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Impulsive coupled systems with generalized jump conditions

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Abstract. This work considers a second-order impulsive coupled system with full nonlinearities, generalized impulse functions and mixed boundary conditions. This is the first time where such coupled systems are considered with nonlinearities with dependence on both unknown functions and their derivatives, together impulsive functions given by more general framework allowing jumps on the both functions and both derivatives.

The arguments apply the fixed point theory, Green's functions technique, L^1-Carathéodory functions theory and Schauder's fixed point theorem.

An application to the transverse vibration system of elastically coupled double-string is presented in the last section.

Keywords: impulsive coupled systems, L^1-Carathéodory functions, Green's functions, Schauder's fixed-point theorem, elastically coupled double-string system.

1 Introduction

In this paper, we consider the second-order impulsive coupled system with mixed boundary conditions

$$\begin{aligned} u''(x) &= f\big(x, u(x), u'(x), v(x), v'(x)\big), \\ v''(x) &= h\big(x, u(x), u'(x), v(x), v'(x)\big), \\ u(a) &= A_1, \quad u'(b) = B_1, \\ v(a) &= A_2, \quad v(b) = B_2, \end{aligned} \tag{1}$$

where $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ are L^1-Carathéodory functions, $A_1, A_2, B_1, B_2 \in \mathbb{R}$, with the generalized impulsive conditions

$$\begin{aligned}
&\Delta u(x_k) = I_{0k}\bigl(x_k, u(x_k), u'(x_k)\bigr),\\
&\Delta u'(x_k) = I_{1k}\bigl(x_k, u(x_k), u'(x_k)\bigr), \quad k = 1, 2, \dots, n,\\
&\Delta v(\tau_j) = J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)),\\
&\Delta v'(\tau_j) = J_{1j}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr), \quad j = 1, 2, \dots, m,
\end{aligned} \tag{2}$$

where, for $i = 0, 1$, $\Delta u^{(i)}(x_k) = u^{(i)}(x_k^+) - u^{(i)}(x_k^-)$, $\Delta v^{(i)}(\tau_j) = v^{(i)}(\tau_j^+) - v^{(i)}(\tau_j^-)$, and being $I_{ik}, J_{ij} \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$ with x_k, τ_j fixed points such that $a < x_1 < x_2 < \cdots < x_n < b$ and $a < \tau_1 < \tau_2 < \cdots < \tau_m < b$.

The theory of impulsive differential equations describes processes in which a sudden change of state occurs at certain moments. Several authors (see, for example, [1, 3, 6–8, 10–14, 17, 20, 25]) have dealt with impulsive differential equations from different points of view and using many techniques.

There are many phenomena and applications related to impulsive differential systems, for example, we can find biological models, population dynamics, neural networks, models in economics, on time scales, on state-dependent delays, on delay-dependent impulsive control, on electrochemical communication between cells in the brain (see for instance, [2, 4, 5, 9, 15, 16, 19, 22–24, 26, 27, 29, 30]), among others.

In [28], the author considers a sufficient conditions for the existence and uniqueness of solutions to the following complex dynamical network in the form of a coupled system of $m + 2$ point boundary conditions for impulsive fractional differential equations

$$\begin{aligned}
&{}^cD^\alpha u(t) = \phi\bigl(t, u(t), v(t)\bigr), \quad t \in [0,1],\ t \neq t_j,\ j = 1, \dots, m,\\
&{}^cD^\beta v(t) = \psi\bigl(t, u(t), v(t)\bigr), \quad t \in [0,1],\ t \neq t_i,\ i = 1, \dots, n,\\
&u(0) = h(u), \quad u(1) = g(u) \quad \text{and} \quad v(0) = k(v), \quad v(1) = f(v),\\
&\Delta u(t_j) = I_j\bigl(u(t_j)\bigr), \quad \Delta u'(t_j) = \bar{I}_j\bigl(u(t_j)\bigr), \quad j = 1, \dots, m,\\
&\Delta v(t_i) = I_i\bigl(v(t_i)\bigr), \quad \Delta v'(t_i) = \bar{I}_i\bigl(v(t_i)\bigr), \quad i = 1, \dots, n,
\end{aligned}$$

where $1 < \alpha, \beta \leqslant 2$, $\phi, \psi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, and $g, h : X \to \mathbb{R}$, $f, k : Y \to \mathbb{R}$ are continuous functionals defined by

$$\begin{aligned}
g(u) &= \sum_{j=1}^{p} \lambda_j u(\xi_j), \qquad h(u) = \sum_{j=1}^{p} \lambda_j u(\eta_j),\\
f(v) &= \sum_{i=1}^{q} \delta_i v(\xi_i), \qquad k(v) = \sum_{i=1}^{q} \delta_i v(\eta_i),
\end{aligned}$$

$\xi_i, \eta_i, \xi_j, \eta_j \in (0,1)$ for $i = 1, \dots, q$ and $j = 1, \dots, q$.

In [18], it is studied the BVP for second-order singular differential system on the whole line with impulse effects, i.e., consisting of the differential system

$$\begin{aligned} \left[\phi_p\left(\rho(t)x'(t)\right)\right]' &= f\left(t, x(t), y(t)\right), \quad \text{a.e. } t \in \mathbb{R}, \\ \left[\phi_q\left(\varrho(t)y'(t)\right)\right]' &= g\left(t, x(t), y(t)\right), \quad \text{a.e. } t \in \mathbb{R}, \end{aligned}$$

subjected to the boundary conditions

$$\lim_{t\to\pm\infty} x(s) = 0, \qquad \lim_{t\to\pm\infty} y(s) = 0$$

and the impulse effects

$$\begin{aligned} \Delta x(t_k) &= I_k\left(t_k, x(t_k), y(t_k)\right), \quad k \in \mathbb{Z}, \\ \Delta y(t_k) &= J_k\left(t_k, x(t_k), y(t_k)\right), \quad k \in \mathbb{Z}, \end{aligned}$$

where

(i) $\rho, \varrho \in C^0(\mathbb{R}, [0, \infty))$, $\rho(t), \varrho(t) > 0$ for all $t \in \mathbb{R}$ with $\int_{-\infty}^{+\infty} \mathrm{d}s/\rho(s) < +\infty$ and $\int_{-\infty}^{+\infty} \mathrm{d}s/\varrho(s) < +\infty$,

(ii) $\phi_p(x) = x|x|^{p-2}$, $\phi_q(x) = x|x|^{q-2}$ with $p > 1$ and $q > 1$ are Laplace operators,

(iii) f, g on $\mathbb{R}^3$ are Carathéodory functions,

(iv) $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$ with $\lim_{k\to-\infty} t_k = -\infty$ and $\lim_{k\to+\infty} t_k = +\infty$, $\Delta x(t_k) = u(t_x^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_x^+) - y(t_k^-)$ ($k \in \mathbb{Z}$), $\mathbb{Z}$ is the set of all integers,

(v) $\{I_k\}$, $\{J_k\}$, with $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$, are Carathéodory sequences.

Motivated by these works, we follow arguments applied in [21] to study problem (1)–(2). We point out that is the first time when second-order coupled differential equations systems include full nonlinearities. That is, they depend on the unknown functions, and their first derivatives, together with generalized impulsive conditions with dependence on the first derivative, too.

The paper is organized as it follows: Section 2 contains the preliminary results: definitions and some auxiliary lemmas. Section 3 contains the main result: an existence solution of the problem. In the last section, our main theorem is illustrated by an example and applications to a real phenomena: the transverse vibrations system of elastically coupled double-string model.

2 Definitions and auxiliary results

Define $u(x_k^\pm) := \lim_{x\to x_k^\pm} u(x)$, consider the set

$$\begin{aligned} PC_1\left([a,b]\right) = \bigl\{&u\colon [a,b] \to \mathbb{R},\ u \text{ is continuous for } x \neq x_k,\ u(x_k) = u\left(x_k^-\right), \\ &u\left(x_k^+\right) \text{ exists for } k = 1, 2, \ldots, n\bigr\} \end{aligned}$$

and the space $X_1 := PC_1^1([a,b]) = \{u\colon u'(t) \in PC_1([a,b])\}$ equipped with the norm $\|u\|_{X_1} = \max\{\|u\|, \|u'\|\}$, where

$$\|w\| := \sup_{x\in[a,b]} \bigl|w(x)\bigr|.$$

Analogously, define the space $X_2 := PC_2^1([a,b]) = \{v\colon v'(x) \in PC_2([a,b])\}$, where

$$PC_2\bigl([a,b]\bigr) = \bigl\{v\colon v[a,b] \to \mathbb{R},\ v \text{ is continuous for } \tau \neq \tau_j,\ v(\tau_j) = v\bigl(\tau_j^- v\bigr),\\ v\bigl(\tau_j^+\bigr) \text{ exists for } j = 1,2,\dots,m\bigr\},$$

equipped with the norm $\|v\|_{X_2} = \max\{\|v\|, \|v'\|\}$.

Denoting $X := X_1 \times X_2$ and the norm $\|(u,v)\|_X = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}$, it is clear that $(X, \|\cdot\|_X)$ is a Banach space.

A pair of functions (u,v) is a solution of problem (1)–(2) if $(u,v) \in X$ and verifies conditions (1) and (2).

Definition 1. A function $g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ is L^1-Carathéodory if

(i) for each $(t,y,z,w) \in \mathbb{R}^4$, $x \mapsto g(x,t,y,z,w)$ is measurable on $[a,b]$;
(ii) for a.e. $x \in [a,b]$, $(t,y,z,w) \mapsto g(x,t,y,z,w)$ is continuous on $\mathbb{R}^4$;
(iii) for each $\rho > 0$, there exists a positive function $\phi_\rho \in L^1([a,b])$, and for $(t,y,z,w) \in \mathbb{R}^4$ such that

$$\max\bigl\{|t|, |y|, |z|, |w|\bigr\} < \rho,$$

one has

$$\bigl|g(x,t,y,z,w)\bigr| \leqslant \phi_\rho(t), \quad a.e.\ x \in [a,b].$$

Lemma 1. *A pair of functions* $(u,v) \in X$ *is a solution of problem* (1)–(2) *if and only if*

$$\begin{aligned} u(x) = {} & A_1 + B_1(x-a) \\ & + \sum_{x_k<x} \bigl[I_{0k}(x_k, u(x_k), u'(x_k)\bigr) + I_{1k}\bigl(x_k, u(x_k), u'(x_k)\bigr)(x - x_k)\bigr] \\ & - (x-a)\sum_{k=1}^{n} I_{1k}\bigl(x_k, u(x_k), u'(x_k)\bigr) \\ & + \int_a^b G_1(x,s) \quad f\bigl(s, u(s), u'(s), v(s), v'(s)\bigr)\,\mathrm{d}s \end{aligned}$$

with $G_1(x,s)$ *given by*

$$G_1(x,s) = \begin{cases} a - s, & a \leqslant x \leqslant s \leqslant b, \\ a - x, & a \leqslant s \leqslant x \leqslant b, \end{cases} \tag{3}$$

and

$$\begin{aligned} v(x) &= A_2 + \frac{B_2 - A_2}{b-a}(x-a) \\ &\quad + \sum_{\tau_j < x} \bigl[J_{0j}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr) + J_{1j}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr)(x-\tau_j)\bigr] \\ &\quad - \frac{x-a}{b-a} \sum_{j=1}^{m} \bigl[J_{0j}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr) + J_{1j}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr)(x-\tau_j)\bigr] \\ &\quad + \int_a^b G_2(x,s) h\bigl(s, u(s), u'(s), v(s), v'(s)\bigr)\,\mathrm{d}s \end{aligned}$$

with $G_2(x,s)$ defined by

$$G_2(x,s) = \frac{1}{a-b} \begin{cases} (a-s)(b-x), & a \leqslant x \leqslant s \leqslant b, \\ (x-a)(b-s), & a \leqslant s \leqslant x \leqslant b. \end{cases} \tag{4}$$

The proof follows standard calculus and it is omitted.

A key tool is the Schauder's fixed point theorem:

Theorem 1. (*See* [31].) *Let Y be a nonempty, closed, bounded, and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

3 Main theorem

The main result will provide the existence of at least one solution of problem (1)–(2).

Theorem 2. *Let $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ be L^1-Carathéodory functions, and let $I_{ik}, J_{ij} : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous functions for $i = 0, 1$, $k = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$. Then there is at least one pair of functions $(u,v) \in X$, which is a solution of* (1)–(2).

Proof. Define the operators $T_1 : X \to X_1$, $T_2 : X \to X_2$, and $T : X \to X$ by

$$T(u,v) = \bigl(T_1(u,v), T_2(u,v)\bigr) \tag{5}$$

with

$$\begin{aligned} \bigl(T_1(u,v)\bigr)(x) &= A_1 + B_1(x-a) \\ &\quad + \sum_{x_k < x} \bigl[I_{0k}\bigl(x_k, u(x_k), u'(x_k)\bigr) + I_{1k}\bigl(x_k, u(x_k), u'(x_k)\bigr)(x-x_k)\bigr] \\ &\quad - (x-a) \sum_{k=1}^{n} I_{1k}\bigl(x_k, u(x_k), u'(x_k)\bigr) \\ &\quad + \int_a^b G_1(x,s) f\bigl(s, u(s), u'(s), v(s), v'(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

$$\begin{aligned}\big(T_2(u,v)\big)(x) = {} & A_2 + \frac{B_2 - A_2}{b-a}(x-a) \\ & + \sum_{\tau_j < x} \big[J_{0j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) + J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big)(x-\tau_j)\big] \\ & - \frac{x-a}{b-a} \sum_{j=1}^{m} \big[J_{0j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) + J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big)(x-\tau_j)\big] \\ & + \int_a^b G_2(x,s) h\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s,\end{aligned}$$

where $G_1(x,s)$ and $G_2(x,s)$ are given by (3) and (4), respectively.

By Lemma 1, it is obvious that the fixed points of T are solutions of (1)–(2), so we shall prove that T has a fixed point, following, for clearness, several steps.

Step 1. T is well defined and continuous in X.

As $f, h : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ are L^1-Carathéodory functions, then $T_1(u,v) \in PC_1^1$ and $T_2(u,v) \in PC_2^1$. In fact, $T(u,v) = (T_1(u,v), T_2(u,v))$ is continuous and

$$\begin{aligned}\big(T_1(u,v)\big)'(x) = {} & B_1 + \sum_{x_k < x} I_{1k}\big(x_k, u(x_k), u'(x_k)\big) - \sum_{k=1}^{n} I_{1k}\big(x_k, u(x_k), u'(x_k)\big) \\ & - \int_a^x f\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s,\end{aligned}$$

$$\begin{aligned}\big(T_2(u,v)\big)'(x) = {} & \frac{B_2 - A_2}{b-a} + \sum_{\tau_j < x} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) \\ & - \frac{1}{b-a} \sum_{j=1}^{m} \big[J_{0j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) + J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big)(x-\tau_j)\big] \\ & - \frac{x-a}{b-a} \sum_{j=1}^{m} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) \\ & + \int_a^b \frac{\partial G_2}{\partial x}(x,s) h\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s\end{aligned}$$

with

$$\frac{\partial G_2}{\partial x}(x,s) = \frac{1}{b-a}\begin{cases} s-a, & a \leqslant x \leqslant s \leqslant b, \\ b-s, & a \leqslant s \leqslant x \leqslant b. \end{cases} \tag{6}$$

Therefore, $T_1(u,v) \in X_1$, $T_2(u,v) \in X_2$, and $T(u,v) \in X$.

Step 2. TB is uniformly bounded in $B \subseteq X$.

Let B be a bounded set of X. As f and h are L^1-Carathéodory functions, there exists $\phi_\rho, \psi_\rho \in L^1([a,b])$ with $\rho_1 > 0$ such that

$$\max\bigl\{\|u\|_{X_1}, \|v\|_{X_2}\bigr\} < \rho_1, \tag{7}$$

we have

$$\begin{aligned}\bigl|f\bigl(x,u(x),u'(x),v(x),v'(x)\bigr)\bigr| &\leqslant \phi_\rho(x),\\ \bigl|h\bigl(x,u(x),u'(x),v(x),v'(x)\bigr)\bigr| &\leqslant \psi_\rho(x), \quad \text{a.e. } x \in [a,b].\end{aligned}$$

For $(u,v) \in B$, let

$$M_1(s) := \sup_{x\in[a,b]} \bigl|G_1(x,s)\bigr|, \qquad M_2(s) := \sup_{x\in[a,b]} \bigl|G_2(x,s)\bigr|. \tag{8}$$

By the continuity of functions I_{ik}, J_{ij} for $i = 0, 1$, $k = 1, 2, \dots, n$, and $j = 1, 2, \dots, m$, there are positive constants φ_{ik} and φ_{ij}^* such that

$$\bigl|I_{ik}\bigl(x_k, u(x_k), u'(x_k)\bigr)\bigr| \leqslant \varphi_{ik} \quad \text{and} \quad \bigl|J_{ij}\bigl(\tau_j, v(\tau_j), v'(\tau_j)\bigr)\bigr| \leqslant \varphi_{ij}^*.$$

Moreover,

$$\begin{aligned}&\bigl\|T_1(u,v)(x)\bigr\|\\ &\quad\leqslant \sup_{x\in[a,b]}\Biggl(|A_1| + |B_1||x-a|\\ &\qquad + \sum_{x_k<x}\bigl|\bigl[I_{0k}\bigl(x_k,u(x_k),u'(x_k)\bigr) + I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr)(x-x_k)\bigr]\bigr|\\ &\qquad + \bigl|(x-a)\bigr|\sum_{k=1}^{n}\bigl|I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr)\bigr|\\ &\qquad + \int_a^b \bigl|G_1(x,s)\bigr|\bigl|f\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\bigr|\,\mathrm{d}s\Biggr)\\ &\quad\leqslant |A_1| + |B_1|(b-a) + \sum_{k=1}^{n}\bigl[\varphi_{0k} + 2(b-a)\varphi_{1k}\bigr] + \int_a^b M_1(s)\phi_\rho(s)\,\mathrm{d}s < +\infty,\end{aligned}$$

$$\begin{aligned}&\bigl\|\bigl(T_1(u,v)\bigr)'(x)\bigr\|\\ &\quad\leqslant \sup_{x\in[a,b]}\Biggl(|B_1| + \sum_{x_k<x}\bigl|I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr)\bigr|\\ &\qquad + \sum_{k=1}^{n}\bigl|I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr)\bigr| + \int_a^b \bigl|f\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\bigr|\,\mathrm{d}s\Biggr)\\ &\quad\leqslant |B_1| + 2\sum_{k=1}^{n}\varphi_{1k} + \int_a^b \phi_\rho(s)\,\mathrm{d}s < +\infty,\end{aligned}$$

$$
\begin{aligned}
&\bigl\|T_2(u,v)(x)\bigr\| \\
&\quad\leqslant \sup_{x\in[a,b]}\Biggl(|A_2| + \frac{|B_2-A_2|}{b-a}|x-a| \\
&\qquad + \sum_{\tau_j<x}\bigl[\bigl|J_{0j}(\tau_j,v(\tau_j),v'(\tau_j)) + J_{1j}(\tau_j,v(\tau_j),v'(\tau_j))(x-\tau_j)\bigr|\bigr] \\
&\qquad + \frac{|x-a|}{b-a}\sum_{j=1}^{m}\bigl|J_{0j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr) + J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(x-\tau_j)\bigr| \\
&\qquad + \int_a^b \bigl|G_2(x,s)\bigr|\bigl|h\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\bigr|\,\mathrm{d}s\Biggr) \\
&\quad\leqslant |A_2| + |B_2-A_2| + 2\sum_{\tau_j<t}\bigl[\varphi_{0j}^* + \varphi_{1j}^*(b-a)\bigr] + \int_a^b M_2(s)\psi_\rho(s)\,\mathrm{d}s < +\infty,
\end{aligned}
$$

and, by (6),

$$
\begin{aligned}
&\bigl\|\bigl(T_2(u,v)\bigr)'(x)\bigr\| \\
&\quad\leqslant \sup_{x\in[a,b]}\Biggl(\frac{|B_2-A_2|}{b-a} + \sum_{\tau_j<x}\bigl|J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)\bigr| \\
&\qquad + \frac{1}{b-a}\sum_{j=1}^{m}\bigl|J_{0j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr) + J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(x-\tau_j)\bigr| \\
&\qquad + \frac{|x-a|}{b-a}\sum_{j=1}^{m}\bigl|J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)\bigr| \\
&\qquad + \frac{1}{b-a}\int_a^b \left|\frac{\partial G_2}{\partial x}(x,s)\right|\bigl|h\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\bigr|\,\mathrm{d}s\Biggr) \\
&\quad\leqslant \frac{|B_2-A_2|}{b-a} + \frac{1}{b-a}\sum_{j=1}^{m}\varphi_{0j}^* + 3\sum_{j=1}^{m}\varphi_{1j}^* + \frac{1}{b-a}\int_a^b\left|\frac{\partial G_2}{\partial x}(x,s)\right|\psi_\rho(s)\,\mathrm{d}s < +\infty.
\end{aligned}
$$

So, TB is uniformly bounded on X.

Step 3. T is equicontinuous on each interval $]x_k,x_{k+1}]\times]\tau_j,\tau_{j+1}]$, that is, T_1B is equicontinuous on each interval $]x_k,x_{k+1}]$ for $k=0,1,\dots,n$ with $x_0=a$ and $x_{n+1}=b$, and T_2B is equicontinuous on each interval $]\tau_j,\tau_{j+1}]$ for $j=0,1,\dots,m$ with $\tau_0=a$ and $\tau_{m+1}=b$.

Consider $J\subseteq]x_k,x_{k+1}]$ and $\iota_1,\iota_2\in J$ such that $\iota_1\leqslant\iota_2$.

So, by the continuity of G_1,

$$\begin{aligned}
&\bigl|T_1(u,v)(\iota_1) - T_1(u,v)(\iota_2)\bigr| \\
&\quad = \Biggl| B_1(\iota_1-\iota_2) + \sum_{x_k<\iota_1} \bigl[I_{0k}\bigl(x_k,u(x_k),u'(x_k)\bigr) \\
&\qquad + I_{1k}(x_k,u(x_k),u'(x_k))(\iota_1-x_k)\bigr] - (\iota_1-\iota_2)\sum_{k=1}^{n} I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr) \\
&\qquad - \sum_{x_k<\iota_2} \bigl[I_{0k}\bigl(x_k,u(x_k),u'(x_k)\bigr) + I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr)(\iota_2-x_k)\bigr] \\
&\qquad + \int_a^b \bigl[G_1(\iota_1,s)-G_1(\iota_2,s)\bigr] f\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\,\mathrm{d}s \Biggr| \\
&\quad \to 0 \quad \text{as } \iota_1\to\iota_2,
\end{aligned}$$

$$\begin{aligned}
&\bigl|\bigl(T_1(u,v)(\iota_1)\bigr)' - \bigl(T_1(u,v)(\iota_2)\bigr)'\bigr| \\
&\quad = \Biggl| \sum_{x_k<\iota_1} I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr) - \sum_{x_k<\iota_2} I_{1k}\bigl(x_k,u(x_k),u'(x_k)\bigr) \\
&\qquad - \int_{\iota_1}^{\iota_2} f\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\,\mathrm{d}s \Biggr| \\
&\quad \to 0 \quad \text{as } \iota_1\to\iota_2,
\end{aligned}$$

$$\begin{aligned}
&\bigl|T_2(u,v)(\iota_1) - T_2(u,v)(\iota_2)\bigr| \\
&\quad = \Biggl| \frac{B_2-A_2}{b-a}(\iota_1-\iota_2) + \sum_{x_k<\iota_1} \bigl[J_{0j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr) \\
&\qquad + J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(\iota_1-\tau_j)\bigr] \\
&\qquad + \frac{\iota_2-\iota_1}{b-a}\sum_{j=1}^{m} J_{0j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr) - \frac{\iota_1-a}{b-a}\sum_{j=1}^{m} J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(\iota_1-\tau_j) \\
&\qquad - \sum_{x_k<\iota_2} \bigl[J_{0j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr) + J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(\iota_2-\tau_j)\bigr] \\
&\qquad + \frac{\iota_2-a}{b-a}\sum_{j=1}^{m} J_{1j}\bigl(\tau_j,v(\tau_j),v'(\tau_j)\bigr)(\iota_2-\tau_j) \\
&\qquad + \int_a^b \bigl[G_2(\iota_1,s)-G_2(\iota_2,s)\bigr] h\bigl(s,u(s),u'(s),v(s),v'(s)\bigr)\,\mathrm{d}s \Biggr| \\
&\quad \to 0 \quad \text{as } \iota_1\to\iota_2,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\big(T_2(u,v)(\iota_1)\big)' - \big(T_2(u,v)(\iota_2)\big)'\right| \\
&\quad = \left| \sum_{\tau_j<\iota_1} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) - \frac{1}{b-a}\sum_{j=1}^{m} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big)(\iota_1-\iota_2) \right. \\
&\qquad + \frac{(\iota_1-\iota_2)}{b-a}\sum_{j=1}^{m} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) - \sum_{\tau_j<\iota_2} J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) \\
&\qquad \left. + \frac{1}{b-a}\int_{\iota_1}^{\iota_2} \frac{\partial G_2}{\partial x}(x,s) h\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s \right| \\
&\quad \to 0 \quad \text{as } \iota_1 \to \iota_2,
\end{aligned}$$

and $\partial G_2/\partial x$ given by (6).

Step 4. TB is equiconvergent, that is, T_1B is equiconvergent at $x = x_k^+$ for $k = 0, 1, \dots, n$, and T_2B is equiconvergent at $\tau = \tau_j^+$ for $\tau = 1, \dots, m$.

$$\begin{aligned}
&\left|T_1(u,v)(x) - T_1(u,v)(x_k^+)\right| \\
&\quad = \left| B_1\big(x - x_k^+\big) + \sum_{x_k<x}\big[I_{0k}\big(x_k, u(x_k), u'(x_k)\big) + I_{1k}\big(x_k, u(x_k), u'(x_k)\big)(x-x_k)\big] \right. \\
&\qquad - \big(x - x_k^+\big)\sum_{k=1}^{n} I_{1k}\big(x_k, u(x_k), u'(x_k)\big) \\
&\qquad - \sum_{x_k<x_k^+}\big[I_{0k}\big(x_k, u(x_k), u'(x_k)\big) + I_{1k}\big(x_k, u(x_k), u'(x_k)\big)\big(x_k^+ - x_k\big)\big] \\
&\qquad \left. + \int_a^b \big[G_1(x,s) - G_1\big(x_k^+, s\big)\big] f\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s \right| \\
&\quad \to 0 \quad \text{uniformly as } x \to x_k^+,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\big(T_1(u,v)(x)\big)' - \big(T_1(u,v)(x_k^+)\big)'\right| \\
&\quad = \left| \sum_{x_k<x} I_{1k}\big(x_k, u(x_k), u'(x_k)\big) - \sum_{x_k<x_k^+} I_{1k}\big(x_k, u(x_k), u'(x_k)\big) \right. \\
&\qquad \left. - \int_{x_k^+}^{x} f\big(s, u(s), u'(s), v(s), v'(s)\big)\,\mathrm{d}s \right| \\
&\quad \to 0 \quad \text{uniformly as } x \to x_k^+.
\end{aligned}$$

So, T_1B is equiconvergent at $x = x_{k^+}$ for $k = 0, 1, \dots, n$.

Similarly,

$$
\begin{aligned}
&\left|T_2(u,v)(\tau) - T_2(u,v)\left(\tau_j^+\right)\right| \\
&\quad = \left|\frac{B_2 - A_2}{b-a}\left(\tau - \tau_j^+\right)\right. \\
&\qquad + \sum_{\tau_j<\tau}\left[J_{0j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) + J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right)(\tau - \tau_j)\right] \\
&\qquad + \frac{\tau_j^+ - \tau}{b-a}\sum_{j=1}^{m} J_{0j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) - \frac{\tau - a}{b-a}\sum_{j=1}^{m} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right)(\tau - \tau_j) \\
&\qquad - \sum_{\tau_j<\tau_j^+}\left[J_{0j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) + J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right)\left(\tau_j^+ - \tau_j\right)\right] \\
&\qquad + \frac{\tau_j^+ - a}{b-a}\sum_{j=1}^{m} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right)\left(\tau_j^+ - \tau_j\right) \\
&\qquad \left. + \int_a^b \left[G_2(\tau,s) - G_2\left(\tau_j^+, s\right)\right] h\left(s, u(s), u'(s), v(s), v'(s)\right) \mathrm{d}s\right| \\
&\quad \to 0 \quad \text{uniformly as } \tau \to \tau_j^+,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\left(T_2(u,v)(\tau)\right)' - \left(T_2(u,v)\left(\tau_j^+\right)\right)'\right| \\
&\quad = \left|\sum_{\tau_j<\tau} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) - \frac{1}{b-a}\sum_{j=1}^{m} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right)\left(\tau - \tau_j^+\right)\right. \\
&\qquad + \frac{\left(\tau - \tau_j^+\right)}{b-a}\sum_{j=1}^{m} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) - \sum_{\tau_j<\tau_j^+} J_{1j}\left(\tau_j, v(\tau_j), v'(\tau_j)\right) \\
&\qquad + \frac{1}{b-a}\left(\int_{\tau_j^+}^{\tau} (b-s) h\left(s, u(s), u'(s), v(s), v'(s)\right) \mathrm{d}s\right. \\
&\qquad \left.\left. + \int_{\tau_j^+}^{\tau} (s-a) h\left(s, u(s), u'(s), v(s), v'(s)\right) \mathrm{d}s\right)\right| \\
&\quad \to 0 \quad \text{uniformly as } \tau \to \tau_j^+.
\end{aligned}
$$

Then T_2B is equiconvergent at $\tau = \tau_j^+$ for $\tau = 1, \dots, m$.

Like this, T_1 and T_2 maps bounded sets into relatively compact sets, that is, $T_1 : X \to X_1$ and $T_2 : X \to X_2$ are compacts. Therefore, $T : X \to X$ is compact (for details, see [18, Lemma 2.4]).

Step 5. $T : X \to X$ has a fixed point.

In order to apply Schauder's fixed point theorem for operator $T(u,v)$, we need to prove that $TD \subset D$ for some closed, bounded, and convex $D \subset X$.

Consider

$$D := \{(u,v) \in X\colon \|(u,v)\|_X \leqslant \rho_2\}$$

with $\rho_2 > 0$ such that

$$\begin{aligned}\rho_2 := \max\Biggl\{&\rho_1,\ |A_1| + |B_1|(b-a) + \sum_{k=1}^{n}\bigl[\varphi_{0k} + 2(b-a)\varphi_{1k}\bigr] + \int_a^b M_1(s)\phi_\rho(s)\,\mathrm{d}s,\\ &|B_1| + 2\sum_{k=1}^{n}\varphi_{1k} + \int_a^b \phi_\rho(s)\,\mathrm{d}s,\\ &|A_2| + |B_2 - A_2| + 2\sum_{\tau_j<x}\bigl[\varphi_{0j}^* + \varphi_{1j}^*(b-a)\bigr] + \int_a^b M_2(s)\psi_\rho(s)\,\mathrm{d}s,\\ &\frac{|B_2-A_2|}{b-a} + \frac{1}{b-a}\sum_{j=1}^{m}\varphi_{0j}^* + 3\sum_{j=1}^{m}\varphi_{1j}^* + \frac{1}{b-a}\int_a^b\left|\frac{\partial G_2}{\partial x}(x,s)\right|\psi_\rho(s)\,\mathrm{d}s\Biggr\}\end{aligned}$$

with ρ_1 given by (7) according to Step 2, and M_1, M_2 are given by (8).

Following similar arguments as in Step 2, we obtain

$$\begin{aligned}\|T(u,v)\|_X &= \|(T_1(u,v), T_2(u,v))\|_X\\ &= \max\{\|T_1(u,v)\|_{X_1}, \|T_2(u,v)\|_{X_2}\}\\ &= \max\{\|T_1(u,v)\|, \|(T_1(u,v))'\|, \|T_2(u,v)\|, \|(T_2(u,v))'\|\}\\ &\leqslant \rho_2\end{aligned}$$

and $TD \subset D$.

By Schauder's fixed point theorem, the operator T given by (5) has a fixed point (u_0, v_0). Thus, problem (1)–(2) has at least one pair solution $(u,v) \in X$. □

4 Example

Consider the coupled system of the second-order differential equations with the mixed boundary conditions

$$\begin{aligned} u''(x) &= \operatorname{sgn}\biggl(x - \frac{1}{2}\biggr)u(x)v(x) + \bigl(u'(x)\bigr)^2 v'(x), \quad x \in]0,1[,\\ v''(x) &= -(x+1)\bigl(v'(x)\bigr)^2 u(x) + \bigl(u'(x)\bigr)^3 \mathrm{e}^{-v(x)},\\ u(0) &= 1, \quad u'(1) = \frac{1}{2},\\ v(0) &= 1, \quad v(1) = 2 \end{aligned} \tag{9}$$

and the generalized impulsive conditions

$$\begin{aligned}\Delta u(x_k) &= \big(u(x_k)\big)^2(1-x_k) + u'(x_k),\\ \Delta u'(x_k) &= \sum_{k=1}^{3} x_k u(x_k) u'(x_k), \quad k=1,2,3,\\ \Delta v(\tau_j) &= \tau_j \big|v(\tau_j)\big| v'(\tau_j),\\ \Delta v'(\tau_j) &= \sum_{j=1}^{2}(1-\tau_j)\frac{v(\tau_j)}{2}\big(v'(\tau_j)\big)^2, \quad j=1,2,\end{aligned} \tag{10}$$

where $0 < x_1 < x_2 < x_3 < 1$, $0 < \tau_1 < \tau_2 < 1$.

This problem is a particular case of system (1)–(2) with

$$f(x,\alpha,\beta,\gamma,\delta) = \operatorname{sgn}\left(x - \frac{1}{2}\right)\alpha\beta + \gamma^2\delta,$$

$$h(x,\alpha,\beta,\gamma,\delta) = -(x+1)\delta^2\alpha + \gamma^3 \mathrm{e}^{-\beta},$$

$$A_1 = 1, \quad B_1 = \frac{1}{2}, \quad A_2 = 1, \quad B_2 = 2,$$

$$I_{0k}(x_k,\alpha,\beta) = \alpha^2(1-x_k) + \gamma, \qquad I_{1k}(x_k,\alpha,\beta) = \sum_{k=1}^{3} x_k\alpha\beta,$$

$$J_{0j}(\tau_j,\gamma,\delta) = \tau_j|\beta|\delta, \qquad J_{1j}(\tau_j,\gamma,\delta) = \sum_{j=1}^{2}(1-\tau_j)\frac{\beta}{2}\delta^2.$$

In fact, f, h are L^1-Carathéodory functions with $\rho > 0$ such that

$$\max\big\{|\alpha|, |\beta|, |\gamma|, |\delta|\big\} < \rho,$$

we have

$$\begin{aligned}\big|f(x,\alpha,\beta,\gamma,\delta)\big| &\leqslant \rho^2 + \rho^3 := \phi_\rho(x),\\ \big|h(x,\alpha,\beta,\gamma,\delta)\big| &\leqslant (x+1)\rho^3 + \rho^4 := \psi_\rho(x).\end{aligned}$$

Moreover,

$$I_{0k}(x_k,\alpha,\beta) = \alpha^2(1-x_k) + \gamma, \qquad I_{1k}(x_k,\alpha,\beta) = \sum_{k=1}^{3} x_k\alpha\beta,$$

$$J_{0j}(\tau_j,\gamma,\delta) = \tau_j|\beta|\delta \quad \text{and} \quad J_{1j}(\tau_j,\gamma,\delta) = \sum_{j=1}^{2}(1-\tau_j)\frac{\beta}{2}\delta^2$$

are continuous functions in $[0,1]\times\mathbb{R}^2$.

Therefore, by Theorem 2, problem (9)–(10) has at least one solution $(u,v) \in X$.

5 The transverse vibration system of elastically coupled double-string

Consider the transverse vibration system of elastically coupled double-string with damping. The strings have the same length L, are attached by a viscoelastic element, and stretched at a constant tension according to Fig. 1.

By [26], the system of elastically coupled double-string stationary model is given by the second-order nonlinear system of differential equations

$$\begin{aligned} S_1u''(x) - K\big(u(x) - v(x)\big) &= -l_1(x),\\ S_2v''(x) - K\big(v(x) - u(x)\big) &= -l_2(x), \end{aligned} \tag{11}$$

where $x \in [0, L]$,

- $u(x)$, $v(x)$ are the transverse deflections of strings u and v, respectively;
- $l_1(x)$ and $l_2(x)$ are the exciting distributed load;
- K is the modulus of Kelvin–Voigt viscoelastic;
- S_1, S_2 are the string tensions of u and v, respectively.

Adding to system (11) the boundary conditions

$$u(0) = 0, \quad u'(L) = B_1, \qquad v(0) = 0, \quad v(L) = 0, \tag{12}$$

we remark that strings u and v have different behaviors at the end points. Moreover, we consider impulsive conditions that may depend on the string deflections and on the slope of the corresponding deflections:

$$\begin{aligned} I_{0k}\big(x_k, u(x_k), u'(x_k)\big) &= \eta_1 u(x_k) + \eta_2 u'(x_k) + x_k,\\ I_{1k}\big(x_k, u(x_k), u'(x_k)\big) &= \eta_3 u(x_k) + \eta_4 u'(x_k) + x_k,\\ J_{0j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) &= \eta_5 v(\tau_j) + \eta_6 v'(\tau_j) + \tau_j,\\ J_{1j}\big(\tau_j, v(\tau_j), v'(\tau_j)\big) &= \eta_7 v(\tau_j) + \eta_8 v'(\tau_j) + \tau_j. \end{aligned} \tag{13}$$

Here B_1, $\eta_i \in \mathbb{R}$, $i = 1, \dots, 8$, $k = 1, \dots, n$ and $j = 1, \dots, m$. In this way, we have a particular case of (1)–(2).

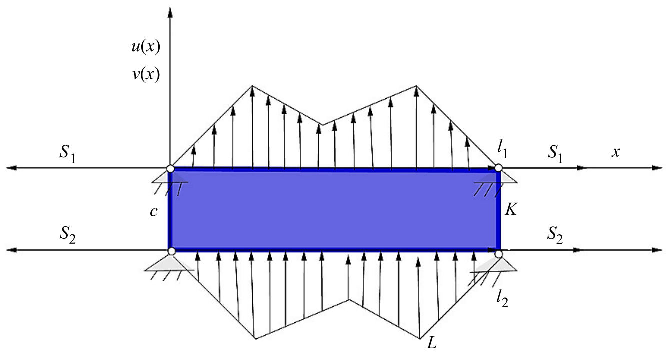

Figure 1. Elastically coupled double-string.

In system (11)–(13), it follows that

$$f(x,\alpha,\beta,\gamma,\delta) = \frac{1}{S_1}\bigl[K(\alpha-\gamma) - l_1(x)\bigr],$$
$$h(x,\alpha,\beta,\gamma,\delta) = \frac{1}{S_2}\bigl[+K(\gamma-\alpha) - l_2(x)\bigr],$$

and

$$I_{0k}(x_k,\alpha,\gamma) = \eta_1\alpha + \eta_2\gamma, \qquad I_{1k}(x_k,\alpha,\gamma) = \eta_3\alpha + \eta_4\gamma + x_k,$$
$$J_{0j}(\tau_j,\beta,\delta) = \eta_5\beta + \eta_6\delta, \qquad J_{1j}(\tau_j,\beta,\delta) = \eta_7\beta + \eta_8\delta + \tau_j$$

with $k = 1,\dots,n$ and $j = 1,\dots,m$.

Notice that f, h are L^1-Carathéodory functions in $[0,L]\times\mathbb{R}^4$ with $\rho > 0$ and

$$\max\bigl\{|\alpha|,|\beta|,|\gamma|,|\delta|\bigr\} < \rho,$$
$$\bigl|f(x,\alpha,\beta,\gamma,\delta)\bigr| \leqslant \frac{1}{|S_1|}\bigl[2|K|\rho + \bigl|l_1(x)\bigr|\bigr] := \phi_\rho(x),$$
$$\bigl|h(x,\alpha,\beta,\gamma,\delta)\bigr| \leqslant \frac{1}{|S_2|}\bigl[2|K|\rho + \bigl|l_2(x)\bigr|\bigr] := \psi_\rho(x).$$

As the impulsive conditions, I_{ik} for $i = 1,2$, $k = 1,\dots,n$ and J_{ij} for $j = 1,\dots,m$ are continuous functions on $[0,L]\times\mathbb{R}^2$. Then the assumptions of Theorem 2 are satisfied, and therefore, problem (11)–(13) has at least one solution $(u,v)\in X$.

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