About a Vanishing Viscosity-Capillarity Method

Joaquim M.C. Correia
DMat, ECT & CIMA, IIFA, University of Évora, Évora, Portugal
CAMGSD, IST, Lisbon, Portugal

Applied Analysis Seminar
Department of Mathematics and Statistics
University of Strathclyde, April 22, 2016

Acknowledgements:

University of Strathclyde Glasgow

UNIVERSIDADE DE ÉVORA

FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA EDUCAÇÃO E CIÊNCIA

UID/MAT/04674/2013

Abstract

We consider a class of nonlinear dissipative-dispersive perturbations of the scalar conservation law $\partial_t u + \operatorname{div} f(u) = 0$[1] and we study the convergence of the approximated solutions to its entropy solution. In particular, we obtain conditions under which the balance between dissipation and dispersion gives rise to the convergence (by DiPerna's measure-valued solution technique).

Do $\varepsilon, \delta \searrow 0$ in

$$\partial_t u + \operatorname{div}\left(f(u) - \varepsilon\,\mathcal{B}(u, \nabla u) + \delta\,\mathcal{C}(u, \nabla u, \nabla^2 u)\right) = 0,$$
$$u(x,0) = u_0^{\varepsilon,\delta}(x).$$

Example. The 1-dimensional Körteweg-de Vries-Burguers' equation (shortly: KdV-B eq.), where $f(u) = u^2/2$, $\mathcal{B}(u, \nabla u) = u_x$ and $\mathcal{C}(u, \nabla u, \nabla^2 u) = u_{xx}$:

$$u_t + (u^2/2)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

[1] Possibly non-convex transport f and non-linear viscosity $\mathcal{B}$ or capillarity $\mathcal{C}$.

Nonlinear hyperbolic conservation laws

- Cauchy Problem (1st order nonlin. pde's) $\Rightarrow$ hyperb. (real eigenvalues $\equiv$ finite velocity) $\Rightarrow$ discontinuities (characteristic lines meet) $\Rightarrow$ weak sol. (global in time) $\Rightarrow$ non uniqueness

- Entropy Methods from Gas Dynamics and 2nd Law of Thermodynamics (for Euler Equations $\equiv$ inviscid and compressible Navier-Stokes Equations)

- Equivalence to the Vanishing Viscosity Method selection: "classical" entropy weak solutions or Kružkov solutions

Traffic Burgers' Inviscid Equation or Arnold's particle/wave duality

In a straight line particles move freely and $u(x,t)$ is the velocity of the particle which is in position x at time t.

Let $x = x(t; 0, x_0)$ be the position at time t of the particle in x_0 at initial time $t_0 = 0$, which we abbreviate as $x = x(t)$.

By Newton's law (particles are moving freely) $x''(t) = 0$, then $x(t) = x_0 + \vec{v}t$ where $\vec{v} = u(x_0, 0)$.

'Particle description': the physical system is described by an infinite set of ODEs, one for each $x_0 \in \mathbb{R}$,

$$\begin{cases} x'(t) = u(x_0, 0), & t \geq 0 \\ x(0) = x_0. \end{cases}$$

Now, $\vec{v} = x'(t) = u(x(t), t)$, then

$$0 = x''(t) = u_t(x(t), t) + x'(t)u_x(x(t), t) = u_t + uu_x.$$

'Wave description': the physical system is described by a single PDE

$$\begin{cases} u_t + uu_x = 0, & x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = u_0(x). \end{cases}$$

Rk. if we reverse that computation, we are solving the PDE by the 'characteristics method'.

A convergence result

Correia [2, 2016??] "Zero Limit for Multi-D Conservation Laws with Nonlinear Diffusion and Dispersion": we have (formal)[2] convergence, if $r \geq \rho + 1 + \vartheta$ and $\delta = o(\varepsilon^{\gamma})$ with $\gamma = \frac{\rho+2}{r+1-\vartheta}(\leq 1)$, when

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}\left(\varepsilon\, b_j\big(u, \nabla u\big) + \delta\, g(u) \sum_{k=1}^{d} \partial_{x_k} c_{jk}\big(g(u)\,\nabla u\big)\right)_{1\leq j\leq d}$$

(A_1) for some $m > 1$, $|f'(u)| = \mathcal{O}\left(|u|^{m-1}\right)$ as $|u| \to \infty$,

(A_2) for some $\mu \geq 0$, $r > 2$, $|b(u,\lambda)| = \mathcal{O}\left(|u|^{\mu}\right)\,\mathcal{O}\left(|\lambda|^{r}\right)$ as $|u|, |\lambda| \to \infty$,

(A_3) for some $\varphi \geq 0$, $\vartheta < r$, $D > 0$, $\lambda \cdot b(u,\lambda) \geq D\,|u|^{\mu\varphi}\,|\lambda|^{r+1-\vartheta}$ $\forall u \in \mathbb{R}$, $\lambda \in \mathbb{R}^d$.

(A_4) for some $\rho > 0$, $\|[c_{jk}(\lambda)]\| = \mathcal{O}(|\lambda|^{\rho})$ as $|\lambda| \to \infty$.

[2] Cf. Bedjaoui-Correia-Mammeri [3, 2015] "Well-Posedness of the Generalized Körteweg-de Vries-Burgers Equation with Nonlinear Dispersion and Nonlinear Dissipation".

Singular limits

- ($\varepsilon = 0$, Lax-Levermore [3, 1983]) The solutions of KdV equation

$$u_t + (u^2/2)_x = -\delta\, u_{xxx}$$

do not converge as $\delta \searrow 0$ in a strong topology (oscillatory effect of capillarity; "zero-dispersion limit". Failure).

- ($\delta = 0$, Kružkov [2, 1970]) The solutions of Burgers' equation

$$u_t + (u^2/2)_x = \varepsilon\, u_{xx}$$

converge as $\varepsilon \searrow 0$ in a strong topology ("vanishing viscosity method").

- ($\delta = K\varepsilon^\gamma$) In the phase transition regime, we can converge to physical solutions different from the entropy solutions. (Reliability.)

Breaking paradigmas

- Truskinovsky [4, 1993]: physical nonclassical solutions (considering dispersive terms; phase transition problems)
- Brenier-Levy [1, 1999]: dissipative KdV-type equations (3rd order equations without the 2nd order viscosity term ($\varepsilon = 0$); conjecture)
- Perthame-Ryzhik [1, 2007]: δ/ε balance in KdV-B equation $\delta = o(\varepsilon^1)$ (Riemann problem; travelling waves ε, δ-limit)

Applied analysis

From the point of view of 'applications', as $\varepsilon, \delta \ll 1$, the equations are simplified by neglecting small scale mechanisms: "$\mathcal{B}$ and $\mathcal{C}$ are spurious terms".
We are then concerned with the Cauchy problem for the hyperbolic (first order) conservation laws

$$\begin{aligned} &\partial_t u + \operatorname{div} f(u) = 0, \\ &u(x, 0) = u_0(x), \end{aligned}$$

which have non-unique solution.

Because of 'singular limits' and of 'nonuniqueness',

- the classification is a practical problem (of practical interest: failure, reliability and integrity);
- the theoretical/applied points of view of approaching/approached equations conduct us to a dilemma, paradoxical situation.

Conclusion

Thus, we are concerned

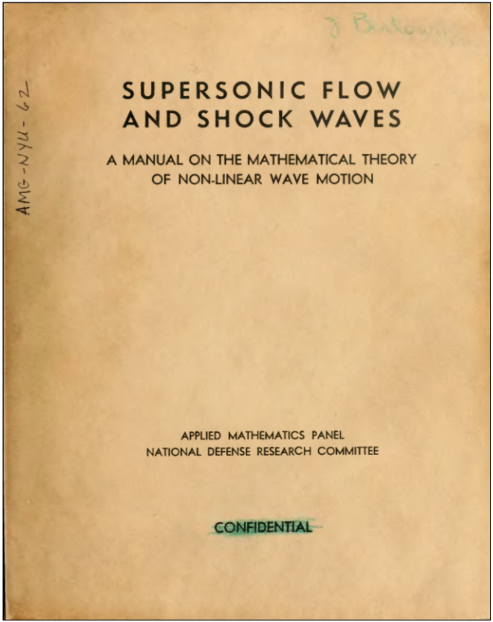
- with a proof of a "vanishing viscosity-capillarity method" relying on
  - the well-posedness of the genKdV-B equations (dispersive techniques),
  - the convergence of their solutions (DiPerna's measure-valued solution techniques),
- with the behaviour and selection of the
  - right models,
  - right solutions.

N.B. In most real phenomena we handle together viscosity and capillarity like mechanisms. Then, we expect reliability if the dissipation effects dominate the dispersion ones and this is given by a δ/ε balance and a ratio of viscosity and capillarity growths. Moreover, according this balance we can select different mathematical solutions: what about the physical solution (Integrity[3])?

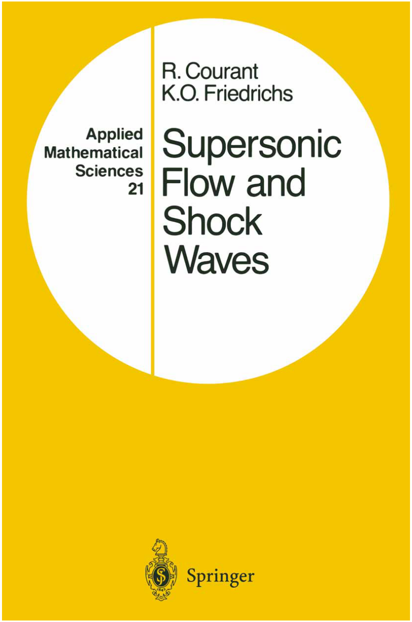
[3] Correia [1, 2010] and Correia-Sasportes [2, 2009]

Shocks

‘Confidential’, 1938!?



“Springer” (Interscience Publ., N.Y, 1948)



In use...

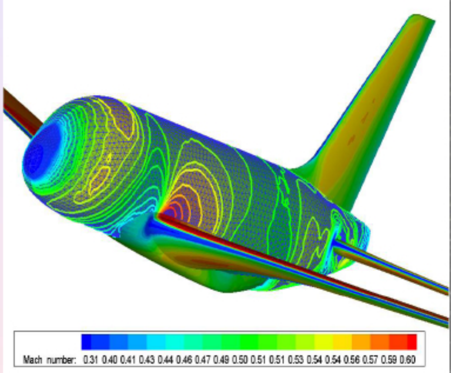
Modelling on continuum physics, chemistry, biology, environment, etc.

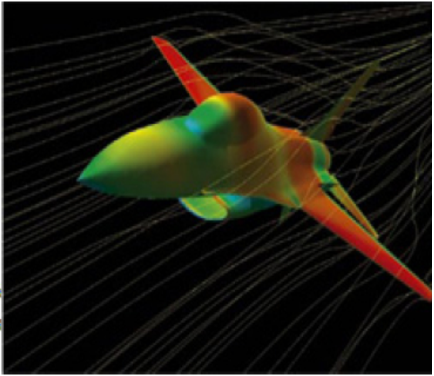
Areas as gas dynamics, nonlinear elasticity, shallow water theory, geometric optics, magneto-fluid dynamics, kinetic theory, combustion theory, cancer medicine, petroleum engineering, irrigation systems, etc.

Applications as optimal shape design (aeronautics, automobiles), noise reduction in cavities and vehicles, flexible structures, seismic waves (earthquakes, tsunamis), laser control in quantum mechanical and molecular systems, chromatography, chemostasis, oil prospection and recovery, cardiovascular system, traffic flow, the Thames barrier, etc.

Optimal design and **active control** in structures: bridges, the Thames barrier, wind towers, aeroplanes or the shuttle and the orbital spatial station...

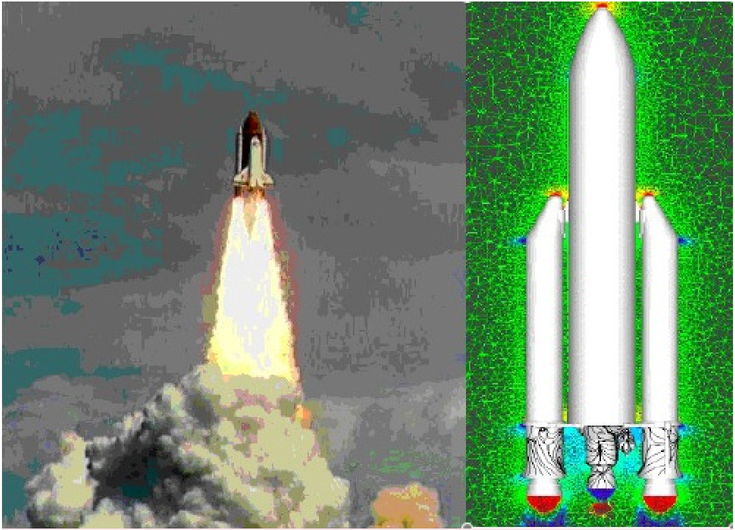
... shocks and oscillations are fundamental issues:



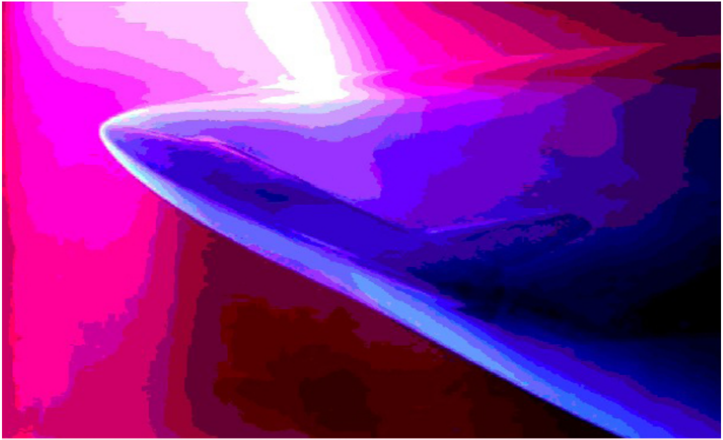


The transonic regime issues:

- control of vibrations and
- shocks strength magnitude



NASA: Hypersonic flow with shocks at the nose



NASA: Visible shocks at the nose in the windtunnel test

- "for the engineer working... in the wind tunnel, design and control problems are much harder as they become inverse problems", see lecture by E. Zuazua at the "1st Porto Meeting on Mathematics for Industry", Porto, 2009).

Our general issues concern:

- the behaviour and selection of the right models and solutions;
- the proof and criteria for a “vanishing viscosity-capillarity method”.

Rk: Numerics is hopeless.

Schonbek [2, 1982]

The (generalised) Körteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}$$

- with linear viscosity and linear capillarity;
- Kružkov solutions only for convex fluxes;
- $\delta = o(\varepsilon^2)$ for $f(u) = u^2/2$ (and $\mathcal{O}(\varepsilon^3)$ for general quadratic flux).

Rk.1: Sharp condition should be $\delta = o(\varepsilon^1)$? (Perthame-Ryzhic [1, 2007]).

Rk.2: Too many technical restrictions (1-D, specific L^p and m growth flux).

LeFloch-Natalini [4, 1997]

Körteweg-de Vries-Burgers type equations

$$u_t + f(u)_x = \varepsilon\, \beta(u_x)_x - \delta\, u_{xxx}$$

- with nonlinear viscosity and linear capillarity;
- Kružkov solutions for possibly non-convex fluxes.

Rk. still with the same technical restrictions of Schonbek [2, 1982].

Correia-LeFloch [4, 1998]

Körteweg-de Vries-Burgers type equations

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div} \Big(\varepsilon\, b_j(\nabla u) + \delta\, \partial_{x_j}^2 u \Big)_{1 \le j \le d}$$

- multi-D equation with general flux;
- nonlinear viscosity but diagonal linear capillarity;
- Kružkov solutions in general L^p spaces;
- new estimates;
- convergence if $\delta = o(\varepsilon^\gamma)$ with $r \ge 2$ and $\gamma = \frac{3}{r+1} \le 1$.

N.B. Here $\rho + 2 = 3$ and γ is better than the previous, but is it sharp?

Rk. $c(\nabla u)$ or $c(\nabla^2 u)$?[4]

[4] cf. Correia-LeFloch [3, 1999]

Brenier-Levy [1, 1999]

Lax-Levermore [3, 1983]) showed that for the Körteweg-de Vries equation

$$u_t + (u^2/2)_x = -\delta\, u_{xxx}$$

their solutions do not converge in a strong topology.

This should not be the paradigma: Brenier-Levy considered the "pure-dispersive equation" with nonlinear capillarity

$$u_t + (u^2/2)_x = -\delta(u_{xx}^2)_x$$

and showed through numerical evidence a dissipative behaviour[5].

[5] See their Conjecture and Bedjaoui-Correia [5, 2012].

"Unexpected" regime [6]

with $r = 1$ and $\rho = 2$ ($\delta = o(\varepsilon^{5/2})$), we proved the well-posedness of the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta(u_{xx}^2)_x,$$
$$u(x,0) = u_0^{\varepsilon,\delta}(x),$$

and as $\varepsilon, \delta \searrow 0$ the convergence of the previous solutions to the entropy weak solution of the initial value problem

$$u_t + f(u)_x = 0,$$
$$u(x,0) = u_0(x).$$

[6] Bedjaoui-Correia-Mammeri [1, 4, 2016, 2014]: "On a Limit of Perturbed Conservation Laws with Diffusion and Non-positive Dispersion" and "On vanishing dissipative-dispersive perturbations of hyperbolic conservation laws".

Slemrod's PDEs Seminar, IST, September 16, 2014

PARTIAL DIFFERENTIAL EQUATIONS SEMINAR [5]

16/09/2014, 15:00 -- Room P4.35, Mathematics Building
Marshall Slemrod, _Department of Mathematics, University of Wisconsin, Madison_

HILBERT'S 6TH PROBLEM REVISITED

In this talk I will outline some thoughts on Hilbert's 6th problem, namely the passage from Boltzmann equation to the classical Euler equations of mass, momentum, and energy for an ideal gas as a small parameter (Knudsen number) tends to zero. The main idea is that via exact summation of the Chapman-Enskog expansion due to Karlin & Gorban the problem can be turned into a limiting problem for PDEs analogous to the KdV-Burgers' equation and for this problem the limit cannot be achieved. Hence, it appears that Hilbert's goal, while admirable, is not attainable.

Hilbert's 6th problem (from Boltzmann to Euler)

4. Implications of Gorban and Karlin's summation for Hilbert's 6th problem

The implication of the exact summation of C–E by Gorban and Karlin now becomes clear. The whole issue may be seen in Eq. (11), the energy balance. If we put the Knudsen number scaling into (11), the coefficient α is actually a term $\alpha_0\varepsilon^2$ and to recover the classical balance of energy of the Euler equation would require the sequence

$$\varepsilon^2 \rho^\varepsilon \partial_i \rho^\varepsilon \partial_i \rho^\varepsilon \to 0$$

in the sense of distributions as $\varepsilon \to 0$. This would require a strong interaction with viscous dissipation. The natural analogy is given by the use of the KdV–Burgers equation:

$$u_t + uu_x = \varepsilon u_{xx} - K\varepsilon^2 u_{xxx} \tag{12}$$

where at a more elementary level we see the competition between viscosity and capillarity. The result in (12) is known but far from trivial. Specifically in the absence of viscosity we have the KdV equation

$$u_t + uu_x = -K\varepsilon^2 u_{xxx} \tag{13}$$

and we know from the results of Lax and Levermore [9] that as $\varepsilon \to 0$ the solution of (13) will not approach the solution of the conservation law

$$u_t + uu_x = 0 \tag{14}$$

after the breakdown time of smooth solutions of (14). On the other hand, addition of viscosity which is sufficiently strong, i.e. K sufficiently small in (12), will allow passage as $\varepsilon \to 0$ to a solution of (14). This has been proven in the paper of Schonbek [10]. So, the next question is whether we are in the Lax–Levermore case (13) or the Schonbek case (12) with K sufficiently small. In my paper [11] I noted the C–E summation of Gorban and Karlin for the Grad 10-moment system leads to a rather weak viscous dissipation, i.e. Eqs. (5.10), (5.11) of [11]. At the moment, this is all we have to go on and I can only conclude that things are not looking too promising for a possible resolution of Hilbert's 6th problem. It appears that in the competition between viscosity and capillarity (mathematically, dissipation of oscillation versus generation of oscillation), capillarity has become a very dogged opponent, and the capillarity energy will not vanish in the limit as $\varepsilon \to 0$. Hilbert's hope may have been justified in 1900, but as a result of the work of Gorban, Karlin, Lax, Levermore, and Schonbek, I think that serious doubts are now apparent.

References

[1] L. Saint-Raymond, Hydrodynamic limits of the Boltzmann equation, in: Lecture Notes in Mathematics 1971, Springer, Berlin, 2009.
[2] A.N. Gorban, I.V. Karlin, Structure and approximation of the Chapman–Enskog expansion for linearized Grad equations, Sov. Phys., JETP 73 (1991) 637–641.
[3] A.N. Gorban, I.V. Karlin, Short wave limit of hydrodynamics: a soluble model, Phys. Rev. Lett. 77 (1996) 282–285.
[4] I.V. Karlin, A.N. Gorban, Hydrodynamics from Grad's equations: what can we learn from exact solutions? Ann. Phys. (Leipzig) 11 (2002) 783–833.
[5] R. Courant, K.O. Friedrichs, Supersonic Flow and Shock Waves, Springer, New York, 1992.
[6] D.J. Korteweg, Sur la forme que prennent les équations du mouvements des fluides si l'on tient compte des forces capillaires causées par des variations de densité, Arch. Néerl. Sci. Exactes Nat. Ser. II 6 (1901) 1–24.
[7] J.E. Dunn, J. Serrin, On the thermomechanics of interstitial working, Arch. Ration. Mech. Anal. 88 (1985) 95–133.
[8] G. Boillat, T. Ruggeri, Hyperbolic principal subsystems: entropy convexity and subcharacteristic conditions, Arch. Ration. Mech. Anal. 137 (1997) 305–320.
[9] P.D. Lax, C.D. Levermore, The small dispersion limit of the Korteweg–de Vries equation, I, II, III, Comm. Pure Appl. Math. 36 (1983) 253–290. 571–593, 809–829.
[10] M.E. Schonbek, Convergence of solutions to nonlinear dispersive equations, Comm. Partial Differential Equations 7 (1982) 959–1000.
[11] M. Slemrod, Chapman–Enskog ⇒ viscosity–capillarity, Quart. Appl. Math. 70 (2012) 613–624.
[12] A.V. Bobylev, The Chapman–Enskog and Grad methods for solving the Boltzmann equation, Sov. Phys. Dokl. 27 (1) (1982) 29–31.

- Bedjaoui, N., Correia, J.M.C., Mammeri, Y. On a Limit of Perturbed Conservation Laws with Diffusion and Non-positive Dispersion, Commun. Math. Sci. (2016), Vol.14, pp.
- Correia, J.M.C. Zero Limit for Multi-D Conservation Laws with Nonlinear Diffusion and Dispersion, in "Modeling, Dynamics, Optimization and Bioeconomics II", Eds. A. Pinto and D. Zilberman, Springer Proc. in Math. & Stat. (submited)
- Bedjaoui, N., Correia, J.M.C., Mammeri, Y. Well-Posedness of the Generalized Körteweg-de Vries-Burgers Equation with Nonlinear Dispersion and Nonlinear Dissipation, Int. J. of Pure Math. (2015), Vol.2, pp. 38-46
- Bedjaoui, N., Correia, J.M.C., Mammeri, Y. On vanishing dissipative-dispersive perturbations of hyperbolic conservation laws, in "Proc. 5th International Conference on Fluid Mechanics and Heat & Mass Transfer" (2014), Lisbon, Portugal, pp.13-18
- Bedjaoui, N., Correia, J.M.C. A Note on Nonlinear KdV-Type Equations, Bol. Soc. Port. Mat. (2012), Special Issue, pp. 27-30

- Correia, J.M.C. Simplifying Some Dissipative-Dispersive Models Bol. Soc. Port. Mat. (2010), Special Issue, pp. 37-41
- Correia, J.M.C., Sasportes, R.S. Nonlinear Hyperbolic Conservation Laws, in "IRF'2009, Integrity, Reliability and Failure (Challenges and Opportunities)", Eds. J.F. Silva Gomes and Shaker A. Meguid, Ed. INEGI, Porto, Portugal (2009)
- Correia, J.M.C., LeFloch, P.G. Nonlinear Hyperbolic Conservation Laws, In "Proc. Luso-Chinese Symposium on Nonlinear Evolution Equations and their Applications", pp. 21-44. World Sci. Publ., River Edge, NJ (1999)
- Correia, J.M.C., LeFloch, P.G. Nonlinear diffusive-dispersive limits for multidimensional conservation laws, in "Advances in Partial Differential Equations and Related Areas (Beijing, 1997)", pp. 103-123. World Sci. Publ., River Edge, NJ (1998) (http://arxiv.org/abs/0810.1880)

- Brenier, Y., Levy, D. Dissipative behavior of some fully non-linear KdV-type equations, Physica D 137, pp. 277-294 (2000)
- Kružkov, S.N. First order quasilinear equations in several independent variables, Mat. Sb. 81, pp. 285-355 (1970); Math. USSR Sb. 10, pp. 217-243 (1970)
- Lax, P.D., Levermore, C.D. The small dispersion limit of the Korteweg-de Vries equation, I, II, III, Comm. Pure Appl. Math. 36, pp. 253-290, pp. 571-593, pp. 809-829 (1983)
- LeFloch, P.G., Natalini, R. Conservation laws with vanishing nonlinear diffusion and dispersion, Nonlinear Analysis 36, pp. 213-230 (1999)

- Perthame, B., Ryzhik, L. Moderate Dispersion in Conservation Laws with Convex Fluxes, Comm. Math. Sci. 5(2), pp. 473-484 (2007)
- Schonbek, M.E. Convergence of solutions to nonlinear dispersive equations, Comm. Part. Diff. Equa. 7, pp. 959-1000 (1982)
- Slemrod, M. From Boltzmann to Euler: Hilbert's 6th problem revisited, Computers and Mathematics with Applications 65, pp. 1497-1501 (2013)
- Truskinovsky, L. Kinks versus shocks, in "Shock Induced Transitions and Phase Structures in General Media (Minnesota, 1990)", pp. 185-229, IMA Series in Mathematics and Its Applications 52, Springer-Verlag (1993)

Have you some suggestions?

Thank you very much!