Numerical semigroups

Tese de Doutoramento

Manuel Baptista Branco

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Introdução

Um semigrupo numérico S é um submonoide de $(\mathbb{N},+)$ tal que o máximo divisor comum dos seus elementos é igual a um. Usando esta definição, S admite um único sistema de geradores $\{n_0,\cdots,n_p\}$ e designamos n_0 e $p+1$ como a multiplicidade e a dimensão de imersão, respectivamente. Além disto o conjunto $\mathbb{N}\setminus S$ é finito e referimos o maior inteiro não pertencente a S como o número de Frobenius e denotamo-lo por $\mathrm{g}(S)$. O estudo dos semigrupos numéricos é um problema clássico equivalente ao estudo do conjunto das soluções das equações lineares com coeficientes em $\mathbb{N}$ (ver [**9, 10, 52, 54**]). A partir de 1970 (ver [**24, 25, 19**]), o estudo dos subsemigrupos era essencialmente motivado pelas suas aplicações em Geometria Algébrica. Como exemplo, temos que se K é um corpo, $K[S]$ é uma K-algebra de tipo finito associada a S e $K[X]=K[X_0,\ldots,X_p]$ é um anel de polinómios em $p+1$ indeterminadas, o epimorfismo K-algebra $\lambda : K[X]\to K[S]$ definido por $X_i\mapsto t^{ni}$ é homomorfismo de anéis S-graduado com grau zero. Assim, o ideal primo associado $P=\mathrm{kernel}(\lambda)$ (chamado ideal associado a S) é homogéneo e define uma curva num espaço afim de dimensão $p+1$. Herzog prova em [**24**] que encontrar um sistema de geradores para P é equivalente a encontrar uma apresentação para S.

Todo o semigrupo numérico S gerado pelo conjunto $\{n_0,\cdots,n_p\}$ é isomorfo ao monoide quociente $\mathbb{N}^{p+1}/\sigma$ (ver [**39**]) com σ uma congruência em $\mathbb{N}^{p+1}$. Rédei mostra em [**28**] que a congruência σ em $\mathbb{N}^{p+1}$ é finitamente gerada e portanto existe ρ um subconjunto de $\mathbb{N}^{p+1}\times\mathbb{N}^{p+1}$ tal que $\sigma=\langle\rho\rangle$. Ao conjunto ρ chamamos apresentação para S e dizemos que ρ é uma apresentação minimal se nenhum subconjunto próprio de ρ gerar σ. No processo de encontrar uma apresentação minimal para σ vamos usar alguma teoria dos grafos. Esta idéia de caracterizar uma apresentação minimal em termos da conexidade de certos grafos foi introduzida por Rosales (ver por exemplo [**30**]). Além disso multiplicidade e a dimensão de imersão desempenham um papel fundamental para uma cota máxima de uma apresentação minimal para S. De facto em [**32**] demonstra-se que o cardinal de qualquer apresentação minimal para S é menor ou igual a $\frac{n_0(n_0-1)}{2} - 2(n_0 - 1 - p)$.

Definimos semigrupo numérico irredutível como um semigrupo numérico que não pode ser expresso como intersecção de dois semigrupos numéricos que o contenham propriamente. Em [**25**] temos que um anel de semigrupo $K[S]$ é de Gorestin se e só se S é simétrico; e em [**5**] temos que um anel $K[S]$ é de Kunz se e só se S é pseudo-simétrico. O capítulo 2 é dedicado ao estudo dos semigrupos numéricos irredutíveis e os seus resultados encontram-se em ([**35, 36, 37, 38**]). Mostramos que S é irredutível se e só se S é maximal no conjunto de todos os semigrupos numéricos com número de Frobenius $g(S)$. Em [**31**] é feito o estudo dos semigrupos irredutíveis com número de Frobenius ímpar. Assim, o nosso objectivo na primeira secção é generalizar este estudo para semigrupos numéricos irredutíveis em geral (com número de Frobenius par ou ímpar). Caracterizamos os semigrupos numéricos irredutíveis dando especial atenção aos seus conjuntos de Apéry. Estabelecemos uma cota para o cardinal de uma apresentação minimal para estes semigrupos em termos da sua multiplicidade e da sua dimensão de imersão. Estudamos também os semigrupos irredutíveis com máxima dimensão de imersão. Sabemos que um semigrupo numérico pode ser expresso como uma intersecção finita de semigrupos numéricos irredutíveis. Donde é natural questionar quando é que um semigrupo numérico pode ser expresso como intersecção de semigrupos numéricos simétricos. Respondemos a esta questão caracterizando a classe dos semigrupos numéricos que podem ser expressos como intersecção finita de semigrupos numéricos simétricos (chamados ISY-semigrupos). A partir do conceito de pseudo-número de Frobenius damos uma nova caracterização de ISY-semigrupo e obtemos um método algoritmo para encontrar uma sua decomposição. Além disto caracterizamos as classes dos semigrupos numéricos que podem ser expressos como intersecção finita de semigrupos simétricos com o mesmo número de Frobenius (chamados ISYG-semigrupos) e com a mesma dimensão de imersão (chamados ISYM-semigrupos). Sejam S um semigrupo numérico e $\mathrm{r}(S)$ o menor inteiro positivo tal que $S = S_1 \cap \cdots \cap S_n$ com S_i semigrupo numérico irredutível. Usando novamente o conceito de pseudo-número de Frobenius damos uma cota superior e uma cota inferior para $\mathrm{r}(S)$. Com estes resultados caracterizamos os semigrupos numéricos que são intersecção de semigrupos numéricos simétricos e os que são intersecção de semigrupos numéricos pseudo-simétricos. Um problema subjacente a decompor um semigrupo em irredutíveis é encontrar uma decomposição com o menor número de elementos. Para resolver esta questão usamos **[48]** o qual nos descreve um algoritmo para uma decomposição minimal em irredutíveis. A finalizar este capítulo completamos os resultados de **[33]**. Provamos que se m e e são inteiros positivos tal $3 \leq e \leq m-1$, então existe um semigrupo numérico irredutível com número de Frobenius par tal que $\mathrm{m}(S) = m$ e $\mu(S) = e$. Esta prova é construtiva e permite-nos obter uma família de semigrupos numéricos irredutíveis com número de Frobenius par com multiplicidade e dimensão de imersão arbitrárias. Mostramos ainda que se $\mu(S) \geq 4$, então o cardinal de uma apresentação minimal para esta família de semigrupos numéricos é igual a $\frac{\mu(S)(\mu(S)-1)}{2} - 1$.

No capítulo 3 estudamos o conjunto dos semigrupos numéricos com multiplicidade m e os seus resultados encontram-se em ([**44**]). Dado um semigrupo numérico S com $m = \mathrm{m}(S)$, o conjunto de Apéry relativamente a m é o conjunto $\mathrm{Ap}(S,m) = \{s \in$

$S \mid s-m \notin S\}$. Suponhamos $w(i)$ o menor elemento em S congruente com i módulo m (denotado por $w(i) \equiv i(\text{mod}\, m)$) com $i \in \{0,\ldots,n-1\}$, então temos que

$$\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \ldots, w(m-1)\}$$

e $w(i) = k_i m + i$ para algum $k_i \in \mathbb{N}$. Por outro lado, [**32**, Lema 3.3] afirma que para $i, j \in \{0,\ldots,m-1\}$ existem $t \in \mathbb{N}$ e $k \in \{0,\ldots,m-1\}$ tal que $w(i)+w(j) = tm + w(k)$. Partindo destes resultados deduzimos que $(k_1,\ldots,k_{m-1})$ é solução do sistema linear de inequações Diophantino.

$$\begin{array}{rc}
x_i \geq 1 & i \in \{1,\ldots,m-1\}, \\
x_i + x_j - x_{i+j} \geq 0 & 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\
x_i + x_j - x_{i+j-m} \geq -1 & 1 \leq i \leq j \leq m-1, i+j > m.
\end{array}$$

Estudamos o conjunto das soluções inteiras positivas de um sistema de equações Diophantino. Mostramos que estas soluções podem ser descritas como um conjunto finito de parâmetros e que estes podem ser calculados algoritmicamente. Neste contexto construimos uma bijecção entre, $\mathcal{S}(m)$, o conjunto dos semigrupos numéricos com multiplicidade m e o conjunto das soluções positivas de um sistema linear de inequações Diophantino. Como vimos anteriormente um sistema linear de inequações Diophantino pode ser descrito por um conjunto finito de parâmetros logo obtemos descrição similar para $\mathcal{S}(m)$. Em seguida estudamos os MED-semigrupos (semigrupos numéricos com máxima dimensão de imersão). Mostramos que o conjunto, $\mathcal{MED}(m)$, de MED-semigrupos com multiplicidade m é bijectivo com um subsemigrupo de $\mathbb{N}^{m-1}$ este surge de uma adaptação das inequações do caso anterior para máxima dimensão de imersão. Particularizamos também estes resultados para o caso dos semigrupos numéricos simétricos. Neste caso os sistemas que aparecem contêm também equações lineares e o conjunto dos semigrupos numéricos é a união do conjunto das soluções inteiras não negativas dos sistemas deste tipo. Dizemos que um semigrupo numérico S tem um **conjunto de Apéry monotónico** se $w(1) < w(2) < \cdots < w(\mathrm{m}(S)-1)$, com $\{0, w(1), \ldots, w(\mathrm{m}(S)-1)\} = \mathrm{Ap}(S, \mathrm{m}(S))$.

Finalizamos este capítulo estudando o conjunto $C(m)$ dos semigrupos numéricos com conjunto de Apéry monotónico e multiplicidade m. Mostramos que existe uma correspondência biunívoca entre o conjunto $C(m)$ e um subsemigrupo de $\mathbb{N}^{m-1}$ finitamente gerado.

Existem na literatura um grande número de resultados referentes ao estudo de domínios locais analiticamente irredutíveis de dimensão um via os valores de um semigrupo (ver [**8, 17, 19, 21, 20, 25, 56, 55**]). Entre as propriedades estudadas para este tipo de aneis, à parte das estudadas anteriormente neste trabalho (Gorestein e Kunz), focamos as seguintes: máxima dimensão de imersão, (ver [**1, 5, 14, 15, 50, 51**]), Arf (ver, [**26, 49, 16**]) e ser saturado (ver, [**57, 27, 12**]). O capítulo 4 é dedicado ao estudo das classes dos MED-semigrupos e dois interessantes tipos destes semigrupos: semigrupos numéricos Arf e saturados. Quando descrevemos e trabalhamos com os MED-semigrupos (respectivamente semigrupos numéricos Arf e semigrupos numéricos saturados) usualmente usamos o seu sistema de geradores. Assim, não obtemos vantagens da estrutura extra MED-semigrupo (respectivamente semigrupo numérico Arf e semigrupo numérico saturado) que têm estes semigrupos numéricos. Começamos por demonstrar que a intersecção de dois semigrupos numéricos Arf (respectivamente saturado) é ainda um semigrupo Arf (respectivamente saturado). No caso dos MED-semigrupos é necessário fixar a multiplicidade para provarmos que a intersecção de dois MED-semigrupos é ainda um MED-semigrupo. Neste contexto, introduzimos o conceito de MED-sistema de geradores (respectivamente Arf sistema de geradores e SAT sistema de geradores) e concluimos que qualquer MED-semigrupo (respectivamente Arf semigrupo e saturado semigrupo) admite um único MED-sistema de geradores (respectivamente Arf sistema de geradores e SAT sistema de geradores). Provamos que se S é um semigrupo numérico Arf (respectivamente saturado) então $S \cup \{g(S)\}$ é Arf (respectivamente saturado). Por outro lado, se S é um semigrupo numérico tal que A o seu sistema minimal de geradores Arf (respectivamente saturado), então $a \in A$ se e somente se $S \setminus \{a\}$ é um semigrupo numérico Arf (respectivamente saturado). Em consequência deste resultado ordenamos o conjunto de todos os semigrupos numéricos Arf (respectivamente saturados) numa árvore binária cuja raíz é $\mathbb{N}$ (no caso saturado sem folhas). No caso dos MED-semigrupos temos que se S é um MED-semigrupo e $g(S) \geq \mathrm{m}(S)$, então $S \cup \{g(S)\}$ é um MED-semigrupo; e se S é um MED-semigrupo com A o seu minimal MED-sistema de geradores, então $a \in A \setminus \{\mathrm{m}(S)\}$ se e somente se $S \setminus \{a\}$ é um MED-semigrupo de multiplicidade m. Análogamente isto permite-nos ordenar o conjunto dos MED-semigrupos com multiplicidade m numa árvore cuja raíz é o semigrupo $\langle m, m+1, \ldots, 2m-1 \rangle$. Finalmente, dado um semigrupo numérico obtemos um método para calcularmos o seu fecho MED, Arf e SAT, que é, o menor semigrupo numérico MED, Arf e saturado, respectivamente, que o contém.

Introduction

A numerical semigroup S is a submonoid of $(\mathbb{N},+)$ such that the greatest common divisor of its elements is equal to one. From this definition, one obtains that S admits a unique minimal system of generators $\{n_0 < \cdots < n_p\}$. We refer to the numbers n_0 and $p+1$ as the multiplicity and embedding dimension of S, and denote them by $\mathrm{m}(S)$ and $\mu(S)$, respectively. Moreover $\mathbb{N} \setminus S$ is finite, and the greatest integer not in S is the Frobenius number of S and it is denoted by $\mathrm{g}(S)$. The study of numerical semigroups is a classical problem, which is equivalent to the study of the sets of natural solutions of linear equations with coefficients in $\mathbb{N}$ (see for instance **[9, 10, 52, 54]**). From 1970 (see for instance **[24, 25, 19]**), the study of subsemigroups of $\mathbb{N}$ has been motivated by its applications to Algebraic Geometry. As an example, if K is a field, $K[S]$ is the finite type K-algebra associated to S and $K[X] = K[X_0,\ldots,X_p]$ is the polynomial ring in $p+1$ unknowns, the K-algebras epimorphism $\lambda : K[X] \to K[S]$ defined by $X_i \mapsto t^{ni}$ is a S-graduate ring homomorphism with degree zero. Therefore, the prime ideal $P =$ kernel(λ) (called the ideal associated to the semigroup) is homogeneous and defines a monomial curve in the $(p+1)$-dimensional affine space on K. Herzog proves in **[24]** that finding a system of generators for P is equivalent to finding a presentation for S.

Every numerical semigroup S with minimal system of generators $\{n_0,\ldots,n_p\}$ is isomorphic to the quotient monoid $\mathbb{N}^{p+1}/\sigma$ (see for instance **[39]**) where σ is a congruence on $\mathbb{N}^{p+1}$. Rédei shows in **[28]** that the congruence σ on $\mathbb{N}^{p+1}$ is finitely generated and therefore there exists ρ a finite subset of $\mathbb{N}^{p+1} \times \mathbb{N}^{p+1}$ such that $\sigma = \langle\rho\rangle$. The set ρ is called presentation of S. We say that ρ is a minimal presentation if no proper subset of ρ generates σ. In the process of finding a minimal presentation for σ, it is used some graph theory. This idea of characterizing a minimal presentation in terms of the connectedness of certain graphs is due to Rosales (see for example **[30]**). Furthermore the multiplicity and the embedding dimension play an important role in order to find bounds for the cardinality of a minimal presentation for S. In fact in **[32]** it is shown that the cardinal of any minimal presentation for S is less than or equal to $\frac{1}{2}n_0(n_0-1)-2(n_0-1-p)$. Bresinsky proves in **[11]** that an upper bound for the cardinality of a minimal presentation of S can not be given by using only the embedding dimension of S.

We say that a numerical semigroup is irreducible if it can not be expressed as an intersection of two numerical semigroups that contain it properly. In **[25]** it is shown that the semigroup ring $K[S]$ is Gorestein if and only if S is symmetric; and in **[5]** it is shown that the semigroup ring $K[S]$ is Kunz if only if S is pseudo-symmetric. From **[5]** and **[20]** we deduce that S is irreducible with odd Frobenius number if only if S is symmetric; and that S is irreducible with even Frobenius number if only if S is pseudo-symmetric. Chapter 2 is devoted to the study of irreducible numerical semigroups and the results presented there can be found in (**[35, 36, 37, 38]**). We prove that S is irreducible if only if S is maximal in the set of all numerical semigroups with Frobenius number $\mathrm{g}(S)$. Rosales in **[31]** gives us a study of symmetric numerical semigroups. Our aim in the first section of this chapter is to generalize this study to irreducible numerical semigroups in general (that is, with even or odd Frobenius number). We characterize irreducible numerical semigroups giving especial attention to their Apéry sets. We give an upper bound for the cardinality of the minimal presentation for this kind of numerical semigroups in terms of their multiplicity and embedding dimension. We also study those irreducible numerical semigroups with maximal embedding dimension. A numerical semigroup can be expressed as an intersection of finitely many irreducible numerical semigroups. Then, it is natural to ask whether or not a numerical semigroup can be expressed as an intersection of only symmetric numerical semigroups. We answer this motivating question, characterizing this class of semigroups that can be expressed as a finite intersection of symmetric semigroups (called ISY-semigroups). From the concept of pseudo-Frobenius numbers we give a new characterization of ISY-semigroups and we derive algorithmic methods to find such decomposition. Moreover we characterize the class of numerical semigroups that can be expressed as a finite intersection of symmetric numerical semigroups with the same Frobenius-number (called ISYG-semigroups) and the same multiplicity (called ISYM-semigroups). Now suppose that S is a numerical semigroup and we denote by $\mathrm{r}(S)$ the least positive integer n such that $S = S_1 \cap \cdots \cap S_n$ with S_i an irreducible numerical semigroup. Using again the concept of pseudo-Frobenius numbers we give an upper bound and lower bound for $\mathrm{r}(S)$. We will use these results to characterize those numerical semigroups that are intersection of symmetric numerical semigroups and those that are intersection of pseudo-symmetric numerical semigroups. A subjacent problem for such decompositions in irreducibles is to found a decomposition with the least possible number of irreducibles. In order to achieve this result we use **[48]** which describes an algorithm for computing a minimal decomposition of a numerical semigroup in terms of irreducible numerical semigroups. Finally, in this chapter, we complete the results given in **[33]**. We prove that if m and e are positive integers such that $3 \leq e \leq m-1$, then there exists an irreducible numerical semigroup with even Frobenius-number such that $\mathrm{m}(S) = m$ and $\mu(S) = e$. The prove we give is constructive and so we can obtain a family of irreducible numerical semigroups with even Frobenius-number and with arbitrary multiplicity and embedding dimension. Also we show that if $\mu(S) \geq 4$, then the cardinality of a minimal presentation for any element of this family is equal to $\frac{\mu(S)(\mu(S)-1)}{2} - 1$.

In Chapter 3 we study the set of numerical semigroups with multiplicity m. The results presented in this chapter are collected in ([**44**]). Given a numerical semigroup and $m = \mathrm{m}(S)$, the Apéry set of S with respect to m is the set $\mathrm{Ap}(S,m) = \{s \in S \mid s - m \notin S\}$. It can be shown that if for every $i \in \{0,\ldots,n-1\}$ we take $w(i)$ to be the least element in S congruent with i modulo m (denoted $w(i) \equiv i(\operatorname{mod} m)$), then

$$\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \ldots, w(m-1)\}$$

and $w(i) = k_i m + i$ for some $k_i \in \mathbb{N}$. Furthermore, [**32**, Lemma 3.3] states that for every $i, j \in \{0,\ldots,m-1\}$ there exist $t \in \mathbb{N}$ and $k \in \{0,\ldots,m-1\}$ such that $w(i) + w(j) = tm + w(k)$. From this fact it can be deduced that $(k_1,\ldots,k_{m-1})$ is a solution of the system of linear Diophantine inequalities

$$\begin{array}{rc} x_i \geq 1 & i \in \{1,\ldots,m-1\}, \\ x_i + x_j - x_{i+j} \geq 0 & 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i + x_j - x_{i+j-m} \geq -1 & 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

We study the set of nonnegative integer solutions of systems of linear Diophantine equations. We show that these solutions can be described with a finite set of parameters and that the coefficients of these can be computed algorithmically. With this result we construct a one to one map between the set $\mathcal{S}(m)$ of all numerical semigroups with multiplicity m and the set of nonnegative integer solutions of a system of linear Diophantine inequalities. Since the above the system of linear Diophantine inequalities can be described by a finite set of parameters that can be computed, we have a similar description of $\mathcal{S}(m)$. Next we study MED-semigroups (numerical semigroups with maximal embedding dimension) and we show that the set $\mathcal{MED}(m)$ of MED-semigroups with multiplicity m is bijective with a subsemigroup of $\mathbb{N}^{m-1}$ arising from the adaptation of the above inequalities to the maximal embedding dimension case. Finally, we particularize these results for symmetric numerical semigroups. In this setting the systems that appear also contain linear equations and the set of symmetric numerical semigroups is a union of the sets of nonnegative integer solutions of systems of this type. We say that a numerical semigroup S has **monotonic Apéry set** if $w(1) < w(2) < \cdots < w(\mathrm{m}(S)-1)$, with $\{0, w(1), \ldots, w(\mathrm{m}(S)-1)\} = \mathrm{Ap}(S, \mathrm{m}(S))$. We finish this chapter studding the set $C(m)$ of the numerical semigroups with monotonic Apéry set and multiplicity m. We show that there is a one-to-one correspondence between $C(m)$ and a finitely generated subsemigroup of $\mathbb{N}^{m-1}$.

In the literature one can find a long list of works dealing with the study of one dimensional analytically irreducible local domains via their value semigroups (see for instance [**8, 17, 19, 21, 20, 25, 56, 55**]). Among the properties studied for this kind of ring apart from the one S studied so far in this work (Gorestein and Kunz), we focus on the following: maximal embedding dimension (see [**1, 5, 14, 15, 50, 51**]), Arf (see [**26, 49, 16**]) and saturated (see [**57, 27, 12**]). Chapter 4 is devoted to the study of the class of MED-semigroups and two interesting kinds of these semigroups: Arf and saturated numerical semigroups. For describing and working with MED-semigroups (respectively Arf numerical semigroups and saturated numerical semigroups) one can use their systems of generators (usually this is the case). In this way one does not take advantage on the extra structure that MED-semigroups (respectively Arf numerical semigroups and saturated numerical semigroups) have over general numerical semigroups. Here we show that the intersection of two Arf (respectively saturated) numerical semigroups is again an Arf (respectively saturated) numerical semigroup. In the case of MED-semigroups we need to fix the multiplicity to prove that the intersection of two MED-semigroups is again a MED-semigroup. From this fact we introduce the concept of MED-system of generators (respectively Arf system of generators and SAT system of generators) and we will see that every MED-semigroup (respectively Arf semigroup and saturated semigroup) admits a unique minimal MED-system of generators (respectively Arf system of generators and SAT system of generators). We show that if S is an Arf (respectively saturated) numerical semigroup then so is $S \cup \{g(S)\}$.

Furthermore, if S is a numerical semigroup and A it is minimal Arf (respectively SAT) system of generators then $a \in A$ if only if $S \setminus \{a\}$ is an Arf (respectively saturated) numerical semigroup. As a consequence of this result we show that the set of all Arf (respectively saturated) numerical semigroups can be arranged in a binary tree with root $\mathbb{N}$ (no leaves in the saturated case). For MED-semigroups, we have that if S is a MED-semigroup and $g(S) \geq \mathrm{m}(S)$, then $S \cup \{g(S)\}$ is again a MED-semigroup; and if S is a MED-semigroup with A its minimal MED-system of generators, then $a \in A \setminus \{\mathrm{m}(S)\}$ if only if $S \setminus \{a\}$ is a MED-semigroup with multiplicity m. This will allow us to show that the set of MED-semigroups with multiplicity m can be arranged in a tree whose root is the semigroup $\langle m, m+1, \ldots, 2m-1\rangle$. We also give a procedure for computing the MED, Arf and SAT closure for a given numerical semigroup, that is, the smallest MED, Arf and saturated numerical semigroup, respectively, containing it.

CHAPTER 1

Preliminaries

In this chapter we give a brief introduction to numerical semigroups and we fix the notation used along this work.

We use $\mathbb{N}$ and $\mathbb{Z}$ to denote the set of nonnegative integers and the set of the integers, respectively.

A **semigroup** is a pair $(S,+)$, with S a non empty set and $+$ a binary operation defined on S verifying the associative law. If there exists an element $t \in S$ such that $t+s=s+t=s$ for all $s \in S$ we say that $(S,+)$ is a **monoid**. This element t is usually referred to as the **identity element** and it is denoted by 0. In addition, S is a **commutative monoid** if for all, $a,b \in S$, $a+b=b+a$. A **submonoid** of a monoid S is a subset A of S such that $0 \in A$ and for every $a,b \in A$ we have that $a+b \in A$. Given a subset B of a monoid S , the **monoid generated** by $\langle B \rangle$, is the least (with respect to set inclusion) submonoid of S containing B, which turns out to be the intersection of all submonoids of S containing B. If $S=\langle B \rangle$ we say that B is a **system of generators** of S or that S is generated by B. Furthermore if $S=\langle B \rangle$ and there exists no proper subset of B that generates S we say that B is a **minimal system of generators** for S. A monoid S is **finitely generated** if it has a finite system of generators. A map $\varphi : F \to S$, where F and S are monoids, is called monoid morphism if $\varphi(0)=0$ and $\varphi(a+b)=\varphi(a)+\varphi(b)$ for all a,b in F.

A **numerical semigroup** is a submonoid of $\mathbb{N}$ such that the greatest common divisor of its elements is equal to one. The following result gives us alternative ways of defining a numerical semigroup.

PROPOSITION 1. *Let S a submonoid of $\mathbb{N}$. The following conditions are equivalent:*

1) *S is a numerical semigroup,*
2) *the group spanned by S is $\mathbb{Z}$,*
3) *$\mathbb{N} \setminus S$ is finite.*

Then it makes sense to take into account the greatest element of $\mathbb{Z}$ not in S. We call this element **Frobenius number** of S and denote it by $\mathrm{g}(S)$.

Suppose that $A = \{a_1, \dots, a_m\} \subseteq \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$ are such that $a_i \not\equiv a_j (\mathrm{mod}\, m)$ for all $1 \leq i < j \leq m$. We say that A is a **complete system modulo** m. For $n \in S \setminus \{0\}$, we define the **Apéry set** of n in S (see **[4]**) as the set

$$\mathrm{Ap}(S,n) = \{x \in S \mid x - n \notin S\}.$$

Next result follows easily.

PROPOSITION 2. *Let S be a numerical semigroup and let $n \in S \setminus \{0\}$. Then $\mathrm{Ap}(S,n)$ is a complete system modulo n.*

Hence $\#\mathrm{Ap}(S,n) = n$ (where $\#A$ stands for cardinality of A).

LEMMA 3. *Let S be a numerical semigroup and let $n \in S \setminus \{0\}$. Then:*

(1) $\mathrm{g}(S) = \max(\mathrm{Ap}(S,n)) - n$,

(2) $S = \langle \{n\} \cup \mathrm{Ap}(S,n) \rangle$,

$\max(A)$ *denotes maximum of A.*

As a consequence of the above lemma every numerical semigroup S is finitely generated. Clearly S has a unique minimal system of generators. Assume that $\{n_0 < n_1 < \cdots < n_p\}$ is a minimal system of generators of S, we refer to the numbers n_0 (the least integer in $S \setminus \{0\}$) and $p+1$ (cardinality of its minimal system of generators) as the **multiplicity** and **embedding dimension** of S, and denote them by $\mathrm{m}(S)$ and $\mu(S)$, respectively.

A binary relation σ on a monoid S is an **equivalence relation** if the following properties hold:

1. for all $a \in S$, $a\ \sigma\ a$ (reflexive),
2. for all $a,b \in S$, if $a\ \sigma\ b$ then $b\ \sigma\ a$ (symmetric),
3. for all $a,b,c \in S$, if $a\ \sigma\ b$ and $b\ \sigma\ c$, then $a\ \sigma\ c$ (transitive).

In addition, if σ is an equivalence relation such that $a\ \sigma\ b$ implies that $a+c\ \sigma\ b+c$ for all $c \in S$ we say σ is a **congruence** on S. An alternative notation for $a\ \sigma\ b$ is $(a,b) \in \sigma$.

For ρ a subset of $\mathbb{N}^n \times \mathbb{N}^n$, there always exists the smallest congruence $\langle\rho\rangle$ containing ρ, which can be described in three steps:

1. $\rho^0 = \rho \cup \rho^{-1} \cap \tau$, where $\rho^{-1} = \{(v,w) \mid (w,v) \in \rho\}$, and $\tau = \{(w,w) \mid w \in N^n\}$,
2. $\rho^1 = \{(v+u, w+u) \mid (v,w) \in \rho^0$, and $u \in \mathbb{N}^n\}$,
3. $(v,w) \in \langle\rho\rangle$ if there exist $v_0 = v, \ldots, v_k = w$ with $(v_i, v_{i+1}) \in \rho^1$ for all $i \in \{0,\ldots,k-1\}$.

We also refer to the congruence $\langle\rho\rangle$ as the congruence generated by ρ.

Let $F = \{a_0X_0 + \cdots + a_pX_p \mid a_0,\ldots,a_p \in \mathbb{N}\}$ be the free commutative monoid generated by $\{X_0,\ldots,X_p\}$ and let $\varphi : F \to S$ be the monoid epimorphism defined by

$$\varphi(a_0X_0 + \cdots + a_pX_p) = a_0n_0 + \cdots + a_pn_p.$$

It is well known that if σ is the kernel congruence of φ (that is, $x\sigma y$ if $\varphi(x) = \varphi(y)$), then S is isomorphic to the quotient monoid F/σ (see **[39]**). Rédei shows in **[28]** that the congruence σ is finitely generated and therefore there exists

$$\rho = \{(x_1,y_1),\ldots,(x_t,y_t)\} \subseteq F \times F$$

such that σ is the congruence on F generated by ρ. The set ρ is called a **presentation** for the numerical semigroup S. We say that ρ is **minimal presentation** if no proper subset of ρ generates σ. In fact the concepts of minimal presentation and presentation with the lowest cardinality coincide for a numerical semigroup (see for instance **[30]**).

Let S be a numerical semigroup with minimal system of generators $\{n_0 < n_1 < \cdots < n_p\}$ and consider again $\varphi : F \to S$ defined as above. Denote by σ the kernel congruence of φ and, for $n \in S \setminus \{0\}$, denote by $[n] = \{x \in F \mid \varphi(x) = n\}$ (the inverse image of n by φ). We define in $[n]$ the following equivalence relation $\mathcal{R}$:

$$a_0X_0 + \cdots + a_pX_p \ \mathcal{R}\ b_0X_0 + \cdots + b_pX_p$$

if there exist elements

$$k_{0_0}X_0 + \cdots + k_{0_p}X_p, k_{1_0}X_0 + \cdots + k_{1_p}X_p, \cdots, k_{j_0}X_0 + \cdots + k_{j_p}X_p \in [n]$$

such that

$$a_0X_0 + \cdots + a_pX_p = k_{0_0}X_0 + \cdots + k_{0_p}X_p$$

and

$$b_0X_0 + \cdots + b_pX_p = k_{j_0}X_0 + \cdots + k_{j_p}X_p$$

and $k_{i_0}k_{i+1_0} + \cdots + k_{i_p}k_{i+1_p} \neq 0$ for all $i \in \{0, \ldots j-1\}$.

Let X be a set, $\mathcal{P} = \{X_1, \ldots, X_t\}$ be a partition of X and $\gamma \subseteq X \times X$ be a binary relation on X. We define the graph G_γ associated to γ with respect to the partition $\mathcal{P}$ as a graph whose vertices are the elements in $\mathcal{P}$ and $\overline{X_iX_j}$, with $i \neq j$, is an edge of G_γ whenever there exist $x \in X_i$ and $y \in X_j$ such that $(x,y) \in \gamma \cup \gamma^{-1}$.

The following results can be found in [**30**].

PROPOSITION 4. *Let $n \in S$ be and let $\mathcal{P} = \{X_1, \ldots, X_t\}$ be the set of $\mathcal{R}$-classes contained in $[n]$. If γ is a presentation of S, and $\gamma_n = \gamma \cap ([n] \times [n])$, then the graph associated to G_{γ_n} with respect to the partition $\mathcal{P}$ of $[n]$ is a connected graph.*

PROPOSITION 5. *Let γ be a subset of σ such that G_{γ_n} is connected for all $n \in S$. Then γ is a presentation for S.*

With these two results together, we obtain the following theorem.

THEOREM 6. *Let γ be a subset of σ. Then γ is presentation of S if only if G_{γ_n} is connected for all $n \in S$.*

Now we show how a minimal presentation of a numerical semigroup can be computed from its minimal system of generators.

For $n \in S$ define the graph G_n with vertices V_n and edges E_n as

$$V_n = \{n_i \in \{n_0, \ldots, n_p\} \mid n - n_i \in S\},$$
$$E_n = \{\overline{n_i n_j} \mid n - (n_i + n_j) \in S, i, j \in \{0, \ldots, p\}, i \neq j\}.$$

The next result illustrates the connection between $\mathcal{R}$-classes and the graphs G_n.

PROPOSITION 7. *([**30**]) For every $n \in S \setminus \{0\}$, there is a bijective map between the set of connected components of G_n and set of $\mathcal{R}$-classes of $[n]$.*

For every $n \in S$, define γ_n as

1) If G_n is not connected and $G_n^1 = (V_n^1, E_n^1), \ldots, G_n^r = (V_n^r, E_n^r)$ are its connected components, then for each $i \in \{1, \ldots, r\}$ we choose a vertex $n_{k_i} \in V_n^i$ and an element $\alpha_i = (a_0, \ldots, a_p)$ such that $\varphi(\alpha_i) = n$ and $a_{k_i} \neq 0$; set $\gamma_n = \{(\alpha_1, \alpha_2), \ldots, (\alpha_1, \alpha_r)\}$.

2) If G_n is connected, set $\gamma_n = \emptyset$.

Using the above results we have the following statement.

THEOREM 8. *([**30**]) The set $\gamma = \cup_{n \in \mathbb{N}} \gamma_n$ is a minimal presentation of S.*

Therefore, the main idea for computing a minimal presentation of a numerical semigroup S consists in finding the elements in S such that the graph G_n is not connected. Next result shows that G_n is not connected only for finitely many $n \in S$.

THEOREM 9. *([**30**]) Let $\{n_0 < n_1 < \cdots < n_p\}$ be a minimal system of generators of the numerical semigroup S. If G_n is not connected for $n \in S$, then there exist $w \in \mathrm{Ap}(S, n) \setminus \{0\}$ and $j \in \{1, \ldots, p\}$ such that $n = w + n_j$.*

And since $\mathrm{g}(S)+n_0=\max(\mathrm{Ap}(S,n_0))$ then there are at most $\mathrm{g}(S)+n_0+n_p$ elements of the form $w+n_j$ with $w\in\mathrm{Ap}(S,n_0)$ and $j\in\{1,\ldots,p\}$.

CHAPTER 2

Irreducible numerical semigroups

In this chapter, we study irreducible numerical semigroups. From **[20]** and **[5]**, we deduce that the class of irreducible semigroups with odd (respectively even) Frobenius number is the same as the class of symmetric (respectively pseudo-symmetric) numerical semigroups. This kind of numerical semigroups have been widely studied in literature, not only from the semigroupist point of view, but also by their applications in Ring Theory. In **[25]** it is shown that the semigroup ring associated to a numerical semigroup S ($K[S] = \oplus_{s\in S} K y_s$) is Gorestein if and only if S is irreducible with odd Frobenius number; and in **[5]** it is shown that the semigroup ring $K[S]$ is Kunz if only if S is irreducible with odd Frobenius number.

Section 1 is devoted to characterize irreducible numerical semigroups paying special attention to their Apéry sets.

In Section 2 we give an upper bound for the cardinality of a minimal presentation for an irreducible numerical semigroup, in function of their multiplicity and embedding dimension. Finally, in this section, we study those irreducible numerical semigroups with maximal embedding dimension.

In Section 3, from the concept of pseudo-Frobenius number of a numerical semigroup, we characterize the class of numerical semigroups that can be expressed as an intersection of irreducible numerical semigroups with odd Frobenius number (that is, symmetric). We construct algorithms for decompositions in symmetric numerical semigroups in general, and then we study the problem of finding such decompositions with the restriction that all symmetric numerical semigroup S involved have the same Frobenius number or multiplicity.

We know that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups. In Section 4, we give lower and upper bounds for the minimal number of irreducibles in such decompositions. Associated to the problem of finding a decomposition into irreducibles with the same Frobenius number, we introduce and study the concept of atomic numerical semigroup. Finally, we use the results given in **[48]** to describe an algorithm for computing a minimal decomposition of a numerical semigroup in terms of irreducible numerical semigroups.

In Section 5, we construct families of irreducible numerical semigroups with even Frobenius number, for arbitrary multiplicity $\mathrm{m}(S)$ and embedding dimension $\mu(S)$. Furthermore, we show who are the presentation with minimal cardinality for this family of numerical semigroups. This section complete the results given in **[33]**, to the case of families of irreducible numerical semigroups with odd Frobenius number.

1. Symmetric and pseudo-symmetric numerical semigroups

In this section we characterize and study symmetric and pseudo-symmetric numerical semigroups. Separately, we study semigroups of this kind with multiplicity 3 and 4.

Throughout this section S denotes a numerical semigroup, such that $S \neq \mathbb{N}$. It is well known (see for instance **[32]**) that $S \cup \{\mathrm{g}(S)\}$ is also a numerical semigroup.

A numerical semigroup S is **irreducible** if it can not be expressed as an intersection of two numerical semigroups containing it properly.

From this definition we have the following characterization of irreducible numerical semigroup.

THEOREM 10. *The following conditions are equivalent:*

1) *S is irreducible,*

2) *S is maximal in the set of all numerical semigroups with Frobenius number* $\mathrm{g}(S)$,

3) *S is maximal in the set of all numerical semigroups that do not contain* $\mathrm{g}(S)$, *where the order taken is set inclusion.*

PROOF. *1) $\Rightarrow$ 2)* Let $\overline{S}$ be a numerical semigroup such that $S \subseteq \overline{S}$ and $g(\overline{S}) = \mathrm{g}(S)$. Then $S = (S \cup \{\mathrm{g}(S)\}) \cap \overline{S}$. Since S is irreducible, we deduce that $S = \overline{S}$.

2) $\Rightarrow$ 3) Let $\overline{S}$ be a numerical semigroup such that $S \subseteq \overline{S}$ and $\mathrm{g}(S) \notin \overline{S}$. Then $\overline{S} \cup \{\mathrm{g}(S)+1, \mathrm{g}(S)+2, \ldots\}$ is a numerical semigroup that contains S with Frobenius number $\mathrm{g}(S)$. Therefore, $S = \overline{S} \cup \{\mathrm{g}(S)+1, \mathrm{g}(S)+2, \cdots\}$ and so $S = \overline{S}$.

3) $\Rightarrow$ 1) Let S_1 and S_2 be two numerical semigroups that contain S properly. Then, by hypothesis, $\mathrm{g}(S) \in S_1$ and $\mathrm{g}(S) \in S_2$. Therefore $S \neq S_1 \cap S_2$ and so S is irreducible.

□

Using **[20]** and **[5]** we deduce the next result.

PROPOSITION 11. 1) *If $\mathrm{g}(S)$ is odd, then S is irreducible if and only if for all $h, h' \in \mathbb{Z}$, such that $h + h' = \mathrm{g}(S)$, we have that either $h \in S$ or $h' \in S$ (that is, S is symmetric).*

2) *If $\mathrm{g}(S)$ is even, then S is irreducible if and only if for all $h, h' \in \mathbb{Z} \setminus \{\frac{\mathrm{g}(S)}{2}\}$, such that $h + h' = \mathrm{g}(S)$, we have that either $h \in S$ or $h' \in S$ (that is, S is pseudo-symmetric).*

The following result is also well known (see **[4]**, **[11]** or **[31]**).

PROPOSITION 12. *Let $n \in S \setminus \{0\}$ with $\mathrm{Ap}(S,n) = \{0 = w(1) < w(2) < \cdots < w(n)\}$. Then S is irreducible with odd Frobenius number (that is, S is symmetric) if and only if $w(i) + w(n-i+1) = w(n)$ for all $i \in \{1, \ldots, n\}$.*

PROOF. For $i \in \{1,\ldots,n\}$ as $w(i) \in \mathrm{Ap}(S,n)$ then $w(i)-n \notin S$ and, by Proposition 11, we obtain that $w(n)-w(i) = \mathrm{g}(S)-(w(i)-n) \in S$. We have that $w(n)-w(i) \in \mathrm{Ap}(S,n)$ because $w(n) \in \mathrm{Ap}(S,n)$.

□

Now we see how is $\mathrm{Ap}(S,n)$ when S is irreducible with an even Frobenius number.

LEMMA 13. *If S is irreducible with even Frobenius number and $n \in S \setminus \{0\}$, then* $\frac{\mathrm{g}(S)}{2}+n \in \mathrm{Ap}(S,n)$.

PROOF. It is enough to prove that $\frac{\mathrm{g}(S)}{2}+n \in S$, since $\frac{\mathrm{g}(S)}{2} \notin S$, but this follows from Proposition 11 (note that $(\frac{\mathrm{g}(S)}{2}+n)+(\frac{\mathrm{g}(S)}{2}-n) = \mathrm{g}(S)$). □

PROPOSITION 14. *Let S be a numerical semigroup with even Frobenius number and let $n \in S \setminus \{0\}$. Then S is irreducible if and only if*

$$\mathrm{Ap}(S,n) = \{0 = w(1) < w(2) < \ldots < w(n-1) = \mathrm{g}(S)+n\} \cup \{\frac{\mathrm{g}(S)}{2}+n\}$$

and $w(i)+w(n-i) = w(n-1)$ for all $i \in \{1,\ldots,n-1\}$.

PROOF. First note that if $\mathrm{g}(S)$ is even, then $\frac{\mathrm{g}(S)}{2}+n \in \mathrm{Ap}(S,n)$ and $\frac{\mathrm{g}(S)}{2}+n < \max \mathrm{Ap}(S,n)$. If $i \in \{1,\ldots,n-1\}$, then $w(i)-n \notin S$ and $w(i)-n \neq \frac{\mathrm{g}(S)}{2}$. By Proposition 11, we have that $\mathrm{g}(S)-(w(i)-n) \in S$ and thus $w(n-1)-w(i) = \mathrm{g}(S)+n-w(i) \in S$. Since $w(n-1) \in \mathrm{Ap}(S,n)$ we deduce that $w(n-1)-w(i) \in \mathrm{Ap}(S,n)$. Furthermore $w(n-1)-w(i) \neq \frac{\mathrm{g}(S)}{2}+n$ because otherwise we would have $w(i) = \frac{\mathrm{g}(S)}{2}$. Hence the reader can check that $w(i)+w(n-i) = w(n-1)$.

Conversely, let x be an integer such that $x \neq \frac{\mathrm{g}(S)}{2}$ and $x \notin S$. Let us show that $\mathrm{g}(S)-x \in S$. Take $w \in \mathrm{Ap}(S,n)$ such that $w \equiv x (mod\ n)$. Then $x = w-kn$ for some $k \in \mathbb{N} \setminus \{0\}$. We distinguish two cases.

(1) If $w = \frac{g(S)}{2} + n$, then $g(S) - x = g(S) - (\frac{g(S)}{2} + n - kn) = \frac{g(S)}{2} + (k-1)n$. Besides, $x \neq \frac{g(S)}{2}$ leads to $k \neq 1$ and therefore $k \geq 2$. Hence we can assert that $g(S) - x \in S$.

(2) If $w \neq \frac{g(S)}{2} + n$, then $g(S) - x = g(S) - (w - kn) = g(S) + n - w + (k-1)n = w(n-1) - w + (k-1)n \in S$, since $w(n-1) - w \in S$ by hypothesis.

□

Note that if S has embedding dimension two, then S is irreducible with odd Frobenius number (that is, S is symmetric); in fact S is a complete intersection (see **[24]**).

Observe also that $\mu(S) \leq \mathrm{m}(S)$ for every numerical semigroup S.

PROPOSITION 15. *Let S be an irreducible numerical semigroup.*

1) *If* $g(S)$ *is odd and* $\mathrm{m}(S) \geq 3$, *then* $\mu(S) \leq \mathrm{m}(S) - 1$.

2) *If* $g(S)$ *is even and* $\mathrm{m}(S) \geq 4$, *then* $\mu(S) \leq \mathrm{m}(S) - 1$.

PROOF. *1.* Suppose that S is a symmetric semigroup with minimal system of generators $\{\mathrm{m}(S), n_1, \cdots, n_{\mu(S)-1}\}$ then $\{0 < n_1 < \cdots < n_{\mu(S)-1}\} \subseteq \mathrm{Ap}(S, \mathrm{m}(S))$ and $n_{\mu(S)-1} \neq W(n)$. Hence $\mu(S) \leq \mathrm{m}(S) - 1$.

2. It is enough to prove that $\mu(S) \neq \mathrm{m}(S)$. If $\mu(S) = \mathrm{m}(S)$, then S is minimally generated by $\{\mathrm{m}(S), n_1, \ldots, n_{\mathrm{m}(S)-1}\}$ and therefore $\mathrm{Ap}(S, n)$ is of the form

$$\mathrm{Ap}(S,n) = \{0 < n_2 < \cdots < n_{\mathrm{m}(S)-1}\} \cup \{n_1 = \frac{g(S)}{2} + \mathrm{m}(S)\}.$$

Since $\mathrm{m}(S) - 1 \geq 3$ then $n_1 \neq n_2 \neq n_{\mathrm{m}(S)-1}$. By Proposition 14 we deduce that $n_{\mathrm{m}(S)-1} - n_2 \in S$, which contradicts the fact that $\{\mathrm{m}(S), n_1, \ldots, n_{\mathrm{m}(S)-1}\}$ is a minimal system of generators for S. □

Note that $S = \langle 3, 7, 11 \rangle$ is an irreducible numerical semigroup with Frobenius number $g(S) = 8$ (it is easy to see that 8 belongs to every numerical semigroup that properly contains S). That is why in 2) of the above proposition we need that $\mathrm{m}(S) \geq 4$ instead of $\mathrm{m}(S) \geq 3$.

Using 1) and 2) of the above proposition we can assert that if S is an irreducible numerical semigroup with $\mathrm{m}(S) \geq 4$, then $\mu(S) \leq \mathrm{m}(S) - 1$.

Next we study irreducible numerical semigroups with multiplicity 3 and 4. By the remark made after Proposition 14, we know that if $\mu(S) = 2$, then S is irreducible. Recall also, that from Proposition 15, if $\mathrm{m}(S) = 4$ and S is irreducible then $\mu(S) \leq 3$.

Therefore, we focus our study in the cases:

1) S is irreducible with $\mathrm{m}(S) = \mu(S) = 3$,
2) S is irreducible with $\mathrm{m}(S) = 4$ and $\mu(S) = 3$.

THEOREM 16. *The following conditions are equivalent:*

1) *S is an irreducible numerical semigroup with $\mathrm{m}(S) = \mu(S) = 3$,*
2) *S is generated by $\{3, x+3, 2x+3\}$ with x a positive integer not a multiple of 3.*

PROOF. *1) $\Rightarrow$ 2)* If $\mathrm{m}(S) = \mu(S) = 3$, then $\{3, n_1, n_2\}$ is a minimal system of generators for S. From Proposition 15 we deduce that $\mathrm{g}(S)$ is even and by Proposition 14 we have that

$$\mathrm{Ap}(S,3) = \{0, n_1 = \frac{\mathrm{g}(S)}{2} + 3, n_2 = \mathrm{g}(S) + 3\}.$$

Taking $x = \frac{\mathrm{g}(S)}{2}$ we have that $n_1 = x+3$ and $n_2 = 2x+3$. Furthermore, $x = \frac{\mathrm{g}(S)}{2} \notin S$ and thus x is not a multiple of 3.

2) $\Rightarrow$ 1) Clearly $\{3, x+3, 2x+3\}$ is a minimal system of generators for S and thus $\mathrm{m}(S) = \mu(S) = 3$. We have that $\mathrm{Ap}(S,3) = \{0, x+3, 2x+3\}$. Hence $2x+3 = \mathrm{g}(S)+3$ and therefore $\frac{\mathrm{g}(S)}{2} + 3 = x+3$. From Proposition 14 we deduce that S is irreducible. $\square$

The semigroup $S = \langle 3, 3+x, 2x+3 \rangle$ is a MED-semigroup (MED stands for Maximal Embedding Dimension, that is a numerical semigroup with $\mu(S) = \mathrm{m}(S)$). Applying the results obtained in **[32]** we deduce that a minimal presentation for S is:

$$\rho = \{(2X_1, X_0 + X_2), (2X_2, xX_0 + X_1), ((x+1)X_0, X_1 + X_2)\}.$$

Now we study irreducible numerical semigroups with multiplicity 4. We distinguish two cases taking into account that the Frobenius number can be odd (a symmetric semigroup) or even (a pseudo-symmetric semigroup).

Herzog proves in **[24]** that a numerical semigroup S with minimal system of generators $\{n_0, n_1, n_2\}$ is irreducible with an odd Frobenius number (i.e. symmetric) if and only if it is a complete intersection. Applying the results obtained in **[19]** this occurs if and only if $n_i \in \langle \frac{n_j}{(n_j,n_k)}, \frac{n_k}{(n_j,n_k)} \rangle$ for some $\{i,j,k\} = \{0,1,2\}$, where (n_j, n_k) denotes the greatest common divisor (gcd for short) of n_j, n_k.

THEOREM 17. *The following conditions are equivalent:*

1) *S is an irreducible numerical semigroup,* $\mathrm{g}(S)$ *is odd,* $\mathrm{m}(S) = 4$ *and* $\mu(S) = 3$,
2) *S is a numerical semigroup generated by* $\{4, 2x, x+2y\}$ *with* $y \in \mathbb{N} \setminus \{0\}$ *and x an odd integer greater than or equal to 3.*

PROOF. *1)* $\Rightarrow$ *2)* If $\mathrm{m}(S) = 4$ and $\mu(S) = 3$, then $\{4, n_1, n_2\}$ is a minimal system of generators for S. From the previous remark we only have distinguish two cases.

a) Assume that $d = \gcd\{4, n_1\}$ and $n_2 \in \langle \frac{4}{d}, \frac{n_1}{d} \rangle$. Notice that $d = 2$ and $n_1 = 2x$ with x an odd number greater than or equal to 3. Furthermore $1 = \gcd\{4, n_1, n_2\}$, then n_2 is an odd number and $n_2 \in \langle 2, x \rangle$ thus $n_2 = x + 2y$ (because all odd numbers in $\langle 2, x \rangle$ are of this kind).
b) Assume that $d = \gcd\{n_1, n_2\}$ and $4 \in \langle \frac{n_1}{d}, \frac{n_2}{d} \rangle$. From here we deduce that $n_1 = 2d$, $n_2 = k_2 d$ with k_2 odd and d an odd integer greater than or equal to 3. Therefore, $n_2 = d + (k_2 - 1)d$ with $(k_2 - 1)d$ even. Taking $x = d$ and $y = \frac{(k_2-1)d}{2}$ we obtain the desired result.

2) $\Rightarrow$ *1)* Clearly, $2 = \gcd\{4, 2x\}$ and $x + 2y \in \langle \frac{4}{2}, \frac{2x}{2} \rangle$. By remark made before this theorem we have that S is an irreducible numerical semigroup with an odd Frobenius number. Now, we need to show that $\{4, 2x, x+2y\}$ is a minimal system of generators for S, but this is clear because:

1) $x+2y \notin \langle 4, 2x \rangle$, since $x+2y$ is odd,
2) $2x \notin \langle 4, x+2y \rangle$, since if $2x = a4 + b(x+2y)$ with $a, b \in \mathbb{N}$, then applying that $2x$ is an even integer not multiple of 4 and that $x+2y$ is odd, we deduce that $b \geq 2$, contradicting that $2(x+2y) > 2x$.

$\square$

The semigroup $S = \langle 4, 2x, x+2y \rangle$ has Frobenius number $\mathrm{g}(S) = 3x+2y-4$. Furthermore using that it is a complete intersection we deduce that a minimal presentation for S is:

$$\rho = \{(2X_1, xX_0), (2X_2, yX_0 + X_1)\}.$$

Finally, we study irreducible numerical semigroups S for which $\mathrm{g}(S)$ is even, $\mathrm{m}(S) = 4$ and $\mu(S) = 3$.

THEOREM 18. *The following conditions are equivalent:*

1) *S is an irreducible numerical semigroup, $\mathrm{g}(S)$ is even, $\mathrm{m}(S) = 4$ and $\mu(S) = 3$,*
2) *S is generated by $\{4, x+2, x+4\}$ with x an odd integer greater than or equal to 3.*

PROOF. *1) $\Rightarrow$ 2)* If $\mathrm{m}(S) = 4$ and $\mu(S) = 3$, then $\{4, n_1, n_2\}$ is a minimal system of generators for S. From Lemma 13 we know that $\frac{\mathrm{g}(S)}{2} + 4 \in \mathrm{Ap}(S, 4)$. We distinguish two cases.

a) If $\frac{\mathrm{g}(S)}{2} + 4$ is a minimal generator then, by Proposition 14, we deduce that
$$\mathrm{Ap}(S,4) = \{0, n_1 = \frac{\mathrm{g}(S)}{2} + 4, n_2, 2n_2 = \mathrm{g}(S) + 4\}.$$
Taking $x = \frac{\mathrm{g}(S)}{2}$, then $n_1 = x+4$ and $n_2 = x+2$. Furthermore $\mathrm{g}(S) \notin S$ and therefore x is odd.

b) If $\frac{g(S)}{2}+4$ is not a minimal generator, then

$$\mathrm{Ap}(S,4)=\{0,n_1,n_2,\frac{g(S)}{2}+4\}.$$

Hence $g(S)+4=n_1$ or $g(S)+4=n_2$. Suppose that $g(S)+4=n_1$ then, by Proposition 14, we deduce that $n_1-n_2\in S$, contradicting that $\{4,n_1,n_2\}$ is a minimal system of generators.

2) $\Rightarrow$ *1)* Clearly, $\{4,x+2,x+4\}$ is a minimal system of generators of S, whence $\mathrm{m}(S)=4$ and $\mu(S)=3$. The reader can check that

$$\mathrm{Ap}(S,4)=\{0,x+2,x+4,2x+4\}.$$

Therefore $g(S)=2x$ and then

$$\mathrm{Ap}(S,4)=\{0,\frac{g(S)}{2}+4,\frac{g(S)+4}{2},g(S)+4\}.$$

Using Proposition 14 we obtain that S is irreducible. □

Note that $S=\langle 4,x+2,x+4\rangle$ has Frobenius number $2x$. Applying **[24]** and that this semigroup is not symmetric (therefore it is not a complete intersection), we can deduce that a minimal presentation for S is:

$$\rho=\{(2X_2,X_0+2X_1),(3X_1,kX_0+X_2),(tX_0,X_1+X_2)\}$$

with $k=\frac{3(x+2)-(x+4)}{4}$ and $t=\frac{(x+4)+(x+2)}{4}$. Observe that $3(x+2)-(x+4)$ is a multiple of 4 if and only if x is odd, and $(x+4)+(x+2)$ is a multiple of 4 if and only if x is odd.

2. Minimal presentations for irreducible numerical semigroups

Our aim in this section is to give an upper bound for the cardinality of a minimal presentation of an irreducible numerical semigroup. We particularize these results for irreducible numerical semigroups with maximal embedding dimension.

Let S be a numerical semigroup with minimal system of generators $\{n_0 < n_1 < \cdots < n_p\}$. In **[32]** it is shown the following result (#MRS denotes the cardinality of a minimal presentation for S).

PROPOSITION 19. *Let S be defined as above. Then*

$$\#MRS \leq \frac{n_0(n_0-1)}{2} - 2(n_0 - 1 - p).$$

In **[31]** this bound is improved in the case S is irreducible with odd Frobenius number. In fact, the following result is given there.

PROPOSITION 20. *If S an irreducible semigroup with odd Frobenius number, $n_0 \geq 3$ and $p \geq 2$, then*

$$\#MRS \leq \frac{(n_0-2)(n_0-1)}{2} - 1 + (p+2-n_0).$$

Now we prove the analogue to this result for S an irreducible semigroup with even Frobenius number.

From **[32]** we can deduce the following result.

PROPOSITION 21. *Let S be an irreducible numerical semigroup with $\mathrm{g}(S)$ even and $p \geq 3$. If $\{n_0, n_1, \ldots, n_p, \mathrm{g}(S)\}$ is a minimal system of generators for $S' = S \cup \{\mathrm{g}(S)\}$, $\mathrm{g}(S) > n_0$ and n_i and n_0 are in the same connected component of $G_{\mathrm{g}(S)+n_0+n_i}$ for all $i \in \{1, \ldots, p\}$, then*

$$\#MRS + p + 2 = \#MRS'.$$

Applying Proposition 14 and using that $p \geq 3$ we deduce that $\mathrm{g}(S) + n_0 \geq n_i + n_j$ for some $i, j \in \{1, \ldots, p\}$ and therefore $\mathrm{g}(S) > n_0$. Furthermore, $\{n_0, n_1, \ldots, n_p, \mathrm{g}(S)\}$ is a minimal system of generators for $S' = S \cup \{\mathrm{g}(S)\}$, since otherwise we would deduce from **[32]** that $n_p = \mathrm{g}(S) + n_0$, which contradicts Proposition 14 for $p \geq 3$.

LEMMA 22. *Let S be an irreducible numerical semigroup with* $\mathrm{g}(S)$ *even and* $p \geq 3$. *If* $i \in \{1,\ldots,p\}$, $w \in \mathrm{Ap}(S,n_0)$ *and* n_0 *and* n_i *are in different connected components of* G_{w+n_i}, *then for all* $w' \in \mathrm{Ap}(S,n_0)$ *such that* $w-w' \in S\setminus\{0\}$ *we have that* $w'+n_i \in \mathrm{Ap}(S,n_0)$

PROOF. Suppose that $w'+n_i \notin \mathrm{Ap}(S,n_0)$, then $w'+n_i-n_0 \in S$. Let $s \in S\setminus\{0\}$ be such that $w = w'+s$ and $j \in \{0,\ldots,p\}$ such that $s-n_j \in S$. Then, $w+n_i-(n_i+n_j) \in S$ and $w+n_i-(n_j+n_0) \in S$. Therefore $\overline{n_in_j}$, $\overline{n_jn_0} \in E_{w+n_i}$ and so n_i and n_0 are in the same connected component of G_{w+n_i}. □

LEMMA 23. *Let S be an irreducible numerical semigroup with* $\mathrm{g}(S)$ *even and* $p \geq 3$. *If* $i \in \{1,\ldots,p\}$, *then* n_0 *and* n_i *are in the same connected component of* $G_{\mathrm{g}(S)+n_0+n_i}$.

PROOF. Suppose that n_0 and n_i are in two different connected components of $G_{\mathrm{g}(S)+n_0+n_i}$. Let $j \in \{1,\ldots,p\}$ be such that $n_j \neq \frac{\mathrm{g}(S)}{2}+n_0$ and $n_i \neq n_j$ (this is possible because $p \geq 3$). By Lemma 22 and Proposition 14 we deduce that $\mathrm{g}(S)+n_0-n_j+n_i \in \mathrm{Ap}(S,n_0)$.

Observe that $\mathrm{g}(S)+n_0-n_j+n_i = \frac{\mathrm{g}(S)}{2}+n_0$, since otherwise using Proposition 14 we would obtain that $\mathrm{g}(S)+n_0-(\mathrm{g}(S)+n_0-n_j+n_i) \in S$ and therefore $n_j-n_i \in S$, contradicting that $\{n_0,\ldots,n_p\}$ is a minimal system of generators for S.

Let us observe that $n_i \neq \frac{\mathrm{g}(S)}{2}+n_0$ because otherwise we would deduce, from $\mathrm{g}(S)+n_0-n_j+n_i = \frac{\mathrm{g}(S)}{2}+n_0$, that $n_j = \mathrm{g}(S)+n_0$ and applying Proposition 14 we can assert that $S = \langle n_0, \frac{\mathrm{g}(S)}{2}+n_0, \mathrm{g}(S)+n_0\rangle$, which contradicts that $p \geq 3$. Now assume that $\mathrm{Ap}(S,n_0) = \{0 = w(1) < \cdots < w(n_0-1)\} \cup \{\frac{\mathrm{g}(S)}{2}+n_0\}$. We distinguish two cases.

1) If $\frac{\mathrm{g}(S)}{2}+n_0 \in \{n_1,\ldots,n_p\}$, then from Proposition 14 and Lemma 22 we have that

$$w(1)+n_i = w(2), w(2)+n_i = w(3), \ldots, w(n_0-2)+n_i = w(n_0-1).$$

Hence,

$$\mathrm{Ap}(S,n_0) = \{0, n_i, 2n_i, \ldots, (n_0-2)n_i\} \cup \{\frac{\mathrm{g}(S)}{2} + n_0\}$$

and thus $S = \langle n_0, n_i, \frac{\mathrm{g}(S)}{2} + n_0 \rangle$, a contradiction because $p \geq 3$.

2) If $\frac{\mathrm{g}(S)}{2} + n_0 \notin \{n_1, \ldots, n_p\}$, then again from Proposition 14 and Lemma 22 we obtain that

$$\mathrm{Ap}(S,n_0) = \{0, n_i, \ldots, kn_i = \frac{\mathrm{g}(S)}{2} + n_0, n_j, n_j + n_i, \ldots, n_j + tn_i = \mathrm{g}(S) + n_0\}$$

for some $k, t \in \mathbb{N}$. Therefore, $S = \langle n_0, n_i, n_j \rangle$, in contradiction again with $p \geq 3$.

□

PROPOSITION 24. *Let S be an irreducible numerical semigroup with $\mathrm{g}(S)$ even and $p \geq 3$. Then*

$$\#MRS \leq \frac{(n_0-2)(n_0-1)}{2} - 1 + (p + 2 - n_0).$$

PROOF. Applying Lemma 23 and Proposition 21 we deduce that $\#MRS = \#MR(S \cup \{\mathrm{g}(S)\}) - (p+2)$. From Proposition 19 we have that

$$\#MR(S \cup \{\mathrm{g}(S)\}) \leq \frac{n_0(n_0-1)}{2} - 2(n_0 - 1 - p - 1).$$

Hence,

$$\#MRS \leq \frac{(n_0-2)(n_0-1)}{2} - 1 + (p + 2 - n_0).$$

□

From Propositions 24 and 20 we can obtain the following result.

THEOREM 25. *If S is an irreducible numerical semigroup with $\mu(S) \geq 4$, then*

$$\#MRS \leq \frac{(\mathrm{m}(S)-2)(\mathrm{m}(S)-1)}{2} - 1 + (\mu(S) + 1 - \mathrm{m}(S)).$$

Note that if $\mu(S) = 2$, then $\#MRS = 1$ and if $\mu(S) = 3$, then $\#MRS = 2$ or 3 depending on the parity of $\mathrm{g}(S)$ (see[**24**]).

A **MEDI-semigroup** (irreducible semigroup with maximal embedding dimension) is an irreducible semigroup with multiplicity $m \geq 5$ and embedding dimension $m-1$. Remember from Proposition 15 that if S is irreducible and $\mathrm{m}(S) \geq 5$, then $\mu(S) \leq \mathrm{m}(S) - 1$ and this is why we use the name MEDI-semigroup.

If $S = \langle \mathrm{m}(S), n_1, \ldots, n_{\mathrm{m}(S)-2} \rangle$ is a MEDI-semigroup, then

$$\mathrm{Ap}(S, \mathrm{m}(S)) = \{0, n_1, \ldots, n_{\mathrm{m}(S)-2}, \mathrm{g}(S) + \mathrm{m}(S)\}.$$

Moreover, from Propositions 12 and 14 we can deduce that $\mathrm{g}(S) + \mathrm{m}(S) = n_i + n_j$ with $i, j \in \{1, \ldots, \mathrm{m}(S) - 2\}$ and $i \neq j$. Applying now [**40**, Theorem 1] we get that

$$\#MRS = \frac{(\mathrm{m}(S)-2)(\mathrm{m}(S)-1)}{2} - 1.$$

Note that for $\mathrm{m}(S) \in \{3, 4\}$, the previous formula is not true (for this reason in the definition of MEDI-semigroup we need that $\mathrm{m}(S) \geq 5$). In fact, for $\mathrm{m}(S) = 3$ applying the previous formula, we have $\#MRS = 0$ but we know that a minimal presentation for $\langle 3, n_1 \rangle$ has cardinality 1. For $\mathrm{m}(S) = 4$ applying the previous formula, we have $\#MRS = 2$ and we know that in this class there are semigroups with minimal presentation of cardinality 3 (see the remark after Theorem 18). If S is a MEDI-semigroup with $\mathrm{g}(S)$ odd, then S is a **MEDSY-semigroup** according to the terminology used in [**31**].

THEOREM 26. *If S is an irreducible numerical semigroup with $\mu(S) \geq 5$, then the following conditions are equivalent:*

1) *S is a MEDI-semigroup,*
2) $\#MRS = \frac{(\mathrm{m}(S)-2)(\mathrm{m}(S)-1)}{2} - 1.$

PROOF. *2)* $\Rightarrow$ *1)* Since $\mu(S) \geq 4$, by Theorem 25 we know that

$$\#MRS \leq \frac{(\mathrm{m}(S)-2)(\mathrm{m}(S)-1)}{2} - 1 + (\mu(S)+1-\mathrm{m}(S)).$$

Since

$$\#MRS = \frac{(\mathrm{m}(S)-2)(\mathrm{m}(S)-1)}{2} - 1,$$

we get that $\mu(S) = \mathrm{m}(S) - 1$ and therefore S is a MEDI-semigroup.

1) $\Rightarrow$ *2)* Proved already (see the beginning of this section). □

The next result appears in [**31**].

LEMMA 27. *Let $A = \{0 = w(1), w(2), \ldots, w(m)\} \subseteq \mathbb{N}$ be a complete system modulo m, and let S be a numerical semigroup generated by $A \cup \{m\}$. Then $Ap(S,m) = A$ if and only if for all $1 \leq i, j \leq m$ there exist $1 \leq k \leq m$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$.*

PROPOSITION 28. *If S is an irreducible numerical semigroup with $\mathrm{m}(S) \geq 5$ and*

$$\mathrm{Ap}(S, \mathrm{m}(S)) = \{0 = w(1) < w(2) < \cdots < w(\mathrm{m}(S))\},$$

then the semigroup S' generated by

$$\{\mathrm{m}(S), w(2) + \mathrm{m}(S), \ldots, w(\mathrm{m}(S)-1) + \mathrm{m}(S)\}$$

is a MEDI-semigroup.

PROOF. In [**31**, Proposition 2.4] it is proved that $\{\mathrm{m}(S), w(2) + \mathrm{m}(S), \ldots, w(\mathrm{m}(S)-1) + \mathrm{m}(S)\}$ is a minimal system of generators for S'. Furthermore, in that proposition it is also shown that if S is symmetric, then S' is MEDSY-semigroup. Therefore it is enough to prove that if S is irreducible with $\mathrm{g}(S)$ even, then S' is irreducible. From Lemma 27 we obtain that

$$\mathrm{Ap}(S', \mathrm{m}(S)) = \{0 < w(2) + \mathrm{m}(S) < \cdots < w(\mathrm{m}(S)-1) + \mathrm{m}(S) < w(\mathrm{m}(S)) + 2\mathrm{m}(S)\}$$

and, by Proposition 14, we get that S' is irreducible. $\square$

As a consequence of the previous proof we have that $g(S') = \mathrm{g}(S) + 2\mathrm{m}(S)$.

PROPOSITION 29. *If S is a MEDI-semigroup with a minimal system of generators $\{\mathrm{m}(S) < n_1 < \cdots < n_{\mathrm{m}(S)-2}\}$, then the semigroup S' generated by $\{\mathrm{m}(S), n_1 - \mathrm{m}(S), \ldots, n_{\mathrm{m}(S)-2} - \mathrm{m}(S)\}$ is irreducible.*

PROOF. In [31, Propositon 2.5] it is proved that if S is a MEDSY-semigroup then S' is symmetric. Therefore, it is enough to prove that if S is a MEDI-semigroup with $\mathrm{g}(S)$ even, then S' is irreducible.

Assume that $n_j = \frac{\mathrm{g}(S)}{2} + \mathrm{m}(S)$ and

$$\mathrm{Ap}(S, \mathrm{m}(S)) = \{0, n_1, \cdots, n_{\mathrm{m}(S)-2}, \mathrm{g}(S) + \mathrm{m}(S) = n_1 + n_{\mathrm{m}(S)-2}\}.$$

Using Lemma 27 it is easy to prove that

$$\mathrm{Ap}(S', \mathrm{m}(S)) = \{0, n_1 - \mathrm{m}(S), \cdots, n_{\mathrm{m}(S)-2} - \mathrm{m}(S), \mathrm{g}(S) - \mathrm{m}(S)\}.$$

From Proposition 14 we conclude that S' is irreducible (note that $g(S') = \mathrm{g}(S) - 2\mathrm{m}(S)$ and $n_j - \mathrm{m}(S) = \frac{g(S')}{2} + \mathrm{m}(S)$). $\square$

Applying Propositions 28 and 29 we obtain the following result.

THEOREM 30. *There is a one to one correspondence between the set of irreducible numerical semigroups with Frobenius number g and multiplicity $m \geq 5$, and the set of MEDI-semigroups with Frobenius number $g + 2m$, multiplicity m and the rest of minimal generators greater than $2m$.*

PROOF. Let $\mathcal{M}(g, m)$ be the set of irreducible numerical semigroups with Frobenius number g and minimum minimal generator m, and let $\mathcal{MEDI}$ $(g+m, m)$ be the set of irreducible numerical semigroups with Frobenius number $g + 2m$, minimum minimal generator m and the rest of minimal generators greater than $2m$.

Let $\phi : \mathcal{M}(g,m) \longrightarrow \mathcal{MEDI}(g+m,m)$ be the map defined by: for $S \in \mathcal{M}(g,m)$ with $\mathrm{Ap}(S,m) = \{0 = w(1),\ldots,w(m-1),w(m)\}$ we make $\phi(S) = \langle m, w(2)+m,\ldots,w(m-1)+m\rangle$. As a consequence of Proposition 28 we have that ϕ is a well defined map and, by Proposition 29, we conclude that ϕ is a bijective map. $\square$

3. Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups

We say that a numerical semigroup is an **ISY-semigroup** if it can be expressed as a finite intersection of symmetric numerical semigroups. We start by proving Theorem 34 which gives a characterization for this kind of semigroup. Later we see that this result can be improved (see Theorem 45) and for this we introduce the concept of pseudo-Frobenius number. We also characterize numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups with the same Frobenius number (ISYG-semigroups) and with the same multiplicity (ISYM-semigroups).

From Theorem 10 and Proposition 11 we deduce the following result.

LEMMA 31. *Let g be an integer number and $\mathcal{S}(g)$ the set of all numerical semigroups with Frobenius number g. Then $S \in \mathcal{S}(g)$ is symmetric if and only if g is odd and S is maximal with respect to set inclusion in $\mathcal{S}(g)$.*

In order to prove Theorem 34 we introduce the following lemmas.

LEMMA 32. *If S is a numerical semigroup and x is an odd positive integer not in S, then there exists a symmetric numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $g(\overline{S}) = x$.*

PROOF. Let $S' = S \cup \{x+1, x+2, \ldots\}$. Clearly, S' is a numerical semigroup and $g(S') = x$. Let $\overline{S}$ be a maximal semigroup in $\mathcal{S}(x)$ such that $S' \subseteq \overline{S}$. By Lemma 31 we can deduce that $\overline{S}$ is symmetric with Frobenius number x and contains S. $\square$

LEMMA 33. *Let S be a numerical semigroup and let x be an even positive integer not in S. Then, the following conditions are equivalent:*

1) *there exists a symmetric semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $x \notin \overline{S}$,*
2) *there exists an odd positive integer y such that $x+y \notin \langle S,y \rangle$.*

PROOF. *1) $\Rightarrow$ 2)* Let $y = g(\overline{S}) - x$. Since x is even and $g(\overline{S})$ is odd, we have that y is odd (note that, by Lemma 31, the Frobenius number of a symmetric semigroup is always odd). Furthermore, $y = g(\overline{S}) - x \in \overline{S}$, since $x \notin \overline{S}$ and $\overline{S}$ is symmetric. Hence, $\langle S,y \rangle \subseteq \overline{S}$ and thus $x+y = g(\overline{S}) \notin \langle S,y \rangle$.

2) $\Rightarrow$ 1) Let $S' = \langle S,y \rangle \cup \{x+y+1, x+y+2, \ldots\}$. Then S' is a numerical semigroup with odd Frobenius number $x+y$. Using Lemma 32 we deduce that there exists a symmetric semigroup $\overline{S}$ such that $S' \subseteq \overline{S}$ and $g(\overline{S}) = x+y$. Then $S \subseteq \overline{S}$ and $x \notin \overline{S}$, because otherwise, since $y \in \overline{S}$, we would obtain that $g(\overline{S}) = x+y \in \overline{S}$, which is impossible. $\square$

THEOREM 34. *Let S be a numerical semigroup. The following conditions are equivalent:*

1) *S is an ISY-semigroup,*
2) *for every even positive integer $x \notin S$, there exists an odd positive integer y such that $x+y \notin \langle S,y \rangle$.*

PROOF. *1) $\Rightarrow$ 2)* Let x be an even positive integer such that $x \notin S$ and let $\overline{S}$ be a symmetric numerical semigroup such that $S \subseteq \overline{S}$ and $x \notin \overline{S}$ (the existence of $\overline{S}$ is guaranteed because S is ISY-semigroup). Applying Lemma 33 we deduce that there exists an odd positive integer y such that $x+y \notin \langle S,y \rangle$.

2) $\Rightarrow$ 1) If x is an odd positive integer such that $x \notin S$, then let S_x be a symmetric numerical semigroup with $S \subseteq S_x$ and $g(S_x) = x$ (Lemma 32 guarantees the existence of S_x). If x is an even positive integer such that $x \notin S$ then, by Lemma 33, we deduce that there exists a symmetric numerical semigroup S_x fulfilling that $S \subseteq S_x$ and $x \notin S_x$. Finally, it is clear that $S = \cap_{x \notin S} S_x$. □

The next result has an immediate proof.

LEMMA 35. *Let $S, S_1, \ldots, S_n$ be numerical semigroups such that $S = S_1 \cap \ldots \cap S_n$. Then* $\mathrm{g}(S) = \max\{g(S_1), \ldots, g(S_n)\}$.

As a consequence of this lemma and from the fact that the Frobenius number of a symmetric numerical semigroup is always odd, we get the next result.

LEMMA 36. *If S is an ISY-semigroup, then* $\mathrm{g}(S)$ *is odd.*

We can see, with the following example, that the converse of this result is not true.

EXAMPLE 37. If $S = \langle 4,5,6,7 \rangle$, then $\mathrm{g}(S) = 3$. Now we see that S is not an ISY-semigroup and for this we use the Theorem 34. In fact, $2 \notin S$ (2 is even) and for every odd positive integer y we have that $2 + y \in \langle S, y \rangle$. □

Arguing as in this example and using Lemma 36, the reader can check the following result.

PROPOSITION 38. *If $m \geq 3$, then $S = \langle m, m+1, \ldots, m+(m-1) \rangle$ is not an ISY-semigroup.*

Note that if S is a numerical semigroup and $\mu(S) \in \{1,2\}$ then S is symmetric (see for instance **[24]**). Then, $\langle 5,7 \rangle \cap \langle 5,8 \rangle = \langle 5,21,24,28,32 \rangle$ is an ISY-semigroup.

Let S be a numerical semigroup. We say that an element of $x \in \mathbb{Z}$ is a **pseudo-Frobenius number** of S if $x \notin S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$. We denote by $\mathrm{Pg}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{Pg}(S)$ is the **type** of S and it is denoted by $\mathrm{type}(S)$. In **[20]** it is proved that a numerical semigroup is symmetric if and only if $\mathrm{type}(S) = 1$ (or equivalently $\mathrm{Pg}(S) = \{\mathrm{g}(S)\}$).

Now, our main goal is to prove Theorem 45 which is an improvement of Theorem 34. The following result is easy to demonstrate.

LEMMA 39. *Let S be a numerical semigroup generated by $\{n_1,\ldots,n_p\}$ and let $x \in \mathbb{Z}$. Then x is a pseudo-Frobenius number of S if and only if $x \notin S$ and $x+n_i \in S$ for all $i \in \{1,\ldots,p\}$.*

Using the previous lemma it is clear that if $S = \langle 5,6,7,8,9\rangle$, then $\mathrm{Pg}(S) = \{1,2,3,4\}$. In general, if $S = \langle m, m+1,\ldots,m+(m-1)\rangle$, then $\mathrm{Pg}(S) = \{1,\ldots,m-1\}$.

Let S be a numerical semigroup, we define in S the following partial order:

$$a \leq_S b \text{ if } b-a \in S.$$

By [**20**, Proposition 7] we deduce the following result.

LEMMA 40. *If S is a numerical semigroup, $n \in S\setminus\{0\}$ and $\{w_{i_1},\ldots,w_{i_t}\} =$ $\mathrm{maximals}_{\leq_S}\mathrm{Ap}(S,n)$, then $\mathrm{Pg}(S) = \{w_{i_1}-n,\ldots,w_{i_t}-n\}$.*

Recall that a MED-semigroup is a numerical semigroup whose multiplicity equals its embedding dimension. From Lemma 40 we get the following result.

LEMMA 41. *Let S be a numerical semigroup. The following conditions are equivalent:*

1) *S is a MED-semigroup,*
2) $\mathrm{type}(S) = \mathrm{m}(S)-1$.

Next we prove that the role that plays $\mathrm{Pg}(S)$ in a numerical semigroup is analogue to the one played by $\mathrm{g}(S)$ when the semigroup is symmetric.

PROPOSITION 42. *Let S be a numerical semigroup, $g_1,\ldots,g_t$ be the pseudo-Frobenius numbers of S and $x \in \mathbb{Z}$. Then $x \notin S$ if and only if $g_i - x \in S$ for some $i \in \{1,\ldots,t\}$*

PROOF. If $x \notin S$ and $n \in S \setminus \{0\}$ then there exists $w \in \mathrm{Ap}(S,n)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $x = w - kn$. Let $\{w_{j1}, \ldots, w_{jt}\} = \mathrm{maximals}_{\leq_S} \mathrm{Ap}(S,n)$ be and let $i \in \{1, \ldots, t\}$ such that $w_{ji} - w \in S$. By Lemma 40 we can assume that $g_i = w_{ji} - n$. Then $g_i - x = w_{ji} - n - (w - kn) = (w_{ji} - w) + (k-1)n \in S$.

Conversely, since $g_i - x \in S$ and $g_i \notin S$ we obtain that $x \notin S$. □

Now we study sufficient conditions for a numerical semigroup to be an ISY-semigroup.

PROPOSITION 43. *Let S be a numerical semigroup whose all pseudo-Frobenius numbers are odd. Then S is an ISY-semigroup.*

PROOF. Suppose that $g_1, \ldots, g_t$ are the pseudo-Frobenius numbers of S. For each $i \in \{1, \ldots, t\}$ let S_{g_i} be a symmetric numerical semigroup such that $S \subseteq S_{g_i}$ and $g(S_{g_i}) = g_i$ (the existence of S_{g_i} follows by Lemma 32). We will see that $S = S_{g_1} \cap \ldots \cap S_{g_t}$. To this purpose it is enough to prove that $S_{g_1} \cap \ldots \cap S_{g_t} \subseteq S$. Assume that $x \notin S$, then by Proposition 42, there exists $i \in \{1, \ldots, n\}$ such that $g_i - x \in S$ and thus $g_i - x \in S_{g_1} \cap \ldots \cap S_{g_t}$. Hence $g_i - x \in S_{g_i}$ and so $x \notin S_{g_i}$ (note that $g_i \notin S_{g_i}$). □

The converse of Proposition 43 is not true in general, as the following example shows.

EXAMPLE 44. Let $S = \langle 5, 21, 24, 28, 32 \rangle = \langle 5, 7 \rangle \cap \langle 5, 8 \rangle$, which is an ISY-semigroup. Then $\mathrm{Ap}(S,5) = \{0, 21, 24, 28, 32\}$ and $\mathrm{maximals}_{\leq_S} \mathrm{Ap}(S,5) = \{21, 24, 28, 32\}$. Using Lemma 40 we obtain that $\mathrm{Pg}(S) = \{16, 19, 23, 27\}$. Note that S has an even pseudo-Frobenius number but it is an ISY-semigroup. □

THEOREM 45. *Let S be a numerical semigroup and let $g_1 \ldots, g_t$ be its pseudo-Frobenius numbers. The following conditions are equivalent:*

1) *S is an ISY-semigroup,*

2) *for all* g_i *even, there exists an odd positive integer* y_i *such that* $g_i + y_i \notin \langle S, y_i \rangle$.

PROOF. *1)* $\Rightarrow$ *2)* It is a consequence of Theorem 34.

2) $\Rightarrow$ *1)* If g_i is even, by Lemma 33, we deduce that there exists a symmetric semigroup S_{g_i} such that $S \subseteq S_{g_i}$ and $g_i \notin S_{g_i}$. The case g_i odd and the proof of $S = S_{g_1} \cap \cdots \cap S_{g_t}$ follows as in Proposition 43. □

As a consequence of the proof of 2) $\Rightarrow$ 1) of the previous theorem we obtain the following result.

COROLLARY 46. *Let* S *be an ISY-semigroup with* $\mathrm{type}(S) = t$. *Then* S *can be expressed as an intersection of* t *symmetric numerical semigroups.*

Now we describe an algorithmic method to express an ISY-semigroup as an intersection of symmetric numerical semigroups. From the proof of 2) $\Rightarrow$ 1) in Theorem 45 it suffices to determine, from a numerical semigroup with odd Frobenius number $\mathrm{g}(S)$, a symmetric numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $g(\overline{S}) = \mathrm{g}(S)$. To this purpose the next result is crucial and has similar proof to the one of [**31**, Lemma 3.2] and it is also contained in the proof of [**20**, Proposition 4].

LEMMA 47. *Let* S *be a non symmetric element of* $\mathcal{S}(g)$ *with* $\mathrm{g}(S) = g$ *odd and set* $h = \max\{x \in \mathbb{N} \mid x \notin S \text{ and } g - x \notin S\}$. *Then* $S \cup \{h\} \in \mathcal{S}(g)$.

Let us consider the sequence of elements in $\mathcal{S}(g)$;

· $S^0 = S$,

· $S^{j+1} = S^j \cup \{h_j\}$, were $h_j = \max\{x \in \mathbb{N} \mid x \notin S^j \text{ and } g - x \notin S^j\}$.

Then there exists $r \in \mathbb{N}$ verifying that $\{x \in \mathbb{N} \mid x \notin S^r \text{ and } g - x \notin S^r\} = \emptyset$. Clearly, S^r is a symmetric numerical semigroup such that $S \subseteq S^r$ and $g(S^r) = g$.

In order to illustrate this method, we give an example.

EXAMPLE 48. Let $S=\langle 5,21,24,28,32\rangle$ be a numerical semigroup. Then $\mathrm{g}(S)=27$. We compute a symmetric numerical semigroup $\overline{S}$ such that $S\subseteq\overline{S}$ and $g(\overline{S})=27$.

Note that $S=\{0,5,10,15,20,21,24,25,26\}\cup\{x\geq 28\}$

$\cdot$ $h_1=\max\{x\in\mathbb{N}\mid x\notin S \text{ and } 27-x\notin S\}=23$ and $S^1=S\cup\{23\}$,

$\cdot$ $h_2=\max\{x\in\mathbb{N}\mid x\notin S_1 \text{ and } 27-x\notin S_1\}=19$ and $S^2=S\cup\{19,23\}$,

$\cdot$ $h_3=\max\{x\in\mathbb{N}\mid x\notin S_2 \text{ and } 27-x\notin S_2\}=18$ and $S^3=S\cup\{18,19,23\}$,

$\cdot$ $h_4=\max\{x\in\mathbb{N}\mid x\notin S_3 \text{ and } 27-x\notin S_3\}=16$ and $S^4=S\cup\{16,18,19,23\}$,

$\cdot$ $h_5=\max\{x\in\mathbb{N}\mid x\notin S_4 \text{ and } 27-x\notin S_4\}=14$ and $S^5=S\cup\{14,16,18,19,23\}$,

$\cdot$ $\{x\in\mathbb{N}\mid x\notin S^5 \text{ and } 27-x\notin S^5\}=\emptyset$. Hence, $\overline{S}=S^5$ is a symmetric numerical semigroup generated by $\{5,14,16,18\}$ with Frobenius number $\mathrm{g}(S)$ containing S.

Now, we express $S=\langle 5,21,24,28,32\rangle$ as an intersection of symmetric numerical semigroups.

Note that the pseudo-Frobenius numbers of S are $g_1=16$, $g_2=19$, $g_3=23$ and $g_4=27$ (see the example before Theorem 45). Note also that $16+7\notin\langle S,7\rangle$ and therefore, by Theorem 45, we obtain that S is an ISY-semigroup. From the proof of 2) $\Rightarrow$ 1) in Theorem 45, we have that $S=S_{16}\cap S_{19}\cap S_{23}\cap S_{27}$.

$\cdot$ S_{16} is a symmetric numerical semigroup which contains $S'=\langle S,7\rangle\cup\{x\geq 24\}$ and $g(S_{16})=g(S')=23$.

$\cdot$ S_{19} is a symmetric numerical semigroup which contains $S'=S\cup\{x\geq 20\}$ and $g(S_{19})=g(S')=19$.

$\cdot$ S_{23} is a symmetric numerical semigroup which contains $S'=S\cup\{x\geq 24\}$ and $g(S_{23})=g(S')=23$.

$\cdot$ S_{27} is a symmetric numerical semigroup which contains $S'=S\cup\{x\geq 28\}=S$ and $g(S_{27})=g(S')=\mathrm{g}(S)=27$.

Using the sequences described after Lemma 47 we obtain that $S_{16}=\langle 5,7\rangle$, $S_{19}=\langle 5,11,12,13\rangle$, $S_{23}=\langle 5,12,14,16\rangle$ and $S_{27}=\langle 5,14,16,18\rangle$. $\square$

REMARK 49. Note that in the preceding example S is expressed as an intersection of four symmetric numerical semigroups, though in Example 44 this same semigroup is expressed as an intersection of only two symmetric numerical semigroups. The algorithmic process described above does not supply the minimal decomposition of an ISY-semigroup.

3.1. ISYG-semigroups. We say that a numerical semigroup is an **ISYG-semigroup** if $S = S_1 \cap \ldots \cap S_r$, where $S_1, \ldots, S_r$ are symmetric numerical semigroups such that $g(S_1) = \cdots = g(S_r) = \mathrm{g}(S)$. In this section we study this kind of semigroups.

LEMMA 50. *Let S be a numerical semigroup with odd Frobenius number g. The following conditions are equivalent:*

1) *S is an ISYG-semigroup,*
2) *for every $x \in \mathbb{Z} \setminus S$, we have that $g \notin \langle S, g-x \rangle$.*

PROOF. *1) $\Rightarrow$ 2)* Take $x \notin S$. Since S is an ISYG-semigroup, there exists a symmetric numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$, $g(\overline{S}) = g$ and $x \notin \overline{S}$. Thus $g - x \in \overline{S}$ and therefore $g \notin \langle S, g-x \rangle$, because $\langle S, g-x \rangle \subseteq \overline{S}$ and $g \notin \overline{S}$.

2) $\Rightarrow$ 1) For $x \in \mathbb{N} \setminus S$, let S_x be a maximal numerical semigroup containing $\langle S, g - x \rangle$ with $g(S_x) = g$. By Lemma 31, we know that S_x is a symmetric numerical semigroup with Frobenius number g and that $x \notin S_x$. It follows that $S = \bigcap_{x \in \mathbb{N} \setminus S} S_x$. □

LEMMA 51. *Let S be a numerical semigroup with* $\mathrm{Pg}(S) = \{g_1, \ldots, g_t\}$ *and* $\mathrm{g}(S) = g$. *Then the following conditions are equivalent:*

1) *for every $x \in \mathbb{Z} \setminus S$, we have that $g \notin \langle S, g-x \rangle$,*
2) *$g \notin \langle S, g - g_i \rangle$ for all $i \in \{1, \ldots, t\}$.*

PROOF. *1) $\Rightarrow$ 2)* It is trivial, because $g_i \in \mathbb{Z} \setminus S$ for all $i \in \{1, \ldots, t\}$.

2) $\Rightarrow$ *1)* If $x \in \mathbb{Z} \setminus S$, then by Proposition 42, we know that there exists $i \in \{1, \ldots, t\}$ such that $g_i - x \in S$. Assume that $s \in S$ is such that $g_i = x + s$. Then, since $g \notin \langle S, g - g_i \rangle = \langle S, g - x - s \rangle \supseteq \langle S, g - x \rangle$, we have that $g \notin \langle S, g - x \rangle$ □

As a consequence of Lemmas 50 and 51 we get the following result.

THEOREM 52. *Let S be a numerical semigroup with odd Frobenius number g and* $\mathrm{Pg}(S) = \{g_1, \ldots, g_t\}$. *The following conditions are equivalent:*

1) *S is an ISYG-semigroup,*
2) $g \notin \langle S, g - g_i \rangle$ *for all* $i \in \{1, \ldots, t\}$.

Assume that S is an ISYG-semigroup and hence it verifies Condition 2) of the previous theorem. We denote by S_{g_i} a symmetric numerical semigroup with Frobenius number g such that $\langle S, g - g_i \rangle \subseteq S_{g_i}$. The existence of S_{g_i} follows by Lemma 31 and furthermore we can construct S_{g_i} using the procedure given after Lemma 47. Then $S = S_{g_1} \cap \ldots \cap S_{g_t}$. In fact, if $x \in \mathbb{Z} \setminus S$, then by Proposition 42, we know that $g_i - x \in S$ for some $i \in \{1, \ldots, t\}$. Hence $g_i - x \in S_{g_i}$, since $S \subseteq S_{g_i}$. Then we can conclude that $g - x \in S_{g_i}$ and thus $x \notin S_{g_i}$.

Note that if $g \notin \langle S, g - g_{i_1}, \ldots, g - g_{i_k} \rangle$ with $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, t\}$, then we can take $S_{g_{i_1}} = S_{g_{i_2}} = \cdots = S_{g_{i_k}}$.

Assume that $t \geq 2$ and $g_t = g$ (recall that $g \in \mathrm{Pg}(S)$). Then using the previous remark we can take $S_{g_t} = S_{g_1}$ and thus $S = S_{g_1} \cap \cdots \cap S_{g_{t-1}}$, whence we have the following result.

COROLLARY 53. *Let S be a non symmetric ISYG-semigroup. Then* $\mathrm{type}(S) = t \geq 3$ *and S can be expressed as an intersection of at most* $t - 1$ *symmetric numerical semigroups with Frobenius number* $\mathrm{g}(S)$.

In [**20**, Theorem 11] the following result is given.

LEMMA 54. *If S is a numerical semigroup with $\mu(S) = 3$, then* $\mathrm{type}(S) \in \{1,2\}$.

As a consequence of Corollary 53 and Lemma 54, we have the following.

COROLLARY 55. *Let S be a numerical semigroup with $\mu(S) = 3$. Then the following conditions are equivalent:*

1) *S is an ISYG-semigroup,*
2) *S is a symmetric numerical semigroup.*

EXAMPLE 56. We prove that $S = \langle 6, 11, 15, 20, 25 \rangle$ is an ISYG-semigroup. Moreover, applying the remark after Theorem 52, we see that S can be expressed as an intersection of symmetric numerical semigroups with Frobenius number equal to 19. Note that

$$\mathrm{Ap}(S,6) = \{0, 11, 15, 20, 22, 25\}$$

and $\mathrm{g}(S) = 19$. Then $\mathrm{maximals}_{\leq_S}\mathrm{Ap}(S,6) = \{15, 20, 22, 25\}$, by Lemma 40, we have that $\mathrm{Pg}(S) = \{g_1 = 9, g_2 = 14, g_3 = 16, g_4 = \mathrm{g}(S) = 19\}$ and therefore $\mathrm{g}(S) - g_1 = 10$, $\mathrm{g}(S) - g_2 = 5$, $\mathrm{g}(S) - g_3 = 3$ and $\mathrm{g}(S) - g_4 = 0$. It is clear that $19 \notin \langle S, 10 \rangle$, $19 \notin \langle S, 5 \rangle$, $19 \notin \langle S, 3 \rangle$ and $19 \notin \langle S, 0 \rangle$. Hence, from Theorem 52, we deduce that S is an ISYG-semigroup.

Note that $19 \notin \langle S, 10, 5, 0 \rangle$, whence we can take $S_{g_1} = S_{g_2} = S_{g_4}$ and this semigroup is symmetric with Frobenius number $g(S_1) = 19$ containing $\langle S, 10, 5, 0 \rangle$. Applying the method given after Lemma 47, we have that $S_{g_1} = S_{g_2} = S_{g_4} = \langle 5, 6 \rangle$.

Note also that S_{g_3} is a symmetric semigroup with $g(S_{g_3}) = 19$ and such that $\langle S, 3 \rangle \subseteq S_{g_3}$. Applying again the previous method we have that $S_{g_3} = \langle 3, 11 \rangle$.

Finally, $S = S_{g_1} \cap S_{g_2} \cap S_{g_3} \cap S_{g_4} = \langle 5, 6 \rangle \cap \langle 3, 11 \rangle$. □

Next we study those numerical semigroups of type 2 that are intersection of symmetric numerical semigroups. Our goal is to prove Theorem 59 which states that the converse of Proposition 43 is true for numerical semigroups of type 2.

LEMMA 57. *Let S be a numerical semigroup, $x,y \in \mathbb{Z}$ and $s \in S$. If $x+y \notin \langle S,y\rangle$ and $x+y+s \notin S$, then $x+y+s \notin \langle S,y+s\rangle$.*

PROOF. If $x+y+s \in \langle S,y+s\rangle$, then there exist $s' \in S$ and $a \in \mathbb{N}$ such that $x+y+s = s' + a(y+s)$. Since $x+y+s \notin S$, $a \neq 0$. Hence $x+y = s' + (a-1)s + ay \in \langle S,y\rangle$ a contradiction. □

LEMMA 58. *Let S be a numerical semigroup, with pseudo-Frobenius numbers $g_1,\ldots,g_t$ and let $y \in \mathbb{Z}$ be such that $g_i + y \notin \langle S,y\rangle$ for some $i \in \{1,\ldots,t\}$. Then there exists $g_j \geq g_i + y$ such that $g_j \notin \langle S, g_j - g_i\rangle$.*

PROOF. Since $g_i + y \notin S$, then by Proposition 42, we deduce that there exists g_j such that $g_j - (g_i + y) \in S$ and so $g_j = g_i + y + s$ for some $s \in S$. Hence, we have that $g_i + y \notin \langle S,y\rangle$, $g_i + y + s \notin S$ and $s \in S$. Using the previous lemma, we obtain that $g_i + y + s \notin \langle S, y+s\rangle$ and therefore $g_j \notin \langle S, g_j - g_i\rangle$ □

Now we can prove the following result.

THEOREM 59. *Let S be a numerical semigroup with* $\mathrm{type}(S) = 2$ *and* $\mathrm{Pg}(S) = \{g_1 < g_2\}$. *The following conditions are equivalent:*

1) *S is an ISY-semigroup,*
2) *g_1 and g_2 are odd.*

PROOF. *2) $\Rightarrow$ 1)* Follows from Proposition 43.

1) $\Rightarrow$ 2) By Lemma 36 we know that $g_2 = \mathrm{g}(S)$ is odd. If g_1 is even , then from Theorem 45, we deduce that there exists an odd number y such that $g_1 + y \notin \langle S,y\rangle$. Using Lemma 58 we obtain that $g_2 \notin \langle S, g_2 - g_1\rangle$. Note that S satisfies the condition 2) of Theorem 52 and so S is an ISYG-semigroup, which contradicts Corollary 53. □

EXAMPLE 60. Using Theorem 59, we deduce that $S = \langle 5,6,7\rangle$ is not ISY-semigroup because $\mathrm{Pg}(S) = \{8,9\}$. Applying again Theorem 59 we have that $S = \langle 5,6,8\rangle$ is an ISY-semigroup since $\mathrm{Pg}(S) = \{7,9\}$. □

3.2. ISYM-semigroups. We say that a numerical semigroup S is an **ISYM-semigroup** if $S = S_1 \cap \cdots \cap S_r$, with $S_1, \ldots, S_r$ symmetric numerical semigroups such that $m(S_1) = \cdots = m(S_r) = \mathrm{m}(S)$.

Suppose that S is a numerical semigroup with $\mathrm{m}(S) \geq 3$. Note that if $\mathrm{m}(S) = 1$, then $\mathbb{N} = S$ and if $\mathrm{m}(S) = 2$ then $S = \langle 2, \mathrm{g}(S)+2\rangle$; in both cases the semigroup S is symmetric.

LEMMA 61. *Let S be a symmetric numerical semigroup with* $\mathrm{m}(S) \geq 3$. *Then* $\mathrm{g}(S) \geq 2\mathrm{m}(S) - 1$.

PROOF. Note that $\mathrm{g}(S)$ is odd and so $\mathrm{g}(S) \geq 3$. If $\mathrm{g}(S) < 2\mathrm{m}(S) - 1$, then there exists $x, y \in \{1, \ldots, \mathrm{m}(S) - 1\}$ such that $x + y = \mathrm{g}(S)$. Applying that S is symmetric we deduce that $x \in S$ or $y \in S$, contradicting that $\mathrm{m}(S) = \min S \setminus \{0\}$. □

LEMMA 62. *If m is an integer greater than or equal to 3, then*

$$S = \langle m, m+1, \ldots, m+(m-2)\rangle$$

is the unique symmetric numerical semigroup with $\mathrm{m}(S) = m$ *and* $\mathrm{g}(S) = 2m - 1$.

PROOF. By definition, it is obvious that S is symmetric. Now we need to show that S is unique. Suppose that $\overline{S}$ is a symmetric semigroup with $m(\overline{S}) = m$ and $g(\overline{S}) = 2m - 1$. We have that $\{1, \ldots, m-1\} \cap \overline{S} = \emptyset$, therefore

$$\{(2m-1)-1, \ldots, (2m-1)-(m-1)\} \subseteq \overline{S}$$

and thus $m, m+1, \ldots, m+(m-2) \in \overline{S}$. Hence we conclude that $S = \overline{S}$. □

LEMMA 63. *Let S be a numerical semigroup such that* $\mathrm{m}(S) \geq 3$ *and* $\mathrm{g}(S)$ *is odd. The following conditions are equivalent:*

1) $\mathrm{g}(S) \geq 2\mathrm{m}(S) - 1$,
2) *there exists a symmetric numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $m(\overline{S}) =$* $\mathrm{m}(S)$.

PROOF. *1) $\Rightarrow$ 2)* Let $\overline{S}$ be the symmetric numerical semigroup obtained from S by using the recurrent method exposed after Lemma 47. Now, it is enough to see that $\mathrm{m}(S) = m(\overline{S})$. In fact,

$$h_j = \max\{x \in \mathbb{N} \mid x \notin S_j \text{ and } \mathrm{g}(S) - x \notin S_j\} > \frac{\mathrm{g}(S)}{2} \geq \frac{2\mathrm{m}(S) - 1}{2}$$

and so $h_j \geq \mathrm{m}(S)$.

2) $\Rightarrow$ 1) Follows from Lemma 61 (note that $\mathrm{g}(S) \geq g(\overline{S})$). □

Using the previous results we can characterize ISYM-semigroups.

THEOREM 64. *Let S be a numerical semigroup with* $\mathrm{m}(S) \geq 3$, $\mathrm{g}(S)$ *odd and* $\mathrm{g}(S) \geq 2\mathrm{m}(S) + 1$. *The following conditions are equivalent:*

1) *S is an ISYM-semigroup,*
2) *for every $x \in \mathbb{N} \setminus S$ with $x > \frac{\mathrm{g}(S)}{2}$, there exists $y \in \mathbb{N}$ such that:*
   i) $x + y \geq 2\mathrm{m}(S) - 1$,
   ii) *$x + y$ is odd,*
   iii) $x + y \notin \langle S, y \rangle$,
   iv) *if $y \neq 0$, then $y \geq \mathrm{m}(S)$.*

PROOF. *1) $\Rightarrow$ 2)* Since S is an ISYM-semigroup, for $x \in \mathbb{N} \setminus S$ there exists a symmetric numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$, $\mathrm{m}(S) = m(\overline{S})$ and $x \notin \overline{S}$. If we choose $y = g(\overline{S}) - x$, then, since $\overline{S}$ is symmetric, $g(\overline{S})$ is odd and $y \in \overline{S}$. Hence ii) and iv) are satisfied. Furthermore, by Lemma 61, i) is fulfilled. Finally, iii) is verified too, since $x+y=g(\overline{S}) \notin \overline{S} \supseteq \langle S,y\rangle$.

2) $\Rightarrow$ 1) Let $\overline{S}$ be a symmetric numerical semigroup obtained from S by using the recurrent method exposed after Lemma 47. In the proof of Lemma 63 we saw that $m(\overline{S}) = \mathrm{m}(S)$. It is clear that if $x \in \overline{S}\setminus S$, then $x \notin S$, $\mathrm{g}(S)-x \notin S$ and $x > \frac{\mathrm{g}(S)}{2}$. We will see that there exists a symmetric numerical semigroup S_x such that $S \subseteq S_x$, $m(S_x) = \mathrm{m}(S)$ and $x \notin S_x$. Let $y \in \mathbb{N}$ verifying i), ii), iii) and iv) and set $S^{'} = \langle S,y\rangle \cup \{x+y+1, x+y+2, \ldots\}$. Then $S^{'}$ is a numerical semigroup with multiplicity $\mathrm{m}(S)$ and Frobenius number $x+y \geq 2\mathrm{m}(S)-1$. Take S_x a symmetric numerical semigroup such that $S^{'} \subseteq S_x$, $m(S_x) = \mathrm{m}(S)$ and $g(S_x) = x+y$ (the proof of 1) $\Rightarrow$ 2) in Lemma 63 guarantees the existence of S_x). Furthermore, $x \notin S_x$, since $y \in S_x$ and $g(S_x) = x+y$. Clearly, $S = \overline{S} \cap (\cap_{x\in\overline{S}\setminus S} S_x)$ and therefore S is an ISYM-semigroup. □

Finally we illustrate the previous results with some examples.

EXAMPLE 65. $S = \langle 5,6,8,9\rangle$ is a numerical semigroup with $\mathrm{m}(S)=5$ and $\mathrm{g}(S)=7$. As $\mathrm{g}(S) < 2\mathrm{m}(S)-1$, then by Lemma 63 we obtain that S is not an ISYM-semigroup. □

EXAMPLE 66. $S = \langle 6,11,15,20,25\rangle$ is a numerical semigroup with $\mathrm{m}(S)=6$ and $\mathrm{g}(S)=19$. Taking $x=16$, we have that $16 \in \mathbb{N}\setminus S$, $19-6=3 \notin S$ and $16 > \frac{19}{2}$. It is clear that the unique natural number y such that $x+y$ is odd and $x+y \notin \langle S,y\rangle$ is $y=3$. Hence S is not an ISYM-semigroup because the condition iv) of Theorem 64 is not satisfied for $y=3$. □

EXAMPLE 67. $S = \langle 5,21,24,28,32\rangle$ is a numerical semigroup with $\mathrm{m}(S)=5$ and $\mathrm{g}(S)=27$. It is easy to see that $\{x \geq 14 \mid x \notin S \text{ and } 27-x \notin S\} = \{14,16,18,19,23\}$. Taking $x=14$ we can use $y=5$ which verifies conditions i), ii), iii) and iv) of Theorem 64. Analogously, for $x=16$ we can use $y=7$, for $x=18$ we can use $y=5$ and for

$x = 23$ we can use $y = 0$. Hence, by Theorem 64 we deduce that S is an ISYM-semigroup. Note that S can be expressed as an intersection of symmetric numerical semigroups with multiplicity 5, for this we apply the method that it is deduced from the proof of 2) $\Rightarrow$ 1) in Theorem 64. $\square$

4. Decomposition of a numerical semigroup as an intersection of irreducible numerical semigroups

We know that every numerical semigroup S admits a decomposition $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible (that is, S_i is symmetric or pseudo-symmetric) for all i and we denote by $\mathrm{r}(S)$ the least positive integer n. Our aim in this section is to give an upper bound and a lower bound for $\mathrm{r}(S)$. We also describe an algorithm for computing a minimal decomposition of a numerical semigroup into irreducibles.

We assume that $S \neq \mathbb{N}$ and therefore $\mathrm{Pg}(S) \subseteq \mathbb{N}$ (see Lemma 40).

LEMMA 68. *If S is a numerical semigroup and $x \in \mathbb{N} \setminus S$, then there exists an irreducible numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$ and $g(\overline{S}) = x$.*

PROOF. Let $S' = S \cup \{x+1, x+2, \ldots\}$. It is clear that S' is a numerical semigroup with $g(S') = x$. Let $\overline{S}$ be a maximal element in the set of all numerical semigroups with Frobenius number x containing S'. From Theorem10 we deduce that $\overline{S}$ is an irreducible numerical semigroup. $\square$

LEMMA 69. *Let $S_1, \ldots, S_n$ be numerical semigroups containing S. The following conditions are equivalent:*

1) $S = S_1 \cap \ldots \cap S_n$,

2) if $g' \in \mathrm{Pg}(S)$, then there exists $i \in \{1, \ldots, n\}$ such that $g' \notin S_i$.

PROOF. *1) $\Rightarrow$ 2)* As $g' \notin S = S_1 \cap \cdots \cap S_n$, then there exist $i \in \{1, \ldots n\}$ such that $g' \notin S_i$.

2) $\Rightarrow$ 1) It is enough to prove that if $x \in \mathbb{N} \setminus S$, then there exists $i \in \{1,\ldots,n\}$ such that $x \notin S_i$. Suppose that $x \notin S$, from Proposition 42, we obtain that there exists $g' \in \mathrm{Pg}(S)$ such that $g' - x \in S$. By hypothesis we can find $i \in \{1,\ldots,n\}$ such that $g' \notin S_i$ and since $g' - x \in S \subseteq S_i$ we obtain that $x \notin S_i$. □

As a consequence of [**31**, Theorem 3.3] we obtain the following result.

LEMMA 70. *If S is a numerical semigroup, then there exists $B \subseteq \{x \in \mathbb{N} : x > \frac{\mathrm{g}(S)}{2}\}$ such that $S \cup B$ is an irreducible numerical semigroup and $\mathrm{g}(S \cup B) = \mathrm{g}(S)$.*

Let S be a numerical semigroup. Define

$$\mathrm{BPg}(S) = \{g' \in \mathrm{Pg}(S) \mid g' > \frac{\mathrm{g}(S)}{2}\}.$$

THEOREM 71. *Let S be a numerical semigroup with $\mathrm{BPg}(S) = \{g_1,\ldots,g_r\}$. Then there exist $S_1,\ldots,S_r$ irreducible numerical semigroups such that $S = S_1 \cap \cdots \cap S_r$ and $\mathrm{g}(S_i) = g_i$ for all $i \in \{1,\ldots,r\}$.*

PROOF. Suppose that $g_1 = \mathrm{g}(S)$ and S_1 is the irreducible numerical semigroup described in Lemma 70. For each $i \in \{2,\ldots,r\}$, let S_i be an irreducible numerical semigroup such that $S \subseteq S_i$ and $g(S_i) = g_i$ (the existence of S_i is guaranteed by Lemma 68). Now for proving that $S = S_1 \cap \cdots \cap S_r$ we use Lemma 69. If $g' \in \mathrm{Pg}(S)$ and $g' \leq \frac{\mathrm{g}(S)}{2}$, then $g' \notin S_1$. If $g' \in \mathrm{Pg}(S)$ and $g' > \frac{\mathrm{g}(S)}{2}$, then $g' = g_i$ for some $i \in \{1,\ldots,r\}$ and therefore $g' \notin S_i$. □

From [**20**] we can deduce that if S is an irreducible numerical semigroup, then

$$\mathrm{Pg}(S) = \begin{cases} \{\mathrm{g}(S)\} & \text{if } \mathrm{g}(S) \text{ is odd,} \\ \{\mathrm{g}(S), \frac{\mathrm{g}(S)}{2}\} & \text{if } \mathrm{g}(S) \text{ is even.} \end{cases}$$

From this remark and Theorem 71 we obtain the following result.

COROLLARY 72. *If S is a numerical semigroup, then the following conditions are equivalent:*

1) *S is irreducible,*

2) $\#\mathrm{BPg}(S) = 1$.

Let S be a numerical semigroup. Recall that $\mathrm{r}(S)$ is the smallest positive integer n such that $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible numerical semigroups for all $i \in \{1, \ldots, n\}$. As a consequence of Theorem 71 we have the following result.

COROLLARY 73. *If S is a numerical semigroup, then* $\mathrm{r}(S) \leq \#\mathrm{BPg}(S)$.

The decomposition given in Theorem 71 is not minimal as the following example illustrates.

EXAMPLE 74. Let $S = \langle 5,7 \rangle \cap \langle 5,8 \rangle = \langle 5,21,24,28,32 \rangle$. Then $\mathrm{Ap}(S,5) = \{0,21,24,28,32\}$, using Lemma 40 we get $\mathrm{Pg}(S) = \{16,19,23,27\}$ and so $\#\mathrm{BPg}(S) = 4$. Note that a numerical semigroup generated by two elements is symmetric (see [**24**]) and thus $S = \langle 5,7 \rangle \cap \langle 5,8 \rangle$ is a decomposition of S as an intersection of irreducibles. □

COROLLARY 75. *If S is a numerical semigroup such that* $\#\mathrm{BPg}(S) = 2$, *then* $\mathrm{r}(S) = 2$.

PROOF. If $\#\mathrm{BPg}(S) = 2$, then by Corollary 72 we have that S is not an irreducible numerical semigroup and thus $\mathrm{r}(S) \geq 2$. Besides, applying Corollary 73 we get that $\mathrm{r}(S) \leq 2$. Hence we have that $\mathrm{r}(S) = 2$. □

Note that, from Example 74, we can see that the converse of Corollary 75 is not true. But there are many semigroups verifying the hypothesis of Corollary 75 as we see in the following example.

EXAMPLE 76. Let m a positive integer greater than or equal to 3 and let $S = (\{x \in \mathbb{N} \mid x \geq m\} \setminus \{2m-2, 2m-1\}) \cup \{0\}$. The reader can prove that S is a numerical semigroup and $\mathrm{Pg}(S) = \{2m-2, 2m-1\}$. Applying Corollary 75 we get that $\mathrm{r}(S) = 2$. $\square$

Now we give a lower bound for $\mathrm{r}(S)$. Suppose that S is a numerical semigroup and $\mathrm{BPg}(S) = \{g_1, \ldots, g_r\}$. For each $i \in \{1, \ldots, r\}$, define

$$\xi(g_i) = \{g_i + x \mid x \in \mathbb{N} \text{ and } g_i + x \notin \langle S, x\rangle\}.$$

THEOREM 77. *Let S be a numerical semigroup with $\mathrm{Pg}(S) = \{g_1, \ldots, g_r\}$ and let $g_i \in \{g_1, \ldots, g_r\}$. If $\overline{S}$ is an irreducible numerical semigroup such that $S \subseteq \overline{S}$ and $g_i \notin \overline{S}$, then $g(\overline{S}) \in \xi(g_i)$. Conversely, if $g_i + x \in \xi(g_i)$ then there exists an irreducible numerical semigroup $\overline{S}$ such that $S \subseteq \overline{S}$, $g_i \notin \overline{S}$ and $g(\overline{S}) = g_i + x$.*

PROOF. If $g_i \notin \overline{S}$, then by Proposition 11 we get that $g(\overline{S}) - g_i \in \overline{S}$ (note that $g(\overline{S}) \le \mathrm{g}(S)$ and that $g_i > \frac{\mathrm{g}(S)}{2}$ and therefore $g_i \neq \frac{g(\overline{S})}{2}$). Since $g_i + (g(\overline{S}) - g_i) = g(\overline{S}) \notin \overline{S} \supseteq \langle S, g(\overline{S}) - g_i\rangle$ we obtain that $g(\overline{S}) \in \xi(g_i)$.

Conversely, if $g_i + x \in \xi(g_i)$, then $g_i + x \notin \langle S, x\rangle$. Let $\overline{S}$ be an irreducible numerical semigroup such that $\langle S, x\rangle \subseteq \overline{S}$ and $g_i + x = g(\overline{S})$ (the existence of $\overline{S}$ is guaranteed by Lemma 68). Since $x \in \overline{S}$ and $g_i + x = g(\overline{S}) \notin \overline{S}$, we obtain that $g_i \notin \overline{S}$. $\square$

COROLLARY 78. *If $S = S_1 \cap \cdots \cap S_n$ with $S_1, \ldots, S_n$ irreducible numerical semigroups, then for each $i \in \{1, \ldots, r\}$ there exists $j \in \{1, \ldots, n\}$ such that $g(S_j) \in \xi(g_i)$.*

PROOF. If $i \in \{1, \ldots, r\}$, then $g_i \notin S = S_1 \cap \cdots \cap S_n$ and therefore there exists $j \in \{1 \ldots, n\}$ such that $g_i \notin S_j$. Using Theorem 77 we get that $g(S_j) \in \xi(g_i)$. $\square$

COROLLARY 79. *Let $x_1, \ldots, x_r \in \mathbb{N}$ be such that $g_i + x_i \in \xi(g_i)$ for all $i \in \{1, \ldots, r\}$. Then there exist irreducible numerical semigroups $S_1, \ldots, S_r$ such that $S = S_1 \cap \cdots \cap S_r$ and $\{g(S_1), \ldots, g(S_r)\} \subseteq \{g_1 + x_1, \ldots, g_r + x_r\}$.*

PROOF. Assume that $g_1 = \mathrm{g}(S)$. Note that $\xi(g_1) = \{g_1\}$ and thus $x_1 = 0$. Let S_1 be the numerical semigroup $S \cup B$ described in Lemma 70. Now, for each $i \in \{2, \ldots, r\}$ let

S_i be an irreducible numerical semigroup such that $S \subseteq S_i$, $g_i \notin S_i$ and $g(S_i) = g_i + x_i$ (the existence of S_i is guaranteed by Theorem 77). Applying Lemma 69 we can deduce that $S = S_1 \cap \cdots \cap S_r$. $\square$

Let A be a subset of $\mathbb{N}$. We say that S is an A – semigroup if S can be expressed as an intersection of irreducible numerical semigroups whose Frobenius numbers are in A (that is, $S = S_1 \cap \cdots \cap S_n$ with S_i irreducible numerical semigroups and $g(S_i) \in A$ for all $i \in \{1, \ldots, n\}$). Denote by $\mathrm{h}(S) = \min\{\#A \mid S \text{ is an } A - \text{semigroup}\}$.

COROLLARY 80. *If A is a subset of $\mathbb{N}$, then the following conditions are equivalent:*

1) *S is an A-semigroup,*

2) *there exist $(a_1, \ldots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)$ such that $\{a_1, \ldots, a_r\} \subseteq A$.*

PROOF. *1) $\Rightarrow$ 2)* This is a consequence of Corollary 78.

2) $\Rightarrow$ 1) Follows from Corollary 79. $\square$

COROLLARY 81. *If S is a numerical semigroup, then* $\mathrm{r}(S) \geq \mathrm{h}(S) = \min\{\#\{a_1, \ldots, a_r\} \mid (a_1, \ldots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)\}$.

PROOF. As a consequence of Corollary 80 we get that

$$\mathrm{h}(S) = \min\{\#\{a_1, \ldots, a_r\} \mid (a_1, \ldots, a_r) \in \xi(g_1) \times \cdots \times \xi(g_r)\}.$$

Now we see that $\mathrm{r}(S) \geq \mathrm{h}(S)$. In fact, if $S_1, \ldots, S_n$ are irreducible numerical semigroups such that $S = S_1 \cap \cdots \cap S_n$, then S is a $\{g(S_1), \ldots, g(S_n)\}$ – semigroup and thus $n \geq \#\{g(S_1), \ldots, g(S_n)\} \geq \mathrm{h}(S)$. Hence $\mathrm{r}(S) \geq \mathrm{h}(S)$. $\square$

Note that if we take again $S = \langle 5,7 \rangle \cap \langle 5,8 \rangle = \langle 5,21,24,28,32 \rangle$ (see Example 74) we know that $\mathrm{r}(S) = 2$. Remember that $\mathrm{BPg}(S) = \{16,19,23,27\}$ and so $\xi(16) = \{16,23\}$, $\xi(19) = \{19,27\}$, $\xi(23) = \{23\}$ and $\xi(27) = \{27\}$. Applying Corollary 81, we obtain that $\mathrm{h}(S) = 2$ and therefore $\mathrm{h}(S) = \mathrm{r}(S)$. Note that there are many examples for which the previous equality does not hold. Observe that if S_1 and S_2 are irreducible numerical semigroups with $g(S_1) = g(S_2)$, then $r(S_1 \cap S_2) = 2$ and $\mathrm{h}(S_1 \cap S_2) = 1$.

4.1. Odd and even numerical semigroups. We say that a numerical semigroup is an **odd numerical semigroup** (respectively **even numerical semigroup**) if it can be expressed as an intersection of irreducible numerical semigroups with odd (respectively even) Frobenius numbers.

Note that odd (respectively even) numerical semigroups are numerical semigroups that are intersection of symmetric (respectively pseudo-symmetric) numerical semigroups. If $S, S_1, \ldots, S_n$ are numerical semigroups and $S = S_1 \cap \cdots \cap S_n$, then $\mathrm{g}(S) = \max\{g(S_1), \ldots, g(S_n)\}$ and therefore if S is an odd (respectively even) numerical semigroup, then $\mathrm{g}(S)$ is odd (respectively even). Note also that every numerical semigroup is odd, even, or an intersection of an odd and an even numerical semigroup.

As a consequence of Corollary 80 we get the following result that is a generalization and an improvement Theorem 45.

COROLLARY 82. *If S is a numerical semigroup and* $\mathrm{BPg}(S) = \{g_1, \ldots, g_r\}$, *then the following conditions are equivalent:*

1) *S is an odd (respectively even) numerical semigroup,*

2) *$\xi(g_i)$ contains at least an odd (respectively even) element for all $i \in \{1, \ldots, r\}$.*

Note that a numerical semigroup is a $\{g\}$ – semigroup if $S = S_1 \cap \cdots \cap S_n$ with S_i an irreducible numerical semigroup and $g(S_i) = g$ for all $i \in \{1, \ldots, n\}$. Observe that S is a $\{g\}$ – semigroup if only if $\mathrm{h}(S) = 1$.

As an immediate consequence of Corollary 80 we obtain the following result.

COROLLARY 83. *If S is a numerical semigroup and* $\mathrm{BPg}(S) = \{g_1, \ldots, g_r\}$, *then the following conditions are equivalent:*

1) *S is a* $\{\mathrm{g}(S)\}$ – semigroup,

2) $g(S) \in \xi(g_i)$ *for all* $i \in \{1,\ldots,r\}$.

4.2. Atoms. Let g be a positive integer. Set

$$\mathcal{L}(g) = \{S \mid S \text{ is a numerical semigroup with } g(S) = g\}$$

and

$$\widehat{\mathcal{L}(g)} = \{S \mid S \text{ is a numerical semigroup with } g(S) \leq g\}.$$

Note that $(\widehat{\mathcal{L}(g)}, \cap)$ is a semigroup and, as a consequence of Theorem 71, the set of irreducible numerical semigroups of $\widehat{\mathcal{L}(g)}$ is a minimal system of generators for it.

Note also that $\mathcal{L}(g)$ is a subsemigroup of $(\widehat{\mathcal{L}(g)}, \cap)$. An element in $\mathcal{L}(g)$ is an **atom** if it is not an intersection of two elements of $\mathcal{L}(g)$ containing it properly. Note that an irreducible numerical semigroup of $\mathcal{L}(g)$ is an atom, but in general the converse is not true (see Example 89).

LEMMA 84. *Let S and $\overline{S}$ be two numerical semigroups with $S \subset \overline{S}$ and let $x =$ max $(\overline{S} \setminus S)$. Then $S \cup \{x\}$ is a numerical semigroup.*

In particular if S and $\overline{S} \in \mathcal{L}(g)$, then $S \cup \{x\} \in \mathcal{L}(g)$.

PROOF. From the definition of x we obtain that $2x \in S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$. Hence $S \cup \{x\}$ is a numerical semigroup. Since $x \in \overline{S}$, $x \neq g(\overline{S}) = g$ and thus $g(S \cup \{x\}) = g$. □

LEMMA 85. *If $S \in \mathcal{L}(g)$ and S is not an atom of $\mathcal{L}(g)$, then there exist $x_1, x_2 \in \mathbb{N} \setminus S$ such that $x_1 \neq x_2$ and $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are elements of $\mathcal{L}(g)$.*

PROOF. If S is not an atom, then there exist $S_1, S_2 \in \mathcal{L}(g)$ such that $S \subset S_1$ and $S \subset S_2$ and $S = S_1 \cap S_2$. Assume that $x_i = \max(S_i \setminus S)$ for $i = 1, 2$. Applying Lemma 84 we obtain that $S \cup \{x_1\}, S \cup \{x_2\} \in \mathcal{L}(g)$. Note that $x_1 \neq x_2$ because otherwise we would have $x_1 = x_2 \in S_1 \cap S_2 = S$, which contradicts $x_1 \notin S$. □

LEMMA 86. *Let S be a numerical semigroup and $x \in \mathbb{N} \setminus S$. Then $S \cup \{x\}$ is a numerical semigroup if only if $x \in \mathrm{Pg}(S)$ and $2x \notin \mathrm{Pg}(S)$.*

PROOF. If $S \cup \{x\}$ is a numerical semigroup, then $x+s \in S$ for all $s \in S \setminus \{0\}$ and thus $x \in \mathrm{Pg}(S)$. Furthermore $2x \in S$ and whence $2x \notin \mathrm{Pg}(S)$.

Conversely, if $x \in \mathrm{Pg}(S)$, then $x+s \in S$ for all $s \in S \setminus \{0\}$. If $2x \notin \mathrm{Pg}(S)$ then, since $x \in \mathrm{Pg}(S)$, we can deduce that $2x \in S$. Hence $S \cup \{x\}$ is a numerical semigroup. □

PROPOSITION 87. *If $S \in \mathcal{L}(g)$, then the following conditions are equivalent:*

1) *S is not an atom of $\mathcal{L}(g)$,*

2) *there exist $x_1, x_2 \in \mathrm{Pg}(S) \setminus \{g\}$ such that $x_1 \neq x_2$ and $\{2x_1, 2x_2\} \cap \mathrm{Pg}(S) = \emptyset$.*

PROOF. *1) $\Rightarrow$ 2)* By Lemma 85 we know that there exist $x_1, x_2 \in \mathbb{N}$ such that $x_1 \neq x_2$ and $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are elements of $\mathcal{L}(g)$. Using Lemma 86 and the fact that $g \notin S \cup \{x_1\}$ and $g \notin S \cup \{x_2\}$, we deduce that $x_i \in \mathrm{Pg}(S) \setminus \{g\}$ and $2x_i \notin \mathrm{Pg}(S)$ for $i = 1, 2$.

2) $\Rightarrow$ 1) From Lemma 86 we deduce that $S \cup \{x_1\}$, $S \cup \{x_2\} \in \mathcal{L}(g)$. Since $S = (S \cup \{x_1\}) \cap (S \cup \{x_2\})$, we have that S is not an atom of $\mathcal{L}(g)$ □

As an immediate consequence of the previous proposition we get the following result.

COROLLARY 88. *If S is an numerical semigroup and* $\mathrm{type}(S) \in \{1, 2\}$*, then S is an atom of $\mathcal{L}(\mathrm{g}(S))$.*

EXAMPLE 89. Let $S = \langle 4, 5, 11 \rangle$. Then $\mathrm{Pg}(S) = \{6, 7\}$ (see Example 76) and therefore $\mathrm{type}(S) = 2$. Applying the previous corollary, we get that S is an atom of $\mathcal{L}(7)$. Note also that S is not irreducible because, using Lemma 86, we have that $S \cup \{6\}$ and $S \cup \{7\}$ are numerical semigroups for which $S = (S \cup \{6\}) \cap (S \cup \{7\})$. □

4.3. Computing minimal decompositions. We finish this section by describing an algorithm for computing a minimal decomposition of a numerical semigroup into irreducibles.

For a numerical semigroup S, we define

$$\mathrm{H}(S) = \mathbb{N} \setminus S \text{ and } \mathrm{EH}(S) = \{x \in H(S) \mid 2x \in S,\ x+s \in S \text{ for all } s \in S \setminus \{0\}\}.$$

And from this definition, it easy to prove the following result.

PROPOSITION 90. *Let S be a numerical semigroup and $x \in \mathrm{H}(S)$. Then $x \in \mathrm{EH}(S)$ if only if $S \cup \{x\}$ is a numerical semigroup.*

The set $\mathrm{EH}(S)$ is a subset of $\mathrm{Pg}(S) = \{x \notin S \mid x+s \in S \text{ for all } s \in S \setminus \{0\}\}$. Remember that, from Lemma 40, we have that $\mathrm{Pg}(S) = \{w_{i_1} - n, \ldots, w_{i_t} - n\}$ with $\{w_{i_1}, \ldots, w_{i_t}\} = \mathrm{maximals}_{\leq_S} \mathrm{Ap}(S,n)$ and that the cardinal of $\mathrm{Pg}(S)$ was called the type of S and denoted by $\mathrm{type}(S)$. Hence, this implies that

$$\#\mathrm{EH}(S) \leq \mathrm{type}(S) \leq \mathrm{m}(S) - 1.$$

As a consequence of Lemma 84, we can see that all numerical semigroups that contain properly the numerical semigroup S must contain an element of $\mathrm{EH}(S)$. In fact S is maximal in the set of all numerical semigroups not cutting $\mathrm{EH}(S)$. From, Theorem 10, we know that S is irreducible if only if S is maximal in the set of all numerical semigroups that do not contain $\mathrm{g}(S)$. So we have the next result.

COROLLARY 91. *Let S be a numerical semigroup. Then S is irreducible if only if $\#\mathrm{EH}(S) = 1$.*

Given two numerical semigroups S and $\overline{S}$ such that $S \subset \overline{S}$ for $n \in \mathbb{N}$, define recursively the semigroup S_n as:

- $S_0 = S$,

- $S_{n+1} = \begin{cases} S_n, & \text{if } S_n = \overline{S}, \\ S_n \cup \{\max(\overline{S} \setminus S_n)\}, & \text{otherwise.} \end{cases}$

If $k = \#(\overline{S} \setminus S)$, then we get the sequence

$$S = S_0 \subset S_1 \subset \cdots \subset S_k = \overline{S}.$$

As a consequence of Lemma 84, we deduce that for all $i \in \{0, \ldots k\}$ S_i is a numerical semigroup. Then we can compute all numerical semigroups $\overline{S}$ containing S. The idea is to proceed as follows: once you have an element S' containing S (you start with $S' = S$), compute $\mathrm{EH}(S') = \{x_1, \ldots, x_r\}$ and thus we obtain $S' \cup \{x_1\}, \ldots, S' \cup \{x_r\}$ which are numerical semigroups containing S; next do the same with each $S' \cup \{x_i\}$. Performing this process as many times as necessary we get all numerical semigroups containing S. Denote by

$$\mathfrak{I}(S) = \{\overline{S} \text{ irreducible} \mid S \subset \overline{S}\}.$$

Its clear that $S = S_1 \cap \cdots \cap S_n$ with $S_i \in \mathfrak{I}(S)$. We can remove those irreducibles that are not minimal with respect to inclusion. As an immediate consequence of this remark we obtain the following result.

PROPOSITION 92. *Let S be a numerical semigroup such that* $\mathrm{minimals}_{\subseteq}\mathfrak{I}(S) = \{S_1, \ldots, S_r\}$. *Then $S = S_1 \cap \ldots \cap S_r$.*

But the decomposition described above is not necessary minimal as we can see in the following example.

EXAMPLE 93. Let $S = \langle 5, 6, 8 \rangle$. We compute $\mathrm{EH}(S) = \{7, 9\}$, by Proposition 90, $S \cup \{7\}$ and $S \cup \{9\}$ are numerical semigroups. For $S \cup \{7\}$ we have that $EH(S \cup \{7\}) = \{9\}$, by Corollary 91, we conclude that $S \cup \{7\})$ is irreducible and thus $S \cup \{7\} \in \mathrm{minimals}_{\subseteq}\mathfrak{I}(S)$. For $S \cup \{9\}$ we have $EH(S \cup \{9\}) = \{3, 4, 7\}$. Using again Proposition 90, we obtain that $S \cup \{9, 3\}$, $S \cup \{9, 4\}$ and $S \cup \{9, 7\}$ are numerical semigroups. Now we deduce that $S \cup \{9, 3\}$, $S \cup \{9, 4\}$ are irreducibles and $S \cup \{9, 7\}$

contains the irreducible $S \cup \{7\}$. Therefore we get that

$$\text{minimals}_{\subseteq}\mathfrak{I}(S) = \{(S \cup \{7\}, S \cup \{9,3\}, S \cup \{9,4\}\}.$$

Hence

$$S = (S \cup \{7\}) \cap (S \cup \{9,3\}) \cap (S \cup \{9,4\}) = (S \cup \{7\}) \cap (S \cup \{9,3\}).$$

□

However, to express a numerical semigroup S as intersection of the least possible number of irreducibles, it suffices to search among the decompositions with elements in $\text{minimals}_{\subseteq}\mathfrak{I}(S)$.

PROPOSITION 94. *Let S be a numerical semigroup. If $S = S_1 \cap \cdots \cap S_r$ with $S_i \in \mathfrak{I}(S)$, then there exists $S_i' \in \text{minimals}_{\subseteq}\mathfrak{I}(S)$ ($i \in \{1, \ldots, r\}$) such that $S = S_1' \cap \cdots \cap S_r'$.*

PROOF. For each $S_i \in \mathfrak{I}(S)$, we take $S_i' \in \text{minimals}_{\subseteq}\mathfrak{I}(S)$ such that $S_i' \subseteq S_i$ and so $S = S_1' \cap \cdots \cap S_r'$. □

The next result sheds some light on which semigroups are required in a decomposition (compare with Lemma 69).

PROPOSITION 95. *Let S be a numerical semigroup and $S_1, \ldots, S_r \in \mathfrak{I}(S)$. The following conditions are equivalent:*

1) $S = S_1 \cap \cdots \cap S_r$,
2) *for each $h \in \mathrm{EH}(S)$ there exists $i \in \{1, \ldots, r\}$ such that $h \notin S_i$.*

PROOF. *1) ⇒ 2)* If $h \in \mathrm{EH}(S)$ then h is not in S and so there exist $i \in \{1, \ldots, r\}$ such that $h \notin S_i$.

2) ⇒ 1) It is clear that $S \subseteq S_1 \cap \cdots \cap S_r$. Suppose that $S \subset S_1 \cap \cdots \cap S_r$. This implies, by Lemma 84, $S \cup \{\max(\cap S_i \setminus S)\}$ is a numerical semigroup and, by Proposition 90,

$h = \{\max(\cap S_i \setminus S)\} \in \mathrm{EH}(S)$. We have that $h \in \mathrm{EH}(S)$ and $h \in S_i$ for all $\{1,\ldots,r\}$ which contradict the hypothesis. □

From the set $\text{minimals}_{\subseteq}\mathfrak{I}(S) = \{S_1,\ldots,S_r\}$ we define for each S_i with $i \in \{1,\ldots,r\}$

$$C(S_i) = \{h \in \mathrm{EH}(S) \mid h \notin S_i\}.$$

Using Proposition 95 we have that

$$S = S_{i_1} \cap \cdots \cap S_{i_r} \text{ if and only if } C(S_{i_1}) \cup \cdots \cup C(S_{i_r}) = \mathrm{EH}(S).$$

With the above results, we can obtain a algorithm for computing a decomposition of S as an intersection of irreducible semigroups using the least possible number of them.

ALGORITHM 96. Let S be a non-irreducible semigroup.

(1) Compute the set $\mathrm{EH}(S)$.

(2) Set $I = \emptyset$ and $C = \{S\}$.

(3) For all $S' \in C$, compute (using Proposition 90) all the semigroups $\overline{S}$ such that $\#(\overline{S} \setminus S') = 1$. Remove S' from C. Let B be the set formed by the semigroups constructed in this way.

(4) Remove from B the semigroups S' fulfilling that $\mathrm{EH}(S) \subseteq S'$.

(5) Remove from B the semigroups S' such that there exists $\tilde{S} \in I$ with $\tilde{S} \subseteq S'$.

(6) Set $C = \{S' \in B \mid S' \text{ is not irreducible}\}$.

(7) Set $I = I \cup \{S' \in B \mid S' \text{ is irreducible}\}$.

(8) If $C \neq \emptyset$, go to Step 3.

(9) For every $S \in I$, compute $C(S)$.

(10) Choose $\{S_1,\ldots,S_r\}$ such that r is minimum fulfilling that

$$C(S_1) \cup \cdots \cup C(S_r) = \mathrm{EH}(S).$$

(11) Return $S_1,\ldots,S_r$. □

Now we illustrate the above method with an example.

EXAMPLE 97. We consider again the semigroup $S = \langle 5,6,8\rangle$. We have that $\mathrm{EH}(S) = \{7,9\}$. Performing the steps of the above algorithm, we get (in the Steps 6 and 7) that $I = \{\langle 5,6,7,8\rangle\}$ and $C = \{\langle 5,6,8,9\rangle\}\}$. Since $C \neq \emptyset$, we go back to Step 3 obtaining that $I = \{\langle 5,6,7,8\rangle, \langle 3,5\rangle, \langle 4,5,6\rangle\}$ and $C = \emptyset$. Step 8 yields

$$C(\langle 5,6,7,8\rangle) = \{9\},\ C(\langle 3,5\rangle) = \{7\},\ C(\langle 4,5,6\rangle) = \{7\}.$$

The minimal decompositions of S are

$$S = \langle 5,6,7,8\rangle \cap \langle 3,5\rangle$$

and

$$S = \langle 5,6,7,8\rangle \cap \langle 4,5,6\rangle.$$

□

5. Irreducible numerical semigroups with arbitrary multiplicity and embedding dimension

In this section we study families of irreducible numerical semigroups with even conductor. Furthermore, we give a minimal presentation for all semigroups in these families.

Let S be a numerical semigroup and $n \in S \setminus \{0\}$. From Lemma 13 and Proposition 14, we will derive the Lemmas 98, 100 and 102 which give families of irreducible numerical semigroups with even conductor.

LEMMA 98. *Let m, $q \in \mathbb{N}$ be such that $m \geq 2q+5$ and let S be the submonoid of $(\mathbb{N},+)$ generated by*

$$\{m, m+1, (q+1)m+q+2, \ldots, (q+1)m+m-q-3, (q+1)m+m-1\}.$$

Then S is an irreducible numerical semigroup with $\mathrm{m}(S) = m$, $\mu(S) = m-2q-1$ and $\mathrm{g}(S) = 2(q+1)m-2$.

PROOF. Since $\gcd\{m,m+1\}=1$, then we have that S generates $\mathbb{Z}$ as a group and therefore S is a numerical semigroup. Note that $m=\min S\setminus\{0\}$ and so $\mathrm{m}(S)=m$. It is easy to see that

$$\{n_0=m, n_1=m+1, n_2=(q+1)m+q+2,\ldots,$$

$$n_{p-1}=(q+1)m+m-q-3, n_p=(q+1)m+m-1\}$$

is a minimal system of generators for S and thus $\mu(S)=m-2q-1$. The reader can prove that

$$\mathrm{Ap}(S,m)=\{0,n_1,2n_1,\ldots,(q+1)n_1,n_2,\ldots,n_{p-1},n_1+n_{p-1},2n_1+n_{p-1},\ldots,$$

$$qn_1+n_{p-1},\mathrm{g}(S)+m=(q+1)n_1+n_{p-1}\}\cup\{n_p\},$$

and if $p\geq 4$, then in addition $\mathrm{g}(S)+m=n_i+n_{p-i}$ for all $i\in\{2,\ldots,\lceil p/2\rceil\}$ ($\lceil q\rceil$ denotes the integer part of the rational number q). Hence, $\mathrm{g}(S)=2(q+1)m-2$ and so $\frac{\mathrm{g}(S)}{2}+m=(q+1)m+(m-1)=n_p$. Applying Proposition 14 we get that S is an irreducible numerical semigroup. □

We give an example that illustrates the previous lemma and its proof.

EXAMPLE 99. We take $q=2$ and $m=11$ (note that $m\geq 2q+5$), then by the previous lemma, we have that $S=\langle 11,12,37,38,39,43\rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S)=11$, $\mu(S)=6$ and $\mathrm{g}(S)=64$. Furthermore, from the proof of this lemma, we obtain that

$$\mathrm{Ap}(S,11)=\{0,12,24,36,37,38,39,51,63,75\}\cup\{43\}.$$

□

LEMMA 100. *Let $m\in\mathbb{N}$ and $q\in\mathbb{N}\setminus\{0\}$ be such that $m\geq 2q+4$ and let S be the submonoid of $(\mathbb{N},+)$ generated by*

$$\{m,m+1,qm+2q+3,\ldots,qm+m-1,(q+1)m+q+2\}.$$

Then S is an irreducible numerical semigroup with $\mathrm{m}(S) = m$, $\mu(S) = m - 2q$ *and* $\mathrm{g}(S) = 2qm + 2q + 2$.

PROOF. Since $\gcd\{m, m+1\} = 1$, then we have that S generates $\mathbb{Z}$ as a group and therefore S is a numerical semigroup. Note also that $m = \min S \setminus \{0\}$ and so $\mathrm{m}(S) = m$. Clearly,

$$\{n_0 = m, n_1 = m+1, n_2 = qm + 2q + 3, \ldots,$$
$$n_{p-1} = qm + (m-1), n_p = (q+1)m + q + 2\}$$

is a minimal system of generators for S and so $\mu(S) = m - 2q$. The reader can prove that

$$\mathrm{Ap}(S,m) = \{0, n_1, 2n_1, \ldots, qn_1, n_2, \ldots, n_{p-1}, n_p, n_1 + n_p, 2n_1 + n_p, \ldots,$$
$$\mathrm{g}(S) + m = qn_1 + n_p\} \cup \{(q+1)n_1\},$$

and $\mathrm{g}(S) + m = n_i + n_{p-i+1}$ for all $i \in \{2, \ldots, \lceil (p+1)/2 \rceil\}$. Then $\mathrm{g}(S) = 2qm + 2q + 2$ and thus $\frac{\mathrm{g}(S)}{2} + m = (q+1)m + q + 1 = (q+1)n_1$. Using Proposition 14, we deduce that S is an irreducible numerical semigroup. □

We also give an example to illustrate the above lemma.

EXAMPLE 101. Let $q = 2$ and $m = 11$ (note that $m \geq 2q + 4$). Then, by the above lemma, we have that $S = \langle 11, 12, 29, 30, 31, 32, 37 \rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S) = 11$, $\mu(S) = 7$ and $\mathrm{g}(S) = 50$. Furthermore, from its proof, we obtain that

$$\mathrm{Ap}(S, 11) = \{0, 12, 24, 29, 30, 31, 32, 37, 49, 61\} \cup \{36\}.$$

□

LEMMA 102. *If m is a positive integer greater than or equal to 4, then there exists an irreducible numerical semigroup S with* $\mathrm{g}(S)$ *even,* $\mathrm{m}(S) = m$ *and* $\mu(S) = 3$.

PROOF. We distinguish two cases depending on the parity of m.

1) If m is even, then $m = 2q+4$ for some $q \in \mathbb{N}$. Let $S = \langle m, m+1, (q+1)m + (m-1)\rangle$. It is clear that $\mathrm{m}(S) = m$ and $\mu(S) = 3$. The reader can prove that

$$\mathrm{Ap}(S,m) = \{0, m+1, 2(m+1), \ldots, (m-2)(m+1)\} \cup \{(q+1)m + (m-1)\}.$$

Therefore, $\mathrm{g}(S) = (m-2)m - 2$ is even and $\frac{\mathrm{g}(S)}{2} + m = (q+1)m + (m-1)$. By Proposition 14 we conclude that S is an irreducible numerical semigroup.

2) If m is odd, then $m = 2q+3$ for some $q \in \mathbb{N}\setminus\{0\}$. Let $S = \langle m, m+1, (q+1)m + q+2\rangle$. Clearly, $\mathrm{m}(S) = m$ and $\mu(S) = 3$. In this setting,

$$\mathrm{Ap}(S,m) = \{0, m+1, 2(m+1), \ldots, q(m+1), (q+1)m + q + 2,$$

$$(m+1) + (q+1)m + q + 2, \ldots, q(m+1) + (q+1)m + q + 2\} \cup \{(q+1)(m+1)\}.$$

Hence, $\mathrm{g}(S) = (2q+1)m - 1$ is even and $\frac{\mathrm{g}(S)}{2} + m = (q+1)(m+1)$. By Proposition 14, we have that S is an irreducible numerical semigroup. □

REMARK 103. *1)* As a consequence of the proof of case 1) in Lemma 102 and since $m = 2q+4$, we have that if m is an even integer greater than or equal to 4, then $S = \langle m, m+1, \frac{m^2-2}{2}\rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S) = m$, $\mathrm{g}(S) = (m-2)m - 2$ and $\mu(S) = 3$.

2) As a consequence of the proof of case 2) in Lemma 102 and since $m = 2q+3$, we have that if m is an odd integer greater than or equal to 5, then $S = \langle m, m+1, \frac{m^2+1}{2}\rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S) = m$, $\mathrm{g}(S) = (m-2)m - 1$ and $\mu(S) = 3$.

EXAMPLE 104. $S = \langle 6, 7, 17\rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S) = 6$, $\mu(S) = 3$ and $\mathrm{g}(S) = 22$. Furthermore,

$$\mathrm{Ap}(S,6) = \{0, 7, 14, 21, 28\} \cup \{17\}.$$

□

EXAMPLE 105. $S = \langle 7, 8, 25 \rangle$ is an irreducible numerical semigroup with $\mathrm{m}(S) = 7$, $\mu(S) = 3$ and $\mathrm{g}(S) = 34$. Furthermore

$$\mathrm{Ap}(S, 7) = \{0, 8, 16, 25, 33, 41\} \cup \{24\}.$$

□

We are ready to prove the main result of this section.

THEOREM 106. *Let m and e positive integers such that $3 \le e \le m-1$. Then there exists an irreducible numerical semigroup with even conductor, multiplicity m and embedding dimension e.*

PROOF. If $e = 3$, then Lemma 102 guarantees the existence of this semigroup. Thus, in sequel, we shall assume that $4 \le e \le m-1$. We distinguish two cases.

· If $m - e$ is odd, then there exists $q \in \mathbb{N}$ such that $m - e = 2q + 1$. Furthermore, since $e \ge 4$, then $m \ge 2q + 5$. By Lemma 98, we deduce that there exists an irreducible numerical semigroup S with $\mathrm{g}(S)$ even, $\mathrm{m}(S) = m$ and $\mu(S) = m - 2q - 1 = e$.

· If $m - e$ is even, then there exists $q \in \mathbb{N} \setminus \{0\}$ such that $m - e = 2q$. Furthermore, since $e \ge 4$, then $m \ge 2q + 4$. By Lemma 100, we deduce that there exists an irreducible numerical semigroup S with $\mathrm{g}(S)$ even, $\mathrm{m}(S) = m$ and $\mu(S) = m - 2q = e$. □

Now we describe minimal presentations for the families of numerical semigroups obtained from Lemmas 98, 100 and 102. Note that the family of numerical semigroups described in Lemma 102 is (see remark 103):

1) $S = \langle m, m+1, \frac{m^2-2}{2} \rangle$, if m is an even positive integer greater than or equal to 4.

2) $S = \langle m, m+1, \frac{m^2+1}{2} \rangle$, if m is an odd positive integer greater than or equal to 4.

In both cases S is a non symmetric numerical semigroup with $\mu(S) = 3$. Using the results of **[24]**, we deduce that the cardinality of a minimal presentation for these numerical semigroups is 3. Furthermore, from this paper, we know that a minimal presentation for a non symmetric numerical semigroup $S = \langle n_0, n_1, n_2 \rangle$ is

$$\rho = \{(c_0X_0, a_{01}X_1 + a_{02}X_2), (c_1X_1, a_{10}X_0 + a_{12}X_2), (c_2X_2, a_{20}X_0 + a_{21}X_1)\}$$

where $c_i = \min\{l \in \mathbb{N} \setminus \{0\} \mid ln_i \in \langle n_j, n_k \rangle\}$ with $\{i, j, k\} = \{0, 1, 2\}$ and $c_in_i = a_{ij}n_j + a_{ik}n_k$.

In order to find minimal presentations for the semigroups belonging to the families given in the preceding section, we must introduce and recall some concepts and results.

If S is a numerical semigroup with minimal system of generators $\{n_0 < \ldots < n_p\}$ and $s \in S$, then there exists $(a_0, \ldots, a_p) \in \mathbb{N}^{p+1}$ such that $s = a_0n_0 + \cdots + a_pn_p$. We say that an element s has **unique expression** when $(a_0, \ldots, a_p)$ is unique.

In **[34]** it is given a method to obtain a minimal presentation for a numerical semigroup fulfilling the condition that all the elements of $\mathrm{Ap}(S, n_0)$ have unique expression. The process is the following: let

$$T = \{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1n_1 + \cdots + a_pn_p \notin \mathrm{Ap}(S, n_0)\}$$

and

$$\{\alpha_1 = (\alpha_{11}, \ldots, \alpha_{1p}), \ldots, \alpha_t = (\alpha_{t1}, \ldots \alpha_{tp})\} = \mathrm{minimals}_{\leq}(T),$$

where $\leq$ is the usual order of $\mathbb{N}^p$ (observe that by Dickson's Lemma this set is finite). For every $i \in \{1, \ldots, t\}$ we define $x_i = 0X_0 + \alpha_{i1}X_1 + \cdots + \alpha_{ip}X_p \in F$. Since $\varphi(x_i) \notin \mathrm{Ap}(S, n_0)$ (recall the definition of φ and F given in the Preliminaries), we deduce that there exists $(\beta_{i_0}, \beta_{i_1}, \ldots, \beta_{i_p}) \in \mathbb{N}^{p+1}$ with $\beta_{i_0} \neq 0$ such that

$$\varphi(x_i) = \beta_{i_0}n_0 + \beta_{i_1}n_1 + \cdots + \beta_{i_p}n_p.$$

For every $i \in \{1, \ldots, t\}$ we define $y_i = \beta_{i_0}X_0 + \beta_{i_1}X_1 + \cdots + \beta_{i_p}X_p$. Note that $\varphi(x_i) = \varphi(y_i)$ for all $i \in \{1, \ldots, t\}$ and so

$$\rho = \{(x_1, y_1), \ldots, (x_t, y_t)\} \subseteq \sigma.$$

In **[34]** it is proved the following result.

PROPOSITION 107. *Under the standing hypothesis,* $\rho = \{(x_1,y_1),\ldots,(x_t,y_t)\}$ *is a minimal presentation for S.*

Now, with these results, we can give a minimal presentation for the family of numerical semigroups obtained from Lemma 102 (or Remark 103).

PROPOSITION 108. 1) *If m is an even positive integer greater than or equal to 4, then a minimal presentation for* $S = \langle m, m+1, \frac{m^2-2}{2}\rangle$ *is*

$$\rho = \left\{(\frac{m+2}{2}X_0, X_1+X_2), ((m-1)X_1, \frac{m}{2}X_0+X_2), (2X_2, X_0+(m-2)X_1)\right\}.$$

2) *If m is an odd positive integer greater than or equal to 5, then a minimal presentation for* $S = \langle m, m+1, \frac{m^2+1}{2}\rangle$ *is*

$$\rho = \left\{(mX_0, \frac{m-1}{2}X_1+X_2), (\frac{m+1}{2}X_1, X_0+X_2), (2X_2, (m-1)X_0+X_1)\right\}.$$

PROOF. *1)* Let $S = \langle n_0 = m, n_1 = m+1, n_2 = \frac{m^2-2}{2}\rangle$. By the proof of case 1) in Lemma 102, we have that

$$\mathrm{Ap}(S, n_0) = \{0, n_1, 2n_1, \ldots, (n_0-2)n_1, n_2\}.$$

It is clear that all elements in $\mathrm{Ap}(S,n_0)$ have a unique expression. Applying Proposition 107 the reader can check that

$$\rho = \left\{(\frac{m+2}{2}X_0, X_1+X_2), ((m-1)X_1, \frac{m}{2}X_0+X_2), (2X_2, X_0+(m-2)X_1)\right\}$$

is a minimal presentation for S.

2) Let $S = \langle n_0 = m, n_1 = m+1, n_2 = \frac{m^2+1}{2}\rangle$. By the proof of case 2) in Lemma 102, we have that

$$\mathrm{Ap}(S, n_0) = \{0, n_1, 2n_1, \ldots, (q+1)n_1, n_2, n_1+n_2, \ldots, qn_1+n_2\}.$$

Clearly, all elements in $\mathrm{Ap}(S,n_0)$ have again unique expression. Using Proposition 107 again the reader can check that

$$\rho = \left\{(mX_0, \frac{m-1}{2}X_1+X_2), (\frac{m+1}{2}X_1, X_0+X_2), (2X_2, (m-1)X_0+X_1)\right\}$$

is a minimal presentation for S. □

We illustrate this proposition with some examples.

EXAMPLE 109. If we take $m = 6$, then a minimal presentation for the numerical semigroup $S = \langle 6, 7, 17\rangle$ is

$$\rho = \{4X_0, X_1 + X_2), (5X_1, 3X_0 + X_2), (2X_2, X_0 + 4X_1)\}.$$

□

EXAMPLE 110. If we take $m = 7$, then a minimal presentation for the numerical semigroup $S = \langle 7, 8, 25\rangle$ is

$$\rho = \{7X_0, 3X_1 + X_2), (4X_1, X_0 + X_2), (2X_2, 6X_0 + X_1)\}.$$

□

Now we describe the minimal presentations for the families of numerical semigroups obtained from Lemmas 98 and 100.

PROPOSITION 111. *Let $m, q \in \mathbb{N}$ be such that $m \geq 2q + 5$ and*

$$S = \langle n_0 = m, n_1 = m + 1, n_2 = (q+1)m + q + 2,$$
$$\ldots, n_{p-1} = (q+1)m + m - q - 3, n_p = (q+1)m + m - 1\rangle.$$

The cardinality of a minimal presentation for S is equal to

$$\frac{p(p+1)}{2} - 1.$$

PROOF. By Lemma 98, we obtain that

$$\mathrm{Ap}(S, n_0) = \{0, n_1, 2n_1, \ldots, (q+1)n_1, n_2, \ldots, n_{p-1}, n_p, n_1 + n_{p-1}, 2n_1 + n_{p-1},$$
$$\ldots, qn_1 + n_{p-1}, (q+1)n_1 + n_{p-1} = n_2 + n_{p-2} = n_3 + n_{p-3} \cdots\}.$$

Note that all the elements the in $\mathrm{Ap}(S,n_0)$, except

$$g(S)+n_0=(q+1)n_1+n_{p-1}=n_2+n_{p-2}=n_3+n_{p-3}\cdots,$$

have a unique expression. It easy to see that $S'=S\cup\{g(S)\}$ is a numerical semigroup with a minimal system of generators $\{n_0,n_1,\ldots,n_p,n_{p+1}=g(S)\}$ and

$$\mathrm{Ap}(S',n_0)=\{0,n_1,2n_1,\ldots,(q+1)n_1,n_2,\ldots,n_{p-1},n_p,n_1+n_{p-1},\\ \ldots,qn_1+n_{p-1},n_{p+1}\}.$$

Since all the elements in $\mathrm{Ap}(S',n_0)$ have unique expression, using Proposition 107, we can compute a minimal presentation ρ' for S'. Then we have that

$$\rho'=\{((q+2)X_1,y_1),(X_1+X_2,y_2),\ldots,((q+1)X_1+X_{p-1},y_{p-1}),(X_1+X_p,y_p),\\ (X_1+X_{p+1},y_{p+1}),(2X_2,y_{p+2}),(X_2+X_3,y_{p+3}),\ldots,(X_2+X_{p+1},y_{2p+1}),\\ \ldots,(2X_p,y_{p+1+p+\cdots+3+1}),(X_p+X_{p+1},y_{p+1+p+\cdots+3+2}),(2X_{p+1},y_{p+1+p\cdots+3+2+1})\}.$$

Therefore,

$$\#\rho'=p+1+p+\cdots 3+2+1=\frac{(p+2)(p+1)}{2}=\frac{p(p+1)}{2}+p+1.$$

By Proposition 21 and Lemma 23, we obtain that, if ρ is a minimal presentation for S, then $\#\rho+p+2=\#\rho'$. Hence,

$$\#\rho=\frac{p(p+1)}{2}-1.$$

□

EXAMPLE 112. If we take $q=0$ and $m=5$, then $S=\langle 5,6,7,9\rangle$. Using the previous proposition we get that the cardinality of any minimal presentation for S is 5. □

PROPOSITION 113. *Let $m\in\mathbb{N}$ and $q\in\mathbb{N}\setminus\{0\}$ be such that $m\geq 2q+4$ and*

$$S=\langle n_0=m,n_1=m+1,n_2=qm+2q+3,\ldots,n_{p-1}=qm+m-1,\\ n_p=(q+1)m+q+2\rangle.$$

Then the cardinality of a minimal presentation for S is equal to

$$\frac{p(p+1)}{2}-1.$$

PROOF. By Lemma 100, we deduce that

$$\mathrm{Ap}(S,n_0)=\{0,n_1,2n_1,\ldots,(q+1)n_1,n_2,\ldots,n_{p-1},n_p,n_1+n_p,2n_1+n_p,$$
$$\ldots,(q-1)n_1+n_p,qn_1+n_p=n_2+n_{p-1}=n_3+n_{p-2}\cdots\}.$$

Note also that all the elements in $\mathrm{Ap}(S,n_0)$, except

$$\mathrm{g}(S)+n_0=qn_1+n_p=n_2+n_{p-1}=n_3+n_{p-2}\cdots,$$

have unique expression. Clearly, $S'=S\cup\{\mathrm{g}(S)\}$ is a numerical semigroup with minimal system of generators $\{n_0,n_1,\ldots,n_p,n_{p+1}=\mathrm{g}(S)\}$ and

$$\mathrm{Ap}(S',n_0)=\{0,n_1,2n_1,\ldots,(q+1)n_1,n_2,\ldots,n_p,n_1+n_p,2n_1+n_p,$$
$$\ldots,(q-1)n_1+n_p,n_{p+1}\}.$$

Since all the elements in $\mathrm{Ap}(S',n_0)$ have unique expression, by Proposition 107, we have that a minimal presentation ρ' for S', is

$$\rho'=\{((q+2)X_1,y_1),(X_1+X_2,y_2),\ldots,(X_1+X_{p-1},y_{p-1}),(qX_1+X_p,y_p),$$
$$(X_1+X_{p+1},y_{p+1}),(2X_2,y_{p+2}),(X_2+X_3,y_{p+3}),\ldots,(X_2+X_{p+1},y_{p+1+p}),$$
$$\ldots,(2X_p,y_{p+1+p+\cdots+3+1}),(X_p+X_{p+1},y_{p+1+p+\cdots+3+2}),(2X_{p+1},y_{p+1+p\cdots+3+2+1})\}.$$

Hence,

$$\#\rho'=p+1+p+\cdots 3+2+1=\frac{(p+2)(p+1)}{2}=\frac{p(p+1)}{2}+p+1.$$

Using again Proposition 21 and Lemma 23, we obtain that if ρ is a minimal presentation for S, then $\#\rho+p+2=\#\rho'$. Hence,

$$\#\rho=\frac{p(p+1)}{2}-1.$$

□

EXAMPLE 114. We take $q = 1$ and $m = 6$. Then $S = \langle 6, 7, 11, 15 \rangle$. Applying the previous proposition we obtain that the cardinality of any minimal presentation for S is 5. $\square$

CHAPTER 3

Systems of inequalities and numerical semigroups

Let S be a numerical semigroup with multiplicity m and $\mathrm{Ap}(S,m) = \{0 = w(0), k_1m+1 = w(1), \ldots, k_{m-1}m+m-1 = w(m-1)\}$. Then for every $i,j \in \{1,\ldots,m-1\}$ there exist $t \in \mathbb{N}$ and $k \in \{0,\ldots,m-1\}$ such that $w(i)+w(j) = tm+w(k)$. Using this fact, in this chapter, we describe a one-to-one correspondence between the set of numerical semigroups with multiplicity m and a subsemigroup of $\mathbb{N}^{m-1}$.

In Section 1, we study the set of nonnegative solutions of systems of linear Diophantine equations. We see that these solutions can be described with a finite set of parameters and the coefficients of these can be computed algorithmically.

In Section 2, we deduce that there is a one-to-one correspondence between the set $\mathcal{S}(m)$ of numerical semigroups with multiplicity m and the set of nonnegative integer solutions of a system of linear Diophantine inequalities. As a consequence of these results, this correspondence infers in $\mathcal{S}(m)$ a semigroup structure with the resulting semigroup isomorphic to a subsemigroup of $\mathbb{N}^{m-1}$.

In Sections 3 and 4, we particularize the previous results to MED-semigroups and to symmetric numerical semigroups. In the symmetric case, the systems that appear also contain linear equations, and the set of symmetric numerical semigroups is a union of sets of nonnegative integer solutions of systems of this type.

We say that S has **monotonic Apéry set** if $\mathrm{Ap}(S,m) = \{0 < w(1) < \ldots < w(m-1)\}$. Denote by $C(m)$ the set of numerical semigroups with monotonic Apéry set and multiplicity m. Our main goal , in Section 5, is to study this particular case of numerical semigroups. We show that there is a one-to-one correspondence, between $C(m)$ and a finitely generated subsemigroup of $\mathbb{N}^{m-1}$. Finally we study the set of symmetric numerical semigroups of $C(m)$.

1. Nonnegative integer solutions to Diophantine linear inequalities

Our aim in this section is to describe the set of nonnegative integer solutions of systems of linear inequalities and equations with integer coefficients. Assume that we are given the system

$$(1)\qquad \left\{\begin{array}{lcl} a_{1_1}x_1+\cdots+a_{1_n}x_n & \geq & b_1, \\ & \vdots & \\ a_{r_1}x_1+\cdots+a_{r_n}x_n & \geq & b_r, \\ a_{r+1_1}x_1+\cdots+a_{r+1_n}x_n & = & b_{r+1}, \\ & \cdots & \\ a_{r+t_1}x_1+\cdots+a_{r+t_n}x_n & = & b_{r+t}, \end{array}\right.$$

with $a_{i_j}, b_j \in \mathbb{Z}$. In order to solve it we will use the following supplementary systems of linear Diophantine equations.

$$(2)\qquad \left\{\begin{array}{lcl} a_{1_1}x_1+\cdots+a_{1_n}x_n-x_{n+1} & = & b_1, \\ & \vdots & \\ a_{r_1}x_1+\cdots+a_{r_n}x_n-x_{n+r} & = & b_r, \\ a_{r+1_1}x_1+\cdots+a_{r+1_n}x_n & = & b_{r+1}, \\ & \cdots & \\ a_{r+t_1}x_1+\cdots+a_{r+t_n}x_n & = & b_{r+t}, \end{array}\right.$$

and

$$(3)\qquad \left\{\begin{array}{lcl} a_{1_1}x_1+\cdots+a_{1_n}x_n-x_{n+1}-b_1x_{n+r+1} & = & 0, \\ & \vdots & \\ a_{r_1}x_1+\cdots+a_{r_n}x_n-x_{n+r}-b_rx_{n+r+1} & = & 0, \\ a_{r+1_1}x_1+\cdots+a_{r+1_n}x_n-b_{r+1}x_{n+r+1} & = & 0, \\ & \cdots & \\ a_{r+t_1}x_1+\cdots+a_{r+t_n}x_n-b_{r+t}x_{n+r+1} & = & 0, \end{array}\right.$$

The variables $x_{n+1},\ldots,x_{n+r}$ are usually known in the literature as slack variables (see for instance **[13]** and the references there). The set of nonnegative integer solutions of (3) is a monoid and it is generated by its set of nonzero minimal elements with respect to the usual partial order $\leq$ in $\mathbb{N}^{n+r+1}$ (this set is finite in view of Dickson's lemma; see for instance [**41**]).

The following result is straightforward to prove.

LEMMA 115. *The element $(s_1,\ldots,s_n) \in \mathbb{N}^n$ is a solution of (1) if and only if there exists $s_{n+1},\ldots,s_{n+r} \in \mathbb{N}$ such that $(s_1,\ldots,s_n,s_{n+1},\ldots,s_{n+r})$ is solution to (2).*

From [**42**, Section 4] we deduce the following result.

LEMMA 116. *Let $A = \{\alpha_1,\ldots,\alpha_t\}$, with $\alpha_i = (\alpha_{i_1},\ldots,\alpha_{i_{n+r+1}})$, be a system of generators of the Diophantine monoid given by the set of nonnegative solutions of (3). Assume that $\alpha_i,\ldots,\alpha_d$ are the elements in A having its last coordinate equal to zero and that $\alpha_{d+1},\ldots,\alpha_g$ are those elements in A with the last coordinate equal to one. Then the set of nonnegative solutions of (2) is*

$$\{\overline{\alpha}_{d+1},\ldots,\overline{\alpha}_g\} + \langle \overline{\alpha}_1,\ldots,\overline{\alpha}_d\rangle,$$

where $\overline{\alpha}_i = (\alpha_{i_1},\ldots,\alpha_{i_{n+r}})$.

PROPOSITION 117. *Let $\{\alpha_1,\ldots,\alpha_t\}$, $\{\alpha_1,\ldots,\alpha_d\}$ and $\{\alpha_{d+1},\ldots,\alpha_g\}$ be as in Lemma 116, and let $\pi : \mathbb{N}^{n+r+1} \to \mathbb{N}^n$ be the projection onto the first n coordinates. Then the set of nonnegative integer solutions of (1) is*

$$\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\} + \langle \pi(\alpha_1),\ldots,\pi(\alpha_d)\rangle.$$

PROOF. Let $(s_1,\ldots,s_n) \in \mathbb{N}^n$ be a solution of (1). By Lemma 115, there exist $s_{n+1},\ldots,s_{n+r} \in \mathbb{N}$ such that $(s_1,\ldots,s_n,s_{n+1},\ldots,s_{n+r})$ is a solution for (2). Applying Lemma 116, we deduce that $(s_1,\ldots,s_{n+r}) \in \{\overline{\alpha}_{d+1},\ldots,\overline{\alpha}_g\} + \langle \overline{\alpha}_1,\ldots,\overline{\alpha}_d\rangle$, whence $(s_1,\ldots,s_n) \in \{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\} + \langle \pi(\alpha_1),\ldots,\pi(\alpha_d)\rangle$.

Conversely, if $(s_1,\ldots,s_n) \in \{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\} + \langle \pi(\alpha_1),\ldots,\pi(\alpha_d)\rangle$, then

$$(s_1,\ldots,s_n) = \pi(\alpha_{d+i}) + a_1\pi(\alpha_1) + \cdots + a_d\pi(\alpha_d)$$

for some $i \in \{1,\ldots,g-d\}$ and $a_1,\ldots,a_d \in \mathbb{N}$. Let $s_{n+1},\ldots,s_{n+r+1} \in \mathbb{N}$ be such that $(s_1,\ldots,s_n,s_{n+1},\ldots,s_{n+1+r}) = \alpha_{i+d} + a_1\alpha_1 + \cdots + a_d\alpha_d$. Using Lemma 116, $(s_1,\ldots,s_{n+r})$ is a nonnegative integer solution to (2) and by Lemma 115 we conclude that $(s_1,\ldots,s_n)$ is a nonnegative integer solution for (1). $\square$

We now sharpen these results a bit more, taking into account some monoid structure arising in the process. Let T be the monoid of nonnegative integer solutions of the system of inequalities

$$
(4)\qquad \begin{cases} a_{1_1}x_1 + \cdots + a_{1_n}x_n & \geq & 0, \\ & \vdots & \\ a_{r_1}x_1 + \cdots + a_{r_n}x_n & \geq & 0, \\ a_{r+1_1}x_1 + \cdots + a_{r+1_n}x_n & = & 0, \\ & \cdots & \\ a_{r+t_1}x_1 + \cdots + a_{r+t_n}x_n & = & 0. \end{cases}
$$

Denote by $\mathcal{T}$ the set of nonnegative integer solutions of the system (1).We define on $\mathcal{T}$ the following binary relation: $x \leq_T y$ if $y - x \in T$.

LEMMA 118. *The binary relation $\leq_T$ is an order relation on $\mathcal{T}$.*

PROOF. Observe that $\leq_T$ is reflexive since $0 \in T$. As T is unit free, $\leq_T$ is antisymmetric. Finally, if $x \leq_T y$ and $y \leq_T z$, then $y - x \in T$ and $z - y \in T$, whence $z - x = z - y + y - x \in T$ and thus $x \leq_T z$. $\square$

LEMMA 119. *Let $\{\alpha_1,\ldots,\alpha_t\}$, $\{\alpha_1,\ldots,\alpha_d\}$ and $\{\alpha_{d+1},\ldots,\alpha_g\}$ be as in Lemma 116. Then*

$$T = \langle \pi(\alpha_1),\ldots,\pi(\alpha_d)\rangle,$$

where π is defined as in Proposition 117.

PROOF. The elements $\alpha_1,\ldots,\alpha_d$ are nonnegative integer solutions of (3) with the last coordinate equal to zero, whence $\{\pi(\alpha_1),\ldots,\pi(\alpha_d)\} \subseteq T$. If $(s_1,\ldots,s_n) \in T$, then there exist $s_1,\ldots,s_{n+r}\in\mathbb{N}$ such that $(s_1,\ldots,s_n,s_{n+1},\ldots,s_{n+r},0)$ is a nonnegative integer solution of (3). Hence there exist $a_1,\ldots,a_t\in\mathbb{N}$ such that $(s_1,\ldots,s_{n+r},0)=a_1\alpha_1+\cdots+a_t\alpha_t$. Observe that for $k>d$, a_k must be zero, since the last coordinate of $(s_1,\ldots,s_{n+r},0)$ is zero. Therefore $(s_1,\ldots,s_n)=a_1\pi(\alpha_1)+\cdots a_d\pi(\alpha_d)\in\langle\pi(\alpha_1),\ldots,\pi(\alpha_d)\rangle$. □

LEMMA 120. *The set* $\mathrm{Minimals}_{\leq_T}(\mathcal{T})$ *has finitely many elements.*

PROOF. By Proposition 117 and Lemma 119, $\mathcal{T}=\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}+T$, whence

$$\mathrm{Minimals}_{\leq_T}(\mathcal{T})\subseteq\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}.$$

□

THEOREM 121. *Let* $\mathcal{T}$ *be the set of nonnegative integer solutions of (1) and* T *be the set of nonnegative integer solutions of (4) Then*

$$\mathcal{T}=\mathrm{Minimals}_{\leq_T}(\mathcal{T})+T$$

and the set $\mathrm{Minimals}_{\leq_T}(\mathcal{T})$ *is finite.*

PROOF. We already know by Proposition 117 and Lemma 119 that $\mathcal{T}=\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}+T$. From the proof of Lemma 120, we have that $\mathrm{Minimals}_{\leq_T}(\mathcal{T})\subseteq\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}$, and from the definition of $\leq_T$, it follows that $\mathcal{T}=\mathrm{Minimals}_{\leq_T}(\mathcal{T})+T$. □

REMARK 122. (1) There are several algorithms for finding the set of elements described in Lemma 116 (see for instance **[13]** or **[39]**). Hence we know how to compute $\{\alpha_1,\ldots,\alpha_t\}$. From this set one can compute a system of generators of T by projecting onto the first n coordinates the elements $\alpha_1,\ldots,\alpha_d$. Using now (4) one can easily check which elements in

$\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}$ belong to $\mathrm{Minimals}_{\leq_T}(\mathcal{T})$, whence we have a complete procedure for describing all the elements in $\mathcal{T}$.

(2) One might wonder why we are introducing and proving Theorem 121 instead of using Proposition 117. The idea is that the set $\mathrm{Minimals}_{\leq_T}(\mathcal{T})$ can be strictly included (and in fact be much smaller than) the set $\{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}$. It may also happen that one can find a smaller set of generators for T by a method not relying on the procedure explained in the above remark. Besides, Theorem 121 gives a description for systems of inequalities similar to the one obtained in Lemma 116 for systems of equations.

(3) In **[2]** it is presented an algorithm for solving (1) without adding slack variables. This algorithm can be used to find $\pi(\alpha_1),\ldots,\pi(\alpha_g)$.

(4) In the literature on can also find implementations relying on Gröbner basis computation for solving (3). Unfortunately in the examples we give in this section the number of variables becomes too large for using this kind of algorithm.

Let us illustrate this process with an example.

EXAMPLE 123. Let $\mathcal{T}$ be the set of nonnegative integer solutions of

$$\begin{array}{lcl} x_1 & \geq & 2, \\ x_2 & \geq & 2, \\ 2x_1 - x_2 & \geq & 1, \\ 2x_2 - x_1 & \geq & 0, \end{array}$$

and T be the set of nonnegative integer solutions to the associated "homogeneous" system of inequalities. A minimal system of generators for the monoid of nonnegative integer solutions of

$$\begin{array}{lcl} x_1 - x_3 - 2x_7 & = & 0, \\ x_2 - x_4 - 2x_7 & = & 0, \\ 2x_1 - x_2 - x_5 - x_7 & = & 0, \\ 2x_2 - x_1 - x_6 & = & 0, \end{array}$$

is

$$\{\alpha_1,\ldots,\alpha_t\} = \{(2,2,0,0,1,2,1),(2,3,0,1,0,4,1),(3,2,1,0,3,1,1),$$
$$(4,2,2,0,5,0,1),(1,2,1,2,0,3,0),(1,1,1,1,1,1,0),(2,1,2,1,3,0,0)\}.$$

Hence

$$\{\alpha_1,\ldots,\alpha_d\} = \{(1,2,1,2,0,3,0),(1,1,1,1,1,1,0),(2,1,2,1,3,0,0)\}$$

and

$$\{\alpha_{d+1},\ldots,\alpha_g\} = \{(2,2,0,0,1,2,1),(2,3,0,1,0,4,1),(3,2,1,0,3,1,1),(4,2,2,0,5,0,1)\}.$$

By Lemma 119,

$$T = \langle (1,2),(1,1),(2,1)\rangle$$

and by Proposition 117

$$\mathcal{T} = \{(2,2),(2,3),(3,2),(4,2)\} + T.$$

In this case $\mathrm{Minimals}_{\leq_T}(\mathcal{T}) = \{\pi(\alpha_{d+1}),\ldots,\pi(\alpha_g)\}$. □

2. Systems of inequalities associated to the set of numerical semigroups with fixed multiplicity

Let $\mathcal{S}(m)$ be the set of all numerical semigroups with multiplicity $m \in \mathbb{N}\setminus\{0\}$. In this section we prove that there is a one-to-one correspondence between this set and the set of nonnegative integer solutions of a system of linear Diophantine inequalities. The key for this correspondence is given in the following result that can be derived from [**32**, Lemma 3.3]. If $m = 1$ the only semigroup with multiplicity m is $\mathbb{N}$, whence the interesting cases arise when $m > 1$. Thus we will assume that $m > 1$.

LEMMA 124. *Let m be an integer greater than one and let*

$$X = \{0 = w(0), w(1),\ldots,w(m-1)\}$$

be a subset of $\mathbb{N}$ *with* m *elements such that* $w(i) \equiv i (\mathrm{mod}\ m)$ *and* $m < w(i)$ *for all* $i \in \{1,\ldots,m-1\}$. *Let* S *be the submonoid of* $\mathbb{N}$ *generated by* $X \cup \{m\}$. *Then* S *is a numerical semigroup with multiplicity* m. *Furthermore* $\mathrm{Ap}(S,m) = X$ *if and only if for all* $i, j \in \{1,\ldots,m-1\}$ *there exist* $k \in \{0,\ldots,m-1\}$ *and* $t \in \mathbb{N}$ *such that* $w(i)+w(j) = w(k)+tm$.

Next lemma associates to $\mathcal{S}(m)$ a system of linear Diophantine inequalities.

LEMMA 125. *Let* m *be an integer greater than one and let* S *be in* $\mathcal{S}(m)$ *with*

$$\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \ldots, w(m-1)\}.$$

For all $i \in \{1,\ldots,m-1\}$ *let* $k_i \in \mathbb{N}$ *be such that* $w(i) = k_i m + i$. *Then*

(1) $k_i \geq 1$ *for all* $i \in \{1,\ldots,m-1\}$,

(2) $k_i + k_j - k_{i+j} \geq 0$ *for all* $1 \leq i \leq j \leq m-1$ *with* $i+j \leq m-1$,

(3) $k_i + k_j - k_{i+j-m} \geq -1$ *for all* $1 \leq i \leq j \leq m-1$ *with* $i+j > m$.

PROOF. Since S is a numerical semigroup of multiplicity m and $w(i) \in S \setminus \{0\}$ for all $i \in \{1,\ldots,m-1\}$, $w(i) \geq m$, whence $k_i \geq 1$. If $1 \leq i \leq j \leq m-1$ and $i+j \leq m-1$, by Lemma 124, there exist $t \in \mathbb{N}$ and $k \in \{0,\ldots,m-1\}$ such that $w(i)+w(j) = tm + w(k)$. Note that $i+j \equiv w(i)+w(j) \equiv w(k) \equiv w(i+j) (\mathrm{mod}\ m)$ and by the definition of the elements in $\mathrm{Ap}(S,m)$ we obtain $w(k) = w(i+j)$ and this leads to $(k_i+k_j-k_{i+j})m = tm$; thus $k_i+k_j-k_{i+j} \geq 0$. Now assume that $i+j > m$, using again Lemma 124, there exist $t \in \mathbb{N}$ and $k \in \{0,\ldots,m-1\}$ such that $w(i)+w(j) = tm + w(k)$. Arguing as above we deduce that $w(k) = w(i+j-m)$ and $(k_i+k_j)m + i + j = tm + k_{i+j-m}m + i + j - m$, which yields $k_i + k_j - k_{i+j-m} \geq -1$. $\square$

Observe that $(k_1,\ldots,k_{m-1}) \in \mathbb{N}^{m-1}$ is determined uniquely for $S \in \mathcal{S}(m)$ and it is a nonnegative integer solution of the system of inequalities

$$\begin{array}{cc} x_i \geq 1 & \text{for all } i \in \{1,\ldots,m-1\}, \\ x_i + x_j - x_{i+j} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i + x_j - x_{i+j-m} \geq -1 & \text{for all } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

Denote by $\mathcal{T}(m)$ the set of nonnegative solutions of this system of inequalities. Next we associate to each element in $\mathcal{T}(m)$ an element in $\mathcal{S}(m)$.

LEMMA 126. *Let m be an integer greater than one. For $(k_1,\ldots,k_{m-1}) \in \mathcal{T}(m)$, the semigroup*

$$S = \langle m, k_1m+1, k_2m+2, \ldots, k_{m-1}m+m-1\rangle$$

has multiplicity m and $\mathrm{Ap}(S,m) = \{0, k_1m+1, \ldots, k_{m-1}m+m-1\}$.

PROOF. We make use of Lemma 124 with

$$X = \{0 = w(0), k_1m+1 = w(1), \ldots, k_{m-1}m+m-1 = w(m-1)\}.$$

Then the monoid $S = \langle X \cup \{m\}\rangle$ is a numerical semigroup of multiplicity $m < w(i)$ for all $i \in \{1,\ldots,m-1\}$ and $w(i) \equiv i \pmod m$ for all $i \in \{0,\ldots,m-1\}$. Now we have to check that for $i,j \in \{1,\ldots,m-1\}$ there exist $k \in \{0,\ldots,m-1\}$ and $t \in \mathbb{N}$ such that $w(i)+w(j) = w(k)+tm$. Given $i,j \in \{1,\ldots,m\}$ we distinguish three cases.

(1) If $i+j \leq m-1$, then $w(i)+w(j) = tm+w(i+j)$ with $t = k_i+k_j-k_{i+j} \in \mathbb{N}$ (here arises the condition $k_i+k_j \geq k_{i+j}$).
(2) If $i+j = m$, then $w(i)+w(j) = tm+w(0)$, with $t = k_i+k_j+1 \in \mathbb{N}$.
(3) If $i+j > m$, then $w(i)+w(j) = tm+w(i+j-m)$, with $t = k_i+k_j+1-k_{i+j-m} \in \mathbb{N}$ (we are using that $k_i+k_j-k_{i+j-m} \geq -1$).

□

As a consequence of Lemmas 125 and 126 we obtain the following result that states the desired correspondence.

THEOREM 127. *Let m be an integer greater than one. The map $\varphi : \mathcal{T}(m) \to \mathcal{S}(m)$ defined by*

$$\varphi(k_1,\ldots,k_{m-1}) = \langle m, k_1m+1, k_2m+2, \ldots, k_{m-1}m+m-1\rangle$$

is one-to-one. Moreover

$$\mathrm{Ap}(\varphi(k_1,\ldots,k_{m-1}),m)=\{0,k_1m+1,\ldots,k_{m-1}+m-1\}.$$

Using the results obtained in Section 1 we know that $\mathcal{T}(m)=\{\beta_1,\ldots,\beta_k\}+\langle\gamma_1,\ldots,\gamma_s\rangle$ for some $\beta_i=(\beta_{i_1},\ldots,\beta_{i_{m-1}})$, $\gamma_i=(\gamma_{i_1},\ldots,\gamma_{i_{m-1}})\in\mathbb{N}^{m-1}$, and we have a procedure for computing them. Hence

$$\mathcal{S}(m)=\{\langle m,k_1m+1,\ldots,k_{m-1}+m-1\rangle \mid (k_1,\ldots,k_{m-1})\in\{\beta_1,\ldots,\beta_k\}+\langle\gamma_1,\ldots,\gamma_s\rangle\}.$$

Next we illustrate these results with a couple of examples.

EXAMPLE 128. The set of all numerical semigroups with multiplicity 3 is

$$\mathcal{S}(3)=\{\langle 3,3k_1+1,3k_2+2\rangle \mid (k_1,k_2)\in\mathcal{T}(3)\}$$

and $\mathcal{T}(3)$ is the set of nonnegative integer solutions of

$$\begin{array}{lcl} x_1 & \geq & 1,\\ x_2 & \geq & 1,\\ 2x_1-x_2 & \geq & 0,\\ -x_1+2x_2 & \geq & -1. \end{array}$$

We proceed as we did in Example 123 and obtain

$$\mathcal{T}(3)=\{(1,1),(1,2),(2,1),(3,1)\}+\langle(1,2),(1,1),(2,1)\rangle.$$

$\square$

EXAMPLE 129. We describe those numerical semigroups with multiplicity 4. The set $\mathcal{T}(4)$ is the set of nonnegative solutions to the system of inequalities:

$$\begin{array}{lcl} x_1 & \geq & 1,\\ x_2 & \geq & 1,\\ x_3 & \geq & 1,\\ 2x_1-x_2 & \geq & 0,\\ x_1+x_2-x_3 & \geq & 0,\\ -x_1+x_2+x_3 & \geq & -1,\\ -x_2+2x_3 & \geq & -1. \end{array}$$

Computing those solutions we obtain

$$\begin{aligned}\mathcal{T}(4) = \{((1,1,1),(1,1,2),(1,2,1),(1,2,3),(1,2,2),(2,1,1),(3,1,1),\\ (2,2,1),(2,3,1),(3,2,1),(4,2,1),(3,3,1),(4,3,1),(5,3,1)\}\\ +\langle(1,0,1),(1,2,3),(1,2,2),(1,2,1),(1,1,2),\\ (1,1,1),(2,1,1),(2,2,1),(3,2,1)\rangle.\end{aligned}$$

Hence

$$\mathcal{S}(4) = \{\langle 4, k_1 4+1, k_2 4+2, k_3 4+3\rangle \mid (k_1,k_2,k_3) \in \mathcal{T}(4)\}.$$

□

The description of $\mathcal{S}(m)$ in terms of $\mathcal{T}(m)$ also allows us to construct all numerical semigroups with given multiplicity and Frobenius number. We illustrate this with an example.

EXAMPLE 130. We construct all numerical semigroups with multiplicity 5 and Frobenius number 13. If S is a numerical semigroup with $\mathrm{m}(S) = 5$ and $\mathrm{g}(S) = 13$, then $n \in S$ for all $n > 13$. Besides

$$\mathrm{Ap}(S,5) = \{0, k_1 5+1, k_2 5+2, k_3 5+3, k_4 5+4\},$$

and we know that $\mathrm{g}(S) = \max(\mathrm{Ap}(S,5)) - 5$, whence $\max(\mathrm{Ap}(S,5)) = 18 \equiv 3 (\mathrm{mod}\, 5)$, which means that k_3 must be equal to 3. From $14, 15, 16, 17 \in S$ we deduce the following conditions on k_1, k_2, k_4.

$14 = 4+2\times 5$	$k_4 \leq 2$
$15 = 0+3\times 5$	
$16 = 1+3\times 5$	$k_1 \leq 3$
$17 = 2+3\times 5$	$k_2 \leq 3$

Hence the set of numerical semigroups with multiplicity 5 and Frobenius number 13 is

$$\left\{ \langle 5, k_1 5+1, k_2 5+2, k_3 5+3, k_4 5+4\rangle \;\middle|\; \begin{array}{c} (k_1,k_2,k_3,k_4) \in \mathcal{T}(5) \\ k_1 \leq 3, k_2 \leq 3, k_3 = 3, k_4 \leq 2 \end{array} \right\}$$
$$= \{\langle 5,11,7,18,9\rangle, \langle 5,11,7,18,14\rangle, \langle 5,11,12,18,9\rangle, \langle 5,11,12,18,14\rangle,$$
$$\langle 5,11,17,18,9\rangle, \langle 5,11,17,18,14\rangle, \langle 5,16,7,18,9\rangle, \langle 5,16,7,18,14\rangle,$$
$$\langle 5,16,12,18,9\rangle, \langle 5,16,12,18,14\rangle, \langle 5,16,17,18,9\rangle, \langle 5,16,17,18,14\rangle\}.$$

□

The same procedure can be used to obtain a description of the set of numerical semigroups with fixed Frobenius number g. One has to look for numerical semigroups with multiplicity $m \in \{2, \ldots, g-1, g+1\}$ and proceed as in Example 130.

3. MED-semigroups

Recall that a numerical semigroup is a MED-semigroup if its multiplicity equals its embedding dimension. Denote by $\mathcal{MED}(m)$ the set of MED-semigroups with multiplicity m. We show that there is a one-to-one correspondence between $\mathcal{MED}(m)$ and a subsemigroup of $\mathbb{N}^{m-1}$ (here is one of the main differences with $\mathcal{S}(m)$; actually $\mathcal{T}(m)$ is not a semigroup, see for instance in Example 128 that $(3,1) \in \mathcal{T}(3)$ but $2(3,1) \notin \mathcal{T}(3)$). The following result plays the same role as Lemma 124 did for $\mathcal{S}(m)$.

LEMMA 131. *Let m be an integer greater than one and let S be a numerical semigroup of multiplicity m and*

$$\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \ldots, w(m-1)\}.$$

Then S is a MED-semigroup if and only if for all $1 \leq i \leq j \leq m-1$ *there exist* $k \in \{0, \ldots, m-1\}$ *and* $t \in \mathbb{N} \setminus \{0\}$ *such that* $w(i) + w(j) = tm + w(k)$.

PROOF. From the definition of MED-semigroup it follows that S is a MED-semigroup if and only if $(\mathrm{Ap}(S,m)\setminus\{0\})\cup\{m\}$ is a minimal system of generators for S. If $1 \leq i \leq j \leq m-1$, by Lemma 125, we deduce that $w(i)+w(j) = tm+w(k)$ for some $t \in \mathbb{N}$ and $k \in \{0,\dots,m-1\}$. Observe that $t \neq 0$, since otherwise $(\mathrm{Ap}(S,m)\setminus\{0\})\cup\{m\}$ would not be a minimal system of generators for S. Conversely, assume that

$$w(i) = a_0m + a_1w(1) + \ldots + a_{i-1}w(i-1) + a_{i+1}w(i+1) + \ldots + a_{m-1}w(m-1)$$

for some $a_i \in \mathbb{N}$. Using the hypothesis several times we obtain that $w(i) = \lambda m + w(q)$ for some $\lambda \in \mathbb{N}\setminus\{0\}$ and $q \in \{0,\dots,m-1\}$, which is in contradiction with the definition of the elements in $\mathrm{Ap}(S,m)$ (observe also that m cannot be written as $a_1w(1)+\cdots+a_{m-1}w(m-1)$, since $m < w(i)$ for all i). □

The proof of the following result is analogous to the one of Lemma 125, but now using Lemma 131.

LEMMA 132. *Let m be an integer greater than one and let $S \in \mathcal{MED}(m)$ with $\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \dots, w(m-1)\}$. For all $i \in \{1,\dots,m-1\}$ let $k_i \in \mathbb{N}$ be such that $w(i) = k_im+i$. Then*

(1) *$k_i \geq 1$ for all $i \in \{1,\dots,m-1\}$,*
(2) *$k_i+k_j-k_{i+j} \geq 1$ for all $1 \leq i \leq j \leq m-1$ with $i+j \leq m-1$,*
(3) *$k_i+k_j-k_{i+j-m} \geq 0$ for all $1 \leq i \leq j \leq m-1$ with $i+j > m$.*

In this setting $(k_1,\dots,k_{m-1}) \in \mathbb{N}^{m-1}$ is a nonnegative integer solution of the system of inequalities

$$\begin{array}{rl} x_i \geq 1 & \text{for all } i \in \{1,\dots,m-1\}, \\ x_i+x_j-x_{i+j} \geq 1 & \text{for all } 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i+x_j-x_{i+j-m} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

Denote by $\mathcal{M}(m)$ the set of nonnegative solutions of this system of inequalities, which is a subsemigroup of $\mathbb{N}^{m-1}$ as the following result (with straightforward proof) shows.

LEMMA 133. *Let $b_1,\ldots,b_r \in \mathbb{N}$ and $a_{i_j} \in \mathbb{Z}$. Then the set of nonnegative integer solutions of the system*

$$\begin{array}{rcl} a_{1_1}x_1+\cdots+a_{1_n}x_n & \geq & b_1, \\ & \vdots & \\ a_{r_1}x_1+\cdots+a_{r_n}x_n & \geq & b_r, \end{array}$$

is a subsemigroup of $\mathbb{N}^n$.

Hence Lemma 132 associates to every element in $\mathcal{MED}(m)$ and element in the semigroup $\mathcal{M}(m)$. The following result, whose proof is analogous to the one of Lemma 126, gives the correspondence in the other direction.

LEMMA 134. *Let m be an integer greater than one. If $(k_1,\ldots,k_{m-1}) \in \mathcal{M}(m)$, then the numerical semigroup*

$$S = \langle m, k_1m+1, k_2m+2, \ldots, k_{m-1}m+m-1\rangle$$

is a MED-semigroup with $\mathrm{Ap}(S,m) = \{0, k_1m+1, \ldots, k_{m-1}m+m-1\}$.

As a consequence of Lemmas 132 and 134 we obtain the correspondence between $\mathcal{M}(m)$ and $\mathcal{MED}(m)$.

THEOREM 135. *Let m be an integer greater than one. The map $\psi : \mathcal{M}(m) \to \mathcal{MED}(m)$ defined by*

$$\psi(k_1,\ldots,k_{m-1}) = \langle m, k_1m+1, \ldots, k_{m-1}m+m-1\rangle$$

is one-to-one. Furthermore

$$\mathrm{Ap}(\psi(k_1,\ldots,k_{m-1}), m) = \{0, k_1m+1, \ldots, k_{m-1}m+m-1\}$$

and

$$\{m, k_1m+1, \ldots, k_{m-1}m+m-1\}$$

is a minimal system of generators for $\psi(k_1,\ldots,k_{m-1})$.

The semigroups $\mathcal{M}(m)$ are not finitely generated as the following two results show (except $\mathcal{M}(2) = \mathbb{N} \setminus \{0\}$).

LEMMA 136. *Let $S = \langle s_1, \ldots, s_r \rangle$ be a submonoid of $\mathbb{N}^p$ for some positive integer p. Assume that there exist $v, w \in \mathbb{N}^p$ such that $v + kw \in S$ for all $k \in \mathbb{N}$. Then there exists $l \in \mathbb{N} \setminus \{0\}$ for which $lw \in S$.*

PROOF. For every $k \in \mathbb{N}$ take $(\lambda_1^k, \ldots, \lambda_r^k) \in \mathbb{N}^r$ such that $v + kw = \sum_{i=1}^r \lambda_i^k s_i$. The set $\{(\lambda_1^k, \ldots, \lambda_r^k) \mid k \in \mathbb{N}\}$ has infinitely many elements, and by Dickson's lemma, it follows that there exist $k_1, k_2 \in \mathbb{N}$ such that $(\lambda_1^{k_1}, \ldots, \lambda_r^{k_1}) < (\lambda_1^{k_2}, \ldots, \lambda_r^{k_2})$. Hence $(k_2 - k_1)w \in S$. □

PROPOSITION 137. *Let m be an integer greater than two. The semigroup $\mathcal{M}(m)$ is not finitely generated.*

PROOF. We already know by Theorem 121 that $\mathcal{M}(m) = \mathrm{Minimals}_{\leq_T}(\mathcal{M}(m)) + T$, where T is the set of nonnegative integer solutions of

$$\begin{array}{rl} x_i \geq 0 & \text{for all } i \in \{1, \ldots, m-1\}, \\ x_i + x_j - x_{i+j} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i + x_j - x_{i+j-m} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

Hence $v + kw \in \mathcal{M}(m)$ for all $v \in \mathrm{Minimals}_{\leq_T}(\mathcal{M}(m))$, $w \in T$ and $k \in \mathbb{N}$. Observe that the element $v = (1, \ldots, 1) \in \mathrm{Minimals}_{\leq_T}(\mathcal{M}(m))$ and $w = v + e_{m-1} \in T$ (as usual, e_i denotes the element all of whose coordinates are zero except the ith which is equal to one). Hence $v + kw \in \mathcal{M}(m)$ for all $k \in \mathbb{N}$. If $\mathcal{M}(m)$ is finitely generated, then by Lemma 136 there must be a positive integer l such that $lw \in \mathcal{M}(m)$, but this is impossible, since this element does not fulfill the equation $x_1 + x_{m-2} - x_{m-1} \geq 1$. □

Let m be an integer greater than one and let $\overline{\mathcal{MED}}(m)$ be the set of MED-semigroups of multiplicity m and with minimal generators greater than $2m$ (except of course m). This condition yields $k_i \geq 2$ for all $i \in \{1, \ldots, m-1\}$ in the definition of

k_i, whence

$$\overline{\mathcal{M}}(m) = \psi^{-1}(\overline{\mathcal{MED}}(m)) = \{(k_1,\dots,k_{m-1}) \in \mathcal{M}(m) \mid k_i \geq 2 \text{ for } i \in \{1,\dots,m-1\}\}.$$

Therefore the system of inequalities that defines $\overline{\mathcal{M}}(m)$ is

$$\begin{array}{rl} x_i \geq 2 & \text{for all } i \in \{1,\dots,m-1\}, \\ x_i + x_j - x_{i+j} \geq 1 & \text{for all } 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i + x_j - x_{i+j-m} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

By Lemma 133, $\overline{\mathcal{M}}(m)$ is a subsemigroup of $\mathbb{N}^{m-1}$. Recall that $\mathcal{T}(m)$ was the set of nonnegative integer solutions of the system

$$\begin{array}{rl} x_i \geq 1 & \text{for all } i \in \{1,\dots,m-1\}, \\ x_i + x_j - x_{i+j} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1, i+j \leq m-1, \\ x_i + x_j - x_{i+j-m} \geq -1 & \text{for all } 1 \leq i \leq j \leq m-1, i+j > m. \end{array}$$

From these two systems of inequalities it follows easily that if $(k_1,\dots,k_{m-1}) \in \mathcal{T}(m)$, then $(k_1+1,\dots,k_{m-1}+1) \in \overline{\mathcal{M}}(m)$; and $(k_1,\dots,k_{m-1}) \in \overline{\mathcal{M}}(m)$ implies $(k_1-1,\dots,k_{m-1}-1) \in \mathcal{T}(m)$. Hence we obtain the following correspondence between $\mathcal{T}(m)$ and $\overline{\mathcal{M}}(m)$.

PROPOSITION 138. *Let m be an integer greater than one. Then*

$$\mathcal{T}(m) = \{(-1,\dots,-1)\} + \overline{\mathcal{M}}(m).$$

Since $\mathcal{T}(m)$ is bijective with $\mathcal{S}(m)$ and $\overline{\mathcal{M}}(m)$ is bijective with $\overline{\mathcal{MED}}(m)$, we obtain the following consequence ([**32**, Theorem 3.5]).

COROLLARY 139. *There is a one-to-one correspondence between the set of numerical semigroups with multiplicity $m > 1$ and the set of MED-semigroups with multiplicity m and minimal generators different from m greater than $2m$.*

Proposition 117 and Theorem 121 yield "finite parametrizations" of $\mathcal{T}(m)$, but now light is shed there on the structure of $\mathcal{S}(m)$. Nevertheless, Proposition 138 describes $\mathcal{T}(m)$ as a translation of a subsemigroup of $\mathbb{N}^{m-1}$. We use this information to give semigroup structure to $\mathcal{S}(m)$ and to get a result stronger than Corollary 139. Given

$$S = \langle m, k_1 m + 1, \ldots, k_{m-1} m + m - 1\rangle, \overline{S} = \langle m, \overline{k}_1 m + 1, \ldots, \overline{k}_{m-1} m + m - 1\rangle \in \mathcal{S}(m),$$

that is, $(k_1, \ldots, k_{m-1}), (\overline{k}_1, \ldots, \overline{k}_{m-1}) \in \mathcal{T}(m)$, define

$$S * \overline{S} = \langle m, (k_1 + \overline{k}_1 + 1)m + 1, \ldots, (k_{m-1} + \overline{k}_{m-1} + 1)m + m - 1\rangle.$$

The reader can check that $(k_1 + \overline{k}_1 + 1, \ldots, k_{m-1} + \overline{k}_{m-1} + 1) \in \mathcal{T}(m)$, whence $S * \overline{S} \in \mathcal{S}(m)$. The pair $(\mathcal{S}(m), *)$ is a semigroup and the map

$$\sigma : \mathcal{S}(m) \to \overline{\mathcal{M}}(m),\ \sigma(S) = (1, \ldots, 1) + \varphi^{-1}(S)$$

is a semigroup isomorphism, where φ is the map given in Theorem 127.

COROLLARY 140. *The set of numerical semigroups with multiplicity m is a semigroup isomorphic to a subsemigroup of* $\mathbb{N}^{m-1}$.

4. Symmetric numerical semigroups

In this section we particularize the results obtained in Section 2 for symmetric numerical semigroups. We see how the defining inequalities of $\mathcal{T}(m)$ are reshaped by the symmetric property.

From Proposition 12, we have that S is symmetric if the set $\mathrm{Ap}(S,m)$ has a maximum with respect to the partial ordering $\leq_S$ induced in S by addition. Given an integer $m > 1$ and $p \in \{1, \ldots, m-1\}$, define

$$\mathcal{S}^p_{\mathrm{sy}}(m) = \{S \in \mathcal{S}(m) \mid S \text{ is symmetric and } \mathrm{g}(S) \equiv p (\mathrm{mod}\ m)\}.$$

We prove that there is a one-to-one correspondence between $\mathcal{S}^p_{\mathrm{sy}}(m)$ and the set of nonnegative integer solutions of a system of linear Diophantine equations and inequalities. This set will depend on $p = m-1$ or $p \neq m-1$, whence we study them separately. The following lemma collects the necessary extra conditions we have add to the ones given in Lemma 125 for general numerical semigroups for the specific case of semigroups in $\mathcal{S}_{\mathrm{sy}}^{m-1}(m)$. Those extra conditions are a direct consequence of Proposition 12 taking into account that for $S \in \mathcal{S}_{\mathrm{sy}}^{m-1}(m)$ with $\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \dots, w(m-1)\}$, we have that $\mathrm{g}(S) + m = w(m-1)$, since $\mathrm{g}(S) + m = \max_{\leq}(\mathrm{Ap}(S,m))$ and $\mathrm{g}(S) \equiv m - 1 \pmod m$. Hence the condition $\mathrm{g}(S) + m - w \in \mathrm{Ap}(S,m)$ for all $w \in \mathrm{Ap}(S,m)$ translates to: for all $i \in \{1, \dots, m-1\}$, $w(m-1) - w(i) = w(j)$ for some $j \in \{1, \dots, m-1\}$.

LEMMA 141. *Let m be an integer greater than one and let $S \in \mathcal{S}_{\mathrm{sy}}^{m-1}(m)$ with $\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \dots, w(m-1)\}$. For every $i \in \{1, \dots, m-1\}$ let $k_i \in \mathbb{N}$ be such that $w(i) = k_i m + i$. Then*

(1) $k_i \geq 1$ for all $i \in \{1, \dots, m-1\}$,
(2) $k_i + k_j - k_{i+j} \geq 0$ for all $1 \leq i \leq j \leq m-1$ with $i + j < m-1$,
(3) $k_i + k_j - k_{m-1} = 0$ for all $1 \leq i \leq j \leq m-1$ with $i + j = m-1$,
(4) $k_i + k_j - k_{i+j-m} \geq -1$ for all $1 \leq i \leq j \leq m-1$ with $i + j > m$.

Thus we obtain that $(k_1, \dots, k_{m-1})$ is a nonnegative integer solution of the following system

$$\begin{array}{ll} x_i \geq 1 & \text{for all } i \in \{1, \dots, m-1\}, \\ x_i + x_j - x_{i+j} \geq 0 & \text{for all } 1 \leq i \leq j \leq m-1 \text{ with } i + j < m-1, \\ x_i + x_j - x_{m-1} = 0 & \text{for all } 1 \leq i \leq j \leq m-1 \text{ with } i + j = m-1, \\ x_i + x_j - x_{i+j-m} \geq -1 & \text{for all } 1 \leq i \leq j \leq m-1 \text{ with } i + j > m. \end{array}$$

We denote by $\mathcal{T}_{\mathrm{sy}}^{m-1}(m)$ the set of nonnegative solutions of this system of linear Diophantine equations and inequalities.

As for $p \neq m-1$, we obtain the following result similar to Lemma 141, using once more Lemma 125 and Proposition 12.

LEMMA 142. *Let m be an integer greater than two and let $p \in \{1, \dots, m-2\}$. Let $S \in \mathcal{S}_{\mathrm{sy}}^{p}(m)$ with $\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \dots, w(m-1)\}$. For every $i \in \{1, \dots, m-1\}$ let $k_i \in \mathbb{N}$ be such that $w(i) = k_i m + i$. Then*

(1) $k_i \geq 1$ for all $i \in \{1, \dots, m-1\}$,

(2) $k_i+k_j-k_{i+j}\geq 0$ *for all* $1\leq i\leq j\leq m-1$ *with* $i+j\leq m-1$ *and* $i+j\neq p$,

(3) $k_i+k_j-k_p=0$ *for all* $1\leq i\leq j\leq m-1$ *with* $i+j=p$,

(4) $k_i+k_j-k_{i+j-m}\geq -1$ *for all* $1\leq i\leq j\leq m-1$ *with* $i+j>m$ *and* $i+j\neq m+p$,

(5) $k_i+k_j-k_p=-1$ *for all* $1\leq i\leq j\leq m-1$ *with* $i+j=m+p$.

The element $(k_1,\ldots,k_{m-1})$ belongs to $\mathcal{T}_{\mathrm{sy}}^{p}(m)$, the set of nonnegative integer solutions of the system

$$\begin{array}{ll} x_i\geq 1 & \text{for all } i\in\{1,\ldots,m-1\},\\ x_i+x_j-x_{i+j}\geq 0 & \text{for all } 1\leq i\leq j\leq m-1 \text{ with } i+j\leq m-1, i+j\neq p,\\ x_i+x_j-x_p=0 & \text{for all } 1\leq i\leq j\leq m-1 \text{ with } i+j=p,\\ x_i+x_j-x_{i+j-m}\geq -1 & \text{for all } 1\leq i\leq j\leq m-1 \text{ with } i+j>m, i+j\neq m+p,\\ x_i+x_j-x_p=-1 & \text{for all } 1\leq i\leq j\leq m-1 \text{ with } i+j=m+p. \end{array}$$

As in Section 2, we now get the following consequence (we omit the proof since it is similar to the one given there).

THEOREM 143. *Let m be an integer greater than one and let $p\in\{1,\ldots,m-1\}$. Then the map $\varphi_{\mathrm{sy}}^{p}:\mathcal{T}_{\mathrm{sy}}^{p}(m)\to\mathcal{S}_{\mathrm{sy}}^{p}(m)$ defined by*

$$\varphi_{\mathrm{sy}}^{p}(k_1,\ldots,k_{m-1})=\langle m,k_1m+1,k_2m+2,\ldots,k_{m-1}m+m-1\rangle$$

is a one-to-one correspondence. Furthermore,

$$\mathrm{Ap}(\varphi_{\mathrm{sy}}^{p}(k_1,\ldots,k_{m-1}),m)=\{0,k_1m+1,\ldots,k_{m-1}m+m-1\}$$

and $\mathrm{g}(\varphi_{\mathrm{sy}}^{p}(k_1,\ldots,k_{m-1}))=(k_p-1)m+p$.

$\mathrm{g}(\varphi_{\mathrm{sy}}^{p}(k_1,\ldots,k_{m-1}))=(k_p-1)m+p$ follows from the definition of $\mathcal{S}_{\mathrm{sy}}^{p}(m)$ and the fact that for every numerical semigroup $\mathrm{g}(S)+m=\max(\mathrm{Ap}(S,m))$.

REMARK 144. (1) It is well known that the Frobenius number of any symmetric numerical semigroup is odd. Hence if both m and p are even, then $\mathcal{T}_{\mathrm{sy}}^{p}(m)$ is empty.

(2) We know by Theorem 121 that $\mathcal{T}_{\mathrm{sy}}^p(m) = \{\beta_1,\ldots,\beta_k\} + \langle\gamma_1,\ldots,\gamma_s\rangle$ for some β_i's and γ_j's in $\mathbb{N}^{m-1}$. We also have procedures for computing them (see the remark after Theorem 121). Hence

$$\mathcal{S}_{\mathrm{sy}}^p(m) = \{\langle m, k_1m+1,\ldots,k_{m-1}m+m-1\rangle \mid (k_1,\ldots,k_{m-1}) \in \mathcal{T}_{\mathrm{sy}}^p(m)\}$$

gives a complete "parametric" description of $\mathcal{S}_{\mathrm{sy}}^p(m)$.

(3) Set $\mathcal{S}_{\mathrm{sy}}(m) = \{S \in \mathcal{S}(m) \mid S \text{ is symmetric}\}$. Clearly $\mathcal{S}_{\mathrm{sy}}(m) = \bigcup_{p=1}^{m-1} \mathcal{S}_{\mathrm{sy}}^p(m)$.

EXAMPLE 145. Let us describe $\mathcal{S}_{\mathrm{sy}}(4)$. By the remarks given above, it suffices to compute $\mathcal{S}_{\mathrm{sy}}^1(4)$ and $\mathcal{S}_{\mathrm{sy}}^3(4)$; $\mathcal{S}_{\mathrm{sy}}(4) = \mathcal{S}_{\mathrm{sy}}^1(4) \cup \mathcal{S}_{\mathrm{sy}}^3(4)$.

For $\mathcal{T}_{\mathrm{sy}}^1(4)$ we obtain the following system of equations and inequalities:

$$\begin{array}{lcl} x_1 & \geq & 1, \\ x_2 & \geq & 1, \\ x_3 & \geq & 1, \\ 2x_1 - x_2 & \geq & 0, \\ x_1 + x_2 - x_3 & \geq & 0, \\ 2x_3 - x_2 & \geq & -1, \\ x_2 + x_3 - x_1 & = & -1, \end{array}$$

and using the procedures explained in Section 1 we obtain

$$\mathcal{T}_{\mathrm{sy}}^1(4) = \{(3,1,1),(4,2,1),(5,3,1)\} + \langle(1,0,1),(2,1,1),(3,2,1)\rangle.$$

As for $\mathcal{T}_{\mathrm{sy}}^3(4)$ we start from the system

$$\begin{array}{lcl} x_1 & \geq & 1, \\ x_2 & \geq & 1, \\ x_3 & \geq & 1, \\ 2x_1 - x_2 & \geq & 0, \\ x_1 + x_2 - x_3 & = & 0, \\ 2x_3 - x_2 & \geq & -1, \\ x_2 + x_3 - x_1 & \geq & -1, \end{array}$$

and obtain

$$\mathcal{T}_{\mathrm{sy}}^3(4) = \{(1,1,2),(1,2,3)\} + \langle(1,0,1),(1,2,3),(1,1,2)\rangle.$$

Hence

$$\mathcal{S}_{\mathrm{sy}}(4) = \{\langle 4, k_14+1, k_24+2, k_34+3\rangle \mid (k_1,k_2,k_3,k_3) \in \mathcal{T}_{\mathrm{sy}}^1(4) \cup \mathcal{T}_{\mathrm{sy}}^3(4)\}.$$

□

The concept of MEDSY-semigroup among symmetric numerical semigroups is the analogous to MED-semigroup in numerical semigroups. Actually, by Proposition 12, the embedding dimension of a symmetric numerical semigroup cannot be m for $m \geq 3$ (symmetric numerical semigroups with multiplicity 2 are of the form $\langle 2, 2k+1\rangle$ for $k \neq 0$), whence a **MEDSY**-semigroup is a symmetric numerical semigroup with multiplicity $m \geq 3$ and embedding dimension equal to $m-1$. If S is a symmetric semigroup with multiplicity m and Frobenius number g, then S is a MEDSY-semigroup if and only if the set $(\mathrm{Ap}(S,m) \setminus \{0, w(p)\}) \cup \{m\}$ is a minimal system of generators for S with $\mathrm{g} \equiv p (\mathrm{mod}\, m)$, $p \in \{1, \ldots, m-1\}$. Using this idea we proof the following result.

LEMMA 146. *Let S be a numerical semigroup with multiplicity $m \geq 3$,* $\mathrm{Ap}(S,m) = \{0 = w(0), w(1), \ldots, w(m-1)\}$ *and Frobenius number* $\mathrm{g} \equiv p(\mathrm{mod}\, m)$, $p \in \{1, \ldots, m-1\}$.

(1) If $p = m-1$, then S is a MEDSY-semigroup if and only if for all $1 \leq i \leq j \leq m-1$ such that $i + j \neq m-1$ there exist $0 \leq k \leq m-1$ and $t \in \mathbb{N} \setminus \{0\}$ such that $w(i) + w(j) = tm + w(k)$ and $w(i) + w(j) = w(m-1)$ for $i + j = m-1$.

(2) If $p \neq m-1$, then S is a MEDSY-semigroup if and only if for all $1 \leq i \leq j \leq m-1$ such that $i + j \not\equiv p(\mathrm{mod}\, m)$ there exist $0 \leq k \leq m-1$ and $t \in \mathbb{N} \setminus \{0\}$ such that $w(i) + w(j) = tm + w(k)$ and $w(i) + w(j) = w(p)$ for $i + j \equiv p(\mathrm{mod}\, m)$.

PROOF. Assume that S is a MEDSY-semigroup. Note that $\mathrm{g} + m = \max(\mathrm{Ap}(S,m))$, whence if $\mathrm{g} \equiv p(\mathrm{mod}\, m)$, then $\mathrm{g} + m = w(p)$. Since S is is symmetric, Proposition 12, states that $w(p) - w(i) = w(j)$, for some $j \in \{0, \ldots, m-1\}$. Thus $w(p) \equiv p \equiv w(i) + w(j) \equiv i + j(\mathrm{mod}\, m)$ forces $i + j \equiv p(\mathrm{mod}\, m)$ (the case $p = m-1$ in this setting leads to $i+j=m-1$). Now for $i+j \not\equiv p(\bmod m)$, in view of Lemma 124 there exist $k \in \{0,\ldots,m-1\}$ and $t \in \mathbb{N}$ such that $w(i)+w(j)=tm+w(k)$. The integer t cannot be zero when S is a MEDSY-semigroup, because $\{m,w(1),\ldots,w(m-1)\} \setminus \{w(p)\}$ would not be a minimal system of generators for S.

Conversely, the condition $w(i)+w(j)=w(p)$ for $i+j \equiv p(\bmod m)$, by Proposition 12, implies that S is symmetric. Now assume that S is not a MEDSY-semigroup, or equivalently, that $\{m,w(1),\ldots,w(m-1)\} \setminus \{w(p)\}$ is not a minimal system of generators for S. Then there exist $i \neq p$ such that $w(i)=\sum_{j=1,j\neq i,p}^{m-1} a_j w(j)$ for some $a_j \in \mathbb{N}$. The reader can check that after using the rules $w(i)+w(j)=tm+w(k)$ this leads to a contradiction. $\square$

Let m be an integer greater than two, let $p \in \{1,\ldots,m-1\}$ and let

$$\mathcal{MEDSY}^p(m)=\{S \in \mathcal{S}(m) \mid S \text{ is a MEDSY-semigroup and } \mathrm{g}(S) \equiv p(\bmod m)\}.$$

Using last lemma it is easy to proof the next result.

LEMMA 147. *Let m be an integer greater than two and let $p \in \{1,\ldots,m-1\}$. Let $S \in \mathcal{MEDSY}^p(m)$ with $\mathrm{Ap}(S,m)=\{0=w(0),w(1),\ldots,w(m-1)\}$. For every $i \in \{1,\ldots,m-1\}$ let $k_i \in \mathbb{N}$ be such that $w(i)=k_im+i$. Then*

(1) If $p=m-1$, then
- *(a) $k_i \geq 1$ for all $i \in \{1,\ldots,m-1\}$,*
- *(b) $k_i+k_j-k_{i+j} \geq 1$ for all $1 \leq i \leq j \leq m-1$ with $i+j<m-1$,*
- *(c) $k_i+k_j-k_{m-1}=0$ for all $1 \leq i \leq j \leq m-1$ with $i+j=m-1$,*
- *(d) $k_i+k_j-k_{i+j-m} \geq 0$ for all $1 \leq i \leq j \leq m-1$ with $i+j>m$.*

(2) If $p \neq m-1$, then
- *(a) $k_i \geq 1$ for all $i \in \{1,\ldots,m-1\}$,*
- *(b) $k_i+k_j-k_{i+j} \geq 1$ for all $1 \leq i \leq j \leq m-1$ with $i+j \leq m-1$ and $i+j \neq p$,*
- *(c) $k_i+k_j-k_p=0$ for all $1 \leq i \leq j \leq m-1$ with $i+j=p$,*

(d) $k_i + k_j - k_{i+j-m} \geq 0$ *for all* $1 \leq i \leq j \leq m-1$ *with* $i+j > m$ *and* $i+j \neq m+p$,

(e) $k_i + k_j - k_p = -1$ *for all* $1 \leq i \leq j \leq m-1$ *with* $i+j > m$ *and* $i+j = m+p$.

In the first case $(k_1, \ldots, k_{m-1})$ belongs to $\mathcal{M}_{\mathrm{sy}}^{m-1}(m)$, the set of nonnegative integer solutions of the system

$$(5) \qquad \begin{cases} x_i \geq 1 & i \in \{1, \ldots, m-1\}, \\ x_i + x_j - x_{i+j} \geq 1 & 1 \leq i \leq j \leq m-1 \text{ with } i+j < m-1, \\ x_i + x_j - x_{m-1} = 0 & 1 \leq i \leq j \leq m-1 \text{ with } i+j = m-1, \\ x_i + x_j - x_{i+j-m} \geq 0 & 1 \leq i \leq j \leq m-1 \text{ with } i+j > m, \end{cases}$$

which by Lemma 133 is a subsemigroup of $\mathbb{N}^{m-1}$. In the second case $(k_1, \ldots, k_{m-1})$ belongs to $\mathcal{M}_{\mathrm{sy}}^{p}(m)$ determined by the system

$$(6) \qquad \begin{cases} x_i \geq 1 & i \in \{1, \ldots, m-1\}, \\ x_i + x_j - x_{i+j} \geq 1 & 1 \leq i \leq j \leq m-1, i+j \leq m-1, i+j \neq p, \\ x_i + x_j - x_p = 0 & 1 \leq i \leq j \leq m-1, i+j = p, \\ x_i + x_j - x_{i+j-m} \geq 0 & 1 \leq i \leq j \leq m-1, i+j > m, i+j \neq m+p, \\ x_i + x_j - x_p = -1 & 1 \leq i \leq j \leq m-1, i+j > m, i+j = m+p. \end{cases}$$

(This set is not a subsemigroup of $\mathbb{N}^{m-1}$.)

Proceeding as we did in Section 2 we obtain the following result which is the restriction of Theorem 143 to $\mathcal{MEDSY}^p(m)$.

THEOREM 148. *Let m be an integer greater than two and let $p \in \{1, \ldots, m-1\}$. The map $\psi_{\mathrm{sy}}^p : \mathcal{M}_{\mathrm{sy}}^p(m) \to \mathcal{MEDSY}^p(m)$, defined as*

$$\psi_{\mathrm{sy}}^p(k_1, \ldots, k_{m-1}) = \langle m, k_1 m + 1, k_2 m + 2, \ldots, k_{m-1} m + m - 1 \rangle$$

is a one-to-one correspondence. Furthermore,

$$\mathrm{Ap}(\psi_{\mathrm{sy}}^p(k_1, \ldots, k_{m-1}), m) = \{0, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$$

and $\mathrm{g}(\psi_{\mathrm{sy}}^p(k_1, \ldots, k_{m-1})) = (k_p - 1)m + p$.

Let m be an integer greater than two and let $p \in \{1,\ldots,m-1\}$. Define $\overline{MEDSY}^p(m)$ to be the set of elements $S \in \mathcal{S}_{\mathrm{sy}}^p(m)$ such that the minimal generators of S different from m are greater than $2m$. Let $\overline{\mathcal{M}}_{\mathrm{sy}}^{m-1}(m)$ be the set of nonnegative solutions of the system (5) obtained replacing $x_i \geq 1$ by $x_i \geq 2$; $\overline{\mathcal{M}}_{\mathrm{sy}}^p(m)$ is obtained performing the same operation in (6). The following result is a direct consequence of Theorem 148.

THEOREM 149. *Let m be an integer greater than two and let $p \in \{1,\ldots,m-1\}$. The map $\overline{\Psi}_{\mathrm{sy}}^p : \overline{\mathcal{M}}_{\mathrm{sy}}^p(m) \to \overline{\mathcal{MEDSY}}^p(m)$, defined as*

$$\overline{\Psi}_{\mathrm{sy}}^p(k_1,\ldots,k_{m-1}) = \langle m, k_1m+1, k_2m+2, \ldots, k_{m-1}m+m-1\rangle$$

is a one-to-one correspondence. Furthermore,

$$\mathrm{Ap}(\overline{\Psi}_{\mathrm{sy}}^p(k_1,\ldots,k_{m-1}),m) = \{0, k_1m+1, \ldots, k_{m-1}m+m-1\}$$

and $\mathrm{g}(\overline{\Psi}_{\mathrm{sy}}^p(k_1,\ldots,k_{m-1})) = (k_p-1)m+p$.

As in Proposition 138, the reader can check, by just comparing the defining systems for $\mathcal{T}_{\mathrm{sy}}^p(m)$ and $\overline{\mathcal{M}}_{\mathrm{sy}}^p(m)$, that the following holds.

PROPOSITION 150. *Let m be an integer greater than two and let $p \in \{1,\ldots,m-1\}$. Then*

$$\mathcal{T}_{\mathrm{sy}}^p(m) = \{(-1,\ldots,-1)-e_p\} + \overline{\mathcal{M}}_{\mathrm{sy}}^p(m).$$

As a consequence of this result we obtain the following Corollary appearing in **[31]**.

COROLLARY 151. *There is a one-to-one correspondence between the set of symmetric numerical semigroups with multiplicity $m > 2$ and the set of MEDSY-semigroups with multiplicity m and minimal generators different from m greater than $2m$.*

Observe also that $\overline{\mathcal{M}}_{\text{sy}}^{m-1}(m)$ is a subsemigroup of $\mathbb{N}^{m-1}$ by Lemma 138, whence $\mathcal{S}_{\text{sy}}^{m-1}(m)$ can be endowed with a semigroup structure with the operation $*$ defined as in Section 3. This semigroup is isomorphic to $\overline{\mathcal{M}}_{\text{sy}}^{m-1}(m)$.

5. Numerical semigroups with monotonic Apéry set

We say that a numerical semigroup S has **monotonic Apéry set** if $w(1) < w(2) < \cdots < w(\mathrm{m}-1)$, with $\{0, w(1), \ldots, w(\mathrm{m}-1)\} = \mathrm{Ap}(S, \mathrm{m})$, $w(i) \equiv i(\bmod m)$ for all $i \in \{1, \ldots, m-1\}$. Our main goal in this section is to study the set $C(m)$ of numerical semigroups with monotonic Apéry set and multiplicity m. We show that there is a one-to-one correspondence between $C(m)$ and a finitely generated subsemigroup of $\mathbb{N}^{m-1}$, and for proving this correspondence we use again Lemma 124.

The main result is Theorem 154, and for its proof we need two lemmas.

LEMMA 152. *Let m be an integer greater that one and let S be in $C(m)$ with*

$$\mathrm{Ap}(S, m) = \{0 = w(0) < w(1) < \cdots < w(m-1)\}.$$

For $i \in \{1, \ldots, m-1\}$, set $k_i \in \mathbb{N}$ to be the element such that $w(i) = k_i m + i$ (observe that $w(i) > i$, since $i < m$). Then $1 \leq k_1 \leq \cdots \leq k_{m-1}$ and $k_i + k_j \geq k_{i+j}$ for all $i, j \in \{1, \ldots, m-1\}$ such that $i + j \leq m-1$.

PROOF. Since S is a numerical semigroup of multiplicity m and $w(1) \in S \setminus \{0\}$, we have that $w(1) > m$ and thus $k_1 \geq 1$. As $w(1) < \cdots < w(m-1)$, we obtain $1 \leq k_1 \leq \cdots \leq k_{m-1}$. Now for $i, j \in \{1, \ldots, m-1\}$ such that $i + j \leq m-1$, Lemma 124 states that $w(i) + w(j) = tm + w(l)$ for some $t \in \mathbb{N}$ and $l \in \{0, \ldots, m-1\}$. Observe that $w(i+j) \equiv i + j \equiv w(i) + w(j) \equiv w(l) (\bmod m)$ and this forces l to be $i + j$, whence $k_i + k_j \geq k_{i+j}$. $\square$

We deduce that $(k_1,\dots,k_{m-1}) \in \mathbb{N}^{m-1}$ is determined uniquely for $S \in C(m)$ and it is a nonnegative integer solution of the system of linear Diophantine inequalities

$$\begin{gathered} x_1 \geq 1 \\ x_{i+1} - x_i \geq 0, \text{ for all } i \in \{1 \dots m-2\} \\ x_i + x_j - x_{i+j} \geq 0, \text{ for all } i,j \in \{1 \dots m-1\},\ i+j \leq m-1. \end{gathered}$$

Denote by $\mathcal{A}(m)$ the set of nonnegative solutions of this system of inequalities. From Lemma 133, we have that $\mathcal{A}(m)$ is a subsemigroup of $\mathbb{N}^{m-1}$.

The next result associates to each element of $\mathcal{A}(m)$ an element in $C(m)$.

LEMMA 153. *Let m be an integer greater than one and let $(k_1,\dots,k_{m-1}) \in \mathbb{N}^{m-1}$ with $1 \leq k_1 \leq \cdots \leq k_{m-1}$ and $k_i + k_j \geq k_{i+j}$ for all $i,j \in \{1,\dots,m-1\}$ such that $i+j \leq m-1$. Then there exists a numerical semigroup S with multiplicity m and* $\mathrm{Ap}(S,m) = \{0, k_1 m + 1, k_2 m + 2, \dots, k_{m-1} m + m - 1\}$.

PROOF. We make use of Lemma 124 with

$$X = \{0 = w(0), k_1 m + 1 = w(1), \dots, k_{m-1} m + 1 = w(m-1)\}.$$

Then the monoid $S = \langle X \cup \{m\} \rangle$ is a numerical semigroup of multiplicity m. Now we have to check that for $i,j \in \{1,\dots,m-1\}$ there exist $k \in \{0,\dots,m-1\}$ and $t \in \mathbb{N}$ such that $w(i) + w(j) = w(k) + tm$. For given $i,j \in \{1,\dots,m\}$ we distinguish three cases.

(1) If $i+j \leq m-1$, then $w(i) + w(j) = tm + w(i+j)$ with $t = k_i + k_j - k_{i+j} \in \mathbb{N}$ (here arises the condition $k_i + k_j \geq k_{i+j}$).
(2) If $i+j = m$, then $w(i) + w(j) = tm + w(0)$, with $t = k_i + k_j + 1 \in \mathbb{N}$.
(3) If $i+j > m$, then $i \geq i+j-m \geq 1$, whence $k_i m + i \geq k_{i+j-m} m + i + j - m$ and this leads to $k_i m + i + k_j m + j \geq k_{i+j-m} m + i + j - m$. Since $k_i m + i + k_j m + j \equiv k_{i+j-m} m + i + j - m \pmod{m}$, we deduce that there exist $t \in \mathbb{N}$ such that $w(i) + w(j) = tm + w(i+j-m)$.

□

With the above lemmas we prove the following result.

THEOREM 154. *Let*

$$\mathcal{C}(m) = \left\{ \begin{array}{c} S \text{ numerical} \\ \text{semigroup} \end{array} \,\middle|\, \begin{array}{c} \mathrm{m}(S) = m, \\ S \text{ has monotonic Apéry set} \end{array} \right\}$$

and

$$\mathcal{A}(m) = \left\{ (k_1, \ldots, k_{m-1}) \in \mathbb{N}^{m-1} \,\middle|\, \begin{array}{c} 1 \le k_1 \le \cdots \le k_{m-1}, \\ k_i + k_j \ge k_{i+j} \text{ for } 2 \le i+j \le m-1 \end{array} \right\}.$$

Then the map $\varphi : \mathcal{A}(m) \to \mathcal{C}(m)$ *defined by*

$$\varphi(k_1, \ldots, k_{m-1}) = \langle m, k_1 m + 1, \ldots, k_{m-1} m + m - 1 \rangle$$

is one-to-one. Moreover,

$$\mathrm{Ap}(\varphi(k_1, \ldots, k_{m-1}), m) = \{0, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}.$$

PROOF. In fact, by Lemma 153, we have that φ is a well defined map with

$$\mathrm{Ap}(\varphi(k_1, \ldots, k_{m-1}), m) = \{0, k_1 m + 1, \ldots, k_{m-1} m + m - 1\}$$

and, from Lemma 152, we can conclude that φ is a bijective map. □

Assume that $A = \{a_1, \ldots, a_r\}$ is a system of generators of $\mathcal{A}(m)$, with $a_i = (a_{i_1}, \ldots, a_{i_{m-1}})$ for $i \in \{1, \ldots, r\}$. Then

$$\mathcal{C}(m) = \{\langle m, (\sum_{i=1}^{r} \lambda_i a_{i_1}) m + 1, \ldots, (\sum_{i=1}^{r} \lambda_i a_{i_{m-1}}) m + m - 1 \rangle \mid (\lambda_1, \ldots, \lambda_r) \in \mathbb{N}^r \setminus \{0\}\}.$$

We illustrate this result with an example.

EXAMPLE 155. Let us describe $\mathcal{C}(5)$. We have that

$$\mathcal{A}(5) = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{N}^4 \,\middle|\, \begin{array}{c} x_1 \ge 0, x_2 \ge x_1, x_3 \ge x_2, x_4 \ge x_3 \\ 2x_1 \ge x_2, x_1 + x_2 \ge x_3, x_1 + x_3 \ge x_4 \\ 2x_2 \ge x_4 \end{array} \right\}.$$

and using the process explained in Section 1, we get that

$$\mathcal{A}(5) = \langle (1,2,2,2), (1,1,1,1), (1,1,1,2), (1,2,2,3),$$
$$(1,1,2,2), (1,2,3,3), (1,2,3,4) \rangle.$$

Hence $C(5)$ consists on all numerical semigroups of the form

$$\langle 5, k_1 5+1, k_2 5+2, k_3 5+3, k_4 5+4\rangle$$

such that $(k_1,k_2,k_3,k_4) \in \mathcal{A}(5)$, or in other words,

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix} + \lambda_4 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix} + \lambda_5 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} + \lambda_6 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 3 \end{pmatrix} + \lambda_7 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix},$$

for some $(\lambda_1,\ldots,\lambda_7) \in \mathbb{N}^7 \setminus \{0\}$. □

Next we give the minimal systems of generators for $\mathcal{A}(m)$ with $m \in \{2,\ldots,8\}$, which describe the set of numerical semigroups with monotonic Apéry set and multiplicity up to 8.

- $\mathcal{A}(2)$ is generated by $\{1\}$,
- $\mathcal{A}(3)$ is generated by $\{(1,1),(1,2)\}$,
- $\mathcal{A}(4)$ is generated by $\{(1,1,1),(1,1,2),(1,2,2),(1,2,3)\}$,
- a system of generators for $\mathcal{A}(5)$ is given in Example 155,
- $\mathcal{A}(6)$ is generated by

$$\{(1,2,2,2,2),(1,1,1,1,1),(1,2,2,2,3),(1,1,1,1,2),(1,1,1,2,2),$$
$$(1,2,2,3,3),(1,2,2,3,4),(1,2,3,3,3),(1,1,2,2,2),(1,1,2,2,3),$$
$$(1,2,3,3,4),(1,2,3,4,4),(1,2,3,4,5)\},$$

- $\mathcal{A}(7)$ is generated by

$$\{(1,2,2,2,2,2),(1,1,1,1,1,1),(1,2,2,2,2,3),(1,1,1,1,1,2),(1,2,2,3,3,3),$$
$$(1,1,1,1,2,2),(1,2,2,2,3,4),(1,2,2,3,3,3),(1,1,1,2,2,2),(1,2,2,3,3,4),$$
$$(1,2,2,3,4,4),(1,2,3,3,3,3),(1,1,2,2,2,2),(1,2,3,3,3,4),(1,1,2,2,2,3),$$
$$(1,2,3,3,4,4),(1,1,2,2,3,3),(1,2,3,3,4,4),(1,2,3,4,4,4),(2,2,3,4,4,5),$$
$$(1,2,3,4,4,5),(2,3,4,6,6,8),(2,2,3,4,5,6),(1,2,3,4,5,5),(1,2,3,4,5,6)\},$$

- $\mathcal{A}(8)$ is generated by

$$\{(1,2,3,3,3,3,3),(1,2,2,2,2,2,2),(1,1,2,2,2,2,2),(1,1,1,1,1,1,1),$$
$$(1,2,2,2,2,2,3),(1,1,1,1,1,1,2),(1,2,3,3,3,3,4),(1,1,2,2,2,2,3),$$
$$(1,2,2,2,2,3,3),(1,1,1,1,1,2,2),(1,2,2,2,2,3,4),(1,2,3,3,3,4,4),$$
$$(1,1,2,2,2,3,3),(1,2,3,3,3,4,5),(1,2,2,2,3,3,3),(1,1,1,1,2,2,2),$$
$$(1,2,2,2,3,3,4),(1,2,2,2,3,4,4),(1,2,3,3,4,4,4),(1,1,2,2,3,3,3),$$
$$(1,2,3,3,4,4,5),(1,1,2,2,3,3,4),(1,2,3,3,4,5,5),(1,2,3,3,4,5,6),$$
$$(1,2,3,4,4,4,4),(1,2,2,3,3,3,3),(1,1,1,2,2,2,2),(1,2,2,3,3,3,4),$$
$$(1,1,1,2,2,2,3),(1,2,3,4,4,4,5),(2,2,3,4,4,6,6),(1,2,2,3,3,4,4),$$
$$(1,2,2,3,3,4,5),(1,2,3,4,4,5,5),(1,2,3,4,4,5,6),(2,3,4,6,6,8,8),$$
$$(2,3,4,6,6,8,9),(2,2,3,4,5,6,6),(2,2,3,4,5,6,7),(1,2,2,3,4,4,4),$$
$$(1,2,2,3,4,4,5),(1,2,3,4,5,5,5),(1,2,3,4,5,5,6),(1,2,3,4,5,6,6),$$
$$(1,2,3,4,5,6,7)\}.$$

We finish this chapter studding the symmetric elements of $C(m)$. If we denote by $C_{\text{sy}}(m) = \{S \in C(m) \mid S \text{ is symmetric}\}$, then we show that $C_{\text{sy}}(m)$ is isomorphic to a subsemigroup of $\mathcal{A}(m)$ which we denote by $\mathcal{A}_{\text{sy}}(m)$.

Let S be an element of $C(m)$ and $\mathrm{Ap}(S,m)=\{0<k_1m+1<\cdots<k_{m-1}m+m-1\}$. Then, from Lemma 12, we deduce that S is symmetric if and only if $k_i+k_{m-1-i}=k_{m-1}$ for all $i\in\{1,\ldots,m-2\}$. As a consequence of this fact and Theorem 154 we obtain the following result.

PROPOSITION 156. *Let m be an integer greater than one, $C_{\mathrm{sy}}(m)$ be the subset of symmetric semigroups of $C(m)$ and $\mathcal{A}_{\mathrm{sy}}(m)$ be the set of elements in $\mathcal{A}(m)$ such that $k_i+k_{m-1-i}=k_{m-1}$ for all $i\in\{1,\ldots,m-2\}$.*

Then the map $\varphi:\mathcal{A}_{\mathrm{sy}}(m)\to C_{\mathrm{sy}}(m)$ defined by

$$\varphi(k_1,\ldots,k_{m-1})=\langle m,k_1m+1,\ldots,k_{m-1}m+m-1\rangle$$

is one-to-one. Moreover,

$$\mathrm{Ap}(\varphi(k_1,\ldots,k_{m-1}),m)=\{0,k_1m+1,\ldots,k_{m-1}m+m-1\}.$$

Thus the element $(k_1,\ldots,k_{m-1})$ in $\mathcal{A}_{\mathrm{sy}}(m)$ is the set of nonnegative integer solutions of the system

$$\begin{gathered}x_1\geq 1\\ x_{i+1}-x_i\geq 0,\ \text{for all } i\in\{1\ldots m-2\}\\ x_i+x_j-x_{i+j}\geq 0,\ \text{for all } i,j\in\{1\ldots m-1\},\ i+j\leq m-1\\ x_i+x_{m-1-i}=x_{m-1},\ \text{for all } i\in\{1,\ldots,m-2\}.\end{gathered}$$

CHAPTER 4

MED, Arf and saturated closure of a numerical semigroup

The purpose of this chapter is the study of the class of numerical semigroups with maximal embedding dimension (MED-semigroups), and two types of this kind of semigroups that are of particular interest: the Arf and saturated numerical semigroups. For describing and working with MED-semigroups (respectively Arf, saturated) one can use their system of generators, which do not take any advantage of their additional structure. As a fundamental result of this chapter we will see that every numerical MED-semigroup (respectively Arf, saturated) admits a unique minimal MED (respectively Arf, SAT) system of generators, which is in general smaller than its classical minimal system of generators.

In Section 1, we deduce that the intersection of two MED-semigroups with the same multiplicity is again a MED-semigroup. This fact allows us introduce the concept of MED system of generators. We see that the set of MED semigroups with multiplicity m can be arranged in a tree whose root is the semigroup $\langle m, m+1, \ldots m+m-1\rangle$. Finally, in this section, from Theorem 170 we can compute the MED closure of any numerical semigroup that is, the minimum (with respect to set inclusion) MED-semigroup with the same multiplicity containing it.

In Section 2, from the concept of Arf semigroup, we deduce that the intersection of two Arf numerical semigroups is again an Arf numerical semigroup. This again is used to introduce the concept of Arf system of generators. This study allows us to arrange the set of all Arf numerical semigroups in a binary tree whose root is $\mathbb{N}$. We also present an algorithmic procedure for computing, from a finite subset X of $\mathbb{N}$ with $\gcd(X) = 1$, the elements of $\mathrm{Arf}(X)$.

In Section 3, we characterize the subsets of $\mathbb{N}$ that are saturated numerical semigroups. We see that the intersection of two saturated numerical semigroups is again saturated, from this we introduce the concept of SAT system of generators for a saturated numerical semigroup. This enables us to present a recursive method for computing the set of all saturated numerical semigroups, and arrange it in a binary tree with no leaves and rooted in $\mathbb{N}$. Theorem 198, allows us to construct a saturated numerical semigroup from one of its SAT system of generators.

1. MED systems of generators

In this section we introduce the concept of MED system of generators for MED-semigroups. From this concept we present a recursive method for computing the set of all MED-semigroups with fixed multiplicity. Also we compute the MED closure of a numerical semigroup.

In the bibliography there are many characterizations of MED-semigroups (see for instance [**5**, Proposition I.2.9] were a series of them have been collected). Condition (v) of the abovementioned proposition tells us that a numerical semigroup S is a MED-semigroup if and only if $(S \setminus \{0\}) - \mathrm{m}(S)$ is a semigroup. As an immediate consequence we obtain the following result.

PROPOSITION 157. *If S is a numerical semigroup, then the following conditions are equivalent:*

(1) S is a MED-semigroup,

(2) for every $x, y \in S$ such that $x \geq y \geq \mathrm{m}(S)$, then $x + y - \mathrm{m}(S) \in S$.

The following example shows that the intersection of two MED-semigroups is not in general a MED-semigroup.

EXAMPLE 158. Let $S_1 = \langle 3,7,11\rangle$ and $S_2 = \langle 5,6,7,8,9\rangle$, which are MED-semigroups. However, $S_1 \cap S_2 = \langle 6,7,9,10,11\rangle$ is not a MED-semigroup, since $\mathrm{m}(S_1 \cap S_2) = 6 \neq 5 = \mu(S_1 \cap S_2)$.

PROPOSITION 159. *Let S_1 and S_2 be two MED-semigroups with multiplicity m. Then $S_1 \cap S_2$ is a MED-semigroup of multiplicity m.*

PROOF. The result follows easily using the fact that $S_1 \cap S_2$ has multiplicity m and then applying Proposition 157. □

Recall that if S is a numerical semigroup, then $\mathbb{N} \setminus S$ has finitely many elements, whence there are only finitely many numerical semigroups containing S.

For a given subset X of $\mathbb{N}$, set $X^* = X \setminus \{0\}$. If $\gcd(X) = 1$, then denote by $\mathrm{MED}(X)$ the smallest MED-semigroup containing X and with multiplicity $\min(X^*)$. Observe that the set of MED-semigroups with multiplicity $\min(X^*)$ containing X is not empty, since $\{0, \min(X^*), \min(X^*)+1, \rightarrow\}$ is in this set. Note also that this set is finite by the above remark. Hence by Proposition 159, $\mathrm{MED}(X)$ is just the intersection of all MED-semigroups with multiplicity $\min(X^*)$ containing X, and we call it the **MED closure** of X.

If S is a MED semigroup and $X \subseteq \mathbb{N}$ is such that $\gcd(X) = 1$ and $\mathrm{MED}(X) = S$, then we will say that X is a **MED system of generators** of S and it is a **minimal** MED system of generators provided that no proper subset of X is a MED system of generators of S. Trivially, if $S = \langle n_1, \ldots, n_p\rangle$ is a MED-semigroup, then $\{n_1, \ldots, n_p\}$ is a MED system of generators of S, whence every MED-semigroup admits a MED system of generators (another trivial MED system of generators is the semigroup itself). Our next goal is to prove that every MED-semigroup admits a unique minimal MED system of generators.

LEMMA 160. *Let S be a nontrivial submonoid of* $\mathbb{N}$ *and let* $m = \min(S^*)$. *Then*

$$S' = \{x_1 + \cdots + x_k - am \mid k \in \mathbb{N} \setminus \{0,1\}, a \in \mathbb{Z}, a \leq k-1, x_1, \ldots, x_k \in S^*\} \cup \{0\}$$

is a submonoid of $\mathbb{N}$, $\min(S'^*) = m$ *and* $S \subseteq S'$.

PROOF. For $x \in S^*$, take $x_1 = x$, $x_2 = m$ ($k = 2$) and $a = 1$. Then $x = x_1 + x_2 - am \in S'$, which yields $S \subseteq S'$. Next we prove that S' is a semigroup. Let $\alpha, \beta \in S'^*$. Then $\alpha = x_1 + \cdots + x_k - am$ and $\beta = y_1 + \cdots + y_l - bm$, with $x_i, y_j \in S^*$, $k, l \in \mathbb{N} \setminus \{0,1\}$, $a \leq k-1$ and $b \leq l-1$. Clearly $\alpha + \beta = x_1 + \cdots + x_k + y_1 + \cdots + y_l - (a+b)m \in S'$. Finally observe that since $x_1, \ldots, x_k \in S^*$, we obtain that $x_i \geq m$, and as $a \leq k-1$, we have that $x_1 + \cdots + x_k - am \geq m$. Hence $\min(S'^*) = m$. □

LEMMA 161. *If S is a numerical semigroup, then* $S' \subseteq \mathrm{MED}(S)$.

PROOF. Let $s = x_1 + \cdots + x_k - am \in S'$. To prove that $s \in \mathrm{MED}(S)$, use induction on k (starting with $k = 2$) and apply Proposition 157 to $\mathrm{MED}(S)$. □

As a consequence of Proposition 157 and Lemmas 160 and 161 we obtain the following result.

PROPOSITION 162. *Let S be a numerical semigroup. The following conditions are equivalent:*

(1) S is a MED-semigroup,
(2) $S' \subseteq S$,
(3) $S' = S$.

Given S a nontrivial submonoid of $\mathbb{N}$, define recursively S^n by

- $S^0 = S$,
- $S^{n+1} = (S^n)'$.

If S is a numerical semigroup, then by Lemmas 160 and 161, we know that $S \subseteq S' \subseteq \mathrm{MED}(S)$, and that $\min(S'^*) = \mathrm{m}(S) = \mathrm{m}(\mathrm{MED}(S))$. Hence $\mathrm{MED}(S) \subseteq \mathrm{MED}(S') \subseteq \mathrm{MED}(S)$, which leads to $\mathrm{MED}(S) = \mathrm{MED}(S')$. Consequently, $\mathrm{MED}(S) = \mathrm{MED}(S^n)$ for all $n \in \mathbb{N}$, whence $S^n \subseteq \mathrm{MED}(S)$ for all $n \in \mathbb{N}$. Therefore we have that

$$S = S^0 \subseteq S' \subseteq \cdots \subseteq S^n \subseteq \cdots \subseteq \mathrm{MED}(S).$$

Since there are finitely many numerical semigroups containing S, at a certain step of this chain, it must happen that $S^p = S^{p+1}$. By definition, $S^{p+1} = S^{p'}$, and in view of Proposition 162, this implies that S^p is a MED-semigroup. Thus $S^p = \mathrm{MED}(S^p)$, which leads to $S^p = \mathrm{MED}(S)$, since $\mathrm{MED}(S^p) = \mathrm{MED}(S)$. We have proved the following result.

PROPOSITION 163. *Let S be a numerical semigroup, then there exists $p \in \mathbb{N}$ such that $S^p = \mathrm{MED}(S)$.*

LEMMA 164. *Let S be a MED-semigroup, and let A be a MED system of generators of S. For every $s \in S$, set*

$$\mathrm{B}(s) = \{a \in A \mid a \leq s\}.$$

For every $n \in \mathbb{N}$, if $s \in \langle A \rangle^n$, then $s \in \langle \mathrm{B}(s) \rangle^n$.

PROOF. We use induction on n. For $n = 0$, the result follows trivially. As induction hypothesis assume that $s \in \langle A \rangle^n$ implies $s \in \langle \mathrm{B}(s) \rangle^n$. Let $s \in \langle A \rangle^{n+1}$. Then $s = x_1 + \cdots + x_k - am$ for some $x_i \in \langle A \rangle^n \setminus \{0\}$, $k \in \mathbb{N} \setminus \{0,1\}$ and $a \leq k-1$. By induction hypothesis we know that $x_i \in \langle \mathrm{B}(x_i) \rangle^n$ for all $i \in \{1,\ldots,k\}$. Since for all i, we get $s = x_i + (\sum_{j \neq i}^k x_j - am)$, $x_j \geq m$ (Lemma 160) and $a \leq k-1$, we have that $s \geq x_i$ for all i and thus $x_i \in \langle \mathrm{B}(s) \rangle^n$. Therefore $s \in \langle \mathrm{B}(s) \rangle^{n+1}$. □

THEOREM 165. *Let A and B be minimal MED systems of generators of a MED-semigroup S. Then $A = B$.*

PROOF. Assume that $A = \{n_1 < n_2 < \cdots\}$ and $B = \{m_1 < m_2 < \cdots\}$. From the definition of MED system of generators, it follows that $n_1 = m_1 = \mathrm{m}(S)$ (since both A and B are minimal systems of generators, they do not contain the zero element). Suppose that $A \neq B$ and let r be the least positive integer such that $n_r \neq m_r$ (observe that this integer exists since $n_1 = m_1$, $A \not\subseteq B$ and $B \not\subseteq A$). Without loss of generality, assume that $m_r < n_r$. By Proposition 163, there exists $p \in \mathbb{N}$ such that $m_r \in \langle A \rangle^p$, and by Lemma 164 we have that $m_r \in \langle n_1, \ldots, n_{r-1} \rangle^p$, which by the definition of r is equal to $\langle m_1, \ldots, m_{r-1} \rangle^p$, whence $m_r \in \mathrm{MED}(B \setminus \{m_r\})$. This also implies that $S = \mathrm{MED}(B \setminus \{m_r\})$, in contradiction with the fact that B is a minimal MED system of generators of S. □

The preceding theorem allows us to introduce the concept of MED rank of a MED-semigroup. Let S be a MED-semigroup. The MED rank of S, denoted by $\mathrm{MED}-\mathrm{rank}(S)$, is the cardinality of its minimal MED system of generators. As we pointed out above, if $S = \langle n_1, \ldots, n_p \rangle$ and S is a MED semigroup, then $\mathrm{MED}(\{n_1, \ldots, n_p\}) = S$, and thus $\mathrm{MED}-\mathrm{rank}(S) \leq \mu(S) = \mathrm{m}(S)$.

Now our goal is to arrange the set of MED-semigroups with multiplicity m in a tree rooted by $\langle m, m+1, \ldots, m+m-1 \rangle$. The purpose of the following results will be to show how to construct this tree. First we describe how to construct the father of any vertex (not being of course the root) in the tree; by repeating the process we get the path from the given vertex to the root.

LEMMA 166. *Let S be a MED-semigroup with* $\mathrm{g}(S) > \mathrm{m}(S)$. *Then* $S \cup \{\mathrm{g}(S)\}$ *is also a MED-semigroup and* $\mathrm{m}(S) = \mathrm{m}(S \cup \{\mathrm{g}(S)\})$.

PROOF. We already know that $S \cup \{\mathrm{g}(S)\}$ is a numerical semigroup and as $\mathrm{g}(S) > \mathrm{m}(S)$, we have that $\mathrm{m}(S) = \mathrm{m}(S \cup \{\mathrm{g}(S)\})$.

- If $x, y \in S$, then as S is MED, we obtain that $x + y - \mathrm{m}(S) \in S \subset S \cup \{\mathrm{g}(S)\}$.

- If $\mathrm{g}(S) \in \{x, y\}$, then $x+y-\mathrm{m}(S) \geq \mathrm{g}(S)$ and thus $x+y-\mathrm{m}(S) \in S \cup \{\mathrm{g}(S)\}$.

□

Observe that the only numerical semigroup with multiplicity m and Frobenius number less than m is $\langle m, m+1, \ldots, m+m-1\rangle$. Given a numerical semigroup S we define S_n recurrently by

- $S_0 = S$,
- if $\mathrm{g}(S_n) > \mathrm{m}(S_n)$, then $S_{n+1} = S_n \cup \{\mathrm{g}(S_n)\}$; $S_{n+1} = S_n$, otherwise.

Clearly, there exists $n \in \mathbb{N}$ such that $S_n = \langle m, m+1, \ldots, m+m-1\rangle$. If S is a MED-semigroup, then Lemma 166 states that

$$S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_n = \{0, m, m+1, \rightarrow\}$$

is a chain of MED-semigroups. Moreover, $S_i = S_{i+1} \setminus \{a\}$ for some $a \in S_{i+1}$. This gives rise to the question: if S is a MED-semigroup, which $a \in S$ can be chosen so that $S \setminus \{a\}$ is a MED-semigroup?

LEMMA 167. *Let S be a MED-semigroup with multiplicity m and let $a \in S \setminus \{m\}$. The following conditions are equivalent:*

(1) *a belongs to the minimal MED system of generators of S,*

(2) *$S \setminus \{a\}$ is a MED-semigroup with multiplicity m.*

PROOF. *(1) implies (2).* $\mathrm{MED}(S \setminus \{a\})$ is properly contained in S, since otherwise the minimal MED system of generators of S would not contain a. Hence $S \setminus \{a\} \subseteq \mathrm{MED}(S \setminus \{a\}) \subset S$ and thus $\mathrm{MED}(S \setminus \{a\}) = S \setminus \{a\}$, which implies that $S \setminus \{a\}$ is a MED-semigroup.

(2) implies (1). If a does not belong to the minimal MED system of generators of S, then $\mathrm{MED}(S \setminus \{a\}) = S$ and therefore $S \setminus \{a\}$ is not a MED-semigroup. □

Recall that for every MED-semigroup S we have a sequence (path) $S = S_0 \subseteq \cdots \subseteq S_n = \{0, m, m+1, \rightarrow\}$ such that $S_i \cup \{\mathrm{g}(S_i)\} = S_{i+1}$. Thus S_i is obtained by removing a certain element of S_{i+1}, which actually becomes its Frobenius number. Next result describes this construction and tells us how to build up the sons of a given vertex in the tree (if there are any).

PROPOSITION 168. *Let S be a MED-semigroup with multiplicity m. The following conditions are equivalent:*

(1) *$S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$ with $\overline{S}$ a MED-semigroup of multiplicity m,*

(2) *the minimal MED system of generators of S contains an element a such that $a \neq m$ and $a > \mathrm{g}(S)$.*

PROOF. *(1) implies (2).* Let $S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$. Then $\overline{S} = S \setminus \{g(\overline{S})\}$ and since $\overline{S}$ is a MED-semigroup, Lemma 167 ensures that $\mathrm{g}(\overline{S})$ belongs to a minimal MED system of generators of S. As $\overline{S} \subseteq S$ and $\mathrm{g}(\overline{S}) \in S$, we deduce that $\mathrm{g}(\overline{S}) > \mathrm{g}(S)$ and clearly $\mathrm{g}(\overline{S}) \neq m = \mathrm{m}(S) = \mathrm{m}(\overline{S})$.

(2) implies (1). Let $a \neq m$ be an element of the minimal MED system of generators of S such that $a > \mathrm{g}(S)$. By Lemma 167, $\overline{S} = S \setminus \{a\}$ is a MED-semigroup with multiplicity m. Since $a > \mathrm{g}(S)$, we have that $a = \mathrm{g}(\overline{S})$. Hence $S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$. $\square$

The results presented so far in this section allow us to construct from $\langle m, m+1, \ldots, m+m-1\rangle$ the set of all MED-semigroups with multiplicity m (see the figure). This construction arranges this set in a tree, and as one gets farther from the root, the obtained MED-semigroups have larger Frobenius numbers. The father of any vertex S in the tree is $S \cup \{\mathrm{g}(S)\}$ provided that $S \neq \{0, m, m+1, \rightarrow\}$, and the possible sons are $S \setminus \{a\}$, with $a > \mathrm{g}(S)$ an element in the minimal MED system of generators of S other than m. By Proposition 168 a vertex has no sons (it is a leaf) if and only if its minimal

FIGURE 1. The tree of MED numerical semigroups with multiplicity 4

$$\begin{tikzcd}
 & & & \begin{array}{c}\mathrm{MED}(4,5),\\ g=3\end{array} \arrow[d] & & \\
 & & & \begin{array}{c}\mathrm{MED}(4,6,7),\\ g=5\end{array} \arrow[dl] \arrow[dr] & & \\
 & & \begin{array}{c}\mathrm{MED}(4,7,9),\\ g=6\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\mathrm{MED}(4,6,9),\\ g=7\end{array} \arrow[d] & \\
 & \begin{array}{c}\mathrm{MED}(4,9,10,11),\\ g=7\end{array} \arrow[dl] \arrow[d] \arrow[dr] & & \begin{array}{c}\mathrm{MED}(4,7),\\ g=9\end{array} & \begin{array}{c}\mathrm{MED}(4,6,11),\\ g=9\end{array} \arrow[d] & \\
\cdots & \cdots & \cdots & & \cdots &
\end{tikzcd}$$

MED system of generators has no elements different from its multiplicity and greater than its Frobenius number.

Next we present an algorithmic procedure for computing the MED closure of a given numerical semigroup. Observe that if S is a numerical semigroup generated by A, then $\mathrm{MED}(S) = \mathrm{MED}(A)$ and $\gcd(A) = 1$. Thus we focus our attention on finding a procedure for computing $\mathrm{MED}(X)$ for a given finite set X with $\gcd(X) = 1$.

LEMMA 169. *Let S be a numerical semigroup and let $m \in S^*$. Then $(m+S)\cup\{0\}$ is a MED-semigroup with multiplicity m.*

PROOF. Clearly, $(m+S)\cup\{0\}$ is a numerical semigroup with multiplicity m. Let $s_1, s_2 \in S$. Then $(m+s_1)+(m+s_2)-m = m+s_1+s_2 \in m+S$, which by Proposition 157 implies that $(m+S)\cup\{0\}$ is a MED-semigroup. $\square$

THEOREM 170. *Let $m, r_1, \ldots, r_p \in \mathbb{N}^*$ be such that* $\gcd\{m, r_1, \ldots, r_p\} = 1$. *Then*

$$\mathrm{MED}(m, m+r_1, \ldots, m+r_p) = (m + \langle m, r_1, \ldots, r_p\rangle)\cup\{0\}.$$

PROOF. By Lemma 169, $(m+\langle m, r_1, \ldots, r_p\rangle)\cup\{0\}$ is a MED-semigroup. Furthermore, $m, m+r_1, \ldots, m+r_p \in (m+\langle m, r_1, \ldots, r_p\rangle)\cup\{0\}$, which implies that

$\mathrm{MED}(m, m+r_1, \ldots, m+r_p) \subseteq (m+\langle m, r_1, \ldots, r_p\rangle) \cup \{0\}$. For the other inclusion, take $i, j \in \{1, \ldots, p\}$. Then as $m, m+r_i, m+r_j \in \mathrm{MED}(m, m+r_1, \ldots, m+r_p)$, by Proposition 157, we have that $m+r_i+r_j = (m+r_i)+(m+r_j)-m \in \mathrm{MED}(m, m+r_1, \ldots, m+r_p)$. Now take $k \in \{1, \ldots, p\}$. Since $m, m+r_i+r_j, m+r_k \in \mathrm{MED}(m, m+r_1, \ldots, m+r_p)$, we have that $m+r_i+r_j+r_k \in \mathrm{MED}(m, m+r_1, \ldots, m+r_p)$. Using this idea one gets that $m+\sum_{i=1}^{p} a_i r_i \in \mathrm{MED}(m, m+r_1, \ldots, m+r_p)$ for all $a_1, \ldots, a_p \in \mathbb{N}$. □

EXAMPLE 171.

$$\begin{aligned}\mathrm{MED}(5,8,9) &= (5+\langle 5,3,4\rangle) \cup \{0\} = (5+\{0,3,4,5,6,\rightarrow\}) \cup \{0\} \\ &= \{0,5,8,9,10,11,\rightarrow\} = \langle 5,8,9,11,12\rangle. \quad \square\end{aligned}$$

EXAMPLE 172. Let m be a positive integer.

$$\begin{aligned}\mathrm{MED}(m, m+1) &= (m+\langle m, 1\rangle) \cup \{0\} = (m+\mathbb{N}) \cup \{0\} \\ &= \langle m, m+1, \ldots, m+m-1\rangle.\end{aligned}$$

If $m \geq 2$,

$$\begin{aligned}\mathrm{MED}(m, m+2, m+3) &= (m+\langle m, 2, 3\rangle) \cup \{0\} = (m+\{0,2,3,\rightarrow\}) \cup \{0\} \\ &= \langle m, m+2, m+3, \ldots, m+m-1, m+m+1\rangle.\end{aligned}$$

□

The following result describes the set of MED-semigroups with MED-rank two.

COROLLARY 173. *Let m, r be two positive integers such that* $\gcd\{m, r\} = 1$. *Then*

$$\mathrm{MED}(m, m+r) = \langle m, m+r, m+2r, \ldots, m+(m-1)r\rangle.$$

PROOF. Applying Theorem 170 we obtain that $\mathrm{MED}(m, m+r) = (m+\langle m, r\rangle) \cup \{0\}$. Clearly $\langle m, m+r, \ldots, m+(m-1)r\rangle \subseteq (m+\langle m, r\rangle) \cup \{0\}$. For the other inclusion, take $a \in m+\langle m, r\rangle$. Then $a = (\lambda+1)m+\mu r$ for some $\lambda, \mu \in \mathbb{N}$. There exist $q, d \in \mathbb{N}$

such that $\mu = qm + d$, $d < m$. Hence $a = (\lambda + q + 1)m + dr \in \langle m, m+r, m+2r, \ldots, m+(m-1)r\rangle$. □

2. Arf systems of generators

In this section we introduce the concept of Arf system of generators for an Arf semigroup. This concept allows us to arrange the set of all Arf numerical semigroup in a binary tree. We also describe an algorithmic method to compute, from a finite subset X of $\mathbb{N}$ with $\gcd(X) = 1$, the elements of $\mathrm{Arf}(X)$.

A numerical semigroup S is an **Arf numerical semigroup** if for every $x, y, z \in S$ such that $x \geq y \geq z$, we have that $x + y - z \in S$ (see [**5**, Theorem I.3.4] for fifteen alternative characterizations of this property).

For $A \subseteq \mathbb{N}$ with $\gcd(A) = 1$, if T is an Arf numerical semigroup containing A, then clearly T must contain $S = \langle A\rangle$. A candidate for the smallest (with respect to set inclusion) Arf numerical semigroup containing A is the intersection of all Arf numerical semigroups containing S, provided that the intersection of a finite set of Arf numerical semigroups is Arf. Actually this is ensured by the next result, which follows easily from the definition.

PROPOSITION 174. *If $S_1, \ldots, S_n$ are Arf numerical semigroups, then $S = S_1 \cap \cdots \cap S_n$ is also Arf.*

This enables us to define the Arf numerical semigroup generated by A ($\gcd(A) = 1$) as the intersection of all Arf numerical semigroups containing A (and thus $\langle A\rangle$), and will be denoted by $\mathrm{Arf}(A)$. Observe that in view of Proposition 174, $\mathrm{Arf}(A)$ is the smallest Arf numerical semigroup containing A. Note also that if S is an Arf semigroup, then clearly $\mathrm{Arf}(S) = S$. If $S = \mathrm{Arf}(A)$, we say that A is an **Arf system of generators** of S, and we will say that A is **minimal** if no proper subset of A is an

Arf system of generators of S. For a numerical semigroup S, $\mathrm{Arf}(S)$ will be also called the **Arf closure** of S.

Next we show that every Arf numerical semigroup has a unique minimal system of generators (Theorem 179). First we give a description of $\mathrm{Arf}(A)$. Observe that if we are given $A \subseteq \mathbb{N}$ with $\gcd(A) = 1$, then $\mathrm{Arf}(A)$ must contain the set of all the elements of the form $x+y-z$ with $x,y,z \in \langle A \rangle$ and $x \geq y \geq z$. It must also contain the set of elements that are derived from those obtained above using the same rule and so on. This motivates the following results and definitions.

LEMMA 175. *Let S be a submonoid of $\mathbb{N}$. Then*

$$S' = \{x+y-z \mid x,y,z \in S, x \geq y \geq z\}$$

is a submonoid of $\mathbb{N}$ and $S \subseteq S'$.

PROOF. Let $x \in S$. Then $x+x-x \in S'$, whence $S \subseteq S'$. Clearly $S' \subseteq \mathbb{N}$. Now take $a,b \in S'$ and let us prove that $a+b \in S'$. By the definition of S', there exist $x_1,x_2,y_1,y_2,z_1,z_2 \in S$, such that $x_i \geq y_i \geq z_i$, $i \in \{1,2\}$, and $a = x_1+y_1-z_1$, $b = x_2+y_2-z_2$. Hence, $a+b = (x_1+x_2)+(y_1+y_2)-(z_1+z_2)$. Clearly $x_1+x_2, y_1+y_2, z_1+z_2 \in S$ and $x_1+x_2 \geq y_1+y_2 \geq z_1+z_2$. Therefore $a+b \in S'$. □

For a given submonoid S of $\mathbb{N}$ and $n \in \mathbb{N}$, define S^n recurrently as follows:

- $S^0 = S$,
- $S^{n+1} = (S^n)'$.

LEMMA 176. *Let S be a numerical semigroup. Then there exists $k \in \mathbb{N}$ such that* $S^k = \mathrm{Arf}(S)$.

PROOF. Using induction on n, it can be easily proved that $S^n \subseteq \mathrm{Arf}(S)$ for all $n \in \mathbb{N}$. By Lemma 175, $S^n \subseteq S^{n+1}$ and $S \subseteq S^n$ for all $n \in \mathbb{N}$. As we pointed out before, the number of numerical semigroups containing S is finite, whence $S^k = S^{k+1}$ for some

$k \in \mathbb{N}$. Clearly S^k is an Arf numerical semigroup and $S^k \subseteq \mathrm{Arf}(S)$, and since $\mathrm{Arf}(S)$ is the smallest Arf numerical semigroup containing S, we obtain $S^k = \mathrm{Arf}(S)$. □

For proving that minimal Arf systems of generators are unique, we first show that every Arf system of generators must contain the multiplicity of the semigroup.

LEMMA 177. *Let S be an Arf numerical semigroup and let A be an Arf system of generators of S. Then $\mathrm{m}(S) \in A$.*

PROOF. For $x, y, z \in S \setminus \{\mathrm{m}(S)\}$ with $x \geq y \geq z$, we get that $x+y-z \in S \setminus \{\mathrm{m}(S)\}$, whence $S \setminus \{\mathrm{m}(S)\}$ is an Arf numerical semigroup. If $\mathrm{m}(S) \notin A$, then $\mathrm{Arf}(A) \subseteq \mathrm{Arf}(S \setminus \{\mathrm{m}(S)\}) = S \setminus \{\mathrm{m}(S)\} \neq S$, which contradicts $\mathrm{Arf}(A) = S$. □

We already know that for a given numerical semigroup $S = \langle A \rangle$, there exists $k \in \mathbb{N}$ such that $S^k = \mathrm{Arf}(A)$. This in particular implies that every element in $\mathrm{Arf}(A)$ can be expressed as a linear combination with integer coefficients of the elements in A. What we basically prove next is that for $s \in \mathrm{Arf}(A)$ the generators that appear in any of the expressions of s must be smaller than s.

LEMMA 178. *Let S be an Arf numerical semigroup and let A be an Arf system of generators of S. For every $s \in S$, set $\mathrm{B}(s) = \{a \in A \mid a \leq s\}$. If $s \in \langle A \rangle^n$, then $s \in \langle \mathrm{B}(s) \rangle^n$.*

PROOF. We use induction on n. For $n = 0$, the result is clear by the definition of $\mathrm{B}(s)$. Now assume that the result is true for $n \in \mathbb{N}$ and let us prove it for $n+1$. Take $s \in \langle A \rangle^{n+1}$. Then there exist $x, y, z \in \langle A \rangle^n$ with $x \geq y \geq z$ and such that $s = x+y-z$. By induction hypothesis $x \in \langle \mathrm{B}(x) \rangle^n$, $y \in \langle \mathrm{B}(y) \rangle^n$ and $z \in \langle \mathrm{B}(z) \rangle^n$. Since $s = x+y-z$ and $x \geq y \geq z$, we have that $z \leq y \leq x \leq s$, whence $\mathrm{B}(z) \subseteq \mathrm{B}(y) \subseteq \mathrm{B}(x) \subseteq \mathrm{B}(s)$. It follows that $x, y, z \in \langle \mathrm{B}(s) \rangle^n$ and this leads to $s = x+y-z \in \langle \mathrm{B}(s) \rangle^{n+1}$. □

THEOREM 179. *Let A and B be to minimal Arf systems of generators of an Arf numerical semigroup S. Then $A = B$.*

PROOF. Assume that $A = \{n_1 < \cdots < n_p < \cdots\}$ and $B = \{m_1 < \cdots < m_q < \cdots\}$. By Lemma 177, we have that $n_1 = m_1 = \mathrm{m}(S)$. If $A \neq B$, then let r be the least integer such that $n_r \neq m_r$. Assume without loss of generality that $m_r < n_r$. As $m_r \in S$, we can apply Lemma 176 and obtain that $m_r \in \langle A \rangle^n$ for some nonnegative integer n. Using Lemma 178 we deduce that $m_r \in \langle n_1, \ldots, n_{r-1} \rangle^n$. Since $m_k = n_k$ for all $k < r$, we have that $m_r \in \langle m_1, \ldots, m_{r-1} \rangle^n$, whence $m_r \in \mathrm{Arf}(B \setminus \{m_r\})$ and $S = \mathrm{Arf}(B \setminus \{m_r\})$, contradicting that B is a minimal Arf system of generators. □

This result allows us to define the **Arf rank** of an Arf numerical semigroup S as the cardinality of its minimal Arf system of generators. This amount will be denoted by $\mathrm{Arf}-\mathrm{rank}(S)$. Hence $\mathrm{Arf}-\mathrm{rank}(S) \leq \mu(S)$, that is, the Arf rank of S is smaller than or equal to its embedding dimension. Clearly, every Arf numerical semigroup has maximal embedding dimension, that is, $\mu(S) = \mathrm{m}(S)$ (MED-semigroup). It follows that for an Arf numerical semigroup S

$$\mathrm{Arf}-\mathrm{rank}(S) \leq \mu(S) = \mathrm{m}(S) = \min_{\leq}(S \setminus \{0\}).$$

A binary tree is a rooted tree in which every vertex has 0, 1 or 2 sons (see **[23]**). Now we describe a recursive procedure that arranges the set of all Arf numerical semigroups in a binary tree whose root is $\mathbb{N}$. The idea is to learn how to construct new Arf numerical semigroups by adding or removing an element from a given Arf numerical semigroup. We will show first that adding the Frobenius number to an Arf numerical semigroup yields a new Arf numerical semigroup, and this operation will enable us to move from one vertex in the tree to its parent. The process of generating the sons of a vertex will be by removing certain elements from the minimal Arf system of generators of the semigroup.

LEMMA 180. *Let S be an Arf numerical semigroup, $S \neq \mathbb{N}$. Then $S \cup \{g(S)\}$ is again an Arf numerical semigroup.*

PROOF. We already know that $S \cup \{g(S)\}$ is a numerical semigroup. Take $x, y, z \in S \cup \{g(S)\}$ such that $x \geq y \geq z$, and let us prove that $x + y - z \in S \cup \{g(S)\}$.

- If $x, y, z \in S$, then as S is Arf, we obtain that $x + y - z \in S \subset S \cup \{g(S)\}$.
- If $g(S) \in \{x, y, z\}$, then $x + y - z \geq g(S)$ and thus $x + y - z \in S \cup \{g(S)\}$.

□

Given a numerical semigroup S, for $n \in \mathbb{N}$, define recursively the semigroup S_n as:

- $S_0 = S$,
- $S_{n+1} = S_n \cup \{g(S_n)\}$, if $S_n \neq \mathbb{N}$; $S_{n+1} = \mathbb{N}$, otherwise.

Clearly for every numerical semigroup there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$. Note also that if S is an Arf numerical semigroup, then by Lemma 180, the chain $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = \mathbb{N}$ is a chain of Arf numerical semigroups, and $S_i = S_{i+1} \setminus \{a\}$ for some $a \in S_{i+1}$. The following result studies the condition that we must impose to an element a in an Arf numerical semigroup S for $S \setminus \{a\}$ to be Arf.

LEMMA 181. *Let S be an Arf numerical semigroup and let $a \in S$. The following conditions are equivalent:*

(1) *a belongs to the minimal Arf system of generators of S,*

(2) *$S \setminus \{a\}$ is an Arf numerical semigroup.*

PROOF. *(1) implies (2).* Since a belongs to the minimal Arf system of generators of S, we have that $\mathrm{Arf}(S \setminus \{a\})$ is strictly contained in S. Hence $S \setminus \{a\} \subseteq \mathrm{Arf}(S \setminus \{a\}) \subseteq S$, and $S \neq \mathrm{Arf}(S \setminus \{a\})$ yields $\mathrm{Arf}(S \setminus \{a\}) = S \setminus \{a\}$, which means that $S \setminus \{a\}$ is an Arf numerical semigroup.

(2) implies (1). If a does not belong to the minimal Arf system of generators of S, then $\mathrm{Arf}(S\setminus\{a\}) = S$, and this in particular implies that $S\setminus\{a\}$ does not have the Arf property. □

With the following result we can detect when an Arf numerical semigroup has been constructed by using the procedure described in Lemma 180.

PROPOSITION 182. *Let S be an Arf numerical semigroup. The following conditions are equivalent:*

(1) $S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$, *with $\overline{S}$ an Arf numerical semigroup,*

(2) *the minimal Arf system of generators of S contains at least one element greater than* $\mathrm{g}(S)$.

PROOF. *(1) implies (2).* Clearly, if $S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$, then $\overline{S} = S \setminus \{\mathrm{g}(\overline{S})\}$. Using Lemma 181, we obtain that $\mathrm{g}(\overline{S})$ must belong to the minimal Arf system of generators of S, and since $\overline{S} \subseteq S$ and $\mathrm{g}(\overline{S}) \in S$, we get that $\mathrm{g}(\overline{S}) > \mathrm{g}(S)$.

(2) implies (1). If a is an element of the minimal Arf system of generators of S, then by Lemma 181 we know that $\overline{S} = S\setminus\{a\}$ is an Arf numerical semigroup. If in addition $a > \mathrm{g}(S)$, then $a = \mathrm{g}(\overline{S})$, whence $S = \overline{S} \cup \{\mathrm{g}(\overline{S})\}$ with $\overline{S}$ an Arf numerical semigroup. □

Proposition 182 together with the remark given just after Lemma 180 allow us to construct recursively from the Arf numerical semigroup $\mathbb{N}$ the set of all Arf numerical semigroups (see the figure). This construction arranges them all in a tree ordering shape. It is also clear that as we move "downwards" the branches of this tree, we encounter semigroups with larger Frobenius numbers.

An Arf numerical semigroup having no sons is a leaf. As a consequence of Proposition 182 we get the following result.

FIGURE 2. The binary tree of Arf numerical semigroups

$$\begin{tikzcd}[column sep=small]
 & & & \begin{matrix}\mathbb{N}=\mathrm{Arf}(1),\\ g=-1\end{matrix} \arrow[d] & \\
 & & & \begin{matrix}\mathrm{Arf}(2,3),\\ g=1\end{matrix} \arrow[dl] \arrow[dr] & \\
 & & \begin{matrix}\mathrm{Arf}(3,4),\\ g=2\end{matrix} \arrow[dl] \arrow[dr] & & \begin{matrix}\mathrm{Arf}(2,5),\\ g=3\end{matrix} \arrow[d] \\
 & \begin{matrix}\mathrm{Arf}(4,5),\\ g=3\end{matrix} \arrow[dl] \arrow[dr] & & \begin{matrix}\mathrm{Arf}(3,5),\\ g=4\end{matrix} \arrow[d] & \begin{matrix}\mathrm{Arf}(2,7),\\ g=5\end{matrix} \arrow[d] \\
\begin{matrix}\mathrm{Arf}(5,6),\\ g=4\end{matrix} \arrow[d, dotted, no head] & & \begin{matrix}\mathrm{Arf}(4,6,7),\\ g=5\end{matrix} \arrow[d, dotted, no head] & \begin{matrix}\mathrm{Arf}(3,7),\\ g=5\end{matrix} \arrow[d, dotted, no head] & \begin{matrix}\mathrm{Arf}(2,9),\\ g=7\end{matrix} \arrow[d, dotted, no head] \\
 {} & & {} & {} & {}
\end{tikzcd}$$

COROLLARY 183. *Let S be an Arf numerical semigroup. Then S is a leaf if and only if the minimal Arf system of generators of S does not contain elements greater than* $\mathrm{g}(S)$.

Finally we show that the tree of Arf numerical semigroups is binary. To this end we need a couple of technical lemmas. The idea is to prove that in a minimal Arf system of generators there are at most two elements greater than the Frobenius number and then use Proposition 182.

LEMMA 184. *Let $x \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ with $\{x, x+1\} \subseteq X$. Then $\{a \in \mathbb{N} \mid a \geq x\} \subseteq$* $\mathrm{Arf}(X)$.

PROOF. We use induction to prove that $x+n \in \mathrm{Arf}(X)$ for all $n \in \mathbb{N}$. For $n = 0$, we get $x \in X \subseteq \mathrm{Arf}(X)$. Now assume that $x+n \in \mathrm{Arf}(X)$. Then $x+n+1 \in \mathrm{Arf}(X)$, since $x+n+1 = (x+n)+(x+1)-x$, $x+n, x+1, x \in \mathrm{Arf}(X)$ and $x+n \geq x+1 \geq x$. $\square$

LEMMA 185. *Let S be an Arf numerical semigroup and let A be its minimal Arf system of generators. Then $\{a \in A \mid \mathrm{g}(S) < a\}$ has at most two elements.*

PROOF. Let $\{a_1, \ldots, a_r\} = \{a \in A \mid a < \mathrm{g}(S)\}$. Using Lemmas 178 and 184, we deduce that $\mathrm{Arf}(a_1, \ldots, a_r, \mathrm{g}(S)+1, \mathrm{g}(S)+2) = S$, whence $\{a_1, \ldots, a_r, \mathrm{g}(S)+1, \mathrm{g}(S)+2\}$ is an Arf system of generators of S. Applying now Theorem 179 we get that $\{a \in \ \mid \mathrm{g}(S) < a\} \subseteq \{\mathrm{g}(S)+1, \mathrm{g}(S)+2\}$. □

PROPOSITION 186. *The tree of Arf numerical semigroups is binary.*

PROOF. It suffices to observe, by Lemma 185 and Proposition 182, that if T is a son of S, then either $T = S \setminus \{\mathrm{g}(S)+1\}$ or $T = S \setminus \{\mathrm{g}(S)+2\}$. Therefore every vertex in the tree has at most two sons. □

Next we present an algorithmic procedure for computing, from a finite subset X of $\mathbb{N}$ with $\gcd(X) = 1$, the elements of $\mathrm{Arf}(X)$ (Arf closure of a numerical semigroup). The reader will find a similitude between the algorithm described here and Euclid's algorithm for computing gcd's. It turns out that finding the elements of $\mathrm{Arf}(X)$ is much easier than computing $\langle X \rangle$.

LEMMA 187. *Let S be an Arf numerical semigroup and take $m \in S$. Then $(m + S) \cup \{0\}$ is also an Arf numerical semigroup.*

PROOF. It is clear that $(m+S) \cup \{0\}$ is a numerical semigroup. Now take $m + s_1, m+s_2, m+s_3 \in m+S$ with $m+s_1 \geq m+s_2 \geq m+s_3$. Then $s_1 \geq s_2 \geq s_3$ and since S is Arf, we get $s_1 + s_2 - s_3 \in S$. It follows that $(m+s_1) + (m+s_2) - (m+s_3) = m + (s_1 + s_2 - s_3) \in m+S$. The reader can check that this proves that $(m+S) \cup \{0\}$ is Arf. □

LEMMA 188. *Let $m, r_1, \ldots, r_p \in \mathbb{N}$ such that* $\gcd(\{m, r_1, \ldots, r_p\}) = 1$. *Then*

$$m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m+r_1, \ldots, m+r_p).$$

PROOF. We use once more induction on n. For $n = 0$ we have to prove that $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$. Let $i, j \in \{1, \ldots, p\}$. Then $m, m + r_i, m + r_j \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$, whence $m + r_i + r_j = (m + r_i) + (m + r_j) - m \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$. Now for $k \in \{1, \ldots, p\}$, $m, m + r_i + r_j, m + r_k \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_k)$ and therefore $m + r_i + r_j + r_k = (m + r_i + r_j) + (m + r_k) - m \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_k)$. Using this idea we obtain that for every $a, a_1, \ldots, a_p \in \mathbb{N}$, we have that $(a + 1)m + a_1 r_1 + \cdots + a_p r_p \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$ and thus $m + \langle m, r_1, \ldots, r_p \rangle \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$.

Now assume that $m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$ and let us prove that $m + \langle m, r_1, \ldots, r_p \rangle^{n+1} \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$. Let $a \in m + \langle m, r_1, \ldots, r_p \rangle^{n+1}$. Then $a = m + b$ with $b \in \langle m, r_1, \ldots, r_p \rangle^{n+1}$. Hence there exist $x, y, z \in \langle m, r_1, \ldots, r_p \rangle^n$ such that $x \geq y \geq z$ and $x + y - z = b$. In this way $a = m + b = m + x + y - z = (m + x) + (m + y) - (m + z) \in \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$, since by induction hypothesis $m + x, m + y, m + z \in m + \langle m, r_1, \ldots, r_p \rangle^n \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$. □

THEOREM 189. *Let $m, r_1, \ldots, r_p$ be nonnegative integers with greatest common divisor one. Then*

$$\mathrm{Arf}(m, m + r_1, \ldots, m + r_p) = (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}.$$

PROOF. Using Lemmas 176 and 188 we obtain that $(m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\} \subseteq \mathrm{Arf}(m, m + r_1, \ldots, m + r_p)$. For the other inclusion observe that $m, m + r_1, \ldots, m + r_p \in (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}$, and since by Lemma 187, $(m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}$ is an Arf numerical semigroup, we get that $\mathrm{Arf}(m, m + r_1, \ldots, m + r_p) \subseteq (m + \mathrm{Arf}(m, r_1, \ldots, r_p)) \cup \{0\}$. □

As an immediate consequence of Theorem 189 we obtain the following result.

COROLLARY 190. *Let $m, r_1, \ldots, r_p$ be nonnegative integers with greatest common divisor one. Then*

$$\mathrm{g}(\mathrm{Arf}(m, m+r_1, \ldots, m+r_p)) = m + \mathrm{g}(\mathrm{Arf}(m, r_1, \ldots, r_p)).$$

Let S be an Arf numerical semigroup. Since every system of generators of S is one of its Arf systems of generators, the above corollary can be applied to any system of generators of S. This in particular yields Proposition I.1.11 a) with $i = 1$ in [5]. Let $X \subseteq \mathbb{N} \setminus \{0\}$ be such that $\gcd(X) = 1$. Define recursively the following sequence of subsets of $\mathbb{N}$:

- $A_1 = X$,
- $A_{n+1} = (\{x - \min_{\leq} A_n \mid x \in A_n\} \setminus \{0\}) \cup \{\min_{\leq} A_n\}$.

As a consequence of Euclid's algorithm for the computation of $\gcd(X)$, we obtain that there exists $q = \min_{\leq}\{k \in \mathbb{N} \mid 1 \in A_k\}$.

THEOREM 191. *Under the standing notation, we have that*

$$0, \min_{\leq} A_1, \min_{\leq} A_1 + \min_{\leq} A_2, \ldots, \min_{\leq} A_1 + \cdots + \min_{\leq} A_{q-1}$$

are the elements in $\mathrm{Arf}(X)$ *that are less than or equal to* $\mathrm{g}(\mathrm{Arf}(X)) + 1$.

PROOF. Since $1 \in A_q$, $\mathrm{Arf}(A_q) = \mathbb{N}$. Hence applying Theorem 189, we get that $\mathrm{Arf}(A_{q-1}) = (\min_{\leq} A_{q-1} + \mathbb{N}) \cup \{0\}$. This implies that the elements $0, \min_{\leq} A_{q-1}$ are the elements that are less than or equal to $\mathrm{g}(\mathrm{Arf}(A_{q-1})) + 1$. Assume as induction hypothesis that $0, \min_{\leq} A_{q-i}, \min_{\leq} A_{q-i} + \min_{\leq} A_{q-i+1}, \ldots, \min_{\leq} A_{q-i} + \cdots + \min_{\leq} A_{q-1}$ are the elements of $\mathrm{Arf}(A_{q-i})$ less than or equal to $\mathrm{g}(\mathrm{Arf}(A_{q-i})) + 1$. We must prove now that $0, \min_{\leq} A_{q-i-1}, \min_{\leq} A_{q-i-1} + \min_{\leq} A_{q-i}, \ldots, \min_{\leq} A_{q-i-1} + \cdots + \min_{\leq} A_{q-1}$ are the elements of $\mathrm{Arf}(A_{q-i-1})$ less than or equal to $\mathrm{g}(\mathrm{Arf}(A_{q-i-1})) + 1$. By Theorem 189, we know that $\mathrm{Arf}(A_{q-i-1}) = (\min_{\leq} A_{q-i-1} + \mathrm{Arf}(A_{q-i})) \cup \{0\}$. Using now the induction hypothesis and Corollary 190, we obtain the desired result. □

EXAMPLE 192. Let us compute $\mathrm{Arf}(7,24,33)$.

$$A_1 = \{7,24,33\}, \min_{\leq} A_1 = 7,$$

$$A_2 = \{7,17,26\}, \min_{\leq} A_2 = 7,$$

$$A_3 = \{7,10,19\}, \min_{\leq} A_3 = 7,$$

$$A_4 = \{7,3,12\}, \min_{\leq} A_4 = 3,$$

$$A_5 = \{4,3,9\}, \min_{\leq} A_5 = 3,$$

$$A_6 = \{1,3,6\},$$

whence $\mathrm{Arf}(7,24,33) = \{0,7,14,21,24,27,\rightarrow\}$. □

3. SAT systems of generators

In this Section we characterize the subsets of $\mathbb{N}$ that are saturated numerical semigroups. From the concept of SAT system of generators, for a saturated numerical semigroup, we arrange the set of all saturated numerical semigroup in binary tree with no leaves.

A numerical semigroup S is **saturated** if the following condition holds: if $s, s_1, \ldots, s_r \in S$ are such that $s_i \leq s$ for all $i \in \{1, \ldots, r\}$ and $z_1, \ldots, z_r \in \mathbb{Z}$ are such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then $s + z_1 s_1 + \cdots + z_r s_r \in S$. For $A \subseteq \mathbb{N}$ and $a \in A$, denote by

$$\mathrm{d}_A(a) = \gcd\{x \in A \mid x \leq a\}.$$

LEMMA 193. *Let S be a saturated numerical semigroup and let $s \in S$. Then $s + \mathrm{d}_S(s) \in S$.*

PROOF. Let $\{s_1, \ldots, s_r\} = \{x \in S \mid x \leq s\}$. By Bezout's identity, there exists $z_1, \ldots, z_r \in \mathbb{Z}$ such that $z_1 s_1 + \cdots + z_r s_r = \mathrm{d}_S(s)$. Using now that S is saturated, we get $s + \mathrm{d}_S(s) \in S$. □

LEMMA 194. *Let A be a nonempty subset of $\mathbb{N}$ such that $\gcd(A) = 1$ and $a + \mathrm{d}_A(a) \in A$ for all $a \in A$. Then $a + k\mathrm{d}_A(a) \in A$ for all $k \in \mathbb{N}$.*

PROOF. For the sake of simplicity, and since there is no possible misunderstanding, we denote $\mathrm{d}_A(a)$ by $\mathrm{d}(a)$. We use induction on $\mathrm{d}(a)$.

If $\mathrm{d}(a) = 0$, then the result follows trivially. Next we see that if $\mathrm{d}(a) = 1$, then $a+k \in A$ for all $k \in \mathbb{N}$. To this end we use induction on k. For $k = 0$, the result is trivial. Assume that $a+k \in A$. Since $0 \neq \mathrm{d}(a+k) \leq \mathrm{d}(a) = 1$, we have that $\mathrm{d}(a+k) = 1$. Hence $a+k+1 = a+k+\mathrm{d}(a+k) \in A$.

By induction hypothesis we assume that if $a' \in A$ and $\mathrm{d}(a') < \mathrm{d}(a)$, then $a' + k\mathrm{d}(a') \in A$ for all $k \in \mathbb{N}$. Thus, suppose that $\mathrm{d}(a) \geq 2$ and let us prove that $a+k\mathrm{d}(a) \in A$ for all $k \in \mathbb{N}$. Note that since $\gcd(A) = 1$, there exists $b \in A$ such that $\mathrm{d}(b) = 1$ and that if $\mathrm{d}(a+k\mathrm{d}(a)) = \mathrm{d}(a)$ and $a+k\mathrm{d}(a) \in A$, then $a+(k+1)\mathrm{d}(a) = a+k\mathrm{d}(a)+\mathrm{d}(a+k\mathrm{d}(a)) \in A$. From these two remarks we deduce that there exists the least positive integer t such that $a+t\mathrm{d}(a) \in A$ and $\mathrm{d}(a+t\mathrm{d}(a)) < \mathrm{d}(a)$. As $\mathrm{d}(a+t\mathrm{d}(a)) < \mathrm{d}(a)$, applying induction hypothesis, we obtain that $(a+t\mathrm{d}(a))+k\mathrm{d}(a+t\mathrm{d}(a)) \in A$ for all $k \in \mathbb{N}$. Clearly, $\mathrm{d}(a+t\mathrm{d}(a))$ divides $\mathrm{d}(a)$, whence $\mathrm{d}(a) = l\mathrm{d}(a+t\mathrm{d}(a))$ for some positive integer l. Consequently $a+t\mathrm{d}(a)+kl/l\mathrm{d}(a+t\mathrm{d}(a)) \in A$ for all $k \in \mathbb{N}$, and thus $a+(t+n)\mathrm{d}(a) \in A$ for all $n \in \mathbb{N}$. From the definition of t, it follows that $a+k\mathrm{d}(a) \in A$ for all $k \in \{0,\ldots,t\}$. We conclude that $a+k\mathrm{d}(a) \in A$ for all $k \in \mathbb{N}$. □

LEMMA 195. *Let A be a nonempty subset of $\mathbb{N}$ such that $\gcd(A) = 1$ and $a+\mathrm{d}_A(a) \in A$ for all $a \in A$. Then $A \cup \{0\}$ is a numerical semigroup.*

PROOF. Since $\gcd(A) = 1$, it suffices to prove that for any $a, b \in A$, one gets $a+b \in A$. Assume that $a \leq b$. Then $\mathrm{d}_A(b)$ divides $\mathrm{d}_A(a)$ and thus there exits $\lambda \in \mathbb{N}$ such that $\mathrm{d}_A(a) = \lambda\mathrm{d}_A(b)$. Note also that $\mathrm{d}(a)$ divides a, whence $a = \mu\mathrm{d}_A(a)$ for some $\mu \in \mathbb{N}$. Therefore $a+b = \mu\mathrm{d}_A(a)+b = \mu\lambda\mathrm{d}_A(b)+b$, which by Lemma 194 is in A. □

THEOREM 196. *Let A be a nonempty subset of $\mathbb{N}$ such that $0 \in A$ and $\gcd(A) = 1$. The following conditions are equivalent:*

(1) *A is a saturated numerical semigroup.*

(2) $a + \mathrm{d}_A(a) \in A$ *for all* $a \in A$.

(3) $a + k\mathrm{d}_A(a) \in A$ *for all* $a \in A$ *and* $k \in \mathbb{N}$.

PROOF. *(1) implies (2).* Follows from Lemma 193.

(2) implies (3). Follows from Lemma 194.

(3) implies (1). By Lemma 195 we already know that A is a numerical semigroup. We see that it is saturated. Let $a, a_1, \ldots, a_r \in A$ with $a_i \leq a$ for all $i \in \{1, \ldots, r\}$ and let $z_1, \ldots, z_r$ be integers such that $z_1 a_1 + \cdots + a_r z_r \geq 0$. Since $a_i \leq a$, it follows that $\mathrm{d}_A(a)$ divides a_i for all $i \in \{1, \ldots, r\}$. Hence there exists $k \in \mathbb{N}$ such that $z_1 a_1 + \cdots + z_r a_r = k\mathrm{d}_A(a)$ and thus $a + z_1 a_1 + \cdots + z_r a_r = a + k\mathrm{d}_A(a) \in A$. □

Next we introduce the concept of SAT system of generators for a saturated numerical semigroup. In order to do this we first need to prove that for a given $X \subseteq \mathbb{N}$ with $\gcd(X) = 1$, there exists the least (with respect to set inclusion) saturated numerical semigroup that contains X. The best candidate as usual is the intersection of all saturated numerical semigroups that contain X.

PROPOSITION 197. *Let S_1 and S_2 be two saturated numerical semigroups. Then $S = S_1 \cap S_2$ is a saturated numerical semigroup.*

PROOF. We make use of Theorem 196 (note that $0 \in S$ and that $\gcd(S) = 1$). It suffices to prove that $s + \mathrm{d}_S(s) \in S$ for all $s \in S$. For a given $s \in S$, we have that $s \in S_i$ and that $\mathrm{d}_{S_i}(s)$ divides $\mathrm{d}_S(s)$ for $i \in \{1, 2\}$. Hence there exits nonnegative integers k_1 and k_2 such that $\mathrm{d}_S(s) = k_1 \mathrm{d}_{S_1}(s) = k_2 \mathrm{d}_{S_2}(s)$. By Theorem 196, $s + k_i \mathrm{d}_{S_i}(s) \in S_i$ for $i \in \{1, 2\}$, whence $s + \mathrm{d}_S(s) \in S$. □

As we already now, the set of numerical semigroups containing S is finite, since $\mathbb{N} \setminus S$ is finite. Let X be a subset of $\mathbb{N}$ such that $\gcd(X) = 1$. Then every saturated numerical semigroup containing X must also contain $\langle X \rangle$, and thus there are finitely many of them. We denote by $\mathrm{Sat}(X)$ the intersection of all saturated numerical semigroups containing X, and call it the **saturated closure** of X. Observe that $\mathrm{Sat}(X) = \mathrm{Sat}(\langle X \rangle)$. As a consequence of Proposition 197 and the above remark, we have that $\mathrm{Sat}(X)$ is the smallest saturated semigroup containing X. If S is a saturated numerical semigroup and X is a subset of $\mathbb{N}$ such that $\gcd(X) = 1$ and $\mathrm{Sat}(X) = S$, then we will say that X is a **SAT system of generators** of S. We say that X is a **minimal** SAT system of generators if in addition no proper subset of X is a SAT system of generators of S. Every numerical semigroup is finitely generated (as a semigroup). Hence for a given numerical semigroup S, there exists $\{n_1, \ldots, n_p\} \subset \mathbb{N}$ such that $S = \langle n_1, \ldots, n_p \rangle$. If S is a saturated numerical semigroup, then clearly $\mathrm{Sat}(n_1, \ldots, n_p) = \mathrm{Sat}(S) = S$, and thus every saturated numerical semigroup admits a finite SAT system of generators.

THEOREM 198. *Let $n_1 < n_2 < \cdots < n_p$ be positive integers such that $\gcd(n_1, \ldots, n_p) = 1$. For every $i \in \{1, \ldots, p\}$, set $d_i = \gcd(n_1, \ldots, n_i)$ and for all $j \in \{1, \ldots, p-1\}$ define $k_j = \max\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\}$. Then*

$$\mathrm{Sat}(n_1, \ldots, n_p) = \{0, n_1, n_1 + d_1, \ldots, n_1 + k_1 d_1, n_2, n_2 + d_2, \ldots, n_2 + k_2 d_2, \\ \ldots, n_{p-1}, n_{p-1} + d_{p-1}, \ldots, n_{p-1} + k_{p-1} d_{p-1}, n_p, n_p + 1, \rightarrow\}.$$

PROOF. Let

$$A = \{0, n_1, n_1 + d_1, \ldots, n_1 + k_1 d_1, n_2, n_2 + d_2, \ldots, n_2 + k_2 d_2, \\ \ldots, n_{p-1}, n_{p-1} + d_{p-1}, \ldots, n_{p-1} + k_{p-1} d_{p-1}, n_p, n_p + 1, \rightarrow\}.$$

Clearly A is not empty, $0 \in A$, $\gcd(A) = 1$ and $a + \mathrm{d}_A(a) \in A$ for all $a \in A$. By Theorem 196, A is a saturated numerical semigroup, and as $\{n_1, \ldots, n_p\} \subset A$, we get that $\mathrm{Sat}(n_1, \ldots, n_p) \subseteq A$. For the other inclusion, take $a \in A$. Then there exists $i \in \{1, \ldots, p\}$ and $k \in \mathbb{N}$ such that $a = n_i + kd_i$ (note that $d_p = 1$). Since $\{n_1, \ldots, n_p\} \subset \mathrm{Sat}(n_1, \ldots, n_p)$, we have that $\mathrm{d}_{\mathrm{Sat}(n_1, \ldots, n_p)}(n_i)$ divides d_i, whence there exists $l \in \mathbb{N}$ such that $d_i = l\mathrm{d}_{\mathrm{Sat}(n_1,\dots,n_p)}(n_i)$. Using Theorem 196, we know that $n + t\mathrm{d}_{\mathrm{Sat}(n_1,\dots,n_p)}(n_i) \in \mathrm{Sat}(n_1,\dots,n_p)$ for all $t \in \mathbb{N}$ and thus $a = n_i + kd_i = n_i + kl\mathrm{d}_{\mathrm{Sat}(n_1,\dots,n_p)}(n_i) \in \mathrm{Sat}(n_1,\dots,n_p)$. □

EXAMPLE 199. Let $\{n_1,n_2,n_3\} = \{4,10,23\}$. Then $d_1 = 4$, $d_2 = 2$, $d_3 = 1$, $k_1 = 1$ and $k_2 = 6$. Hence

$$\mathrm{Sat}(4,10,23) = \{0,4,8,10,12,14,16,18,20,22,23,24,\rightarrow\}.$$

□

It may happen that one is interested in the minimal system of generators (as a semigroup) of $\mathrm{Sat}(X)$. It is well known (see for instance **[18]**) that any saturated numerical semigroup has the Arf property, whence it is of maximal embedding dimension (see **[5]**). From **[32]** one can deduce that if $m = \min(X \setminus \{0\})(= \min(\mathrm{Sat}(X) \setminus \{0\}))$, then the minimal system of generators of $S = \mathrm{Sat}(X)$ is

$$A = \{m\} \cup (\{s \in S \mid s - m \notin S\} \setminus \{0\}).$$

Since we know that the cardinality of A is m, once we have computed $\mathrm{Sat}(X)$ as explained in Theorem 198, in order to calculate $\{s \in S \mid s - m \notin S\}$ it suffices to find the first m elements in the list such that subtracting m to them the result is not in the list. In the preceding example, $S = \mathrm{Sat}(4,10,23)$, $m = 4$ and

$$\{s \in S \mid s - m \notin S\} = \{0,10,23,25\},$$

and thus $\mathrm{Sat}(4,10,23) = \langle 4,10,23,25 \rangle$.

Next we show that every saturated numerical semigroup has a unique minimal SAT system of generators.

LEMMA 200. *Let S be a saturated numerical semigroup and let $s \in S \setminus \{0\}$. The following conditions are equivalent:*

(1) $\overline{S} = S \setminus \{s\}$ *is a saturated numerical semigroup,*

(2) $\mathrm{d}_S(s) \neq \mathrm{d}_S(s')$ *for all* $s' \in S$ *with* $s' < s$.

PROOF. *(1) implies (2).* Assume that $\mathrm{d}_S(s) = \mathrm{d}_S(s')$ for some $s' \in S$ such that $s' < s$. Since $s' < s$, there exists $a \in \mathbb{N} \setminus \{0\}$ such that $s = s' + a$, and as $\mathrm{d}_S(s) = \mathrm{d}_S(s')$, we have that $\mathrm{d}_S(s')$ divides both s and s', whence it also divides a. Thus, $a = k\mathrm{d}_S(s')$ for some $k \in \mathbb{N}$. From $\overline{S} = S \setminus \{s\}$ and $s' < s$, we deduce that $\mathrm{d}_S(s') = \mathrm{d}_{\overline{S}}(s')$. Using now Theorem 196 for $\overline{S}$, we get that $s = s' + a = s' + k\mathrm{d}_{\overline{S}}(s') \in \overline{S} = S \setminus \{s\}$, which is impossible.

(2) implies (1). By Theorem 196, it suffices to show that if $a \in S$ and $a \neq s$, then $a + \mathrm{d}_{\overline{S}}(a) \neq s$. Note that $\mathrm{d}_S(a)$ divides $\mathrm{d}_{\overline{S}}(a)$, whence $a + \mathrm{d}_{\overline{S}}(a) = a + k\mathrm{d}_S(a)$ for some $k \in \mathbb{N}$. If $a + k\mathrm{d}_S(a) = s$, then $a < s$. But this leads to $\mathrm{d}_S(s) = \mathrm{d}_S(s')$ with $s' = \max\{x \in S \mid x < s\}$, in contradiction with the hypothesis. □

LEMMA 201. *Let S be a saturated numerical semigroup and let $s \in S \setminus \{0\}$ be such that* $\mathrm{d}_S(s) \neq \mathrm{d}_S(s')$ *for all $s' < s$. Then s belongs to every SAT system of generators of S.*

PROOF. Let X be a SAT system of generators of S and assume that $s \notin X$. Then $\mathrm{Sat}(X) \subseteq \mathrm{Sat}(S \setminus \{s\}) = S \setminus \{s\}$ by Lemma 200. Hence $\mathrm{Sat}(X) \neq S$, contradicting that X is a SAT system of generators of S. □

Let S be a saturated numerical semigroup. Since $\mathbb{N} \setminus S$ has finitely many elements, there exists $s \in S$ such that $\mathrm{d}_S(s) = 1$, and $\mathrm{d}_S(s') = 1$ for all $s' \in S$, $s' > s$. Hence the set $\{s \in S \setminus \{0\} \mid \mathrm{d}_S(s) \neq \mathrm{d}_S(s') \text{ for all } s' < s, s' \in S\}$ is finite.

LEMMA 202. *Let S be a saturated numerical semigroup and let* $\{s_1, \ldots, s_r\} = \{s \in S \setminus \{0\} \mid \mathrm{d}_S(s) \neq \mathrm{d}_S(s') \text{ for all } s' < s, s' \in S\}$. *Then* $\mathrm{Sat}(s_1, \ldots, s_r) = S$.

PROOF. Since $s_1, \ldots, s_r \in S$, we have that $\mathrm{Sat}(s_1, \ldots, s_r) \subseteq S$. Let $s \in S$ and assume that $s_1 < \cdots < s_k \leq s < s_{k+1} < \cdots < s_r$. It follows that $\mathrm{d}_S(s) = \mathrm{d}_S(s_k)$ and $s = s_k + a$

for some $a \in \mathbb{N}$, which implies that $\mathrm{d}_S(s_k)$ divides a. Hence $s = s_k + t\mathrm{d}_S(s_k)$ for some $t \in \mathbb{N}$. As $\mathrm{Sat}(s_1,\ldots,s_r) \subseteq S$, we have that $\mathrm{d}_S(s_k)$ divides $\mathrm{d}_{\mathrm{Sat}(s_1,\ldots,s_r)}(s_k)$ and thus $s = s_k + l\mathrm{d}_{\mathrm{Sat}(s_1,\ldots,s_k)}(s_k)$ for some $l \in \mathbb{N}$. Using now Theorem 196 we get that $s \in \mathrm{Sat}(s_1,\ldots,s_r)$. □

As an immediate consequence of Lemmas 201 and 202, we obtain the following result.

THEOREM 203. *Let S be a saturated numerical semigroup. Then*

$$\{s_1,\ldots,s_r\} = \{s \in S \setminus \{0\} \mid \mathrm{d}_S(s) \neq \mathrm{d}_S(s') \text{ for all } s' < s, s' \in S\}$$

is the minimal SAT system of generators of S.

EXAMPLE 204. Let S be the saturated numerical semigroup

$$S = \{0,4,8,10,12,14,16,18,20,22,23,24,\rightarrow\}.$$

It follows that $\mathrm{d}_S(4) = 4 = \mathrm{d}_S(8)$, $\mathrm{d}_S(10) = \cdots = \mathrm{d}_S(22) = 2$ and $\mathrm{d}_S(23) = 1 = \mathrm{d}_S(23+n)$ for all $n \in \mathbb{N}$. By Theorem 203 the minimal SAT system of generators is $\{4,10,23\}$.

□

Using Theorem 203 it makes sense to define the **SAT rank** of a saturated numerical semigroup S by the cardinality of its minimal SAT system of generators, which we will denote by $\mathrm{SAT}-\mathrm{rank}(S)$. Note that $\mathrm{SAT}-\mathrm{rank}(S) \leq \mu(S) = \mathrm{m}(S) = \min(S \setminus \{0\})$. The following result describing those saturated numerical semigroups of SAT rank two is a direct consequence of Theorem 198.

COROLLARY 205. *Let n_1, n_2 be two integers such that $n_1 < n_2$ and* $\gcd(n_1,n_2) = 1$. *Then*

$$\begin{aligned}\mathrm{Sat}(n_1,n_2) &= \langle n_1,n_2,n_2+1,n_2+2,\rightarrow\rangle \\ &= \{0,n_1,n_1+n_1,\ldots,n_1+kn_1,n_2,n_2+1,\rightarrow\}\end{aligned}$$

with $k = \max\{l \in \mathbb{N} \mid n_1 + ln_1 < n_2\}$.

Next result gives us a sharper upper bound for the SAT rank of a saturated numerical semigroup in terms of its multiplicity.

COROLLARY 206. *Let* $n_1 < n_2 < \cdots < n_p$ *be positive integers such that its greatest common divisor is one. Then* $\{n_1,\ldots,n_p\}$ *is a minimal SAT system of generators of* $\mathrm{Sat}(n_1,\ldots,n_p)$ *if and only if* $\gcd(n_1,\ldots,n_i) \neq \gcd(n_1,\ldots,n_i,n_{i+1})$ *for all* $i \in \{1,\ldots,p-1\}$.

PROOF. Use Theorem 198 for the description of $\mathrm{Sat}(n_1,\ldots,n_p)$ and Theorem 203. □

PROPOSITION 207. *Let* m *be a positive integer and* $m = p_1^{a_1}\cdots p_r^{a_r}$ *be its decomposition into primes. If* S *is a saturated numerical semigroup with multiplicity* m*, then* $\mathrm{SAT}-\mathrm{rank}(S) \leq a_1 + \cdots + a_r + 1$.

PROOF. If $\{m = n_1 < \cdots < n_p\}$ is the minimal SAT system of generators of S (by Lemma 201, $m = n_1$) be and set $d_i = \gcd(n_1,\ldots,n_i)$ for all $i \in \{1,\ldots,p\}$. Corollary 206 states that $m = d_1 > d_2 > \cdots > d_p = 1$ and as d_{i+1} divides d_i, the proof follows easily. □

COROLLARY 208. *Every saturated numerical semigroup with multiplicity a prime number has SAT rank two.*

We finish this section by showing that the set of saturated numerical semigroups is a binary tree with no leaves and rooted in $\mathbb{N}$. We first show how to construct the father of any non root vertex (actually, repeating the process yields the path connecting the given vertex to the root; compare with the binary tree of Arf numerical semigroups).

PROPOSITION 209. *Let* $S \neq \mathbb{N}$ *be a saturated numerical semigroup. Then* $\overline{S} = S \cup \{\mathrm{g}(S)\}$ *is also saturated.*

PROOF. In view of Theorem 196 it suffices to show that if $s \in \overline{S}$, then $s + \mathrm{d}_{\overline{S}}(s) \in \overline{S}$. If $s < \mathrm{g}(S)$, then $s \in S$ and $\mathrm{d}_S(s) = \mathrm{d}_{\overline{S}}(s)$, whence $s + \mathrm{d}_{\overline{S}}(s) = s + \mathrm{d}_S(s) \in S \subset \overline{S}$. If $s \geq \mathrm{g}(S)$, then $s + \mathrm{d}_{\overline{S}}(s) \geq \mathrm{g}(S)$, and thus $s + \mathrm{d}_{\overline{S}}(s) \in \overline{S}$. □

For a given numerical semigroup S, define recursively S_n by

- $S_0 = S$,
- If $S_n = \mathbb{N}$, then $S_{n+1} = S_n$; $S_{n+1} = S_n \cup \{\mathrm{g}(S_n)\}$, otherwise.

Clearly, there exists $k \in \mathbb{N}$ such that $S_k = \mathbb{N}$. If in addition S is a saturated numerical semigroup, Proposition 209 states that $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = S$ is a chain of saturated numerical semigroups. Moreover, $S_i = S_{i+1} \setminus \{a\}$ for a some $a \in S_{i+1}$ (a becomes the Frobenius number of S_i). This idea motivates the next result, which explains how the sons of a vertex in the tree are constructed.

PROPOSITION 210. *Let S be a saturated numerical semigroup. The following conditions are equivalent.*

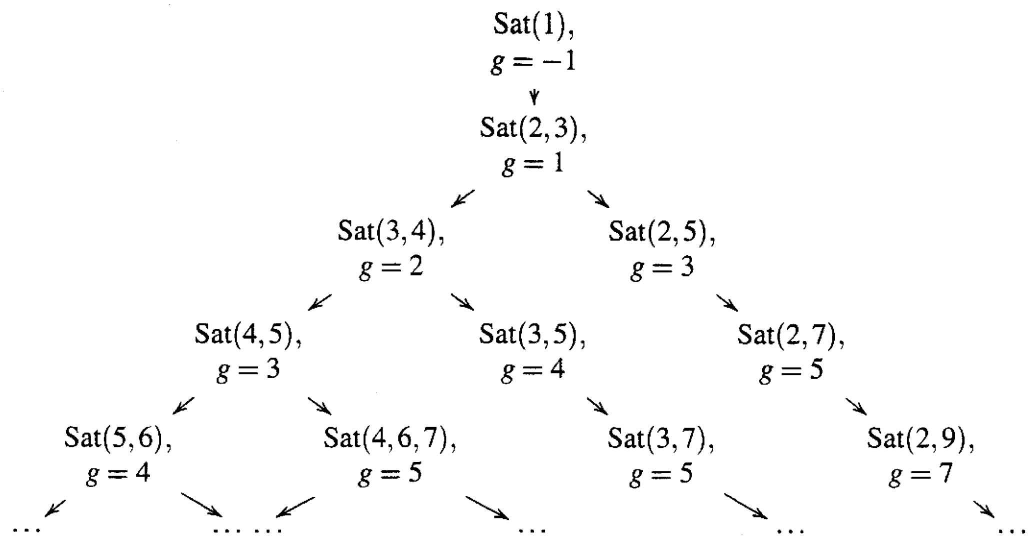
(1) $S = S' \cup \{\mathrm{g}(S')\}$ *with S' a saturated numerical semigroup,*
(2) *the minimal SAT system of generators of S contains an element greater than* $\mathrm{g}(S)$.

PROOF. *(1) implies (2).* If $S = S' \cup \{\mathrm{g}(S')\}$ with S' a saturated numerical semigroup, then $S' = S \setminus \{\mathrm{g}(S')\}$, which by Lemma 200 and Theorem 203, implies that $\mathrm{g}(S')$ belongs to the minimal SAT system of generators of S. As $S' \subsetneq S$ and $\mathrm{g}(S') \in S$, we get that $\mathrm{g}(S') > \mathrm{g}(S)$.

(2) implies (1). By Lemma 200 and Theorem 203, if a belongs to the minimal SAT system of generators of S, then $S' = S \setminus \{a\}$ is a saturated numerical semigroup. If in addition $a > \mathrm{g}(S)$, then $a = \mathrm{g}(S')$, whence $S = S' \cup \{\mathrm{g}(S')\}$, with S' a saturated numerical semigroup. □

This proposition allows us to construct recursively (starting from $\mathbb{N}$) the set of all saturated numerical semigroups. This construction arranges this set in a tree. It is clear that once we move downwards along the branches of the tree, the semigroups we encounter have greater Frobenius numbers.

FIGURE 3. The tree of saturated numerical semigroups



PROPOSITION 211. *The tree of saturated numerical semigroups is a binary tree with no leaves.*

PROOF. Let $A=\{n_1<\cdots<n_p\}$ be the minimal SAT system of generators of a saturated numerical semigroup S. By Theorem 198, we know that $\{n_p, n_p+1, \rightarrow\} \subseteq S$, whence $n_p > \mathrm{g}(S)$, and thus S cannot be a leaf by Proposition 210. Now consider the set $\{s_1,\ldots,s_r\} = \{s \in S \setminus \{0\} \mid s < \mathrm{g}(S)\}$. By Theorem 198, $\mathrm{Sat}(s_1,\ldots,s_r,\mathrm{g}(S)+1,\mathrm{g}(S)+2) = S$, whence $A \subseteq \{s_1,\ldots,s_r,\mathrm{g}(S)+1,\mathrm{g}(S)+2\}$. Using Proposition 210, we get that if S' is a son of S, then either $S' = S\setminus\{\mathrm{g}(S)+1\}$ or $S' = S\setminus\{\mathrm{g}(S)+2\}$. Therefore S has at most two sons, and the tree of saturated numerical semigroups is binary. $\square$

Bibliography

[1] S. S. ABHYANKAR, Local rings of high embedding dimension, Amer. J. Math., **89** (1967), 1073-1077.

[2] F. AJILI AND E. CONTEJEAN, Avoiding slack variables in the solving of linear Diophantine equations and inequations, Theoret. Comput. Sci., **173** (1997), 183-201.

[3] C. ARF, Une interprétation algébrique de la suite des ordres de multiplicité d'une branche algébrique, Proc. London Math. Soc., **20** (1949), 256-287.

[4] R. APÉRY, Sur les branches superlinéaires de courbes algébriques, C.R. Acad. Sci. Paris, **222** (1946), 1198-2000.

[5] V. BARUCCI, D. E. DOBBS AND M. FONTANA, "Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains", Memoirs of the Amer. Math. Soc., **598** (1997).

[6] V. BARUCCI AND R. FRÖBERG, One-Dimensional Almost Gorenstein Rings, J. Algebra, **188** (1997), 418-442.

[7] H. BASS, On the ubiquity of Gorenstein rings, Math. Z, **82** (1963), 8-28.

[8] J. BERTIN AND P. CARBONE, Semi-groups d'entiers et application aux branches, J. Algebra, **49** (1977), 81-95.

[9] A. BRAUER, On a problem of partitions, Amer. J. Math, **64** (1942), 299-312.

[10] A. BRAUER AND J. E. SCHOCKLEY, On a problem of Frobenius, J. Reine Amgey Math, **211** (1962), 215-220.

[11] H. BRESINSKY, On prime ideals with generic zero $x_i = t^{n_i}$, Proc. Amer. Soc., **47** (1975), 329-332.

[12] A. CAMPILLO, On saturation of curve singularities (any characteristic), Proc. of Symp. in Pure Math., **40** (1983), 211-220.

[13] E. CONTEJEAN AND H. DEVIE, An efficient incremental algorithm for solving systems of linear diophantine equations, Inform. and Comput., **113** (1994), 143-172.

[14] W. C. BROWN AND F. CURTIS, Numerical semigroups of maximal and almost maximal length, Semigroup Forum, **42** (1991), 219-235.

[15] W. C. BROWN, J. HERZOG, One dimensional local rings of maximal and almost maximal length, J. Algebra, **151**, 332-347.

[16] A. CAMPILLO AND J. M. RUIZ, Some remarks on Pythagorean real curve germs, J. Algebra, **128** (1990), 271-275.

[17] J. CASTELLANOS, A relation between the sequence of multiplicities and the semigroups of values of an algebroid curve, J. Pure Appl. Algebra, **43** (1986), 119-127.

[18] F. DELGADO AND A. NUÑEZ, Monomial rings and saturated rings, Géométrie algébrique et applications, I (La Rábida, 1984), 23–34, Travaux en Cours, **22**, Hermann, Paris, 1987.

[19] C. DELORME, Sous-monoides d'intersion complete de $\mathbb{N}$, Ann. Scient. Ec. Norm. Sup., 4-série, t.9 (1976), 145-154.

[20] R. FRÖBERG, G. GOTTLIEB AND R. HÄGGKVIST, On numerical semigroups, Semigroup Forum, **35** (1987), 63-83.

[21] R. FRÖBERG, C. GOTTLIEB, R. Häggkvist, Semigroups, semigroup rings and analytically irreducible rings, Report Univ. Stockholm, **1** (1986).

[22] R. GILMER, "Commutative semigroup rings", The University of Chicago Press, 1984.

[23] R. P. GRIMALDI, "Discrete and combinatorial mathematics. An applied introduction", Addison-Wesley, Reading, Massachusetts, 1994.

[24] J. HERZOG, Generators and relations of abelian semigroups and semigroup rings, Manuscripta Math, **3** (1970), 175-193.

[25] E. KUNZ, The value semigroup of one dimensional Gorenstein ring, Proc. Amer. Math. Soc., **25** (1970), 748-751.

[26] J. LIPMAN, Stable ideals and Arf rings, Amer. J. Math., **93** (1971), 649-685.

[27] F. Pham and B. Teissier, Fractions lipschitziennes et saturations de Zariski des algébres analytiques complexes, Centre Math. École Polytech., Paris, 1969. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 649–654. Gauthier-Villars, Paris, 1971.

[28] L. RÉDEI, "The theory of finitely generated commutative semigroups", Pergamon, Oxford-Edinburgh, New York , 1965.

[29] S. M. JOHNSON, A linear diofhantine problem, Can. J. Math., **12** (1960), 390-398.

[30] J. C. ROSALES, An algorithm for determining a minimal relation associated to a numerical semigroup, Int. J. Algebra and Comput., **6** (1996), 441-455.

[31] J. C. ROSALES, On Symmetric numerical semigroups, Algebra, **182** (1996), 422-434.

[32] J. C. ROSALES, On numerical semigroups, Semigroup Forum, **52** (1996), 307-318.

[33] J. C. ROSALES, Symmetric numerical semigroups with arbitrary multiplicity and embedding dimension, Proc. Amer. Math. Soc., **129** (2001), 2197-2203.

[34] J. C. ROSALES. Numerical semigroups with Apéry Sets of Unique Expression, J. Algebra, **226** (2000), 479-487.

[35] J. C. ROSALES AND M. B. BRANCO, Irreducible numerical semigroups, to appear in Pacific Journal of Mathematics.

[36] J. C. ROSALES AND M. B. BRANCO, Irreducible numerical semigroups with arbitrary multiplicity and embeding dimension, to appear in J. Algebra.

[37] J. C. ROSALES AND M. B. BRANCO, Numerical semigroups can be expressed as an intersection of symmetric numerical semigroups, J. Pure and Applied Algebra, **171** (2002), 303-314.

[38] J. C. ROSALES AND M. B. BRANCO, Decomposition of a numerical semigroup as an intersection of irreducible numerical semigroups, B. Belg. Math. Soc., **9** (3) (2002), 373-381.

[39] J. C. ROSALES AND P. A. GARCÍA-SÁNCHEZ, "Finitely generated commutative monoids", Nova Science Publishers, New York, 1999.

[40] J. C. ROSALES AND P. A. GARCÍA-SÁNCHEZ, On numerical semigroups with high embedding dimension, J. Algebra, **203** (1998), 567-578.

[41] J. C. ROSALES AND P. A. GARCÍA-SÁNCHEZ, Nonnegative elements of subgroups of $\mathbb{N}^n$, Linear Algebra Appl., **270** (1998), 351-357.

[42] J. C. ROSALES Y P. A. GARCÍA-SÁNCHEZ, On normal affine semigroups, Linear Algebra Appl., **286** (1999), 175-186.

[43] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND M. B. BRANCO, Numerical semigroups with monotonic Apéry, preprint.

[44] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND M. B. BRANCO, Systems of ineaqualities and numerical semigroups, J. Lond. Math. Soc., **65** (2) (2002), 611-623.

[45] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND M. B. BRANCO, Saturated numerical semigroups, to appear in Hounston J. of Mathematics.

[46] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND M. B. BRANCO, Arf numerical semigroups, submited.

[47] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND M.B.BRANCO, Numerical semigroups with maximal embedding dimension, to appear in Intern. J. of comutative rings.

[48] J. C. ROSALES, P. A. GARCÍA-SÁNCHEZ, J. I. GARCIA-GARCIA AND J. A. JIMÉNEZ MADRID, The oversemigroups of a numerical semigroup, submitted.
[49] J. M. RUIZ, Pythagorean real curve germs, J. Algebra, **94** (1985), 126-144.
[50] J. D. SALLY, On the associated graded ring of a local Cohen-Macaulay ring, J. Math. Kyoto Univ., **17** (1977), 19-21.
[51] J. SALLY Cohen-Macaulay local rings of maximal embedding dimension, J. Algebra, **56** (1979), 168-183.
[52] E. S. SELMER AND O. BEYER, On linear diophantine problem of Frobenius in three variables, J. Reine Angew. Math., **301** (1978), 161-170.
[53] E. SIERING. Uber lineare formen und ein problem von Frobenius, Dissertation, Joh. Gutenberg-Universitad, Mainz, 1974.
[54] J. J. SYLVESTER, Mathematical questions with their solutions, Educacional Times, **41** (1884), 21.
[55] B. TEISSIER, Appendice à "Le probléme des modules pour le branches planes", cours donné par O. Zariski au Centre de Math. de L'École Polytechnique, Paris (1973).
[56] K. WATANABE, Some examples of one dimensional Gorenstein domains, Nagoya Math. J., **49** (1973), 101-109.
[57] O. ZARISKI, General theory of saturation and saturated local rings I, II, III, Amer. J. Math., **93** (1971), 573-684, 872-964, **97**(1975), 415-502.

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