\to\infty}\left(\int_A f(x,u_n,\nabla u_n)\,dx+\int_{S(u_n)\cap A} g\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}\right)$$
$$=\lim_{n\to\infty}\int_A f(x,u_n,\nabla u_n)\,dx+\lim_{n\to\infty}\int_{S(u_n)\cap A} g\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}<\infty.$$

By the coercivity conditions (F_2) and (G_2), up to a subsequence (not relabeled), there exists a nonnegative Radon measure $\mu:\mathcal{B}(A)\to[0,\infty)$, where $\mathcal{B}(A)$ is the family of all Borel subsets of A, such that

$$f(x,u_n,\nabla u_n)\,\mathcal{L}^N\lfloor A+g\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\mathcal{H}^{N-1}\lfloor(S(u_n)\cap A)\overset{\star}{\rightharpoonup}\mu$$

as $n\to\infty$, weakly star in the sense of measures.

By the Radon-Nikodym and Lebesgue decomposition theorems (see [29] Theorems 1.101 and 1.115, respectively), we can write μ as a sum of three mutually singular nonnegative measures

$$\mu=\frac{d\mu}{d\mathcal{L}^N\lfloor A}\mathcal{L}^N\lfloor A+\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}\mathcal{H}^{N-1}\lfloor(S(u)\cap A)+\mu_s.$$

By the Besicovitch derivation theorem (see [29] Theorem 1.153)

$$\frac{d\mu}{d\mathcal{L}^N\lfloor A}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))}<\infty\qquad\text{for }\mathcal{L}^N\text{-a.e. }x_0\in A,$$
$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(Q_\nu(x_0,\varepsilon))}{\mathcal{H}^{N-1}(S(u)\cap Q_\nu(x_0,\varepsilon))}<\infty\qquad\text{for }\mathcal{H}^{N-1}\text{-a.e. }x_0\in S(u)\cap A.$$

By Theorem 2.83 in [5], it follows that

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mu(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}$$

for $\mathcal{H}^{N-1}$-a.e. $x_0\in S(u)\cap A$.

We claim that

$$\frac{d\mu}{d\mathcal{L}^N\lfloor A}(x_0)\ge Q_T f(x_0,u(x_0),\nabla u(x_0))\quad\text{for }\mathcal{L}^N\text{-a.e. }x_0\in A,\tag{3.14}$$
$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)\ge Rg\left(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)\right)\tag{3.15}$$

for $\mathcal{H}^{N-1}$-a.e. $x_0\in S(u)\cap A$.

If (3.14) and (3.15) hold, then the conclusion of the theorem follows immediately. Indeed, since $\mu_n\overset{*}{\rightharpoonup}\mu$ in the sense of measures,

$$\liminf_{n\to\infty}\left(\int_A f(x,u_n,\nabla u_n)\,dx+\int_{S(u_n)\cap A} g\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}\right)$$
$$=\liminf_{n\to\infty}\mu_n(A)\ge\mu(A)\ge\int_A\frac{d\mu}{d\mathcal{L}^N\lfloor A}\,dx+\int_{S(u)\cap A}\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S(u)}\,d\mathcal{H}^{N-1}$$
$$\ge\int_A Q_T f(x,u,\nabla u)\,dx+\int_{S(u)\cap A} Rg\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{N-1},$$

where we have used the fact that $\mu_s\ge0$.

Step 1- Let $\varphi : [0,\infty) \to [0,1]$ be a continuous function such that $\varphi = 0$ in $\left[0, \frac{1}{2}\right]$ and $\varphi = 1$ in $[1,\infty)$. Define $\widetilde{f} : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0,\infty)$ by

$$\widetilde{f}(x,z,\xi) := \begin{cases} \varphi(|z|) f\left(x, \frac{z}{|z|}, P_z\xi\right) + (1-\varphi(|z|))|\xi|^p & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases} \tag{3.16}$$

where $P_z\xi$ is defined in (2.8). Observe that $\widetilde{f}$ is a Carathéodory function satisfying

$$0 \leq \widetilde{f}(x,z,\xi) \leq C(1+|\xi|^p)$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $z \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$. Moreover, by (2.6),

$$Q\widetilde{f}(x,z,\xi) = Q_T f(x,z,\xi) \tag{3.17}$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$, for all $z \in S^{d-1}$ and $\xi \in \mathbb{R}^{d\times N}$.

We denote by B_1 the unit ball in $\mathbb{R}^N$. Fix $x_0 \in A$ satisfying $a)$, $b)$, $c)$ and $d)$ in the proof of Theorem 5.29 in [5] and such that

$$(\nabla u(x_0))^T u(x_0) = 0, \tag{3.18}$$

where we have used Remark 3.4.

Choosing $\varepsilon_k \searrow 0^+$ such that $\mu(\partial B(x_0,\varepsilon_k)) = 0$, we have

$$\begin{aligned} \frac{d\mu}{d\mathcal{L}^N \lfloor A}(x_0) &= \lim_{k\to\infty} \frac{\mu(B(x_0,\varepsilon_k))}{\omega_N \varepsilon_k^N} \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\omega_N\varepsilon_k^N}\left(\int_{B(x_0,\varepsilon_k)} f(x,u_n,\nabla u_n)\,dx + \int_{S(u_n)\cap B(x_0,\varepsilon_k)} g\left(x,u_n^+,u_n^-,\nu_{u_n}\right)d\mathcal{H}^{N-1}\right) \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{\omega_N}\left(\int_{B_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_{n,k}, \nabla w_{n,k})\,dy\right. \\ &\quad \left. + \frac{1}{\varepsilon_k}\int_{S(w_{n,k})\cap B_1} g\left(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_{n,k}^+, u(x_0)+\varepsilon_k w_{n,k}^-, \nu_{w_{n,k}}\right)d\mathcal{H}^{N-1}\right), \end{aligned}$$

where

$$w_{n,k}(y) := \frac{u_n(x_0+\varepsilon_k y) - u(x_0)}{\varepsilon_k}.$$

Clearly, $w_{n,k} \in SBV_p\left(B_1;\mathbb{R}^d\right)$ and $w_{n,k} \to w_0$ in $L^1\left(B_1;\mathbb{R}^d\right)$, where $w_0(y) := \nabla u(x_0)\,y$, $y \in B_1$. By the choice of x_0 and the coercivity conditions (F_2) and (G_2) we have

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_{B_1}|\nabla w_{n,k}|^p\,dy < \infty, \qquad \limsup_{k\to\infty}\limsup_{n\to\infty}\frac{\mathcal{H}^{N-1}(S(w_{n,k})\cap B_1)}{\varepsilon_k} < \infty. \tag{3.19}$$

By a standard diagonalization argument we can extract a subsequence $w_k := w_{n_k,k}$ that converges to w_0 in $L^1\left(B_1;\mathbb{R}^d\right)$ and such that

$$\lim_{k\to\infty}\mathcal{H}^{N-1}(S(w_k)\cap B_1) = 0, \qquad \sup_k\left(\int_{B_1}|\nabla w_k|^p\,dy + \int_{S(w_k)\cap B_1}|[w_k]|\,d\mathcal{H}^{N-1}\right) < \infty, \tag{3.20}$$

$$\frac{d\mu}{d\mathcal{L}^N\lfloor A}(x_0) \geq \lim_{k\to\infty}\frac{1}{\omega_N}\int_{B_1} f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k w_k, \nabla w_k)\,dy,$$

where we used the facts that $g \geq 0$, $|[w_{n,k}]| \leq \frac{2}{\varepsilon_k}$, and $(3.19)_2$. Since $|u(x_0)+\varepsilon_k w_k(y)| = |u_{n_k}(x_0+\varepsilon_k y)| = 1$ for $\mathcal{L}^N$-a.e. $y \in B_1$, then by (3.13),

$$(\nabla(u(x_0)+\varepsilon_k w_k(y)))^T(u(x_0)+\varepsilon_k w_k(y)) = \varepsilon_k(\nabla w_k(y))^T(u(x_0)+\varepsilon_k w_k(y)) = 0.$$

Hence, using the fact that for $z \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{d\times N}$, $(z \otimes z)\,\xi = z \otimes \left(\xi^T z\right)$, we have that

$$[\mathbb{I} - (u(x_0) + \varepsilon_k w_k(y)) \otimes (u(x_0) + \varepsilon_k w_k(y))]\, \nabla w_k(y) = \nabla w_k(y),$$

and so $(3.20)_3$ may be written as

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N \lfloor A}(x_0) &\geq \limsup_{k\to\infty} \frac{1}{\omega_N} \int_{B_1} \widetilde{f}\,(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k, \nabla w_k)\, dy \\ &\geq \limsup_{k\to\infty} \frac{1}{\omega_N} \int_{B_1} Q\widetilde{f}\,(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k, \nabla w_k)\, dy,\end{aligned}$$

where we have used the fact that $\widetilde{f} \geq Q\widetilde{f}$ and $\widetilde{f}$ is the function defined in (3.16).

In view of $(3.20)_2$, it follows that

$$\frac{d\mu}{d\mathcal{L}^N \lfloor A}(x_0) \geq \limsup_{\delta\to 0^+} \limsup_{k\to\infty} \frac{1}{\omega_N} \int_{B_1} \left[Q\widetilde{f}\,(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k, \nabla w_k) + \delta\, |\nabla w_k|^p\right] dy. \quad (3.21)$$

For each $\delta > 0$ fixed, we proceed as in the proof of Theorem 5.29 in [5] (applied to the quasiconvex integrand $Q\widetilde{f}\,(x, z, \xi) + \delta\,|\xi|^p$) to obtain

$$\begin{aligned}&\limsup_{k\to\infty} \frac{1}{\omega_N} \int_{B_1} \left[Q\widetilde{f}\,(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k w_k, \nabla w_k) + \delta\,|\nabla w_k|^p\right] dy \\ &\geq Q\widetilde{f}\,(x_0, u(x_0), \nabla u(x_0)) + \delta\,|\nabla u(x_0)|^p = Q_T f\,(x_0, u(x_0), \nabla u(x_0)) + \delta\,|\nabla u(x_0)|^p,\end{aligned}$$

where we have used (2.6) and the fact that $[\mathbb{I} - u(x_0) \otimes u(x_0)]\, \nabla u(x_0) = \nabla u(x_0)$, since $(\nabla u(x_0))^T u(x_0) = 0$ by (3.18). This, together with (3.21), yields (3.14).

Step 2- To prove (3.15), fix $x_0 \in S(u) \cap A$ such that

$$\begin{aligned}&\frac{d\mu}{d\mathcal{H}^{N-1} \lfloor S(u)}(x_0) = \lim_{\varepsilon\to 0^+} \frac{\mu(Q_\nu(x_0, \varepsilon))}{\varepsilon^{N-1}} < \infty, \\ &\lim_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{Q_\nu(x_0,\varepsilon_k)} |u(x) - u_{x_0,\nu}(x)|\, dx = 0,\end{aligned} \quad (3.22)$$

where $\nu := \nu_u(x_0)$ and

$$u_{x_0,\nu}(y) := \begin{cases} u^+(x_0) & \text{if } y\cdot\nu > 0, \\ u^-(x_0) & \text{if } y\cdot\nu \leq 0. \end{cases}$$

Using the fact that μ is a Radon measure, we may choose $\varepsilon_k \searrow 0^+$ such that $\mu(\partial Q_\nu(x_0, \varepsilon_k)) = 0$. Then

$$\begin{aligned}&\lim_{k\to\infty} \frac{\mu(Q_\nu(x_0, \varepsilon_k))}{\varepsilon_k^{N-1}} = \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{\varepsilon_k^{N-1}} \left(\int_{Q_\nu(x_0,\varepsilon_k)} f(x, u_n, \nabla u_n)\, dx + \int_{S(u_n)\cap Q_\nu(x_0,\varepsilon_k)} g\left(x, u_n^+, u_n^-, \nu_{u_n}\right) d\mathcal{H}^{N-1}\right) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \left(\int_{Q_\nu} \varepsilon_k f\left(x_0 + \varepsilon_k y, v_{n,k}, \frac{1}{\varepsilon_k} \nabla v_{n,k}\right) dy + \int_{S\left(v_{n,k}\right)\cap Q_\nu} g\left(x_0 + \varepsilon_k y, v_{n,k}^+, v_{n,k}^-, \nu_{v_{n,k}}\right) d\mathcal{H}^{N-1}\right),\end{aligned} \quad (3.23)$$

where

$$v_{n,k}(y) := u_n(x_0 + \varepsilon_k y), \qquad y \in Q_\nu.$$

Note that $v_{n,k} \in SBV_p\left(Q_\nu; S^{d-1}\right)$, and by $(3.22)_2$, $\lim_{k\to\infty} \lim_{n\to\infty} \|v_{n,k} - u_{x_0,\nu}\|_{L^1\left(Q_\nu; S^{d-1}\right)} = 0$. Moreover, by $(3.22)_1$, (3.23), and the coercivity hypotheses on f and g, we have that

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \frac{1}{\varepsilon_k^{p-1}} \int_{Q_\nu} |\nabla v_{n,k}|^p\, dy < \infty, \qquad \limsup_{k\to\infty} \limsup_{n\to\infty} \mathcal{H}^{N-1}(S(v_{n,k})) < \infty. \quad (3.24)$$

By a standard diagonalization argument, we may extract a subsequence $v_k := v_{n_k,k}$ that converges to $u_{x_0,\nu}$ in $L^1\left(Q_\nu; S^{d-1}\right)$ such that

$$\lim_{k\to\infty}\int_{Q_\nu}|\nabla v_k|^p\ dy=0,\qquad C_0:=\sup_k\mathcal{H}^{N-1}\left(S\left(v_k\right)\right)<\infty, \tag{3.25}$$

and

$$\lim_{k\to\infty}\frac{\mu\left(Q_\nu\left(x_0,\varepsilon_k\right)\right)}{\varepsilon_k^{N-1}}\geq\lim_{k\to\infty}\int_{S(v_k)}g\left(x_0+\varepsilon_k y, v_k^+, v_k^-, \nu_{v_k}\right)\, d\mathcal{H}^{N-1}. \tag{3.26}$$

By Lemma 3.2 there exists $\{\overline{v}_k\}\subset SBV_p\left(Q_\nu; S^{d-1}\right)$ such that $\overline{v}_k\to u_{x_0,\nu}$ in $L^1\left(Q_\nu;\mathbb{R}^d\right)$, $\overline{v}_k=u_{x_0,\nu}$ in a neighborhood of ∂Q_ν,

$$\lim_{k\to\infty}\int_{Q_\nu}|\nabla \overline{v}_k|^p\ dy=0,$$

and

$$\liminf_{k\to\infty}\int_{S(\overline{v}_k)}g\left(x_0+\varepsilon_k y,\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\,d\mathcal{H}^{N-1}\leq\liminf_{k\to\infty}\int_{S(v_k)}g\left(x_0+\varepsilon_k y, v_k^+, v_k^-,\nu_{v_k}\right)\,d\mathcal{H}^{N-1}. \tag{3.27}$$

Since g is uniformly continuous on $\overline{Q\left(x_0,r\right)}\times S^{d-1}\times S^{d-1}\times S^{N-1}$, where $\overline{Q\left(x_0,r\right)}\subset\Omega$, for $\eta>0$ fixed there exists $\delta\in(0,1)$ such that

$$\left|g\left(x,\lambda,\theta,\nu\right)-g\left(x_1,\lambda_1,\theta_1,\nu\right)\right|\leq\eta \tag{3.28}$$

for all x, $x_1\in\overline{Q\left(x_0,r\right)}$, λ, λ_1, θ, $\theta_1\in S^{d-1}$, $\nu\in S^{N-1}$, with $|x-x_1|<\delta$, $|\lambda-\lambda_1|<\delta$, $|\theta-\theta_1|<\delta$.

By Lemma 3.3, we can find a sequence $\{\widetilde{v}_k\}\subset SBV_0\left(Q_\nu;S^{d-1}\right)$ such that

$$\lim_{k\to\infty}\|\widetilde{v}_k-\overline{v}_k\|_{L^\infty\left(\Omega;\mathbb{R}^d\right)}=0,\qquad\lim_{k\to\infty}\mathcal{H}^{N-1}\left(\left(S\left(\widetilde{v}_k\right)\setminus S\left(\overline{v}_k\right)\right)\right)=0, \tag{3.29}$$

and $\widetilde{v}_k=u_{x_0,\nu}$ on ∂Q_ν. Using the facts that $g\geq0$ and $\left(\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\backsim\left(\widetilde{v}_k^+,\widetilde{v}_k^-,\nu_{\widetilde{v}_k}\right)$ $\mathcal{H}^{N-1}$-a.e. in $S\left(\overline{v}_k\right)\cap S\left(\widetilde{v}_k\right)$ (see Proposition 3.73(b) in [5]) in the sense of (2.1), (G_3), (3.28), $(3.25)_2$, and (3.29), we have

$$\begin{aligned}\int_{S(\overline{v}_k)}g\left(x_0+\varepsilon_k y,\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\,d\mathcal{H}^{N-1}&\geq\int_{S(\overline{v}_k)}g\left(x_0,\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\,d\mathcal{H}^{N-1}-\eta C_0\\&\geq\int_{S(\overline{v}_k)\cap S(\widetilde{v}_k)}g\left(x_0,\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\,d\mathcal{H}^{N-1}-\eta C_0\\&\geq\int_{S(\overline{v}_k)\cap S(\widetilde{v}_k)}g\left(x_0,\widetilde{v}_k^+,\widetilde{v}_k^-,\nu_{\widetilde{v}_k}\right)\,d\mathcal{H}^{N-1}-2\eta C_0.\end{aligned}$$

On the other hand, by the growth condition (G_2) and $(3.29)_2$, we obtain,

$$\int_{(S(\widetilde{v}_k)\setminus S(v_k))}g\left(x_0,\widetilde{v}_k^+,\widetilde{v}_k^-,\nu_{\widetilde{v}_k}\right)\,d\mathcal{H}^{N-1}\leq C\mathcal{H}^{N-1}\left(\left(S\left(\widetilde{v}_k\right)\setminus S\left(\overline{v}_k\right)\right)\right)\to0.$$

Hence,

$$\begin{aligned}\limsup_{k\to\infty}\int_{S(\overline{v}_k)}g\left(x_0+\varepsilon_k y,\overline{v}_k^+,\overline{v}_k^-,\nu_{\overline{v}_k}\right)\,d\mathcal{H}^{N-1}&\geq\limsup_{k\to\infty}\int_{S(\widetilde{v}_k)}g\left(x_0,\widetilde{v}_k^+,\widetilde{v}_k^-,\nu_{\widetilde{v}_k}\right)\,d\mathcal{H}^{N-1}-2\eta C_0\\&\geq Rg\left(x_0,u^+\left(x_0\right),u^-\left(x_0\right),\nu_{u(x_0)}\left(x_0\right)\right)-2\eta C_0,\end{aligned} \tag{3.30}$$

where in the last inequality we have used (2.10), the facts that $\widetilde{v}_k=u_{x_0,\nu}$ on ∂Q_ν and $\widetilde{v}_k\in SBV_0\left(Q_\nu;S^{d-1}\right)$. Combining (3.26), (3.27), and (3.30) yields

$$\frac{d\mu}{d\mathcal{H}^{N-1}\lfloor S\left(u\right)}\left(x_0\right)\geq Rg\left(x_0,u^+\left(x_0\right),u^-\left(x_0\right),\nu_u\right)-2\eta C_0.$$

It now suffices to let $\eta \to 0^+$. □

4. Upper bound

In this section we prove the opposite inequality of (3.2) for functions $u \in SBV_p\left(\Omega; S^{d-1}\right)$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Assume that*

$$p = 2$$

and that $f : \Omega \times S^{d-1} \times \mathbb{R}^{d\times N} \to [0,\infty)$ and $g : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0,\infty)$ satisfy hypotheses (F_1), (F_2) and (G_1), (G_2), (G_3), in the introduction, respectively. Then for every $u \in SBV_2\left(\Omega; S^{d-1}\right)$, $A \in \mathcal{A}(\Omega)$, there exists a sequence $\{u_n\} \subset SBV_2\left(A; S^{d-1}\right)$ converging to u in $L^1\left(\Omega; \mathbb{R}^d\right)$ and such that

$$\liminf_{n\to\infty} F(u_n; A) \le \overline{F}(u; A), \tag{4.1}$$

where $\overline{F}$ is the functional defined in (3.1).

To prove (4.1), we first show that $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure for all functions $u \in SBV_p\left(\Omega; S^{d-1}\right)$ with the property that there exist a closed $(N-1)$-rectifiable set K and a constant $C > 0$ such that $u \in C\left(\Omega \setminus K; S^{d-1}\right)$ and for every compact set $K' \subset K$,

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(\{x \in \Omega : \operatorname{dist}(x, K') < \varepsilon\}\right)}{\varepsilon} \le C\mathcal{H}^{N-1}\left(K'\right). \tag{4.2}$$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Assume that $f : \Omega \times S^{d-1} \times \mathbb{R}^{d\times N} \to [0,\infty)$ and $g : \Omega \times S^{d-1} \times S^{d-1} \times S^{N-1} \to [0,\infty)$ satisfy hypotheses (F_1), (F_2) and (G_1), (G_2), (G_3), in the introduction, respectively and let $u \in SBV_p\left(\Omega; S^{d-1}\right)$, $p > 1$, be such that $u \in C\left(\Omega \setminus K; S^{d-1}\right)$, where $K \subset \mathbb{R}^N$ is a closed $(N-1)$-rectifiable set satisfying* (4.2). *Then $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure absolutely continuous with respect to the measure*

$$(1 + |\nabla u|^p)\,\mathcal{L}^N \lfloor \Omega + \mathcal{H}^{N-1}\lfloor S(u).$$

The following lemma plays a central role in the proof of Theorem 4.2.

Lemma 4.3. *Under the hypotheses of Theorem 4.2, let $u \in SBV_p\left(\Omega; S^{d-1}\right)$ be such that $u \in C\left(\Omega \setminus K; S^{d-1}\right)$, where K is a closed $(N-1)$-rectifiable set satisfying* (4.2), *and let $\{u_n\} \subset SBV_p\left(A; S^{d-1}\right)$ converge to u in $L^1(A;\mathbb{R}^d)$ for some $A \in \mathcal{A}(\Omega)$. Given an open set $B_0 \subset\subset A$ with polyhedral boundary and such that $\mathcal{H}^{N-1}(\partial B_0 \cap K) = 0$, there exists a sequence $\{v_n\} \subset SBV_p\left(A; S^{d-1}\right)$, converging to u in $L^1\left(A; \mathbb{R}^d\right)$, and such that $v_n = u$ in a neighborhood of ∂B_0 (depending on n) and*

$$\limsup_{n\to\infty} F(v_n; A) \le \liminf_{n\to\infty} F(u_n; A).$$

Proof. By extracting subsequences, if necessary, we may assume that

$$\liminf_{n\to\infty} F(u_n; A) = \lim_{n\to\infty} F(u_n; A) < \infty, \tag{4.3}$$

and, by (F_2) and (G_2), that the sequence of measures

$$\mu_n := (1 + |\nabla u_n|^p)\,\mathcal{L}^N \lfloor A + \mathcal{H}^{N-1}\lfloor(S(u_n) \cap A)$$

weakly star converges in the sense of measures to some nonnegative Radon measure $\mu : \mathcal{B}(A) \to [0,\infty)$.

Since B_0 has polyhedral boundary, we may write $\partial B_0 = \bigcup_{i=1}^M P_i$, where

$$P_i \subset \left\{x \in \mathbb{R}^N : (x - a_i)\cdot \nu_i = 0\right\},$$

with $a_i \in \mathbb{R}^N, \nu_i \in S^{N-1}$, $i = 1, \ldots, M$.

For $t>0$ set

$$E_t^1 := \{x \in A:\ \mathrm{dist}\,(x, \partial B_0 \cap K) \le t\}\,, \qquad E_t^2 := \{x \in A:\ \mathrm{dist}\,(x, \cup_{i \ne j}(P_i \cap P_j)) \le t\}\,, \tag{4.4}$$

and

$$E_t := E_t^1 \cup E_t^2.$$

By (4.2),

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{L}^N(\{x \in A:\ \mathrm{dist}\,(x, \partial B_0 \cap K) < \varepsilon\})}{\varepsilon} \le C\mathcal{H}^{N-1}(\partial B_0 \cap K) = 0. \tag{4.5}$$

In particular,

$$\mathcal{L}^N\left(E_t^1\right) \to 0 \qquad \text{as } t \to 0^+. \tag{4.6}$$

Consider the function

$$f(x) := \mathrm{dist}\,(x, \partial B_0 \cap K)\,, \qquad x \in \mathbb{R}^N.$$

It is well-known that f is Lipschitz, and thus by the coarea formula (see (2.74) in [5]) we have that

$$\int_{\{x \in A:\ 0<f(x)<\varepsilon\}} |\nabla f(x)|\ dx = \int_0^\varepsilon \mathcal{H}^{N-1}(\{x \in A:\ f(x) = s\})\ ds. \tag{4.7}$$

By Corollary 3.4.5 in [17] we have that $|\nabla f(x)| = 1$ for all $x \in \mathbb{R}^N \setminus (\partial B_0 \cap K)$ such that f is differentiable at x. Hence, also by Rademacker's theorem (see Theorem 2.14 in [5]) we have that

$$\int_{\{x \in A:\ 0<f(x)<\varepsilon\}} |\nabla f(x)|\ dx = \mathcal{L}^N(\{x \in A:\ 0 < f(x) < \varepsilon\})\,,$$

which, together with (4.7), yields

$$\mathcal{L}^N(\{x \in A:\ \mathrm{dist}\,(x, \partial B_0 \cap K) < \varepsilon\}) = \int_0^\varepsilon \mathcal{H}^{N-1}(\{x \in A:\ \mathrm{dist}\,(x, \partial B_0 \cap K) = s\})\ ds = \int_0^\varepsilon \mathcal{H}^{N-1}\left(\partial E_s^1\right)\ ds.$$

Thus, by (4.5) there exists $s_\varepsilon \in \left(\frac{\varepsilon}{2}, \varepsilon\right)$ such that

$$\mathcal{H}^{N-1}\left(\partial E_{s_\varepsilon}^1\right) \to 0 \qquad \text{as } \varepsilon \to 0^+. \tag{4.8}$$

Set

$$B_\varepsilon := \left\{x \in A:\ \mathrm{dist}\,(x, K) \ge \frac{\varepsilon}{4}\right\} \cap \{x \in A:\ \mathrm{dist}\,(x, \partial B_0) \le 2\varepsilon\}\,.$$

Since $B_0 \subset\subset A$, by taking ε sufficiently small, we have that $B_\varepsilon \subset\subset A$, and so B_ε is compact. Using the fact that $u \in C\left(\Omega \setminus K; S^{d-1}\right)$, we have that $u \in C\left(B_\varepsilon; S^{d-1}\right)$. Hence there exists $0 < \delta_\varepsilon < \frac{\varepsilon}{2}$ such that for every x, $x' \in B_\varepsilon$, with $|x - x'| < \delta_\varepsilon$,

$$\left|u(x) - u(x')\right| < \frac{1}{2\sqrt{d}}. \tag{4.9}$$

For every $i = 1, \dots, M$, let $\{R_{i,j,\varepsilon}\}_{j=1}^{M_{i,\varepsilon}}$ be a grid of closed rectangles, with mutually disjoint interiors, with centers in $P_i \setminus E_{s_\varepsilon}$ and with two sides parallel to ν_i covering $P_i \setminus E_{s_\varepsilon}$ and such that the sides parallel to ν_i have length δ_ε^2 and the sides orthogonal to ν_i have length δ_ε. Note that since the center of each $R_{i,j,\varepsilon}$ does not belong to E_{s_ε} and $\delta_\varepsilon < \frac{\varepsilon}{2}$ we have that $R_{i,j,\varepsilon} \subset B_\varepsilon$. As a consequence, (4.9) holds in each rectangle $R_{i,j,\varepsilon}$. Let

$$P_{i,\varepsilon} := \mathrm{int}\left(\bigcup_{j=1}^{M_{i,\varepsilon}} R_{i,j,\varepsilon}\right). \tag{4.10}$$

Observe that

$$M_{i,\varepsilon} \le \frac{\mathcal{H}^{N-1}(P_i)}{\delta_\varepsilon^{N-1}}. \tag{4.11}$$

Indeed,

$$\mathcal{H}^{N-1}(P_i) \ge \mathcal{H}^{N-1}(P_{i,\varepsilon} \cap P_i) = \sum_{j=1}^{M_{i,\varepsilon}} \mathcal{H}^{N-1}(R_{i,j,\varepsilon} \cap P_i) = M_{i,\varepsilon}\delta_\varepsilon^{N-1}.$$

Step 1- We now modify the sequence $\{u_n\}$ in each rectangle $R_{i,j,\varepsilon}$. Without loss of generality, we may assume that $\nu_i = e_N$ and the center of the rectangle is the origin. To simplify the notation, we denote this rectangle by R.

We will use the same argument as in the proof of Lemma 3.2. Since $|u(0)| = 1$, there is $i = 1, \dots, d$ such that $\left|u^i(0)\right| \geq \frac{1}{\sqrt{d}}$. We may assume that $i = d$ and, further, that $u^d(0) \geq \frac{1}{\sqrt{d}}$ (the case $u^d(0) \leq -\frac{1}{\sqrt{d}}$ is similar). By (4.9), we have

$$u^d(x) \geq u^d(0) - \frac{1}{2\sqrt{d}} \geq \frac{1}{2\sqrt{d}} \qquad \text{for all } x \in R. \tag{4.12}$$

For $x \in R$ define

$$\widehat{u}_n(x) := \left(u_n'(x), \max\left(u_n^d(x), \frac{1}{4\sqrt{d}}\right)\right). \tag{4.13}$$

Reasoning as in the proof of Lemma 3.2, by Corollary 3.1 in [3] with $\psi(s) := \max\left\{s, \frac{1}{4\sqrt{d}}\right\}$, we have

$$\begin{aligned} \nabla\widehat{u}_n^d &= \begin{cases} 0 & \mathcal{L}^N\text{-a.e. on } R \cap \left\{u_n^d \leq \frac{1}{4\sqrt{d}}\right\}, \\ \nabla u_n^d & \text{otherwise,} \end{cases} \\ \nabla\widehat{u}_n^i &= \nabla u_n^i, \quad i = 1, \dots, d-1, \end{aligned} \tag{4.14}$$

and

$$S\left(\widehat{u}_n^d\right) \cap R \subset S(u_n) \cap R,$$

$$\begin{aligned} \left[\widehat{u}_n^d\right] &= \psi\left(\left(u_n^d\right)^+\right) - \psi\left(\left(u_n^d\right)^-\right) \qquad \text{on } S(u_n) \cap R \\ \left[\widehat{u}_n^i\right] &= \left[u_n^i\right] \qquad \text{for } i = 1, \dots, d-1. \end{aligned} \tag{4.15}$$

Observe that since $u_n \to u$ in $L^1\left(A; \mathbb{R}^d\right)$, by (4.12),

$$\mathcal{L}^N\left(\{\widehat{u}_n \neq u_n\} \cap R\right) \leq \mathcal{L}^N\left(\left\{u_n^d < \frac{1}{4\sqrt{d}}\right\} \cap R\right) \leq \mathcal{L}^N\left(\left\{|u_n - u| > \frac{1}{2\sqrt{d}}\right\} \cap R\right) \to 0.$$

Let $R^+ := R' \times \left[0, \frac{\delta_\varepsilon^2}{2}\right]$ and $R^- := R' \times \left[-\frac{\delta_\varepsilon^2}{2}, 0\right]$. By Fubini's theorem we deduce that

$$\mathcal{L}^N\left(\{\widehat{u}_n \neq u_n\} \cap R\right) = \int_{-\frac{\delta_\varepsilon^2}{2}}^{\frac{\delta_\varepsilon^2}{2}} \mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_s\right) \, ds \to 0 \qquad \text{as } n \to \infty,$$

where $Y_s := R' \times \{x_N = s\}$. Hence,

$$\mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_s\right) \to 0 \qquad \text{as } n \to \infty \text{ for } \mathcal{L}^1\text{-a.e. } s \in \left(-\frac{\delta_\varepsilon^2}{2}, \frac{\delta_\varepsilon^2}{2}\right).$$

Choose $s := s(\varepsilon) \in \left(0, \frac{\delta_\varepsilon^2}{2}\right)$ such that

$$\begin{cases} \mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_s\right) + \mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_{-s}\right) \to 0 \qquad \text{as } n \to \infty, \\ \mu(Y_s) + \mu(Y_{-s}) = 0, \qquad \mathcal{H}^{N-1}\left(S(u) \cap (Y_s \cup Y_{-s})\right) = 0. \end{cases} \tag{4.16}$$

Consider $m \in \mathbb{N}$ so large that $\frac{1}{m} < s$, and let $\varphi_m \in C_c^\infty(R; [0,1])$ be such that $\varphi_m \equiv 1$ in $R' \times \left[\left(-\frac{\delta_\varepsilon^2}{2}, -s\right) \cup \left(s, \frac{\delta_\varepsilon^2}{2}\right)\right]$, $\varphi_m \equiv 0$ in $R' \times \left(-s + \frac{1}{m}, s - \frac{1}{m}\right)$, and $\|\nabla\varphi_m\|_\infty \leq Cm$.

Define $\overline{u}_{m,n} : R \to S^{d-1}$ by

$$\overline{u}_{m,n} := \frac{\varphi_m \widehat{u}_n + (1 - \varphi_m) u}{\left|\varphi_m \widehat{u}_n + (1 - \varphi_m) u\right|}.$$

Note that $\overline{u}_{m,n}$ is well-defined since by (4.12) and (4.13) in R,

$$\varphi_m \widehat{u}_n^d + (1-\varphi_m)\, u^d \geq \varphi_m \frac{1}{4\sqrt{d}} + (1-\varphi_m)\,\frac{1}{2\sqrt{d}} \geq \frac{1}{4\sqrt{d}}.$$

Using the fact that the projection $P : \mathbb{R}^d \setminus B_d\left(0, \frac{1}{4\sqrt{d}}\right) \to S^{d-1}$ is Lipschitz, by Corollary 3.1 in [3], we have that $\overline{u}_{m,n} \in SBV_p\left(R; S^{d-1}\right)$ with

$$\begin{aligned}|\nabla \overline{u}_{m,n}| &\leq C \operatorname{Lip}\left(P|_{\mathbb{R}^d \setminus B_d\left(0,\frac{1}{4\sqrt{d}}\right)}\right) |\nabla\left(\varphi_m \widehat{u}_n + (1-\varphi_m)\, u\right)| \\ &\leq C\left(m\,|\widehat{u}_n - u| + \varphi_m\,|\nabla u_n| + (1-\varphi_m)\,|\nabla u|\right)\end{aligned} \tag{4.17}$$

in R, where we used (4.14),

$$S\left(\overline{u}_{n,m}\right) \cap R \subset \left(S\left(u_n\right) \cup S\left(u\right)\right) \cap R, \tag{4.18}$$

and

$$|[\overline{u}_{m,n}]| \leq C\left(|[u_n]| + |[u]|\right) \leq C \tag{4.19}$$

in $S\left(\overline{u}_{m,n}\right) \cap R$ by (4.15) and the fact that ψ is Lipschitz.

Let

$$\begin{aligned} R_s &:= R' \times \left(\left(-\tfrac{\delta_\varepsilon^2}{2}, -s\right) \cup \left(s, \tfrac{\delta_\varepsilon^2}{2}\right)\right), \\ L_{s,m} &:= R' \times \left(s - \tfrac{1}{m}, s\right), \qquad L_{-s,m} := R' \times \left(-s, -s + \tfrac{1}{m}\right), \end{aligned}$$

and define $u_{m,n} : R \to S^{d-1}$ by

$$u_{m,n} := \begin{cases} \overline{u}_{m,n} & \text{on } R \setminus R_s, \\ u_n & \text{in } R_s. \end{cases} \tag{4.20}$$

Note that

$$u_{m,n} = u \qquad \text{in } R' \times \left(-s + \frac{1}{m}, s - \frac{1}{m}\right). \tag{4.21}$$

We claim that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} I_{m,n} \leq C \int_{R'\times(-s,s)} \left(1 + |\nabla u|^p\right)\, dx + C\mathcal{H}^{N-1}\left(S\left(u\right) \cap \left(R' \times (-s,s)\right)\right), \tag{4.22}$$

where

$$I_{m,n} := F\left(u_{m,n}; R\right) - F\left(u_n; R\right). \tag{4.23}$$

To prove this, note that by Corollary 3.89 in [5], $u_{m,n} \in SBV_p\left(R; S^{d-1}\right)$, with

$$\nabla u_{m,n} = \begin{cases} \nabla \overline{u}_{m,n} & \mathcal{L}^N\text{-a.e. on } R \setminus R_s, \\ \nabla u_n & \mathcal{L}^N\text{-a.e. in } R_s, \end{cases} \tag{4.24}$$

$$S\left(u_{m,n}\right) \cap R \subset \left(S\left(u_n\right) \cup S\left(u\right) \cup \left(Y_s \cup Y_{-s}\right)\right) \cap R, \tag{4.25}$$

and

$$|[u_{m,n}]| = \begin{cases} |[u_n]| & \text{on } S\left(u_n\right) \cap R_s, \\ |[\overline{u}_{m,n}]| & \text{on } S\left(\overline{u}_{m,n}\right) \cap \left(R \setminus R_s\right), \\ |\operatorname{tr}\left(u_n\right) - \operatorname{tr}\left(\widehat{u}_n\right)| & \text{on } \left\{\operatorname{tr}\left(u_n\right) \neq \operatorname{tr}\left(\widehat{u}_n\right)\right\} \cap \left(Y_s \cup Y_{-s}\right), \end{cases} \tag{4.26}$$

where we have used (4.19) and the fact that

$$\operatorname{tr}\left(\overline{u}_{m,n}\right) = \operatorname{tr}\left(\widehat{u}_n\right) \qquad \text{on } Y_s \cup Y_{-s}.$$

By (F_2), (G_2), and (4.17)-(4.26), we have

$$\begin{aligned} I_{m,n} = F(u_{m,n}; R) - F(u_n; R) &\leq C \int_{R' \times \left(-s+\frac{1}{m}, s-\frac{1}{m}\right)} \left(1 + |\nabla u|^p\right) \, dx \\ &+ C\mathcal{H}^{N-1}\left(S(u) \cap \left(R' \times (-s,s)\right)\right) \\ &+ C \int_{L_{-s,m} \cup L_{s,m}} \left(1 + m^p |\widehat{u}_n - u|^p + |\nabla u_n|^p + |\nabla u|^p\right) \, dx \\ &+ C\mathcal{H}^{N-1}\left(S(u_n) \cap (L_{-s,m} \cup L_{s,m})\right) + C\mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_{-s}\right) \\ &+ C\mathcal{H}^{N-1}\left(\{\widehat{u}_n \neq u_n\} \cap Y_s\right). \end{aligned}$$

Since $\widehat{u}_n \to u$ in $L^p\left(R; \mathbb{R}^d\right)$ (since $\widehat{u}_n \to u$ in $L^1\left(R; \mathbb{R}^d\right)$ and the sequence is bounded in $L^\infty\left(R; \mathbb{R}^d\right)$) and $\mu_n \overset{*}{\rightharpoonup} \mu$ in the sense of measures, we have that

$$\begin{aligned} \limsup_{m\to\infty} \limsup_{n\to\infty} I_{m,n} &\leq \limsup_{m\to\infty} C \left(\int_{L_{-s,m} \cup L_{s,m} \cup R' \times \left(-s+\frac{1}{m}, s-\frac{1}{m}\right)} \left(1 + |\nabla u|^p\right) \, dx \right. \\ &\left. + \mathcal{H}^{N-1}\left(S(u) \cap R' \times (-s,s)\right) + \mu\left(\overline{L_{-s,m}} \cup \overline{L_{s,m}}\right) \right) \\ &= C \int_{R' \times (-s,s)} \left(1 + |\nabla u|^p\right) \, dx + C\mathcal{H}^{N-1}\left(S(u) \cap \left(R' \times (-s,s)\right)\right), \end{aligned} \tag{4.27}$$

where we have used (4.16). This proves the claim.

Step 2- For every $i = 1, \ldots, M$ and $j = 1, \ldots, M_{i,\varepsilon}$, let $u^{i,j}_{\varepsilon,m,n} : R_{i,j,\varepsilon} \to S^{d-1}$ be the sequence defined in (4.20), and let $v_{\varepsilon,m,n} : \Omega \to S^{d-1}$ be given by

$$v_{\varepsilon,m,n}(x) := \begin{cases} u(x) & \text{if } x \in E_{s_\varepsilon}, \\ u^{i,j}_{\varepsilon,m,n}(x) & \text{if } x \in R_{i,j,\varepsilon} \setminus E_{s_\varepsilon}, \\ u_n(x) & \text{elsewhere.} \end{cases}$$

By Corollary 3.89 in [5], we have that $v_{\varepsilon,m,n} \in SBV_p\left(\Omega; S^{d-1}\right)$. Moreover, since $u^{i,j}_{\varepsilon,m,n} = u_n$ on the top and on the bottom of each rectangle $R_{i,j,\varepsilon}$, the only new jumps created are contained on the lateral sides of each rectangle $R_{i,j,\varepsilon}$ and on the boundary of E_{s_ε}. Thus by (F_2) and (G_2), (4.22), we have

$$\begin{aligned} F(v_{\varepsilon,m,n}; A) &\leq F\left(u_n; A \setminus \cup_{i=1}^M P_{i,\varepsilon}\right) + F(u; E_{s_\varepsilon}) + \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} F\left(u^{i,j}_{\varepsilon,m,n}; R_{i,\varepsilon}\right) \\ &+ C \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} \sum_{\substack{l=1 \\ l\neq j}}^{M_{i,\varepsilon}} \mathcal{H}^{N-1}\left(\partial R_{i,j,\varepsilon} \cap \partial R_{i,l,\varepsilon}\right) + C\mathcal{H}^{N-1}\left(\partial E_{s_\varepsilon}\right) \\ &\leq F(u_n; A) + F(u; E_{s_\varepsilon}) + \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} I^{i,j}_{\varepsilon,m,n} \\ &+ C \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} \sum_{\substack{l=1 \\ l\neq j}}^{M_{i,\varepsilon}} \mathcal{H}^{N-1}\left(\partial R_{i,j,\varepsilon} \cap \partial R_{i,l,\varepsilon}\right) + C\mathcal{H}^{N-1}\left(\partial E_{s_\varepsilon}\right), \end{aligned} \tag{4.28}$$

where in the last inequality we have used (4.27) and $I^{i,j}_{\varepsilon,m,n}$ is the expression $I_{m,n}$ defined in (4.23) for each rectangle $R_{i,j,\varepsilon}$. For each $i = 1, \ldots, M$ and $j = 1, \ldots, M_{i,\varepsilon}$ the number of rectangles $R_{i,l,\varepsilon}$

that have a side in common with $R_{i,j,\varepsilon}$, $j \neq l$, depends only on N, and so we have that

$$\sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} \sum_{\substack{l=1 \\ l\neq j}}^{M_{i,\varepsilon}} \mathcal{H}^{N-1}\left(\partial R_{i,j,\varepsilon} \cap \partial R_{i,l,\varepsilon}\right) = C\left(N\right) \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} \mathcal{H}^{N-2}\left(R'_{i,j,\varepsilon}\right) \delta_\varepsilon^2 \tag{4.29}$$

$$= CMM_{i,\varepsilon}\delta_\varepsilon^{N-2}\delta_\varepsilon^2 \leq C\delta_\varepsilon,$$

where in the last inequality we have used (4.11). By (4.8), $\mathcal{H}^{N-1}\left(\partial E^1_{s_\varepsilon}\right) \to 0$ as $\varepsilon \to 0^+$, while from the fact that B_0 has polyhedral boundary it follows that

$$\mathcal{H}^{N-1}\left(\partial E^2_{s_\varepsilon}\right) \leq C\varepsilon,$$

where the set E^2_s is defined in (4.4). Hence $\mathcal{H}^{N-1}\left(\partial E_{s_\varepsilon}\right) \to 0$. By (4.6) and again the fact that B_0 has polyhedral boundary

$$\mathcal{L}^N\left(E_{s_\varepsilon}\right) = \mathcal{L}^N\left(E^1_{s_\varepsilon}\right) + \mathcal{L}^N\left(E^2_{s_\varepsilon}\right) \to 0. \tag{4.30}$$

Finally, by (F_2) and (G_2),

$$F\left(u; E_{s_\varepsilon}\right) \leq C \int_{E_{s_\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + C\mathcal{H}^{N-1}\left(S\left(u\right) \cap E_{s_\varepsilon}\right). \tag{4.31}$$

Combining (4.28), (4.29), and (4.31) yields

$$F\left(v_{\varepsilon,m,n}; A\right) \leq F\left(u_n; A\right) + C \int_{E_{s_\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + C\mathcal{H}^{N-1}\left(S\left(u\right) \cap E_{s_\varepsilon}\right)$$
$$+ \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} I^{i,j}_{\varepsilon,m,n} + C\delta_\varepsilon + O\left(\varepsilon\right).$$

By (4.3) and (4.27),

$$\limsup_{m\to\infty} \limsup_{n\to\infty} F\left(v_{\varepsilon,m,n}; A\right) \leq \lim_{n\to\infty} F\left(u_n; A\right) + C \int_{E_{s_\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + C\mathcal{H}^{N-1}\left(S\left(u\right) \cap E_{s_\varepsilon}\right)$$
$$+ C \sum_{i=1}^{M} \sum_{j=1}^{M_{i,\varepsilon}} \left[\int_{R_{i,j,\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + \mathcal{H}^{N-1}\left(S\left(u\right) \cap R_{i,j,\varepsilon}\right)\right]$$
$$\leq \lim_{n\to\infty} F\left(u_n; A\right) + C \int_{E_{s_\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + C\mathcal{H}^{N-1}\left(S\left(u\right) \cap E_{s_\varepsilon}\right)$$
$$+ C \sum_{i=1}^{M} \left[\int_{P_{i,\varepsilon}} \left(1 + |\nabla u|^p\right)\, dx + \mathcal{H}^{N-1}\left(S\left(u\right) \cap P_{i,\varepsilon}\right)\right] + O\left(1\right),$$

where $P_{i,\varepsilon}$ is the set defined in (4.10). Using (4.30) and the fact that $\mathcal{H}^{N-1}\left(S\left(u\right) \cap \partial B_0\right) = 0$, by letting $\varepsilon \to 0^+$, it follows that

$$\limsup_{\varepsilon\to 0^+} \limsup_{m\to\infty} \limsup_{n\to\infty} F\left(v_{\varepsilon,m,n}; A\right) \leq \liminf_{n\to\infty} F\left(u_n; A\right).$$

By a diagonalization argument, we obtain a subsequence $v_k := v_{\varepsilon_k,m_k,n_k} \in SBV_p\left(\Omega; S^{d-1}\right)$ converging to u in $L^1\left(\Omega; \mathbb{R}^d\right)$ and such that

$$\limsup_{k\to\infty} F\left(v_k; A\right) \leq \liminf_{n\to\infty} F\left(u_n; A\right).$$

By construction, $v_k = u$ on a neighborhood of ∂B_0. □

We now turn to the proof of Theorem 4.2.

Proof of Theorem 4.2. We prove that $\mathcal{F}(u;\cdot)$ satisfies the hypotheses of Proposition 5.2 in the appendix. Property (1) follows from the fact that admissible sequences for $A_1 \cup A_2$ are exactly those obtained by pairing admissible sequences for A_1 and A_2.

Next we prove that

$$\mathcal{F}(u;A) \leq \mathcal{F}(u;B) + \mathcal{F}\left(u;A\setminus\overline{C}\right) \tag{4.32}$$

for every $A,\ B,\ C \in \mathcal{A}(\Omega)$ such that $C \subset\subset B \subset\subset A$. By (1.1), for every $\eta > 0$ one can find $\{u_n\} \subset SBV_p\left(B;S^{d-1}\right)$, $\{v_n\} \subset SBV_p\left(A\setminus\overline{C};S^{d-1}\right)$ such that $u_n \to u$ in $L^1\left(B;\mathbb{R}^d\right)$, $v_n \to u$ in $L^1\left(A\setminus\overline{C};\mathbb{R}^d\right)$ and

$$\lim_{n\to\infty} F(u_n;B) \leq \mathcal{F}(u;B) + \eta, \qquad \lim_{n\to\infty} F\left(v_n;A\setminus\overline{C}\right) \leq \mathcal{F}\left(u;A\setminus\overline{C}\right) + \eta. \tag{4.33}$$

Choose $B_0 \in \mathcal{A}_\infty(\Omega)$ such that B_0 has polyhedral boundary, $C \subset\subset B_0 \subset\subset B$, and $\mathcal{H}^{N-1}(S(u)\cap\partial B_0) = 0$. Applying Lemma 4.3 we may find $\{u_n'\} \subset SBV_p\left(B;S^{d-1}\right)$, $\{v_n'\} \subset SBV_p\left(A\setminus\overline{C};S^{d-1}\right)$, converging to u in $L^1\left(B;\mathbb{R}^d\right)$ and $L^1\left(A\setminus\overline{C};\mathbb{R}^d\right)$, respectively, such that $u_n' = v_n' = u$ in a neighborhood of ∂B_0 (depending on n), and

$$\limsup_{n\to\infty} F\left(u_n';B\right) \leq \lim_{n\to\infty} F(u_n;B), \qquad \limsup_{n\to\infty} F\left(v_n';A\setminus\overline{C}\right) \leq \lim_{n\to\infty} F\left(v_n;A\setminus\overline{C}\right).$$

Define

$$w_n := \begin{cases} u_n' & \text{in } B_0, \\ v_n' & \text{in } A\setminus B_0. \end{cases}$$

Then $w_n \in SBV_p\left(A;S^{d-1}\right)$ and $w_n \to u$ in $L^1\left(A;\mathbb{R}^d\right)$. Hence, by (1.1), (4.33), and the fact that $f, g \geq 0$,

$$\begin{aligned}\mathcal{F}(u;A) &\leq \liminf_{n\to\infty} F(w_n;A) \leq \limsup_{n\to\infty} F\left(u_n';B\right) + \limsup_{n\to\infty} F\left(v_n';A\setminus\overline{C}\right) \\ &\leq \lim_{n\to\infty} F(u_n;B) + \lim_{n\to\infty} F\left(v_n;A\setminus\overline{C}\right) \leq \mathcal{F}(u;B) + \mathcal{F}\left(u;A\setminus\overline{C}\right) + 2\eta.\end{aligned}$$

Letting $\eta \to 0^+$, we obtain (4.32).

Finally, let

$$\mu := C\left(1+|\nabla u|^p\right)\mathcal{L}^N\lfloor\Omega + C\mathcal{H}^{N-1}\lfloor S(u).$$

By considering the sequence $u_n \equiv u$, by (1.1) and (F_2), (G_2), we have that

$$\mathcal{F}(u;A) \leq \mu(A)$$

for all $A \in \mathcal{A}(\Omega)$.

Thus, all the hypotheses of Proposition 5.2 in the appendix are satisfied, and so the result follows. □

To establish (4.1) for a general $u \in SBV_p\left(\Omega;S^{d-1}\right)$, we will use the regularity results of Carriero and Leaci [18] for sphere-valued minimizers of the Mumford-Shah functional, to approximate any u in $SBV_p\left(\Omega;S^{d-1}\right)$ in a strong sense by a sequence $\{u_n\}$ of functions satisfying the hypotheses of Theorem 4.2. The proof follows essentially the one of Braides and Chiadò-Piat (see Lemma 5.2 in [15]) for the unconstrained case.

Lemma 4.4. *If $u \in SBV_p\left(\Omega;S^{d-1}\right)$, $p > 1$, then there exists a sequence $\{u_n\}$ in $SBV_p\left(\Omega;S^{d-1}\right)$ strongly converging to u in $SBV_p\left(\Omega;S^{d-1}\right)$ with the property that for each $n \in \mathbb{N}$ there exist a closed $(N-1)$-rectifiable set K_n and a constant $C_n > 0$ such that $u_n \in C^1\left(\Omega\setminus K_n;S^{d-1}\right)$ and for every compact set $K \subset K_n$,*

$$\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N\left(\{x\in\Omega:\ \operatorname{dist}(x,K)<\varepsilon\}\right)}{\varepsilon} \leq C_n\mathcal{H}^{N-1}(K). \tag{4.34}$$

Moreover, if $p = 2$, then $u_n \in C^\infty\left(\Omega\setminus K_n;S^{d-1}\right)$.

Proof. Since $S(u)$ is $(N-1)$-rectifiable, for every $n \in \mathbb{N}$, we may find a finite union of closed subsets R_n of hypersurfaces of class C^1 such that

$$\mathcal{H}^{N-1}(S(u) \setminus R_n) \leq \frac{1}{n}.$$

Extend u to be zero outside Ω and let $\tilde{u}_n \in C_c^\infty(\mathbb{R}^N; \mathbb{R}^d)$ be the mollification of this extension. Without loss of generality, we may assume that

$$\lim_{n\to\infty} n \int_\Omega |u - \tilde{u}_n|^p \, dx = 0.$$

Let I be the functional defined on $SBV_p(\Omega; S^{d-1})$ by

$$\begin{aligned} I(v) := & \int_\Omega |\nabla v|^p \, dx + \mathcal{H}^{N-1}(S(v) \setminus R_n) + n \int_\Omega |v - \tilde{u}_n|^p \, dx \\ & + \int_{R_n} \left(1 + \left|v^+ - u^+\right| + \left|v^- - u^-\right|\right) \, d\mathcal{H}^{N-1}. \end{aligned} \tag{4.35}$$

Here we choose the orientation $\nu_v = \nu_u$ on $S(u) \cap S(v) \cap R_n$.

Following the proof of Lemma 5.2 in [15], we have that this functional is coercive in SBV_p and it is lower semicontinuous with respect to strong convergence in $L^1_{\text{loc}}(\Omega; S^{d-1})$. Hence, for each n there exists a minimizer $u_n \in SBV_p(\Omega; S^{d-1})$ for (4.35). Again by Lemma 5.2 in [15], we obtain that $u_n \to u$ strongly in $SBV_p(\Omega; S^{d-1})$.

We claim that the restriction of u_n to $\Omega \setminus R_n$ is a local minimizer for the functional

$$J(v) := \int_{\Omega\setminus R_n} |\nabla v|^p \, dx + \mathcal{H}^{N-1}(S(v) \setminus R_n) + n \int_{\Omega\setminus R_n} |v - \tilde{u}_n|^p \, dx,$$

$v \in SBV_p(\Omega \setminus R_n; S^{d-1})$. Indeed, fix n and let $v \in SBV_p(\Omega \setminus R_n; S^{d-1})$ be such that $v = u_n$ $\mathcal{L}^N$-a.e. $(\Omega \setminus R_n) \setminus K$ for some compact set $K \subset \Omega \setminus R_n$, and define

$$w(x) := \begin{cases} v(x) & \text{for } x \in K, \\ u_n(x) & \text{for } x \in \Omega \setminus K, \end{cases}$$

we have that $w \in SBV_p(\Omega; S^{d-1})$, and so $I(u_n) \leq I(w)$, or, equivalently,

$$J\left(u_n|_{\Omega\setminus R_n}\right) \leq J(v),$$

where we have used the fact $w^+ = u_n^+$ and $w^- = u_n^-$ $\mathcal{H}^{N-1}$-a.e. on R_n.

Since Lemma 4.5 in [18] still holds for local minimizers, we deduce that u_n belongs to the space $C^1\left((\Omega \setminus R_n) \setminus \overline{S(u_n)}; S^{d-1}\right)$ and

$$\mathcal{H}^{N-1}\left(\left(\overline{S(u_n)} \cap (\Omega \setminus R_n)\right) \setminus S(u_n)\right) = 0.$$

Observing that Lemmas 4.8 and 4.9 in [18] are still valid for local minimizers, as in the proof of Proposition 5.3 in [7] (see also Theorem 4.10 in [18]), we have that for every compact set $K \subset \overline{S(u_n)} \cap (\Omega \setminus R_n)$,

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{x \in \Omega \setminus R_n : \operatorname{dist}(x, K) < \varepsilon\})}{2\varepsilon} = \mathcal{H}^{N-1}(K).$$

Letting $K_n := \overline{S(u_n)} \cup R_n$, we have $u_n \in C^1(\Omega \setminus K_n; S^{d-1})$. Fix a compact set $K \subset K_n$. Using the fact that for every $x \in \mathbb{R}^N$,

$$\operatorname{dist}(x, K) = \min\{\operatorname{dist}(x, K \setminus R_n), \operatorname{dist}(x, K \cap R_n)\},$$

we obtain that

$$\{x \in \Omega : \operatorname{dist}(x, K) < \varepsilon\} \subset \{x \in \Omega \setminus R_n : \operatorname{dist}(x, K \setminus R_n) < \varepsilon\} \cup \{x \in \Omega : \operatorname{dist}(x, K \cap R_n) < \varepsilon\}.$$

Since R_n is a finite union of hypersurfaces of class C^1, it satisfies (4.34) (with R_n in place of K_n) for some constant $C_n' > 0$ (see by Theorem 3.2.39 in [28]). Hence,

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{x \in \Omega : \operatorname{dist}(x, K) < \varepsilon\})}{\varepsilon} &\le \limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{x \in \Omega \setminus R_n : \operatorname{dist}(x, K \setminus R_n) < \varepsilon\})}{\varepsilon} \\ &\quad + \limsup_{\varepsilon\to 0^+} \frac{\mathcal{L}^N(\{x \in \Omega : \operatorname{dist}(x, K \cap R_n) < \varepsilon\})}{\varepsilon} \\ &\le 2\mathcal{H}^{N-1}(K \setminus R_n) + C_n'\mathcal{H}^{N-1}(K \cap R_n) \le \left(2 + C_n'\right)\mathcal{H}^{N-1}(K),\end{aligned}$$

which shows (4.34).

Finally, if $p = 2$, then for any open ball $B \subset (\Omega \setminus R_n) \setminus \overline{S(u_n)}$, we have that u_n is a minimizer of the functional

$$J_1(v) := \int_B |\nabla v|^2 \, dx + n \int_B |v - \tilde{u}_n|^2 \, dx$$

among all functions $v \in u_n + W_0^{1,2}\left(B; S^{d-1}\right)$. In view of the continuity of u_n in B, we have that the singular set (i.e. the set of discontinuity points) of u_n is empty in B, and reasoning as in Theorem 2.2.4 in [40], we obtain that $u_n \in C^\infty\left(B; S^{d-1}\right)$. This shows that $u_n \in C^\infty\left((\Omega \setminus R_n) \setminus \overline{S(u_n)}; S^{d-1}\right)$. □

Lemma 4.5. *Under the hypotheses of Theorem 4.2, let $u \in SBV_p\left(\Omega; S^{d-1}\right)$ and let $\{u_n\} \subset SBV_p\left(\Omega; S^{d-1}\right)$ converge to u strongly in $SBV_p\left(\Omega; S^{d-1}\right)$. Then for every $A \in \mathcal{A}(\Omega)$,*

$$\limsup_{n\to\infty} \int_A Q_T f(x, u_n, \nabla u_n)\, dx \le \int_A Q_T f(x, u, \nabla u)\, dx, \tag{4.36}$$

$$\limsup_{n\to\infty} \int_{S(u_n)\cap A} Rg\left(x, u_n^+, u_n^-, \nu_{u_n}\right) d\mathcal{H}^{N-1} \le \int_{S(u)\cap A} Rg\left(x, u^+, u^-, \nu_u\right) d\mathcal{H}^{N-1}. \tag{4.37}$$

Proof. **Step 1-** By extracting a subsequence, if necessary, we have that

$$\lim_{n\to\infty} \int_A Q_T f(x, u_n, \nabla u_n)\, dx = \limsup_{n\to\infty} \int_A Q_T f(x, u_n, \nabla u_n)\, dx.$$

Since $\{u_n\}$ converges strongly to u in $SBV_p\left(\Omega; S^{d-1}\right)$ (see (2.3)), by extracting a further subsequence, without loss of generality, we may assume that $\{u_n\}$ and $\{\nabla u_n\}$ converge pointwise to u and ∇u $\mathcal{L}^N$-a.e. in Ω and that there exists $h \in L^1(\Omega)$ such that $|\nabla u_n|^p \le h$ $\mathcal{L}^N$-a.e. in Ω and for all $n \in \mathbb{N}$. By Definition 2.5 and (F_2), we have that

$$0 \le Q_T f(x, y, \xi) \le f(x, y, \xi) \le C(1 + |\xi|^p)$$

for all $x \in \Omega$, $y \in S^{d-1}$, and $\xi \in \mathbb{R}^{d\times N}$. In particular,

$$Q_T f(x, u_n(x), \nabla u_n(x)) \le C(1 + |h(x)|)$$

for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for all $n \in \mathbb{N}$.

Moreover, by (3.17) and the fact that $\widetilde{f}$ is a Carathéodory function, we have that $Q\widetilde{f}$ is upper semicontinuous in y and continuous in ξ (see Proposition 9.5 in [19]), and so we are in a position to apply Fatou's lemma to the sequence of functions

$$x \in A \mapsto C(1 + |h(x)|) - Q_T f(x, u_n(x), \nabla u_n(x))$$

to obtain (4.36).

Step 2- By extracting a subsequence, if necessary, we have that

$$\lim_{n\to\infty}\int_{S(u_n)\cap A} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}=\limsup_{n\to\infty}\int_{S(u_n)\cap A} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}.$$

Using the fact that

$$\int_{S(u)}\left(\left|u_n^+-u^+\right|+\left|u_n^--u^-\right|\right)\,d\mathcal{H}^{N-1}\to 0$$

by (2.3), by extracting a further subsequence, without loss of generality, we may assume that $u_n^{\pm}(x)\to u^{\pm}(x)$ for $\mathcal{H}^{N-1}$-a.e. $x\in S(u)$. Since $0\le Rg\le g\le C$ by (G_2),

$$\begin{aligned}
&\lim_{n\to\infty}\int_{S(u_n)\cap A} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\,d\mathcal{H}^{N-1}\\
&\le\limsup_{n\to\infty}\left(\int_{S(u_n)\cap S(u)\cap A} Rg\left(x,u_n^+,u_n^-,\nu_u\right)\,d\mathcal{H}^{N-1}+C\mathcal{H}^{N-1}\left(\left(S\left(u_n\right)\setminus S\left(u\right)\right)\cap A\right)\right)\\
&\le\limsup_{n\to\infty}\int_{S(u)\cap A} Rg\left(x,u_n^+,u_n^-,\nu_u\right)\,d\mathcal{H}^{N-1},
\end{aligned}$$

where we have used the fact that $\nu_{u_n}=\nu_u$ $\mathcal{H}^{N-1}$-a.e. in $S(u_n)\cap S(u)$ (see (2.4)), (G_3), and (2.3). Since Rg is nonnegative and upper semicontinuous (see Proposition 5.1 in the appendix), applying Fatou's lemma to the sequence of functions

$$x\in S(u)\cap A\mapsto C-Rg\left(x,u_n^+,u_n^-,\nu_u\right),$$

we obtain (4.37). □

We now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. **Step 1-** Assume first that $u\in SBV_2\left(\Omega;S^{d-1}\right)$ is such that $u\in C^{\infty}\left(\Omega\setminus K;S^{d-1}\right)$, where $K\subset\mathbb{R}^N$ is a closed $(N-1)$-rectifiable set satisfying (4.2). Then, in view of Theorem 4.2, to establish the upper bound (4.1), it is enough to prove that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N\lfloor\Omega}(x_0)\le Q_Tf\left(x_0,u(x_0),\nabla u(x_0)\right)\qquad\text{for }\mathcal{L}^N\text{-a.e. }x_0\in\Omega,\tag{4.38}$$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)\le Rg\left(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)\right)\qquad\text{for }\mathcal{H}^{N-1}\text{-a.e. }x_0\in S(u).\tag{4.39}$$

Substep 1a- We prove (4.38). Since $f:\Omega\times S^{d-1}\times\mathbb{R}^{d\times N}\to[0,\infty)$ is a Carathéodory function, by the Scorza–Dragoni theorem, for each $j\in\mathbb{N}$ there exists a compact set $K_j\subset\Omega$ with $\mathcal{L}^N\left(\Omega\setminus K_j\right)\le\frac{1}{j}$ such that $f:K_j\times S^{d-1}\times\mathbb{R}^{d\times N}\to[0,\infty)$ is continuous. Let K_j^* be the set of Lebesgue points of χ_{K_j} and set

$$\omega:=\bigcup\nolimits_j\left(K_j\cap K_j^*\right).$$

Fix $x_0\in\omega\setminus\overline{S(u)}$ such that x_0 is a Lebesgue point of χ_{K_j},

$$\left(\nabla u(x_0)\right)^T u(x_0)=0,\tag{4.40}$$

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N\lfloor\Omega}(x_0)=\lim_{\varepsilon\to0^+}\frac{\mathcal{F}\left(u;Q(x_0,\varepsilon)\right)}{\varepsilon^N}<\infty,\qquad\lim_{\varepsilon\to0^+}\frac{\mathcal{H}^{N-1}\left(S(u)\cap Q(x_0,\varepsilon)\right)}{\varepsilon^N}=0.\tag{4.41}$$

Since $\Omega\setminus K$ is open there exists $\varepsilon_0>0$ such that $\overline{Q(x_0,\varepsilon_0)}\subset\Omega\setminus K$. Using the fact that $u\in C^1\left(\Omega\setminus K;S^{d-1}\right)$, we have that

$$M:=\|\nabla u\|_{L^{\infty}\left(\overline{Q(x_0,\varepsilon_0)};\mathbb{R}^{d\times N}\right)}<\infty.\tag{4.42}$$

In view of (4.40), $\nabla u(x_0)\in\left[T_{u(x_0)}\left(S^{d-1}\right)\right]^N$, and so by (2.6)

$$Q_T f\left(x_0,u\left(x_0\right),\nabla u\left(x_0\right)\right)=Q\overline{f}\left(x_0,u\left(x_0\right),\nabla u\left(x_0\right)\right), \tag{4.43}$$

where

$$\overline{f}(x,z,\xi):=f\left(x,z,\left(\mathbb{I}_{d\times d}-z\otimes z\right)\xi\right)$$

for all $(x,z,\xi)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times N}$. By (4.43), for any fixed $\eta>0$ there exists $\varphi\in W_0^{1,\infty}\left(Q;\mathbb{R}^d\right)$ such that

$$Q_T f\left(x_0,u\left(x_0\right),\nabla u\left(x_0\right)\right)+\eta\geq\int_Q\overline{f}\left(x_0,u\left(x_0\right),\nabla u\left(x_0\right)+\nabla\varphi(y)\right)dy. \tag{4.44}$$

Extend φ periodically to $\mathbb{R}^N$ with period Q, and for $x\in\Omega$ define

$$u_n(x):=\frac{u(x)+\psi_n(x)}{\left|u(x)+\psi_n(x)\right|},$$

where $\psi_n(x):=\frac{1}{n}\varphi\left(n\left(x-x_0\right)\right)$. Note that for n large enough

$$\left|u+\psi_n\right|\geq|u|-\frac{1}{n}\|\varphi\|_\infty\geq\frac{1}{2}. \tag{4.45}$$

Thus by Corollary 3.1 in [3], $u_n\in SBV_2\left(\Omega;S^{d-1}\right)$ and

$$\nabla u_n=\left(\mathbb{I}-\frac{u+\psi_n}{\left|u+\psi_n\right|}\otimes\frac{u+\psi_n}{\left|u+\psi_n\right|}\right)\frac{\nabla\left(u+\psi_n\right)}{\left|u+\psi_n\right|}, \tag{4.46}$$

so that in view of (4.45) and (4.42) in $Q\left(x_0,\varepsilon_0\right)$,

$$\begin{aligned}\left|\nabla u_n\right|&\leq 2\left|\mathbb{I}-\frac{u+\psi_n}{\left|u+\psi_n\right|}\otimes\frac{u+\psi_n}{\left|u+\psi_n\right|}\right|\left(M+\|\nabla\varphi\|_\infty\right)\\&\leq 2\left(|\mathbb{I}|+1\right)\left(M+\|\nabla\varphi\|_\infty\right)=2\left(\sqrt{d}+1\right)\left(M+\|\nabla\varphi\|_\infty\right)=:L.\end{aligned} \tag{4.47}$$

Since $S\left(u_n\right)\cap\Omega\subset S(u)\cap\Omega$, by (F_2), (4.47), and (G_2) we have

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N\lfloor\Omega}\left(x_0\right)&\leq\liminf_{\varepsilon\to0^+}\liminf_{n\to\infty}\frac{1}{\varepsilon^N}\left(\int_{Q(x_0,\varepsilon)\cap K_j}f\left(x,u_n,\nabla u_n\right)\,dx+C\int_{Q(x_0,\varepsilon)\setminus K_j}\left(1+L^2\right)\,dx\right)\\&+C\lim_{\varepsilon\to0^+}\frac{\mathcal{H}^{N-1}\left(S(u)\cap Q\left(x_0,\varepsilon\right)\right)}{\varepsilon^N}\leq\liminf_{\varepsilon\to0^+}\liminf_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)\cap K_j}f\left(x,u_n,\nabla u_n\right)\,dx,\end{aligned} \tag{4.48}$$

where we have used $(4.40)_3$ and the fact that x_0 is a Lebesgue point of χ_{K_j}.

Since $f:K_j\times S^{d-1}\times B_{d\times N}(0;L)\to[0,\infty)$ is uniformly continuous, there exists $\delta>0$ such that

$$\left|f\left(x,z_1,\xi_1\right)-f\left(x_0,z_2,\xi_2\right)\right|<\eta \tag{4.49}$$

for all $x\in K_j$, z_1, $z_2\in S^{d-1}$, ξ_1, $\xi_2\in\overline{B_{d\times N}(0;L)}$, with $|x-x_0|<\delta$, $|z_1-z_2|<\delta$, $|\xi_1-\xi_2|<\delta$.

Using the fact that $\|\psi_n\|_\infty\leq\frac{1}{n}\|\varphi\|_\infty$ we have that

$$\left(\mathbb{I}-\frac{u+\psi_n}{\left|u+\psi_n\right|}\otimes\frac{u+\psi_n}{\left|u+\psi_n\right|}\right)\frac{1}{\left|u+\psi_n\right|}\to\left(\mathbb{I}-u\otimes u\right)$$

uniformly on $\overline{Q\left(x_0,\varepsilon_0\right)}$. Thus

$$\left|\left(\mathbb{I}-\frac{u+\psi_n}{\left|u+\psi_n\right|}\otimes\frac{u+\psi_n}{\left|u+\psi_n\right|}\right)\frac{1}{\left|u+\psi_n\right|}-\left(\mathbb{I}-u\otimes u\right)\right|\leq\frac{\delta}{3(1+L)}$$

for all n sufficiently large. On the other hand, since u and ∇u are continuous at x_0, there exists $0<\varepsilon_1<\varepsilon_0$ such that

$$|u(x)-u(x_0)|\le\frac{\delta}{2},\qquad |u(x)\otimes u(x)-u(x_0)\otimes u(x_0)|\le\frac{\delta}{3(1+L)},$$
$$|\nabla u(x)-\nabla u(x_0)|\le\frac{\delta}{3\left(2+\sqrt{N}\right)}$$

for all $x\in Q(x_0,\varepsilon_1)$. Hence in $Q(x_0,\varepsilon_1)$, for all n sufficiently large, we have $|u_n-u(x_0)|\le\delta$ and

$$\left|\left(\mathbb{I}-\frac{u+\psi_n}{|u+\psi_n|}\otimes\frac{u+\psi_n}{|u+\psi_n|}\right)\frac{\nabla u+\nabla\psi_n}{|u+\psi_n|}-(\mathbb{I}-u(x_0)\otimes u(x_0))(\nabla u(x_0)+\nabla\psi_n)\right|\le\delta.$$

In turn, by (4.46), (4.49), in $K_j\cap Q(x_0,\varepsilon_1)$ for all n sufficiently large we have

$$f(x,u_n,\nabla u_n)\le f(x_0,u(x_0),(\mathbb{I}-u(x_0)\otimes u(x_0))(\nabla u(x_0)+\nabla\psi_n))+\eta=\overline{f}(x_0,u(x_0),\nabla u(x_0)+\nabla\psi_n)+\eta.$$

It follows from (4.48) and the fact that $f\ge 0$ that

$$\begin{aligned}\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{L}^N\lfloor\Omega}(x_0)&\le\liminf_{\varepsilon\to 0^+}\liminf_{n\to\infty}\frac{1}{\varepsilon^N}\int_{Q(x_0,\varepsilon)}\overline{f}(x_0,u(x_0),\nabla u(x_0)+\nabla\psi_n(x))\,dx+\eta\\&=\int_Q\overline{f}(x_0,u(x_0),\nabla u(x_0)+\nabla\varphi(y))\,dy+\eta\le Q_Tf(x_0,u(x_0),\nabla u(x_0))+2\eta,\end{aligned}$$

where we have used the Riemann-Lebesgue lemma (see Lemma 2.85 in [29]) and (4.44). Letting $\eta\to 0^+$, one attains (4.38).

Substep 1b- To obtain (4.39), let $x_0\in S(u)$ be such that

$$\frac{d\mathcal{F}(u;\cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(x_0)=\lim_{\varepsilon\to 0^+}\frac{\mathcal{F}\left(u;Q_{\nu_u(x_0)}(x_0,\varepsilon)\right)}{\varepsilon^{N-1}}<\infty,\tag{4.50}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_{\nu_u(x_0)}(x_0,\varepsilon)}|\nabla u|^2\,dx=0,\tag{4.51}$$

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N-1}}\int_{Q_{\nu_u(x_0)}(x_0,\varepsilon)\cap S(u)}Rg\left(x,u^+,u^-,\nu_u\right)\,d\mathcal{H}^{N-1}=Rg\left(x_0,u^+(x_0),u^-(x_0),\nu_u(x_0)\right).\tag{4.52}$$

For simplicity, in what follows we assume that $x_0=0$ and $\nu=e_N$. We divide the proof into 4 cases.

Case 1- Assume first that $u\in SBV_2\left(\Omega;S^{d-1}\right)$ has the form

$$u(x)=c_1\chi_E(x)+c_2\chi_{\Omega\setminus E}(x),\quad x\in\Omega,$$

where $c_1,c_2\in S^{d-1}$ and the set $E\subset\mathbb{R}^N$ is a polyhedral set, that is,

$$\partial E\subset\bigcup_{i=1}^M P_i,$$

where

$$P_i=\left\{x\in\mathbb{R}^N:(x-a_i)\cdot\eta_i=0\right\}$$

for some $a_i\in\mathbb{R}^N$, $\eta_i\in S^{N-1}$, $i=1,\dots,M$. Since $S(u)=\partial E\cap\Omega\subset\bigcup_{i=1}^M P_i$, it is enough to study the case in which 0 belongs to the (relative) interior of one of the P_i.

Fix $\rho>0$. By definition of BV-elliptic envelope we may find a function $\varphi\in SBV_0\left(Q;S^{d-1}\right)$ such that

$$\varphi=u_{u^+(0),u^-(0),e_N}\qquad\text{on }\partial Q\tag{4.53}$$

and

$$\int_{S(\varphi)} g\left(0, \varphi^{+}(y), \varphi^{-}(y), \nu_{\varphi(y)}\right) d\mathcal{H}^{N-1}(y) \leq R g\left(0, u^{+}(0), u^{-}(0), e_N\right)+\rho. \tag{4.54}$$

In view of (4.53) for every $y_N \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ we may extend $\varphi(\cdot, y_N)$ to $\mathbb{R}^{N-1}$ as a periodic function with period Q_{N-1}. Since 0 belongs to the (relative) interior of P_i, for $\varepsilon>0$ small we have that $u=u_{u^{+}(0), u^{-}(0), e_N}$ in $Q(0, \varepsilon)$ and

$$S(u) \cap Q(0, \varepsilon)=P_i \cap Q(0, \varepsilon),$$

and consider the sequence $v_n: Q(0, \varepsilon) \rightarrow S^{d-1}$ defined by

$$v_n(x):=\begin{cases}\varphi\left(\frac{n}{\varepsilon} x\right) & \text{if } |x_N| \leq \frac{\varepsilon}{2n}, \\ u(x) & \text{otherwise.}\end{cases}$$

By Corollary 3.89 in [5], $v_n \in SBV_0\left(Q(0, \varepsilon); S^{d-1}\right)$. Moreover, $|v_n|=1$ $\mathcal{L}^N$-a.e. in $Q(0, \varepsilon)$, $v_n \rightarrow u$ in $L^1\left(Q(0, \varepsilon); \mathbb{R}^d\right)$, and so

$$\mathcal{F}(u; Q(0, \varepsilon)) \leq \liminf_{n \rightarrow \infty}\left(\int_{Q(0, \varepsilon)} f(x, v_n, 0)\, dx+\int_{S(v_n) \cap Q_{N-1}(0, \varepsilon) \times \left]-\frac{\varepsilon}{2n}, \frac{\varepsilon}{2n}\right[} g\left(x, v_n^{+}, v_n^{-}, \nu_{v_n}\right) d\mathcal{H}^{N-1}\right), \tag{4.55}$$

where we have used the facts that $v_n \in SBV_0\left(Q(0, \varepsilon); S^{d-1}\right)$ and that, by choice of ε,

$$S(v_n)=S(v_n) \cap\left(Q_{N-1}(0, \varepsilon) \times \left]-\frac{\varepsilon}{2n}, \frac{\varepsilon}{2n}\right[\right)$$

for all n sufficiently large. By (F_2), (4.55) becomes

$$\mathcal{F}(u; Q(0, \varepsilon)) \leq C \varepsilon^N+\liminf_{n \rightarrow \infty} \int_{S(v_n) \cap Q_{N-1}(0, \varepsilon) \times \left]-\frac{\varepsilon}{2n}, \frac{\varepsilon}{2n}\right[} g\left(x, v_n^{+}, v_n^{-}, \nu_{v_n}\right) d\mathcal{H}^{N-1}.$$

Since g is uniformly continuous, there exists $\delta>0$ such that $|g(x, a, b, \nu)-g(y, a, b, \nu)|<\rho$ for all $x,\ y \in Q(0, \varepsilon)$ with $|x-y|<\delta$, all $a,\ b \in S^{d-1}$ and all $\nu \in S^{N-1}$. Hence, if $\varepsilon<\delta$ we have that

$$\begin{aligned}
\mathcal{F}(u; Q(0, \varepsilon)) & \leq C \varepsilon^N+\liminf_{n \rightarrow \infty} \int_{S(v_n) \cap Q_{N-1}(0, \varepsilon) \times \left]-\frac{\varepsilon}{2n}, \frac{\varepsilon}{2n}\right[}\left(g\left(0, v_n^{+}, v_n^{-}, \nu_{v_n}\right)+\rho\right) d\mathcal{H}^{N-1} \\
& \leq C \varepsilon^N+\liminf_{n \rightarrow \infty} \frac{\varepsilon^{N-1}}{n^{N-1}} \int_{S(\varphi) \cap Q_{N-1}(0, n) \times \left]-\frac{1}{2}, \frac{1}{2}\right[}\left(g\left(0, \varphi^{+}, \varphi^{-}, \nu_{\varphi}\right)+\rho\right) d\mathcal{H}^{N-1} \\
& =C \varepsilon^N+\varepsilon^{N-1} \int_{S(\varphi)} g\left(0, \varphi^{+}, \varphi^{-}, \nu_{\varphi}\right) d\mathcal{H}^{N-1}+\rho \varepsilon^{N-1} \\
& \leq C \varepsilon^N+\varepsilon^{N-1} R g\left(0, u^{+}(0), u^{-}(0), e_N\right)+2 \rho \varepsilon^{N-1}
\end{aligned}$$

where we have used the change of variables $x=\frac{\varepsilon}{n} y$, Fubini's theorem, the periodicity of $\varphi(\cdot, y_N)$ (see (4.53)), and (4.54). In turn, by (4.50),

$$\frac{d\mathcal{F}(u; \cdot)}{d\mathcal{H}^{N-1}\lfloor S(u)}(0) \leq R g\left(0, u^{+}(0), u^{-}(0), e_N\right)+2 \rho.$$

Letting ρ go to zero we obtain (4.39).

Case 2- Next assume that $u \in SBV_2\left(\Omega; S^{d-1}\right)$ has the form

$$u(x)=c_1 \chi_E(x)+c_2 \chi_{\Omega \backslash E}(x), \quad x \in \Omega, \tag{4.56}$$

where $c_1, c_2 \in S^{d-1}$ and the set $E \subset \mathbb{R}^N$ is such that ∂E is contained in a closed $(N-1)$-rectifiable set $K \subset \mathbb{R}^N$ satisfying (4.2). Fix $\varepsilon>0$ sufficiently small. By standard approximation results (see, e.g., Lemma 3.1 in [9] or [20]), there exists a sequence $\{E_n\}_{n\in\mathbb{N}} \subset Q(0,\varepsilon)$ such that each E_n is a polyhedral set and

$$\chi_{E_n} \to \chi_E \text{ in } L^1(Q(0,\varepsilon)), \qquad |D\chi_{E_n}|(Q(0,\varepsilon)) \to |D\chi_E|(Q(0,\varepsilon)).$$

Set $u_n := c_1\chi_{E_n} + c_2\chi_{Q(0,\varepsilon)\setminus E_n}$. Then $u_n \to u$ in $L^1(Q(0,\varepsilon);S^{d-1})$, and so by Case 1 applied to each u_n, we have

$$\begin{aligned}\mathcal{F}(u;Q(0,\varepsilon)) &\le \liminf_{n\to\infty} \mathcal{F}(u_n;Q(0,\varepsilon)) \\ &\le \liminf_{n\to\infty}\left(\int_{Q(0,\varepsilon)} f(x,u_n,0)\ dx + \int_{S(u_n)\cap Q(0,\varepsilon)} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\ d\mathcal{H}^{N-1}\right).\end{aligned}$$

Since Rg is upper semicontinuous, there exists a decreasing sequence of continuous functions $g_k : \Omega\times S^{d-1}\times S^{d-1}\times S^{N-1}\to[0,\infty)$ such that $Rg = \inf_k g_k$. Fix $k\in\mathbb{N}$. Then

$$\begin{aligned}&\limsup_{n\to\infty}\int_{S(u_n)\cap Q(0,\varepsilon)} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\ d\mathcal{H}^{N-1} \\ &\le \lim_{n\to\infty}\int_{\partial^* E_n\cap Q(0,\varepsilon)} g_k\left(x,c_1,c_2,\nu_{E_n}\right)\ d\mathcal{H}^{N-1} \\ &= \int_{\partial^* E\cap Q(0,\varepsilon)} g_k\left(x,c_1,c_2,\nu_E\right)\ d\mathcal{H}^{N-1},\end{aligned}$$

where in the equality we have used Reshetnyak continuity theorem (see Theorem 2.39 in [5], see also [42]). Therefore,

$$\limsup_{n\to\infty}\int_{S(u_n)\cap Q(0,\varepsilon)} Rg\left(x,u_n^+,u_n^-,\nu_{u_n}\right)\ d\mathcal{H}^{N-1} \le \int_{S(u)\cap Q(0,\varepsilon)} Rg\left(x,u^+,u^-,\nu_u\right)\ d\mathcal{H}^{N-1},$$

by Lebesgue monotone convergence theorem and (G_2). Hence, also by (F_2), we have that

$$\mathcal{F}(u;Q(0,\varepsilon)) \le C\varepsilon^N + \int_{S(u)\cap Q(0,\varepsilon)} Rg\left(x,u^+,u^-,\nu_u\right)\ d\mathcal{H}^{N-1}.$$

Dividing the previous inequality by ε^{N-1} and letting $\varepsilon\to0^+$, (4.39) follows from (4.50) and (4.52). In view of Step 1, this shows that (4.1) holds for all functions u as in (4.56).

Case 3- Consider next the case in which $u \in SBV_2(\Omega;S^{d-1})$ has the form

$$u(x) = \sum_{i=1}^{M} c_i\chi_{E_i}(x), \quad x\in\Omega, \tag{4.57}$$

where $c_i\in S^{d-1}$, the sets $E_i\subset\mathbb{R}^N$ are pairwise disjoint, $\{E_i\cap\Omega\}_i$ is a partition of Ω, and $\bigcup_{i=1}^M \partial E_i$ is contained in a closed $(N-1)$-rectifiable set $K\subset\mathbb{R}^N$ satisfying (4.2). Then by Theorem 4.2, $\mathcal{F}(u;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure defined on $\mathcal{B}(\Omega)$ and still denoted $\mathcal{F}(u;\cdot)$. As in the proof of Proposition 4.8 in [6], we have that

$$\begin{aligned}\mathcal{F}(u;Q(0,\varepsilon)) &= \mathcal{F}(u;Q(0,\varepsilon)\setminus S(u)) + \mathcal{F}(u;S(u)\cap Q(0,\varepsilon)) \\ &= \mathcal{F}(u;Q(0,\varepsilon)\setminus S(u)) + \sum_{i=1}^{M}\sum_{j>i}\mathcal{F}(u;\partial^*E_i\cap\partial^*E_j\cap Q(0,\varepsilon)).\end{aligned}\tag{4.58}$$

Note that for $\mathcal{H}^{N-1}$-a.e. $x\in\partial^*E_i\cap\partial^*E_j\cap Q(0,\varepsilon)$ the function u coincides with the function

$$v(x) := c_i\chi_{E_i}(x) + c_j\chi_{\Omega\setminus E_i}(x), \quad x\in\Omega.$$

Repeating word-for-word the proof of Step 1 of Proposition 4.4 in [6], we have that for every compact set $K_1 \subset \partial^* E_i \cap \partial^* E_j \cap Q(0,\varepsilon)$,

$$\mathcal{F}(u; K_1) = \mathcal{F}(v; K_1). \tag{4.59}$$

Indeed, all the hypotheses of that proposition are satisfied with the exception of hypothesis (4.12) in [6], that is,

$$\mathcal{F}(w; A) \leq C\,|Dw|(A)$$

for all $w \in BV\left(\Omega; \mathbb{R}^d\right)$ and all $A \in \mathcal{A}(\Omega)$, and for some $C > 0$. Note however, that by (F_2) and (G_2), we have that

$$\mathcal{F}(w; A) \leq C\left(\mathcal{L}^N(A) + |Dw|(A)\right)$$

for all $w \in SBV_0\left(\Omega; \mathbb{R}^d\right)$ and all $A \in \mathcal{A}(\Omega)$. An inspection of the proof of Step 1 of Proposition 4.4 in [6] shows that this latter condition is all is needed. Hence, (4.59) holds, and so

$$\mathcal{F}(u; Q(0,\varepsilon) \cap \partial^* E_i \cap \partial^* E_j) = \mathcal{F}(v; Q(0,\varepsilon) \cap \partial^* E_i \cap \partial^* E_j).$$

Since the function v is of the type (4.56), in view of Step 1 and Case 1, we have that (4.1) holds for v. In turn, also by Theorem 4.2 applied to v,

$$\begin{aligned}\mathcal{F}(v; \partial^* E_i \cap \partial^* E_j \cap Q(0,\varepsilon)) &\leq \int_{\partial^* E_i \cap \partial^* E_j \cap Q(0,\varepsilon)} Rg\left(x, v^+, v^-, \nu_v\right)\, d\mathcal{H}^{N-1}\\ &= \int_{\partial^* E_i \cap \partial^* E_j \cap Q(0,\varepsilon)} Rg\left(x, c_i, c_j, \nu_{E_i}\right)\, d\mathcal{H}^{N-1}.\end{aligned}$$

Hence, also by (4.58) and (F_2), we obtain

$$\begin{aligned}\mathcal{F}(u; Q(0,\varepsilon)) &\leq C\varepsilon^N + \sum_{i=1}^{M}\sum_{j>i}\int_{\partial^* E_i \cap \partial^* E_j \cap Q(0,\varepsilon)} Rg\left(x, c_i, c_j, \nu_{E_i}\right)\, d\mathcal{H}^{N-1}\\ &= C\varepsilon^N + \int_{S(u)\cap Q(0,\varepsilon)} Rg\left(x, u^+, u^-, \nu_u\right)\, d\mathcal{H}^{N-1}.\end{aligned}$$

Dividing the previous inequality by ε^{N-1} and letting $\varepsilon \to 0^+$, (4.39) follows from (4.50) and (4.52). In view of Step 1, this shows that (4.1) holds for all functions u as in (4.57).

Case 4- Finally, assume that $u \in SBV_2\left(\Omega; S^{d-1}\right)$ is such that $u \in C^\infty\left(\Omega \setminus K; S^{d-1}\right)$, where K is a closed $(N-1)$-rectifiable set satisfying (4.2). Since $0 \in \Omega$, we may find $\varepsilon_0 > 0$ such that $\overline{Q(0,\varepsilon_0)} \subset \Omega$. Let

$$B_n := \left\{x \in \Omega :\ \operatorname{dist}\left(x, K \cap \overline{Q(0,\varepsilon_0)}\right) < \frac{1}{n}\right\}.$$

By (4.2) there exists $c > 0$ such that for any $n \in \mathbb{N}$, $\mathcal{L}^N(B_n) \leq \frac{c}{n}$. Since $\nabla u \in L^2\left(\Omega; \mathbb{R}^{d\times N}\right)$, we have that $\int_{B_n} |\nabla u|^2\, dx \to 0$ as $n \to \infty$. Let $k_n \in \mathbb{N}$ be chosen such that $k_n \to \infty$,

$$\frac{n}{k_n^2} + k_n \int_{B_n} |\nabla u|\, dx \to 0 \qquad \text{as } n \to \infty. \tag{4.60}$$

(If $\int_{B_n} |\nabla u|^2\, dx \neq 0$ we may take k_n to be the integer part of $\frac{n^{1/2}}{\left(\int_{B_n}|\nabla u|^2\, dx\right)^{1/4}}$). For every $l \in \mathbb{Z}$, with $-2k_n \leq l < 2k_n$, and for every $j = 1, \ldots, d$, by the coarea formula, we have

$$\begin{aligned}\int_{B_n \cap \left\{\frac{l}{k_n} < u^j \leq \frac{l+1}{k_n}\right\}} \left|\nabla u^j\right|\, dx &= |Du_j|\left((B_n \setminus S(u)) \cap \left\{\frac{l}{k_n} < u^j \leq \frac{l+1}{k_n}\right\}\right)\\ &= \int_{\frac{l}{k_n}}^{\frac{l+1}{k_n}} \mathcal{H}^{N-1}\left((B_n \setminus S(u)) \cap \partial^*\left(\left\{u^j > t\right\}\right)\right)\, dt.\end{aligned}$$

By Sard's theorem (see, e.g., Theorem 1.2 in [12]), we have that for $\mathcal{L}^1$-a.e. $t \in [-1,1]$,

$$(\Omega \setminus K) \cap \left(u^j\right)^{-1}(\{t\}) \text{ is a } C^\infty \text{ hypersurface.} \tag{4.61}$$

Let $\Gamma \subset [-1,1]$ be the set of t for which (4.61) holds. Then

$$\int_{B_n \cap \left\{\frac{l}{k_n} < u^{(j)} \leq \frac{l+1}{k_n}\right\}} \left|\nabla u^j\right| \, dx = \int_{\left(\frac{l}{k_n}, \frac{l+1}{k_n}\right) \cap \Gamma} \mathcal{H}^{N-1}\left((B_n \setminus S(u)) \cap \partial^*\left(\left\{u^j > t\right\}\right)\right) \, dt,$$

and so there exists $t_{j,l} \in \left(\frac{l}{k_n}, \frac{l+1}{k_n}\right) \cap \Gamma$ such that

$$\int_{B_n \cap \left\{\frac{l}{k_n} < u^{(j)} \leq \frac{l+1}{k_n}\right\}} \left|\nabla u^j\right| \, dx \geq \frac{1}{k_n} \mathcal{H}^{N-1}\left((B_n \setminus S(u)) \cap \partial^*\left(\left\{u^j > t_{j,l}\right\}\right)\right).$$

Summing over l yields

$$\int_{B_n} \left|\nabla u^j\right| \, dx \geq \frac{1}{k_n} \sum_{l=-2k_n}^{2k_n - 1} \mathcal{H}^{N-1}\left((B_n \setminus S(u)) \cap \partial^*\left(\left\{u^j > t_{j,l}\right\}\right)\right). \tag{4.62}$$

Define $\widehat{v}_n : B_n \to \mathbb{R}^d$ as

$$\widehat{v}_n^j(x) := \widehat{c}_l^j \qquad \text{if } x \in \left\{y \in B_n : \, t_{j,l} < u^j(y) \leq t_{j,l+1}\right\}, \tag{4.63}$$

for $l = -2k_n, \ldots, 2k_n - 1$, where the numbers $\widehat{c}_l^j \in (t_{j,l}, t_{j,l+1}]$ are chosen so that $\frac{\widehat{c}_l^j}{|\widehat{c}_l^j|} \neq -\frac{\widehat{c}_s^j}{|\widehat{c}_s^j|}$ for $l \neq s$. Since $t_{j,l+1} - t_{j,l} \leq \frac{2}{k_n}$, we have that $\left\|\widehat{v}_n^j - u^j\right\|_{L^\infty(B_n)} \leq \frac{2}{k_n}$, and so

$$\frac{1}{2} \leq 1 - \frac{2\sqrt{d}}{k_n} \leq |u(x)| - |\widehat{v}_n(x) - u(x)| \leq |\widehat{v}_n(x)| \leq |u(x)| + |\widehat{v}_n(x) - u(x)| \leq 1 + \frac{2\sqrt{d}}{k_n} \tag{4.64}$$

for $\mathcal{L}^N$-a.e. $x \in B_n$ and for all n sufficiently large. Thus we may define $v_n : B_n \to S^{d-1}$ as

$$v_n := \frac{\widehat{v}_n}{|\widehat{v}_n|}.$$

Then

$$\begin{aligned} \|v_n - u\|_{L^\infty(B_n)} &\leq \|v_n - \widehat{v}_n\|_{L^\infty(B_n)} + \|\widehat{v}_n - u\|_{L^\infty(B_n)} \leq \|1 - |\widehat{v}_n|\|_{L^\infty(B_n)} + \frac{2\sqrt{d}}{k_n} \\ &\leq \frac{2\sqrt{d}}{k_n} + \frac{2\sqrt{d}}{k_n} \leq \frac{C(d)}{k_n}, \end{aligned} \tag{4.65}$$

where we have used (4.64). Moreover, by construction

$$S(v_n) \cap B_n \subset \textstyle\bigcup_{j=1}^d \bigcup_{l=-2k_n}^{2k_n-1} \partial^*\left(\left\{u^j > t_{j,l}\right\}\right) \cap B_n, \tag{4.66}$$

and so by (4.62),

$$\begin{aligned} \mathcal{H}^{N-1}\left((S(v_n) \cap B_n) \setminus S(u)\right) &\leq \sum_{j=1}^d \sum_{l=-2k_n}^{2k_n-1} \mathcal{H}^{N-1}\left((B_n \setminus S(u)) \cap \partial^*\left(\left\{u^j > t_{j,l}\right\}\right)\right) \\ &\leq k_n \sum_{j=1}^d \int_{B_n} \left|\nabla u^j\right| \, dx. \end{aligned} \tag{4.67}$$

Since $v_n \in SBV_0\left(B_n; S^{d-1}\right)$ takes only a finite number of values, we may write

$$v_n(x) = \sum_{i=1}^{M_n} c_{i,n} \chi_{E_{i,n}}(x), \qquad x \in B_n, \tag{4.68}$$

where $\{E_{i,n}\}_{i=1}^{M_n}$ is a partition of B_n. Let B_n' be an open subset with Lipschitz boundary such that

$$B_{2n} \cap Q(0,\varepsilon_0) \subset B_n' \subset B_n \cap Q(0,\varepsilon_0).$$

Define

$$\bar{v}_n(x) := \begin{cases} v_n(x) & \text{if } x \in B_n', \\ e_1 & \text{if } x \in \Omega \setminus B_n'. \end{cases}$$

Note that in view of (4.61), (4.63), the properties of K, and the fact B_n' is Lipschitz, the function $\bar{v}_n$ is of the type (4.57).

We now modify v_n to match u in the region $B_n' \setminus B_{3n}$. Let $\varphi_n \in C^\infty\left(\overline{B_n'}\right)$ be such that $\varphi_n = 1$ in $B_{3n} \cap Q(0,\varepsilon_0)$, $\varphi_n = 0$ outside B_n', $0 \le \varphi_n \le 1$, and $\|\nabla\varphi_n\|_{L^\infty(B_n';\mathbb{R}^N)} \le Cn$ and define

$$u_n := \frac{\varphi_n \bar{v}_n + (1-\varphi_n)u}{|\varphi_n \bar{v}_n + (1-\varphi_n)u|} = \frac{\varphi_n v_n + (1-\varphi_n)u}{|\varphi_n v_n + (1-\varphi_n)u|}.$$

Since $|v_n - u| \le \frac{C(d)}{k_n}$ $\mathcal{L}^N$-a.e. in B_n, we have that for n large enough,

$$|\varphi_n v_n + (1-\varphi_n)u| = |u + \varphi_n(v_n - u)| \ge |u| - |\varphi_n(v_n-u)| \ge 1 - \frac{C(d)}{k_n} \ge \frac{1}{2} \;\; \mathcal{L}^N\text{-a.e. in } B_n'.$$

Using the fact that the projection $P : \mathbb{R}^d \setminus B_d\left(0, \frac{1}{2}\right) \to S^{d-1}$ is Lipschitz, by Corollary 3.1 in [5], we deduce that $u_n \in SBV_2\left(\Omega; S^{d-1}\right)$. Moreover, since $\bar{v}_n$ is of the type (4.57), we have that $u_n \in C^1\left(\Omega \setminus K_n; S^{d-1}\right)$, where K_n is a closed $(N-1)$-rectifiable set satisfying (4.2).

By Theorem 4.2, $\mathcal{F}(u_n;\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a Radon measure, and so for $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned} \mathcal{F}(u_n; Q(0,\varepsilon)) &= \mathcal{F}(u_n; B_{3n} \cap Q(0,\varepsilon)) + \mathcal{F}(u_n; Q(0,\varepsilon) \setminus B_{3n}) \\ &\le \mathcal{F}(u_n; B_{3n} \cap Q(0,\varepsilon)) + C\int_{Q(0,\varepsilon)\setminus B_{3n}} \left(1 + |\nabla u_n|^2\right) dx \\ &\quad + C\mathcal{H}^{N-1}(S(u_n) \cap (Q(0,\varepsilon) \setminus B_{3n})). \end{aligned} \tag{4.69}$$

Since $u_n = v_n$ in $B_{3n} \cap Q(0,\varepsilon)$, and since Case 3 applies to $\bar{v}_n$, by the locality of $\mathcal{F}(\cdot; B_{3n} \cap Q(0,\varepsilon))$, we have that

$$\begin{aligned} \mathcal{F}(u_n; B_{3n} \cap Q(0,\varepsilon)) &= \mathcal{F}(v_n; B_{3n} \cap Q(0,\varepsilon)) \\ &\le \int_{B_{3n}\cap Q(0,\varepsilon)} Q_T f(x, v_n, 0)\, dx + \int_{Q(0,\varepsilon)\cap B_{3n}\cap S(v_n)} Rg\left(x, v_n^+, v_n^-, \nu_{v_n}\right) d\mathcal{H}^{N-1} \\ &\le C\varepsilon^N + \int_{Q(0,\varepsilon)\cap S(u)} Rg\left(x, v_n^+, v_n^-, \nu_u\right) d\mathcal{H}^{N-1} \\ &\quad + C\mathcal{H}^{N-1}(Q(0,\varepsilon) \cap B_{3n} \cap S(v_n) \setminus S(u)) \\ &\le C\varepsilon^N + \int_{Q(0,\varepsilon)\cap S(u)} Rg\left(x, v_n^+, v_n^-, \nu_u\right) d\mathcal{H}^{N-1} + Ck_n \int_{B_n} |\nabla u|\, dx, \end{aligned} \tag{4.70}$$

where we have used Proposition 3.73 in [5], the fact that $Rg \ge 0$, and (4.67).

Similarly, since $S(u) \subset B_{3n}$, by (4.67),

$$\mathcal{H}^{N-1}(S(u_n) \cap (Q(0,\varepsilon) \setminus B_{3n})) \le \mathcal{H}^{N-1}\left((S(v_n) \cap B_n') \setminus S(u)\right) \le Ck_n \int_{B_n} |\nabla u|\, dx, \tag{4.71}$$

while by (4.60) and (4.65),

$$\begin{aligned}\int_{Q(0,\varepsilon)\setminus B_{3n}}\left(1+|\nabla u_n|^2\right)\,dx &\leq \int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx + C\int_{B_n'\setminus B_{3n}} n^2\,|v_n-u|^2\,dx \\ &\leq \int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx + C\frac{n^2}{k_n^2}\mathcal{L}^N\left(B_n\right) \qquad (4.72)\\ &= \int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx + o\left(1\right),\end{aligned}$$

where we have used the facts that $\nabla v_n = 0\,\mathcal{L}^N$-a.e. in B_n' and that $|\nabla u_n| \leq C\,|\nabla\left(\varphi_n v_n + (1-\varphi_n)\,\nabla u\right)|$, since $|\varphi_n v_n + (1-\varphi_n)\,u| \geq \frac{1}{2}$. Combining (4.69)-(4.72), we obtain

$$\mathcal{F}\left(u_n; Q\left(0,\varepsilon\right)\right) \leq C\int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx + \int_{Q(0,\varepsilon)\cap S(u)} Rg\left(x, v_n^+, v_n^-, \nu_u\right)\,d\mathcal{H}^{N-1} + o\left(1\right).$$

Since $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^d\right)$, it follows that

$$\begin{aligned}\mathcal{F}\left(u; Q\left(0,\varepsilon\right)\right) \leq \liminf_{n\to\infty}\mathcal{F}\left(u_n; Q\left(0,\varepsilon\right)\right) &\leq \int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx \\ &+ \liminf_{n\to\infty}\int_{Q(0,\varepsilon)\cap S(u)} Rg\left(x, v_n^+, v_n^-, \nu_u\right)\,d\mathcal{H}^{N-1}.\end{aligned}$$

By (4.65), using the fact that $S\left(u\right) \subset B_n$, the upper semicontinuity of $Rg\left(x,\cdot,\cdot,\nu_u\right)$, (G_2), and Fatou's lemma, we conclude that

$$\mathcal{F}\left(u; Q\left(0,\varepsilon\right)\right) \leq \int_{Q(0,\varepsilon)}\left(1+|\nabla u|^2\right)\,dx + \int_{Q(0,\varepsilon)\cap S(u)} Rg\left(x, u^+, u^-, \nu_u\right)\,d\mathcal{H}^{N-1}.$$

By dividing the previous inequality by ε^{N-1} and letting $\varepsilon \to 0^+$, (4.39) follows from (4.50)-(4.52).

Step 2- We establish (4.1) for a general $u \in SBV_2\left(\Omega; S^{d-1}\right)$ and $A \in \mathcal{A}\left(\Omega\right)$. Let $\{u_n\} \subset SBV_2\left(\Omega; S^{d-1}\right)$ be the sequence given in Lemma 4.4. By the lower semicontinuity of $\mathcal{F}\left(\cdot; A\right)$, we have that

$$\begin{aligned}\mathcal{F}\left(u; A\right) &\leq \liminf_{n\to\infty}\mathcal{F}\left(u_n; A\right) \\ &\leq \liminf_{n\to\infty}\left(\int_A Q_T f\left(x, u_n, \nabla u_n\right)\,dx + \int_{S(u_n)\cap A} Rg\left(x, u_n^+, u_n^-, \nu_{u_n}\right)\,d\mathcal{H}^{N-1}\right),\end{aligned}$$

where in the last inequality we have applied Step 1 to each u_n. By Lemma 4.5, (4.1) now follows. □

Proof of Theorem 1.1. Theorem 1.1 follows from Theorems 3.1 and 4.1. □

5. Appendix

Proposition 5.1. *Let $g : S^{d-1}\times S^{d-1}\times S^{N-1} \to [0,\infty)$ be continuous. Then Rg is upper semicontinuous.*

Proof. Let $\{a_n\}$, $\{b_n\} \subset S^{d-1}$, $\{\nu_n\} \subset S^{N-1}$ be such that $a_n \to a$, $b_n \to b$ and $\nu_n \to \nu$. For $\varepsilon > 0$, choose $u \in SBV_0\left(Q; S^{d-1}\right)$ such that $u = u_{a,b,\nu}$ on ∂Q_ν and

$$Rg\left(a, b, \nu\right) \geq \int_{S(u)\cap Q_\nu} g\left(u^+, u^-, \nu\right)\,d\mathcal{H}^{N-1} - \varepsilon,$$

where $u_{a,b,\nu}$ is given by (2.9). Since g is uniformly continuous, there exists $0 < \delta < 1$ such that

$$|g\left(\alpha_1, \beta_1, \nu\right) - g\left(\alpha_2, \beta_2, \nu\right)| \leq \varepsilon \qquad (5.1)$$

for all $\alpha_i,\ \beta_i \in S^{d-1}$, $\nu \in S^{N-1}$, $i = 1, 2$ with $|\alpha_1 - \alpha_2|$, $|\beta_1 - \beta_2| \leq \delta$.

Let $n_0 \in \mathbb{N}$ be such that for all $n \geq n_0$,

$$\max\{|a_n - a|, |b_n - a|\} \leq \frac{\delta}{4}.$$

Then for every $\theta \in (0,1)$ and $z \in S^{d-1}$,

$$|z + \theta(a_n - a)| \geq |z| - |a_n - a| \geq \frac{1}{2}, \qquad |z + \theta(b_n - b)| \geq |z| - |b_n - b| \geq \frac{1}{2}. \tag{5.2}$$

Let $r > 0$ be so small that $B(a,r) \cap B(b,r) = \emptyset$ and let $\varphi \in C_c^\infty(B(0,r);[0,1])$ be such that $\varphi(0) = 1$. Define

$$\Psi_n(z) := \begin{cases} \frac{z+\varphi(z-a)(a_n-a)}{|z+\varphi(z-a)(a_n-a)|} & \text{if } z \in B(a,r), \\ \frac{z+\varphi(z-b)(b_n-b)}{|z+\varphi(z-b)(b_n-b)|} & \text{if } z \in B(b,r), \\ z & \text{otherwise.} \end{cases}$$

Then $\Psi_n : S^{d-1} \to S^{d-1}$ is C^∞ and by (5.2)

$$|\Psi_n(z) - z| \leq 4\max\{|a_n - a|, |b_n - a|\} \leq \delta \tag{5.3}$$

for all $n \geq n_0$.

Define

$$u_n(x) := (\Psi_n \circ u)(R_n x), \qquad x \in Q_{\nu_n},$$

where R_n is a rotation such that $R_n^T \nu = \nu_n$. Since $\Psi_n \in C^\infty(S^{d-1};S^{d-1})$, by Corollary 3.1 in [3] we have $u_n \in SBV_0(Q_\nu;S^{d-1})$, $u_n = u_{a_n,b_n,\nu_n}$ on ∂Q_{ν_n}, $S(u_n) = R_n^T S(u)$, and

$$u_n^\pm(x) = (\Psi_n \circ u^\pm)(R_n x)$$

for $x \in S(u_n)$.

Since u_n is admissible for $Rg(a_n,b_n,\nu_n)$, we have

$$\begin{aligned} Rg(a_n,b_n,\nu_n) &\leq \int_{S(u_n)} g\left(u_n^+(x), u_n^-(x), \nu_{u_n}(x)\right) d\mathcal{H}^{N-1}(x) \\ &= \int_{S(u)} g\left((\Psi_n \circ u^+)(y), (\Psi_n \circ u^-)(y), \nu_u(y)\right) d\mathcal{H}^{N-1}(y) \\ &\leq \int_{S(u)} g\left(u^+(y), u^-(y), \nu_u(y)\right) d\mathcal{H}^{N-1}(y) + \varepsilon\mathcal{H}^{N-1}(S(u)), \end{aligned}$$

where we have used the change of variables $y = R_n x$, (5.1), and (5.3). Letting first $n \to \infty$ and then $\varepsilon \to 0^+$ we obtain the desired result. □

The following proposition provides sufficient conditions for an increasing set function be a Radon measure. It was used in the proof of Theorem 4.2 and is a consequence of De Giorgi-Letta's criterion (see [23]). The proof may be found in [25] (see Corollary 5.2), and is an adaptation of that of Theorem 4.3 in [6].

Proposition 5.2. *Let (X,d) be a locally compact metric space such that every open set $A \subset X$ is σ-compact. Assume that $\rho : \mathcal{A}(X) \to [0,\infty)$ is an increasing set function such that*

1. *(additivity on disjoint sets) $\rho(A_1 \cup A_2) = \rho(A_1) + \rho(A_2)$ for all A_1, $A_2 \in \mathcal{A}(X)$, with $A_1 \cap A_2 = \emptyset$;*
2. *for all A, B, $C \in \mathcal{A}(X)$, with $C \subset\subset B \subset\subset A$ we have*
$$\rho(A) \leq \rho(B) + \rho(A \setminus \overline{C});$$
3. *there exists a measure $\mu : \mathcal{B}(X) \to [0,\infty)$ such that*
$$\rho(A) \leq \mu(A) < +\infty$$
for every $A \in \mathcal{A}(X)$.

Then ρ is the restriction to $\mathcal{A}(X)$ of a measure defined on $\mathcal{B}(X)$.

Acknowledgments. This paper is part of the first author's Ph.D. thesis, Carnegie Mellon University. The authors thank the Center for Nonlinear Analysis (NSF Grants No. DMS-0405343 and DMS-0635983) for its support during the preparation of this paper. The research of G. Carita was partially supported by the Portuguese Fundação para a Ciência e a Tecnologia (FCT) under Grant No. PRAXiS XXI\BD\8137\2002 and by the center CIMA-UE at the University of Évora (Portugal), that of I. Fonseca by the National Science Foundation under Grants No. DMS-0401763 and DMS-0905778 and that of G. Leoni under Grants No. DMS-0708039 and DMS-1007989.

The authors would like to thank the referee for his/her thorough reading of the manuscript and his/her very pertinent comments.

References

[1] Alicandro R., Esposito A., and Leone C., *Relaxation in BV of integral functionals defined on Sobolev functions with values in the unit sphere*, J. Convex Anal., **14** (2007), 69–98.
[2] Ambrosio L., *Existence theory for a new class of variational problems*, Arch. Rat. Mech. Anal., **111** (1990), 291–322.
[3] Ambrosio L. and Dal Maso G., *A general chain–rule for distributional derivatives*, Proc. Am. Math. Soc., **108** (1990), 691–702.
[4] Ambrosio L. and Dal Maso G., *On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasiconvex integrals*, J. Funct. Anal., **109** (1992), 76–97.
[5] Ambrosio L., Fusco N., and Pallara D., *Functions of Bounded Variation and Free Discontinuity Problems.* Mathematical Monographs, Oxford University Press, 2000.
[6] Ambrosio L., Mortola S., and Tortorelli V., *Functionals with linear growth defined on vector valued BV functions,* J. Math. Pures Appl., **70** (1991), 269–323.
[7] Ambrosio L. and Tortorelli V., *Approximation of functionals depending on jumps by elliptic functionals via Γ-convergence,* Comm. Pure Appl. Math., **43** (1990), 999–1036.
[8] Babadjian J. F. and Millot V., *Homogenization of variational problems in manifold valued BV-spaces,* Calc. Var. Partial Differential Equations **36** (2009), 7–47.
[9] Baldo S., *Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids,* Annales de l' I.H.P., **7** (1990), 67–90.
[10] Bethuel F., *The approximation problem for Sobolev maps between two manifolds,* Acta Math., **167** (1991), 153-206.
[11] Bethuel F. and Zheng X., *Density of smooth functions between two manifolds in Sobolev spaces,* J. Functional Anal., **80** (1988), 60-75.
[12] Bojarski B., Hajłasz P., and Strzelecki P., *Sard's theorem for mappings in Hölder and Sobolev spaces,* Manuscripta Math. **118** (2005), no. 3, 383–397.
[13] Bouchitté G., Fonseca I., and Mascarenhas L., *A global method for relaxation*, Arch. Rat. Mech. Anal., **145** (1998), 51-98.
[14] Bouchitté G., Fonseca I., Leoni G., and Mascarenhas L., *A global method for relaxation in $W^{1,p}$ and SBV_p*, Arch. Rat. Mech. Anal., **165** (2002), 187–242.
[15] Braides A. and Chiadò Piat V., *Integral representation results for functionals defined on $SBV(\Omega;\mathbb{R}^m)$*, J. Math. Pures Appl., **75** (1996), 595–626.
[16] Brezis H., Coron J. M., and Lieb E. H., *Estimations d'énergie pour des applications de R^3 à valeurs dans S^2*, C. R. Acad. Sci. Paris Sér. I Math. **303** (1986), 207–210.
[17] Cannarsa P. and Sinestrari C., *Semiconcave Functions, Halmiton-Jacobi Equations, and Optimal Control.* Birkhäuser, 2004.
[18] Carriero M. and Leaci A., *S^k-valued maps minimizing the L^p norm of the gradient with free discontinuities*, Ann. Scuola Norm. Sup. Pisa, **18** (1991), 321–352.
[19] Dacorogna B. *Direct methods in the calculus of variations.* Second edition. Applied Mathematical Sciences, 78. Springer, New York, 2008.
[20] De Giorgi E. *Selected papers.* Edited by L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda and S. Spagnolo. Springer-Verlag, Berlin, 2006.
[21] De Giorgi E. and Ambrosio, L., *Un nuovo funzionale del calcolo delle variazoni*, Atti. Accad. Naz. Lincei Cl., Sci. Fis. Mat. Natur. Rend. Rend. Lincei (8) Mat. appl., **82** (1988), 199–210.
[22] De Giorgi E., Carriero M., and Leaci A., *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rat. Mech. Anal., **108** (1989), 195–218.

[23] De Giorgi E. and Letta, G., *Une notion générale de convergence faible pour des fonctions croissantes d ensemble,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **4** (1977), 61–99.

[24] Dacorogna B., Fonseca I., Malý J., and Trivisa K. *Manifold constrained variational problems,* Calc. Var. **9**, (1999), 185–206.

[25] Dal Maso G., Fonseca I., and Leoni G. *Nonlocal characther of the reduced theory of thin films with higher order perturbations,* to appear in Advances in Calculus of Variations.

[26] Ericksen J. L., *Liquid crystals with variable degree of orientation*, Arch. Rational Mech. Anal. **113** (1990), 97–120.

[27] Evans L.C. and Gariepy R.F., *Measure Theory and fine properties of functions,* CRC Press, 1992.

[28] Federer H., *Geometric Measure Theory* Springer Verlag, Berlin 1969.

[29] Fonseca I. and Leoni, G. *Modern Methods in the Calculus of Variations: L^p spaces,* Springer Verlag, 2007.

[30] Fonseca I. and Leoni G. *Modern Methods in the Calculus of Variations: Sobolev spaces* in preparation.

[31] Fonseca I. and Müller S., *Quasi–convex integrands and lower semicontinuity in L^1*, SIAM J. Math. Anal., **23** (1992), 1081–1098.

[32] Fonseca I. and Müller S., *Relaxation of quasiconvex functionals in $BV(\Omega, \mathbb{R}^d)$ for integrands $f(x, u, \nabla u)$*, Arch. Rat. Mech. Anal., **123** (1993), 1–49.

[33] Giaquinta M., Modica L., and Mucci D., *The relaxed Dirichlet energy of manifold constrained mappings*, Adv. Cal. Var., **1** (2008), 1–51.

[34] Giaquinta M., Modica G., and Souček J., *Cartesian currents and variational problems for mappings into spheres*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1989), 393–485.

[35] Giaquinta M., Modica G., and Souček J., *Liquid crystals: relaxed energies, dipoles, singular lines and singular points*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), 415–437.

[36] Giusti E., *Minimal surfaces and functions of bounded variation*. Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.

[37] Hang F. and Lin F., *Topology of Sobolev mappings. II*, Acta Math. **191** (2003), 55–107.

[38] Hardt R. and Lin F. H., *Mappings minimizing the L^p norm of the gradient,* Comm. Pure Appl. Math., **40** (1987), 555–588.

[39] Hardt R., Kinderlehrer D., and Lin F. H., *Existence and partial regularity of static liquid crystals configurations,* Comm. Math. Phys., **105** (1986), 547–570.

[40] Lin F. and Wang D., *The analysis of harmonic maps and their heat flows*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.

[41] Mucci D., *Relaxation of isotropic functionals with linear growth defined on manifold values constrained Sobolev mappings,* ESAIM Cont. Opt. Optim. Calc. Var. **2** (2009), 295–321.

[42] Spector D., *Simple proofs of some results of Reshetnyak*, submitted.

[43] Virga E. G., *Variational theories for liquid crystals*. Applied Mathematics and Mathematical Computation, 8. Chapman & Hall, London, 1994.

[44] Ziemer W. P., *Weakly differentiable functions,* Springer Verlag, Berlin, 1989.

Department of Mathematics, University of Évora, Évora, Portugal
E-mail address: gcarita@uevora.pt

Department of Mathematical Sciences, Carnegie-Mellon University, Pittsburgh, PA, U.S.A.
E-mail address: fonseca@andrew.cmu.edu

Department of Mathematical Sciences, Carnegie-Mellon University, Pittsburgh, PA, U.S.A.
E-mail address: giovanni@andrew.cmu.edu