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Abstract

The Strong Maximum Principle (SMP) is a well known property, which can be recognized as a kind of uniqueness result for solutions of Partial Differential Equations. Through the necessary conditions of optimality it is applicable to minimizers in some classes of variational problems as well. The work is devoted to various versions of SMP in such variational setting, which hold also if the respective Euler-Lagrange equations are no longer valid. We prove variational SMP for some types of integral functionals in the traditional sense as well as obtain an extension of this principle, which can be seen as an extremal property of a series of specific functions.

Algumas versões do Princípio do Máximo para funcionais integrais elípticos

Resumo

Algumas verses do Princpio do Mximo para funcionais integrais elpticos

O Princípio do Máximo Forte (PMF) é uma propriedade bem conhecida que pode ser vista como um resultado de unicidade para soluções de Equações Diferenciais Parciais. Através das condições necessárias de optimalidade, é também aplicável a algumas classes de problemas variacionais. O trabalho é dedicado a várias versões do PMF em tal contexto variacional, que se verificam mesmo quando as respectivas equações de Euler-Lagrange não são válidas. Provamos PMF variacionais para algum tipo de funcionais integrais no sentido tradicional, e obtemos uma extensão deste princípio, que pode ser visto como uma propriedade extremal de uma série de funções específicas.

Extended abstract

The Strong Maximum Principle (SMP) is a well-known property of some classes of Partial Differential Equations. In the classical form it reduces to the comparison of an arbitrary solution of the equation with the fixed one, which is identical zero. Taking into account that the elliptic partial differential equations are often necessary conditions of optimality in variational problems, SMP can be formulated in the variational setting. The first result in this direction was obtained by A. Cellina, who proposed the conditions guaranteeing the validity of this property even in the case when the respective Euler-Lagrange equation in no longer valid.

The purpose of the Thesis is to extend the known results on SMP for elliptic equations and variational problems to the new classes of integral functionals, as well as to the new types of comparison functions (not necessarily identical zero). On the other hand, the comparison results appear also as a technique for proving SMP, where the comparison function is constructed as a solution of the variational problem with some specific properties. In particular, in Chapters 1 and 2, this technique admits the form of a local estimate for minimizers.

The work consists of three parts. In the first one we consider the case of a convex langrangean depending only on the gradient, avoiding the rotational symmetry assumption. Namely, we suppose dependence on the gradient through an arbitrary convex compact *gauge* and prove the validity of SMP under A. Cellina's conditions (smoothness and strict convexity of the lagrangean at the origin). We give more extensions of SMP to the case when one of Cellina's assumptions is dropped.

In the second part we obtain some estimates on minimizers in the case when the lagrangean has an additive term linearly depending on the state variable, which can be seen as a (approximate) version of SMP considered in Chapter 1.

In the third final part we deal with lagrangeans splitted into two parts: one is rotationally invariant with respect to the gradient and the other is a nonlinear function of the state variable. We prove the traditional version of SMP adapting the comparison technique based on Leray-Schauder fixed point theorem, which was applied earlier to elliptic partial differential equations in the works by P. Pucci and J. Serrin.

In the Thesis we use some methods of Calculus of Variations together with the modern technique of Convex and Nonlinear Functional Analysis. On one hand, the results develop the known technique of the Strong Maximum Principle and Comparison Theorems for new classes of problems, while, on the other hand, extend our knowledge on variational problems, emphasizing their properties, which can be treated as the general properties of convex functions.

The results exposed in Thesis are new and some of them were presented on various national and international meetings and seminars, such as:

1. Workshop on Variational Analysis and Applications, Universidade de Évora, October 28, 2011, Évora, Portugal;

2. 8^{th} ISAAC Congress, Peoples' Friendship University of Russia, August 22-27, 2011, Moscow, Russia;

3. 51^{st} Workshop of the International School of Mathematics Guido Stampacchia "Variational Analysis and Applications", May 2009, Erice, Italy;

4. Seminar of the Mathematical Department, University of Padua, by invitation of prof. G.Colombo, October 2009, Padua, Italy;

5. Seminar for PhD students at the University of Milan-Biccoca, October 2009, Milan, Italy;

6. Mini-Symposium on Functional Optimization, May 2009, Évora - Portugal;

7. Meeting CIMA-CEOC, December 2008, Aveiro, Portugal.

The first part of the work is contained in the papers

1. V.V. Goncharov and T.J. Santos, Local estimates for minimizers of some convex integral functional of the gradient and the Strong Maximum Principle, *Set-Valued and Variational Analysis*, Vol. 19 (2011), 179-202.

2. V.V. Goncharov and T.J. Santos, An extremal property of the inf- nd sup- convolutions regarding the Strong Maximum Principle, 8^{th} *ISAAC Congress*, Moscow, 2011, submitted.

The work is written in 93 pages and the bibliography consists of 65 items.

Key words: Calculus of Variations; Strong Maximum Principle; comparison theorems; convex functions; subdifferential; Legendre-Fenchel transform; Leray-Schauder fixed point theorem.

Mathematical Subject classification (2000): 49J10, 49J53; 49N15.

Algumas versões do Princípio do Máximo para funcionais integrais elípticos

Resumo Alargado

O Princípio do Máximo Forte (PMF) é uma propriedade bem conhecida de algumas classes de Equações Diferenciais Parciais. Na forma clássica reduz-se à comparação de uma solução arbitrária da equação com uma fixada, que é identicamente nula. Tendo em conta que as equações diferenciais parciais são muitas vezes condições de optimalidade em problemas variacionais, o PMF pode ser formulado no contexto variacional. O primeiro resultado nesta direcção foi obtido por A. Cellina, que propôs as condições que garantem a validade desta propriedade mesmo no caso em que a respectiva equação de Euler-Lagrange não é válida.

O objectivo da Tese é extender os resultados conhecidos no que respeita o PMF para equações diferenciais elípticas e problemas variacionais a novas classes de funcionais integrais, bem como a novos tipos de funções de comparação (não necessariamente identicamente nulas). Por outro lado, os resultados de comparação aparecem também como uma técnica de demonstração do PMF, onde que a função de comparação é construída como uma solução do problema variacional com algumas propriedades específicas. Em particular, no Capítulos 1 e 2, esta técnica admite a forma de uma estimativa local para minimizantes.

O trabalho consiste em três partes. Na primeira consideramos o caso de um lagrangeano convexo dependente apenas do gradiente, evitando a simetria rotacional. Nomeadamente, supomos a dependência do gradiente através de um calibre convexo e compacto e mostramos a validade do PMF sob as condições de A. Cellina (suavidade e convexidade estrita na origem). Fornecemos ainda outras extensões do PMF para o caso em que uma das hipóteses de A. Cellina é abandonada.

Na segunda parte obtemos algumas estimativas dos minimizantes no caso

em que o lagrangeano tem um termo aditivo linearmente dependente da variável estado, que podem ser vistos como uma versão (aproximada) do PMF considerado no Capítulo 1.

Na terceira e última parte lidamos com lagrangeanos que consistem em duas partes: uma é rotacionalmente invariante com respeito ao gradiente e a outra é uma função não linear da variável estado. Provamos uma versão tradicional do PMF adaptando a técnica de comparação baseada no teorema do ponto fixo de Leray-Schauder, que foi antes utilizada para equações diferenciais elípticas nos trabalhos de P. Pucci e J. Serrin.

Na Tese usamos alguns métodos do Cálculo das Variações com a técnica moderna de Análise Convexa e Análise Funcional Não Linear. Por um lado, os resultados desenvolvem a técnica conhecida do Princípio do Máximo forte e Teoremas de Comparação para novas classes de problemas, enquanto, por outro lado, aumentamos o conhecimento sobre problemas variacionais, enfatizando as suas propriedades, que podem ser tratadas como propriedades gerais de funções convexas.

Os resultados apresentados na Tese são novos e alguns deles foram apresentados em vários encontros e seminários nacionais e internacionais, tais como:

1. Workshop on Variational Analysis and Applications, Universidade de Évora, October 28, 2011, Évora, Portugal;

2. 8^{th} ISAAC Congress, People's Friendship University of Russia, 22 a 27 de Agosto, 2001, Moscovo, Rússia.

3. 51^{st} Workshop - International School of Mathematics Guido Stampacchia "Variational Analysis and Applications", Maio 2009, Erice, Itália;

4. Seminário da Departamento de Matemtica, Universidade de Padua, por convite do prof. G.Colombo, Outubro 2009, Pádua, Itália;

5. Seminário para os alunos de Doutoramento em Matemática, na Universidade de Milão - Bicocca, Outubro de 2009, Milão, Itália;

6. Mini-Simpósio em Optimização Funcional, Maio 2009, Évora, Portugal;

7. Encontro CIMA-CEOC, Dezembro 2008, Aveiro, Portugal.

A primeira parte do trabalho está contida nos artigos

1. V.V. Goncharov and T.J. Santos, Local estimates for minimizers of some convex integral functional of the gradient and the Strong Maximum Principle, *Set-Valued and Variational Analysis*, Vol. 19 (2011), 179-202.

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O trabalho está escrito em 93 páginas e a bibliografia consiste em 65 items

Palavras Chave: Cálculo das Variações; Princípio do Máximo Forte; teoremas de comparação; funções convexas; subdiferencial; transformada de Legendre-Fenchel; teorema do ponto fixo de Leray-Schauder.

Mathematical Subject classification (2000): 49J10, 49J53; 49N15.

Introduction

The purpose of the Thesis is to study several problems from the Calculus of Variations concerning the validity of the Strong Maximum Principle, which is a well-known qualitative property of solutions to Partial Differential Equations and can be extended to variational context. Let us start with some historical notes and the state of art.

Maximum Principles were first stated for harmonic functions, i.e., solutions to the Laplace equation

$$\Delta u(x) = 0, \quad x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is open bounded and connected. Roughly speaking, a maximum principle states that the maximum of a solution of (1) is attained on the boundary of Ω. One usually distinguishes weak and strong maximum principles. Whereas the weak maximum principle allows the maximum of a solution to be attained in the interior of the domain as well, the strong one states that it is possible only in the trivial case when the solution is a constant.

The first version of the *Strong Maximum Principle* (SMP) for harmonic functions apparently belongs to C. Gauss. Afterwards it was extended by many authors. The most general result in this direction was obtained by E. Hopf who proved in 1927 (see [40]) SMP for elliptic partial differential equations of the type

$$\sum_{i,k} a_{ik}(x) \frac{\partial u}{\partial x_i \partial x_k} + \sum_i b_i(x) \frac{\partial u}{\partial x_i} = 0, \quad x \in \Omega, \tag{2}$$

where $a_{ik}(\cdot), b_i(\cdot)$ are continuous functions such that the symmetric matrix $(a_{ik}(x))_{ik}$ is positive definite for all x (the ellipticity condition on the operator in (2)). The idea used by E. Hopf - a comparison technique - led to an enormous range of important applications and generalizations. A comparison result states that an inequality between two solutions of (2) taking place on the boundary $\partial\Omega$ remains valid also inside Ω. It has been extended to more general partial differential equations and in variational problems by many authors (see, e.g., [16, 14, 44, 46, 53]).

D. Gilbarg and N. Trudinger summarized around 1977 the general theory of second order elliptic equations in the book [36], where the results concerning the SMP were also included.

In 1984 J.L. Vazquez studied conditions guaranteeing the validity of SMP for the nonlinear elliptic equation

$$\Delta u(x) = \beta(u(x)) + f(x), \qquad x \in \Omega, \tag{3}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function with $\beta(0) = 0$. Namely, he proved that SMP for (3) holds if and only if the improper Riemann integral

$$\int_0^\delta (s\beta(s))^{-\frac{1}{2}}\ \mathrm{d}s$$

diverges for each small $\delta > 0$. Vazquez's result was extended to a wider class of equations by many authors (see, e.g., [24, 54, 53]). In particular, in 1999 P. Pucci, J. Serrin and H. Zou (see [52]) considered general elliptic nonlinear equations of the form

$$\operatorname{div}(A(\|\nabla u\|)\nabla u) = \beta(u), \tag{4}$$

where $A(\cdot)$ is a continuous function such that $t \to tA(t)$ is continuously differentiable on $(0, +\infty)$, strictly increases and tends to zero as $t \to 0$. Denoting by

$$H(t) = t^2 A(t) - \int_0^t sA(s)\ \mathrm{d}s,$$

the authors gave two conditions for the validity of the SMP:

1. $\liminf_{t\to 0} \frac{H(t)}{t^2 A(t)} > 0$;

2. either $\beta(s) = 0$ on $[0, \delta]$ for some $\delta > 0$ or the improper integral

$$\int_0^\delta \frac{ds}{H^{-1}(\int_0^s \beta(t)\ \mathrm{d}t)} \tag{5}$$

   diverges.

Later on they succeeded in avoiding the first technical assumption by using the comparison technique inspired by E. Hopf (see [55]). In [54] the authors improved their proof by choosing the comparison function as a fixed point of some continuous operator. We refer to [36] and to [42] for more details.

P. Felmer, M. Montenegro and A. Quaas (see [25, 27, 47]) extended the SMP to more general equations containing also a term depending on the norm of the gradient.

The weak and strong maximum principles were also studied for parabolic partial differential equations, starting from the work by L. Nirenberg (see [48]). In this direction we refer also to works [2, 3, 34, 50, 51].

Let us now pass to the variational problems. In modern setting they are formulated in terms of Sobolev spaces; so we refer to [1, 60, 61] for the general theory of these spaces. Consider a classical problem of the Calculus of Variations consisting in minimizing the integral functional

$$I(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) \ \mathrm{d}x \tag{6}$$

on the set of Sobolev functions $u(\cdot) \in u_0(\cdot) + W_0^{1,1}(\Omega)$, where $u^0(\cdot) \in W_0^{1,1}(\Omega)$ is fixed. The mapping $(x, u, \xi) \in \Omega \times R \times R^n \to L(x, u, \xi) \in \mathbb{R}$ is called *lagrangean*. We say that $u(\cdot)$ is a *minimizer* of (6) if it gives a minimum value to $I(\cdot)$ on the class of functions with the same boundary conditions.

As well known, when the lagrangean is sufficiently regular a necessary condition of optimality in the problem above can be written in the form of Euler-Lagrange equation:

$$\mathrm{div} \nabla_{\xi} L(x, u(x), \nabla u(x)) = L_u(x, u(x), \nabla u(x)). \qquad (E - L)$$

In some particular cases $(E - L)$ is also a sufficient condition. Historically this was formulated for the first time by P. Dirichlet (the so called *Dirichlet Principle*). Namely, the problem of minimizing the energy functional

$$I(\omega) = \frac{1}{2} \int_{\Omega} \|\nabla \omega\|^2 \ \mathrm{d}x, \tag{7}$$

was considered, where $\Omega \subset \mathbb{R}^n$ is an open bounded set. It was proved that $\omega(\cdot)$ is a minimizer of (7) if and only if $\Delta \omega = 0$ on Ω.

Indeed, the *Laplace equation* is nothing else than the Euler-Lagrange equation for the functional (7). Hence, the SMP for harmonic functions can be seen as SMP also for the minimizers of (7).

The main task now is to formulate the SMP in the variational setting, which would hold even if the respective Euler-Lagrange equation is not valid. Such formulation was first given by A. Cellina in 2002 (see [16]). He considered the case of lagrangean $L(\cdot)$, which is a convex lower semicontinuous function of the gradient only with $L(0) = 0$, and gave the following definition: $L(\cdot)$ is said to satisfy the *Strong Maximum Principle* if for any open bounded connected domain $\Omega \subset \mathbb{R}^n$ a nonnegative continuous admissible solution $\bar{u}(\cdot)$ of the problem

$$\min \left\{ \int_\Omega L(\nabla u(x))\ \mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega) \right\} \qquad (P)$$

can be equal to zero at some point $x^* \in \Omega$ only in the case $\bar{u} \equiv 0$ on Ω. In rotationally invariant case, i.e., $L(\xi) = f(\|\xi\|)$ for a lower semicontinuous convex function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$, $f(0) = 0$, A. Cellina proved that smoothness and strict convexity of $f(\cdot)$ at the origin are the necessary and sufficient conditions for the validity of the SMP. Along the Thesis we generalize the symmetry assumption and establish other versions of SMP also under the lack of one of Cellina's hypotheses. Besides that we consider more general lagrangeans depending also on u.

The material of the Thesis is distributed as follows. In Chapter 1 we deal with a lagrangean of the form $L(\xi) = f(\rho_F(\xi))$, where $\rho_F(\cdot)$ is the Minkowski functional associated to some convex gauge F. Observe that variational problems with such type integrands were recently considered in [12], where the authors proved a comparison theorem assuming the strict convexity of the gauge F. They constructed a comparison function as a solution of the associated Euler-Lagrange equation written in the classic divergence form, essentially using for this the differentiability of the dual Minkowski functional $\rho_{F^0}(\cdot)$, or, in other words, the rotundity of F itself. We do not suppose instead the set F to be either strictly convex or smooth, or symmetric.

Based on duality arguments of Convex Analysis we obtain first some kinds of estimates for minimizers close to their nonextremum points and show that the assumptions of strict convexity and smoothness of $f(\cdot)$ at the origin are also sufficient and necessary for the validity of the SMP.

Further in this chapter (Sections 1.4 - 1.7) we consider the case when $f(\cdot)$ is not strictly convex at the origin, and, hence, the SMP in the traditional sense is no longer valid. We enlarge this property by considering some specific functions (called *test* or *comparison functions*) in the place of identical zero.

These functions are themselves minimizers in (P) and can be written through the dual Minkowski functional $\rho_{F^0}(\cdot)$. However, for this enlargement we need to assume that the boundary ∂F is smooth. In the simplest case the test function can be chosen as

$$\theta + a\rho_{F^0}(x - x_0) \tag{8}$$

(or

$$\theta - a\rho_{F^0}(x_0 - x)), \tag{9}$$

where $x_0 \in \Omega$; θ is some real number, and the constant a is associated to the lagrangean. Then a "one-point" version of the extended SMP takes place on the class of regions Ω, which are *star-shaped* with respect to the unique minimum (maximum) point x_0. We give also an example showing the importance of the latter hypothesis.

In Section 1.5 we extend SMP to the case of various local extremum points. To this end we consider as a test function the envelope of a finite number of functions (8) (respectively, (9)) subject to the natural consistensy condition.

Finally, in the last sections we generalize SMP in order to cover the case of infinite envelopes. Namely, we take an arbitrary real function $\theta(\cdot)$ defined on a closed set $\Gamma \subset \Omega$ and satisfying a bounded slope condition. Then we can consider as a test function in SMP the *infimum* (respectively, *supremum*) *convolution* of $\theta(\cdot)$ with the gauge function ρ_{F^0}, and the SMP admits the form of the *uniqueness extremal extension principle*.

In Chapter 2 we apply the technique developped in the previous chapter to the lagrangeans depending not only on the gradient through a gauge function but containing an additive term, linear with respect to u. Here we obtain local estimates of minimizers near two kinds of points. First we prove that close to nonextremum points a minimizer can be estimated by an *a priori* solution to the same variational problem (obtained by adding the linear term $\sigma u(x)$ to the lagrangean studied in Chapter 1). Furthermore, it turns out that near points of local minimum (maximum) similar estimates hold, which can be seen as an (approximate) version of SMP. It should be mentioned that this property has asymmetric character unlike the situation before. Namely, each result of Chapter 1 has the symmetric counterpart, which is sometimes called the *Strong Minimum Principle*, while in the case of linear perturbation σu validity of upper (respectively, lower) estimates hold whenever $\sigma > 0$ (respectively, $\sigma < 0$). Moreover, we observe also a "cross" effect: the function, which estimates minimizers of the integral with the perturbation σu is a solution of the variational problem with additive term $-\sigma u$. However, all of these effects disappear when σ tends to zero, and the estimates above reduce to (local) Strong Maximum (Minimum) Principle in usual sense.

In Chapter 3 we consider the case when the additive term depending on u is essentially nonlinear. However, due to technical difficulties we assume lagrangean to be rotationally invariant with respect to the gradient. Namely, we study the functional

$$\int_\Omega [f(\|\nabla u(x)\|) + g(u(x))]\, \mathrm{d}x \qquad (I_g)$$

where $f(\cdot)$ and $g(\cdot)$ are continuous convex nonnegative functions such that $f(0) = g(0) = 0$, $\partial g(0) = \{0\}$, and the derivative $f'(t)$ is continuous and tends to zero as $t \to 0^+$. We see that the identical zero is a minimizer of (I_g) among all Sobolev functions with the zero boundary data, and we can formulate for (I_g) the SMP in the traditional sense (as in [16]). Notice that if the functions $f(\cdot)$ and $g(\cdot)$ are more regular (in particular, $g(\cdot)$ is differentiable), all the minimizers of (I_g) satisfy the Euler-Lagrange equation

$$div \nabla_\xi f(\|\nabla u(x)\|) = g'(u(x)),$$

which is a special case of equations studied by P. Pucci and J. Serrin (see [53, 54]) and for which a necessary and sufficient condition of the validity of SMP was found. By using a similar technique combined with the comparison argument due to A. Cellina we prove that the same condition (divergence of a kind of improper Riemann integral) is sufficient also for the validity of the SMP in variational setting even if the respective Euler-Lagrange equation is no longer valid.

In Conclusion we summarize the obtained results.

Chapter 1

Lagrangeans depending on ∇u through a gauge function

In this chapter we consider the variational problem

$$\min\left\{\int_\Omega L(\nabla u(x))\ \mathrm{d}x : u(\cdot)\in u^0(\cdot)+W_0^{1,1}(\Omega)\right\} \qquad (P)$$

where $L:\mathbb{R}^n\to\mathbb{R}^+\cup\{+\infty\}$ is a convex lower semicontinuous mapping with $L(0)=0$ and $u^0(\cdot)\in W^{1,1}(\Omega)$. In Introduction we already formulated the Strong Maximum Principle for the problem (P) in the following form (due to A. Cellina [16]):

> *given an open bounded connected region $\Omega\subset\mathbb{R}^n$ an arbitrary admissible continuous nonnegative solution $u(\cdot)$ of* (P) *can be equal to zero at some point $x^*\in\Omega$ only in the case $u\equiv 0$.*

Recall that *admissible solutions* of (P) are those which give finite values to the integral. In the case of rotationally invariant lagrangeans, i.e., $L(\xi)=f(\|\xi\|)$ with a lower semicontinuous convex function $f:\mathbb{R}^+\to\mathbb{R}^+\cup\{+\infty\}$, $f(0)=0$, smoothness and strict convexity of the function $f(\cdot)$ at the origin are necessary and sufficient conditions for the validity of SMP (see [16]).

Here we assume the lagrangean to be symmetric in a more general sense, namely, $L(\xi)=f(\rho_F(\xi))$, where $F\subset\mathbb{R}^n$ is a compact convex set containing the origin in its interior, and $\rho_F(\cdot)$ means the *Minkowski functional* (*gauge function*) associated to F,

$$\rho_F(\xi) := \inf\{\lambda > 0 : \xi \in \lambda F\}. \tag{1.1}$$

In Section 1.1 we give some notions and recall definitions needed throughout the Thesis. Based on the duality arguments of Convex Analysis we obtain in Section 1.2 some kind of estimates of minimizers in (P) near points which are distant from their local extremums, emphasizing specially the one-dimensional case. Then, in Section 1.3, we show that the Cellina's conditions are necessary and sufficient for the validity of the Strong Maximum (Minimum) Principle with no supplementary assumptions on F.

On the other hand, in the lack of one of the hypotheses (strict convexity of $f(\cdot)$ at the origin) SMP can be extended as follows. Let us fix a continuous function $\hat{u}(\cdot)$ (called further *test function*) giving minimum to the functional $\int_\Omega L(\nabla u(x))dx$ on $\hat{u}(\cdot)+W_0^{1,1}(\Omega)$ for each appropriate region $\Omega \subset \mathbb{R}^n$. We say that the lagrangean $L(\cdot)$ satisfies the *extended Strong Minimum (Maximum) Principle* with respect to $\hat{u}(\cdot)$ if

for any $\Omega \subset \mathbb{R}^n$ (belonging to a suitable class of regions) each solution $u(\cdot)$ of (P) with $u(x) \geq \hat{u}(x)$ (respectively, with $u(x) \leq \hat{u}(x)$) for all $x \in \Omega$ admitting the same local minimal (respectively, maximal) values as $\hat{u}(\cdot)$ at common points, necessarily coincides with $\hat{u}(\cdot)$.

It turns out that such generalized property holds for the lagrangean $L(\xi) = f(\rho_F(\xi))$, where F is supposed to be smooth, and the test function is chosen by some special way.

In Section 1.4 we set $\hat{u}(x) = \theta + a\rho_{F^0}(x - x_0)$ or $\hat{u}(x) = \theta - a\rho_{F^0}(x_0 - x)$ and assume Ω to be star-shaped to respect to the unique extremal point $x_0 \in \Omega$ (in particular, Ω can be convex).

Further, in Section 1.5, we extend SMP to the case of a test function $\hat{u}(\cdot)$ having a finite number of local minimum (maximum) points, while in Section 1.6 the inf- and sup-convolutions of $\rho_{F^0}(\cdot)$ with some lipschitzean function $\theta(\cdot)$ are defined and in Section 1.7 they are considered as test functions for the Strong Maximum Principle.

1.1 Preliminaries

Let $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ be a convex lower semicontinuous function with $f(0) = 0$, but not identical zero, and $F \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a convex closed bounded set with $0 \in \mathrm{int}F$ (int$F$ means interior of F). Given an open bounded region $\Omega \subset \mathbb{R}^n$, we are interested in the behaviour of continuous solutions of the variational problem

$$(P_F) \qquad \min \left\{ \int_\Omega f(\rho_F(\nabla u(x)))\, \mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega) \right\}.$$

Since the function $f(\cdot)$ is clearly nondecreasing, the lagrangean $f \circ \rho_F$ in (P_F) is convex. In what follows, without loss of generality, we can assume that $\mathrm{dom}f := \{t : f(t) < +\infty\}$ is different from $\{0\}$, because otherwise each solution of (P_F) is constant, and all the results below hold trivially.

Together with the *Minkowski functional* $\rho_F(\xi)$ defined by (1.1) we introduce the *support function* $\sigma_F : \mathbb{R}^n \to \mathbb{R}^+$,

$$\sigma_F(v) := \sup\{\langle v, \xi\rangle : \xi \in F\},$$

and recall that

$$\rho_F(\xi) = \sigma_{F^0}(\xi),\ \xi \in \mathbb{R}^n, \tag{1.2}$$

where

$$F^0 := \{v \in \mathbb{R}^n : \sigma_F(v) \le 1\}$$

is the *polar set* associated to F. Here $\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{R}^n$ (the norm is denoted by $\|\cdot\|$). By the hypotheses on F we have obviously that $(F^0)^0 = F$, and it follows from (1.2) that

$$\frac{1}{\|F\|}\|\xi\| \le \rho_F(\xi) \le \|F^0\|\|\xi\|, \quad \xi \in \mathbb{R}^n, \tag{1.3}$$

where $\|F\| := \sup\{\|\xi\| : \xi \in F\}$.

Furthermore, for a convex lower semicontinuous function $L : \mathbb{R}^n \to \mathbb{R}^+ \cup \{+\infty\}$ (in particular, for $L = f \circ \rho_F$) we denote by L^* the *Legendre-Fenchel conjugate* or *polar* of L and by $\partial L(\xi)$ the *subdifferential* of L at ξ. These operations can be applied to the function $f(\cdot)$ whenever we extend it to the whole real line by setting, e.g., $f(t) := +\infty$ for $t < 0$. It is well known that $v \in \partial L(\xi)$ *iff* $\xi \in \partial L^*(v)$, and for each $\xi \in \mathbb{R}^n$ the equality

$$\partial \rho_F(\xi) = \mathbf{N}_F\left(\frac{\xi}{\rho_F(\xi)}\right) \cap \partial F^0 \tag{1.4}$$

holds (see, e.g., Corollary 2.3 in [21]), where ∂F^0 is the boundary of F^0, and $\mathbf{N}_F(\xi)$ is the *normal cone* to the set F at $\xi \in \partial F$, i.e., the subdifferential of the indicator function $\mathbf{I}_F(\cdot)$ ($\mathbf{I}_F(x)$ is equal to 0 on F and to $+\infty$ elsewhere). For the basic facts of Convex Analysis we refer to [49] or to [57]. Let us now recall only a pair of dual properties, which will be used later on. We say that the set F is *smooth* (has *smooth boundary*) if for each $\xi \in \partial F$ there exists a unique $v \in \mathbf{N}_F(\xi)$ with $\|v\| = 1$. By (1.4) this property is equivalent to the differentiability of $\rho_F(\cdot)$ at each $\xi \neq 0$. On the other hand, we say that F is *rotund* (*strictly convex*) if for each $x, y \in \partial F$, $x \neq y$, and $0 < \lambda < 1$ we have $(1-\lambda)x + \lambda y \in \operatorname{int} F$. Given $r > 0$ and $0 < \alpha < \beta < 1$ let us define the following *modulus of rotundity*:

$$\begin{aligned}\mathfrak{M}_F(r;\alpha,\beta) := \inf\{1 - \rho_F(\xi + \lambda(\eta - \xi)) :\\ \xi, \eta \in \partial F,\ \rho_F(\xi - \eta) \geq r;\ \alpha \leq \lambda \leq \beta\}.\end{aligned} \tag{1.5}$$

Since in a finite-dimensional space a closed and bounded set is compact, the set F is rotund if and only if $\mathfrak{M}_F(r;\alpha,\beta) > 0$ for all $r > 0$ and $0 < \alpha < \beta < 1$. The rotundity can be also interpreted in terms of nonlinearity of the gauge function $\rho_F(\cdot)$. Namely, F is rotund iff the equality $\rho_F(x+y) = \rho_F(x) + \rho_F(y)$ holds only in the case when $x = \lambda y$ with $\lambda \geq 0$. It is well-known that the polar set F^0 is rotund if and only if F is smooth.

Since $f(0) = 0$, $f \neq 0$ and $\operatorname{dom} f \neq \{0\}$, we also have $f^*(0) = 0$, $f^* \neq 0$ and $\operatorname{dom} f^* \neq \{0\}$. Therefore, due to the elementary properties of convex functions there exist $k, a \in [0, +\infty)$ and $0 < b \leq +\infty$ such that

$$\begin{aligned}\partial f(0) &= [0, k];\\ \partial f^*(0) &= \{t : f(t) = 0\} = [0, a];\\ \operatorname{dom} f^* &= [0, b] \text{ (or } [0, b)).\end{aligned} \tag{1.6}$$

Let us define the function $\varphi : [0, b) \to \mathbb{R}^+$ by

$$\varphi(t) := \sup \partial f^*(t) < +\infty. \tag{1.7}$$

It is nondecreasing by monotonicity of the subdifferential. We also introduce the number $\gamma_{n,f}$ that equals k whenever both $n = 1$ and f is not affine in a neighbourhood of zero (i.e., $f(t)/t \neq \text{const}$ near 0), and $\gamma_{n,f} = 0$ in all other cases.

For an open bounded connected domain $\Omega \subset \mathbb{R}^n$ we introduce the functions

$$\mathfrak{r}^{\pm}(x) = \mathfrak{r}^{\pm}_{\Omega}(x) := \sup\{r > 0 : x \pm rF^0 \subset \Omega\},\ x \in \Omega. \tag{1.8}$$

We have that

$$\mathfrak{r}^{+}(x) \leq \mathfrak{r}^{+}(y) + \rho_{F^0}(y - x) \tag{1.9}$$

for all $x, y \in \Omega$. Indeed, given $\varepsilon > 0$ let us take $r > 0$ such that

$$\mathfrak{r}^{+}(x) \leq r + \varepsilon \tag{1.10}$$

and $x + rF^0 \subset \Omega$. Then for each $y \in \Omega$ due to the convexity of F^0 we successively have

$$y - x + (r - \rho_{F^0}(y - x))F^0 \subset \rho_{F^0}(y - x)F^0 + (r - \rho_{F^0}(y - x))F^0 \subset rF^0.$$

Consequently,

$$y + (r - \rho_{F^0}(y - x))F^0 \subset x + rF^0 \subset \Omega,$$

and, hence, $r - \rho_{F^0}(y - x) \leq \mathfrak{r}^{+}(y)$. Combining this with (1.10) and taking into account arbitrarity of ε we arrive at (1.9). Similarly,

$$\mathfrak{r}\ \ (x) \leq \mathfrak{r}\ \ (y) + \rho_{F^0}(x - y), \tag{1.11}$$

$x, y \in \Omega$. The inequalities (1.9) and (1.11) imply in particular the Lipschitz continuity of the functions $\mathfrak{r}^{\pm} : \Omega \to \mathbb{R}^{+}$.

Given $x_0 \in \Omega$ the set

$$\text{St}(x_0) = \text{St}_{\Omega}(x_0) := \{x : [x_0, x] \subset \Omega\} \tag{1.12}$$

is said to be the *star* in Ω associated to the point x_0, where

$$[x_0, x] := \{(1 - \lambda)x_0 + \lambda x : 0 \leq \lambda \leq 1\}$$

is the *closed segment* connecting x_0 and x. It follows immediately from (1.12) that $\text{St}_{\Omega}(x_0)$ is open. The set Ω is said to be *star-shaped* with respect to $x_0 \in \Omega$ if $\Omega = \text{St}(x_0)$. For instance, a convex domain Ω is star-shaped with respect to each point $x \in \Omega$. We say also that Ω is *densely star-shaped* with respect to x_0 if $\Omega \subset \overline{\text{St}(x_0)}$, where over bar means the closure in $\mathbb{R}^n$.

1.2 Local estimates for minimizers

In this section we formulate a result on estimates of minimizers close to their non extremum points. Roughly speaking, it states that if $\bar{x} \in \Omega$ is not a point of local minimum (maximum) of a solution $\bar{u}(\cdot)$ to (P_F), then $\bar{u}(\cdot)$ is minorized (majorized) near $\bar{x}$ by a linear function associated to ρ_{F^0}. This function thus controls the deviation of the value $\bar{u}(x)$ from the respective extremal level.

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Let $\bar{u}(\cdot)$ be a continuous admissible solution of (P_F). Then the following statements hold where the number $a > 0$ and the nondecreasing function $\varphi(\cdot)$ are taken from (1.6) and (1.7), respectively*

(i) *Assume that a point $\bar{x} \in \Omega$ and numbers $\beta > 0$ and $\mu \in \mathbb{R}$ are such that*

$$\bar{u}(x) \geq \mu \quad \forall x \in \bar{x} - \beta F^0 \subset \Omega \tag{1.13}$$

and

$$\bar{u}(\bar{x}) > \mu + a\beta. \tag{1.14}$$

Then there exists $\eta > 0$ such that

$$u(x) \geq \mu + \varphi(\gamma_{n,f} + \eta)(\beta - \rho_{F^0}(\bar{x} - x)) \tag{1.15}$$

for all $x \in \bar{x} - \beta F^0$.

(ii) *Similarly, if in the place of (1.13) and (1.14) a point $\bar{x} \in \Omega$ and numbers $\beta > 0$, $\mu \in \mathbb{R}$ satisfy the inequalities*

$$\bar{u}(x) \leq \mu \quad \forall x \in \bar{x} + \beta F^0 \subset \Omega \tag{1.16}$$

and

$$\bar{u}(\bar{x}) < \mu - a\beta, \tag{1.17}$$

then there exists $\eta > 0$ such that

$$\bar{u}(x) \leq \mu - \varphi(\gamma_{n,f} + \eta)(\beta - \rho_{F^0}(x - \bar{x}))F^0 \tag{1.18}$$

for all $x \in \bar{x} + \beta F^0$.

Proof. **(i)** If $k > 0$ but $\gamma_{n,f} = 0$, i.e., either $n > 1$ or $f(\cdot)$ is affine near 0, then the result is trivial because $\varphi(\eta) = 0$ for each $0 < \eta < k$.

Let us suppose now $k = 0$. Then $\gamma_{n,f} = 0$ and $\varphi(t) > 0$ for all $t > 0$. Since the function $\bar{u}(\cdot)$ is continuous and $\varphi(\cdot)$ is upper semicontinuous, it follows from (1.14) that for some small $\delta > 0$ and $\alpha \in (0, \beta)$ the inequality

$$\bar{u}(x) \geq \mu + \varphi(t)(\beta - \rho_{F^0}(\bar{x} - x)) \tag{1.19}$$

holds whenever $\rho_{F^0}(\bar{x}-x) \leq \alpha$ and $0 < t \leq \delta$. Let us consider the real-valued function $s \mapsto \varphi(\delta(\frac{\alpha}{s})^{n-1})$, which is (Riemann) integrable on the interval $[\alpha, \beta]$. Denoting by

$$R_\delta(r) := \mu + \int_r^\beta \varphi\left(\delta(\frac{\alpha}{s})^{n-1}\right) ds, \tag{1.20}$$

we deduce from (1.19) that

$$u(x) \geq R_\delta(\rho_{F^0}(\bar{x} - x)) \tag{1.21}$$

for all $x \in \Omega$ with $\rho_{F^0}(\bar{x} - x) = \alpha$. Observe that $R_\delta(\beta) = \mu$, and by (1.13) the inequality (1.21) holds trivially for $x \in \Omega$ with $\rho_{F^0}(\bar{x} - x) = \beta$. This inequality is thus valid on $\partial A_{\alpha,\beta}$, where

$$A_{\alpha,\beta} := \{x \in \mathbb{R}^n : \alpha \leq \rho_{F^0}(\bar{x} - x) \leq \beta\}.$$

Our goal now is to extend this inequality to the interior of $A_{\alpha,\beta}$, using its validity on the boundary, i.e., to prove a *comparison result.* Denote by $S_\delta(x) := R_\delta(\rho_{F^0}(\bar{x} - x))$ and assume that the (open) set

$$U := \{x \in A_{\alpha,\beta} : \bar{u}(x) < S_\delta(x)\}$$

is nonempty. Let us extend the Lipschitz continuous function $S_\delta : A_{\alpha,\beta} \to \mathbb{R}^+$ to the whole $\overline{\Omega}$ by setting $S_\delta(x) = R_\delta(\beta) = \mu$ for $x \in \overline{\Omega}$ with $\rho_{F^0}(\bar{x} - x) > \beta$, and $S_\delta(x) = R_\delta(\alpha)$ whenever $\rho_{F^0}(\bar{x} - x) < \alpha$. Consider the function $w(x) := \max\{\bar{u}(x), S_\delta(x)\}$, which is equal to $S_\delta(x)$ on U and to $\bar{u}(x)$ elsewhere. Let us show that $w(\cdot)$ minimizes the functional in (P_F) on $\bar{u}(\cdot) + W_0^{1,1}(\Omega, \mathbb{R})$, taking into account that $\bar{u}$ is a solution of (P_F).

Since $\bar{u}(\cdot)$ is a solution, we have

$$\int_\Omega [f(\rho_F(\nabla \bar{u}(x))) - f(\rho_F(\nabla w(x)))] \, \mathrm{d}x \leq 0. \tag{1.22}$$

If $p(\cdot)$ is an arbitrary (measurable) selection of $x \mapsto \partial(f \circ \rho_F(\nabla w(x)))$, then, by definition of the subdifferential of a convex function,

$$f(\rho_F(\nabla \bar{u}(x))) - f(\rho_F(\nabla w(x))) \geq \langle p(x), \nabla \bar{u}(x) - \nabla w(x) \rangle \tag{1.23}$$

for a.e. $x \in \Omega$, and hence

$$\int_\Omega [f(\rho_F(\nabla \bar{u}(x))) - f(\rho_F(\nabla w(x)))] \, \mathrm{d}x \geq$$

$$\geq \int_\Omega \langle p(x), \nabla \bar{u}(x) - \nabla w(x) \rangle \, \mathrm{d}x. \tag{1.24}$$

Now we construct a measurable selection $p(\cdot)$ in such a way that the last integral is equal to zero. Since $\nabla w(x) = \nabla \bar{u}(x)$ for a.e. $x \in \Omega \setminus U$, and $\nabla w(x) = \nabla S_\delta(x)$ for a.e. $x \in U$ (see [41, p. 50]), we choose first $p(x)$ as an arbitrary measurable selection of the mapping $x \mapsto \partial(f \circ \rho_F)(\nabla u(x)) \neq \varnothing$ on $\Omega \setminus U$. On the set U instead we define $p(x)$ according to the differential properties of $S_\delta(x)$. Observe first of all that $R'_\delta(r) = -\varphi(\delta(\frac{\alpha}{r})^{n-1})$ for all $\alpha < r < \beta$ except for at most a countable number of points. Furthermore, by the Lipschitz continuity of $x \mapsto \rho_{F^0}(\bar{x} - x)$ the function $S_\delta(\cdot)$ is almost everywhere differentiable on Ω, and

$$\nabla S_\delta(x) = -R'_\delta(\rho_{F^0}(\bar{x} - x)) \nabla \rho_{F^0}(\bar{x} - x) \tag{1.25}$$

for a.e. $x \in U$. Due to (1.4) the gradient $\nabla \rho_{F^0}(\bar{x} - x)$ belongs to ∂F, and, consequently,

$$\rho_F(\nabla S_\delta(x)) = \| R'_\delta(\rho_{F^0}(\bar{x} - x)) \| = \varphi(\delta(\frac{\alpha}{\rho_{F^0}(\bar{x} - x)})^{n-1}).$$

Hence, by the definition of $\varphi(\cdot)$, on one hand, we obtain

$$\delta \left(\frac{\alpha}{\rho_{F^0}(\bar{x} - x)} \right)^{n-1} \in \partial f(\rho_F(\nabla S_\delta(x))) \tag{1.26}$$

for a.e. $x \in U$. On the other hand (see (1.4) and (1.25)), $\nabla S_\delta(x)$ is a normal vector to F^0 at the point $\frac{\bar{x} - x}{\rho_{F^0}(\bar{x} - x)}$, that means (see also (1.2))

$$\left\langle \nabla S_\delta(x), \frac{\bar{x} - x}{\rho_{F^0}(\bar{x} - x)} \right\rangle = \sigma_{F^0}(\nabla S_\delta(x)) = \rho_F(\nabla S_\delta(x)),$$

or, in the dual form,

$$\frac{\bar{x}-x}{\rho_{F^0}(\bar{x}-x)} \in \mathbf{N}_F\left(\frac{\nabla S_\delta(x)}{\rho_F(\nabla S_\delta(x))}\right) \cap \partial F^0 = \partial\rho_F(\nabla S_\delta(x)).$$

Recalling (1.26), let us now define the (continuous) function

$$p(x) := \delta\left(\frac{\alpha}{\rho_{F^0}(\bar{x}-x)}\right)^{n-1} \frac{\bar{x}-x}{\rho_{F^0}(\bar{x}-x)}, \tag{1.27}$$

which is a selection of $x \mapsto \partial(f \circ \rho_F)(\nabla S_\delta(x))$ almost everywhere in U.

The next step is to prove that

$$\int_\Omega \langle p(x), \nabla u(x) - \nabla w(x)\rangle \; \mathrm{d}x = 0. \tag{1.28}$$

We use polar coordinates $r = \|x-\bar{x}\|$ and $\omega = (x-\bar{x})/\|x-\bar{x}\|$. Note that for each ω, $\|\omega\| = 1$, on the boundary of the (open) linear set $l_\omega := \{r \in (\alpha,\beta) : (r,\omega) \in U\}$ the equality $\bar{u}(x) = S_\delta(x)$ holds. Therefore,

$$\int_{l_\omega} \langle \omega, \nabla\bar{u}(x) - \nabla S_\delta(x)\rangle \, dr = \int_{l_\omega} \frac{d}{dr}(\bar{u}(x) - S_\delta(x)) \, dr = 0,$$

and by Fubini theorem we obtain

$$\begin{aligned}&\int_U \langle p(x), \nabla u(x) - \nabla S_\delta(x)\rangle \, dx = \\ &= -\delta\alpha^{n-1} \int_{\|\omega\|=1} \frac{d\omega}{\rho^n_{F^0}(-\omega)} \int_{l_\omega} \frac{1}{r^n}\langle r\omega, \nabla\bar{u}(x) - \nabla S_\delta(x)\rangle r^{n-1} \, dr = 0,\end{aligned}$$

proving thus the equality (1.28).

Hence,

$$\int_\Omega f(\rho_F(\nabla\bar{u}(x))) \, dx = \int_\Omega f(\rho_F(\nabla w(x))) \, dx. \tag{1.29}$$

Furthermore, (1.23) together with (1.29) and (1.28) imply that

$$f(\rho_F(\nabla u(x))) - f(\rho_F(\nabla w(x))) = \langle p(x), \nabla u(x) - \nabla w(x)\rangle \tag{1.30}$$

for a.e. $x \in \Omega$. Let $E \subset U$ be a set of null measure such that for all $x \in U \setminus E$ the equality (1.30) takes place, and the gradient $\nabla\sigma_F(\bar{x} - x)$ (coinciding, by homogeneity, with $\nabla\sigma_F(p(x))$ (see (1.27)) exists. In accordance with (1.30) both $\nabla\bar{u}(x)$ and $\nabla w(x)$ belong to $\partial(f\circ\rho_F)^*(p(x)) = \partial(f^*\circ\sigma_F)(p(x))$, and for each $x\in U\setminus E$ the latter subdifferential admits the form

$$\partial f^*(\rho_{F^0}(p(x)))\nabla\sigma_F(\bar{x}-x),$$

where (see (1.27))

$$\rho_{F^0}(p(x)) = \delta\left(\frac{\alpha}{\rho_{F^0}(\bar{x}-x)}\right)^{n-1}. \tag{1.31}$$

Notice that there is at most a countable family of disjoint open intervals $J_1, J_2, ... \subset [\alpha,\beta]$ such that the real function $f(\cdot)$ is affine on each J_i with a slope $\tau_i > 0$, $i = 1, 2, ...$. In other words, τ_i are discontinuity points of $\varphi(\cdot)$. Denoting by $E_i := \bar{x} - \alpha\left(\frac{\delta}{\tau_i}\right)^{\frac{1}{n-1}}\partial F^0$ we see that for all $x \notin \bigcup_{i=1}^{\infty} E_i$ (the set of null measure) $\partial f^*(\rho_{F^0}(p(x)))$ is a singleton. So that $\nabla w(x) = \nabla\bar{u}(x)$ for a.e. $x\in\Omega$ contradicting the assumption $U\neq\varnothing$. Thus, we have proved the inequality (1.21) on $\{x\in\Omega : \alpha\le\rho_{F^0}(\bar{x}-x)\le\beta\}$. Combining (1.21) and (1.20) by the mean value theorem we obtain (1.15) with $\eta = \delta\left(\frac{\alpha}{\beta}\right)^{n-1}$.

Finally, assume that $\gamma_{n,f} = k > 0$. In this case $n = 1$, the function $f(\cdot)$ admits positive slope at zero but it is not affine near 0 (consequently, $\varphi(k) = 0$), and obviously $a = 0$. Then, for a given $\beta > 0$ with $\bar{x} - \beta F^0 \subset \Omega$ we have $\bar{u}(\bar{x}) > \mu + \varphi(k)\beta$, and, since $\bar{u}(\cdot)$ and $\varphi(\cdot)$ are continuous, there exist $\delta > 0$ and $0 < \alpha < \beta$ such that

$$\bar{u}(x) \ge \mu + \varphi(k+t)(\beta - \rho_{F^0}(\bar{x}-x))$$

for all $x\in\Omega$ with $\rho_{F^0}(\bar{x}-x)\le\alpha$ and $0 < t \le \delta$. Taking into account the monotonicity of the subdifferential $\partial f^*(\cdot)$ and the nonaffinity of $f(\cdot)$ in a neighbourhood of zero, we can choose $\delta > 0$ such that $\partial f^*(k+\delta)$ is a singleton (equivalently, $k+\delta$ is a slope of $f(\cdot)$ different from slopes of its affine pieces near zero). Now we can proceed as in the first part of the proof by using the comparison argument with the linear function

$$R_\delta(r) := \mu + \varphi(k+\delta)(\beta - r), \quad \alpha\le r\le\beta.$$

The selection $p(x)\in\partial(f\circ\rho_F)(\nabla(R_\delta\circ\rho_{F^0})(\bar{x}-x))$ appearing in the equality (1.28) takes the form

$$p(x) = (k+\delta)\frac{\bar{x}-x}{\rho_{F^0}(\bar{x}-x)}.$$

Observe that the final part of the proof should be omitted here because the subdifferential $\partial f^*(\rho_{F^0}(p(x))) = \partial f^*(k+\delta)$ is already a singleton by construction. Thus we have

$$\bar{u}(x) \geq \mu + \varphi(k+\delta)(\beta - \rho_{F^0}(\bar{x} - x))$$

for all $x \in \Omega$ with $\rho_{F^0}(\bar{x} - x) \leq \beta$, and the first part of theorem is proved.

(ii) This can be proved using the same reasoning as in **(i)** with some evident modifications. For instance, the inequality (1.21) here has the form

$$\bar{u}(x) \leq \tilde{R}_\delta(\rho_{F^0}(x - \bar{x})),$$

where

$$\tilde{R}_\delta(r) := \mu - \int_r^\beta \varphi\left(\delta\left(\frac{\alpha}{s}\right)^{n-1}\right) ds.$$

□

In the case $a = 0$, i.e., when the function $f(\cdot)$ is strictly convex at the origin, we immediately obtain a consequence of Theorem 1.2.1, which will be exploited in Section 1.3. Here $\mathfrak{r}^{\pm}(\cdot)$ are the functions defined by (1.8).

Corollary 1.2.1. *Assume that $a = 0$, and one of the following hypotheses holds:*

(**a**) $k = 0$;

(**b**) $n = 1$ *and* $f(\cdot)$ *is not affine near* 0.

If $\bar{u}(\cdot)$ is a continuous solution of (P_F), which does not attain its minimal (maximal) value at a point $x \in \Omega$, then the whole set $x - \mathfrak{r}^-(\bar{x})F^0$ (respectively, $\bar{x} + \mathfrak{r}^+(\bar{x})F^0$) does not contain in its interior points of minimum (respectively, maximum) of $\bar{u}(\cdot)$.

Proof. It is enough to take $\beta := \mathfrak{r}^-(\bar{x})$ (respectively, $\mathfrak{r}^+(\bar{x})$) and observe that under the hypothesis (**a**), (**b**) we have $\varphi(\gamma_{n,f} + \eta) > 0$ for all $\eta > 0$. The result follows now from the estimate (1.15) or (1.18), respectively. □

Otherwise, if $a > 0$ then the estimates of Theorem 1.2.1 can be applied to obtain the generalizations of SMP presented in Sections 1.4, 1.5 and 1.6. Those results are essentially based on the following assertion.

Corollary 1.2.2. *Assume that $a > 0$, and let $\bar{u}(\cdot)$ be a continuous solution of (P_F) such that for some $x_0 \in \Omega$ and $\delta > 0$*

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \quad \forall x \in x_0 + \delta F^0 \subset \Omega. \tag{1.32}$$

Then

$$\bar{u}(x) = \bar{u}x_0) + a\rho_{F^0}(x - x_0) \tag{1.33}$$

holds for all $x \in x_0 + \frac{\delta}{\|F\|\|F^0\|+1}F^0$.

Symmetrically, if

$$\bar{u}(x) \leq \bar{u}(x_0) - a\rho_{F^0}(x_0 - x) \quad \forall x \in x_0 - \delta F^0 \subset \Omega, \tag{1.34}$$

then

$$\bar{u}(x) = \bar{u}(x_0) - a\rho_{F^0}(x_0 - x) \tag{1.35}$$

for all $x \in x_0 - \frac{\delta}{\|F\|\|F^0\|+1}F^0$

Proof. We prove the first part by using the statement **(i)** of Theorem 1.2.1, while the symmetric assertion can be proved similarly (it is enough only to apply **(ii)** in the place of **(i)**). Assume that

$$\bar{u}(\bar{x}) > \mu + a\rho_{F^0}(\bar{x} - x_0),$$

where $\mu := \bar{u}(x_0)$, for some $\bar{x} \in \Omega$ with

$$\rho_{F^0}(\bar{x} - x_0) < \frac{\delta}{\|F\|\|F^0\| + 1}.$$

Let us choose $\varepsilon > 0$ so small that

$$\rho_{F^0}(\bar{x} - x_0)(\|F\|\|F^0\| + 1) + \varepsilon\|F\|\|F^0\| < \delta \tag{1.36}$$

and

$$\bar{u}(\bar{x}) > \mu + a(\rho_{F^0}(\bar{x} - x_0) + \varepsilon). \tag{1.37}$$

Setting $\beta := \rho_{F^0}(\bar{x} - x_0) + \varepsilon$, we have $\bar{x} - \beta F^0 \subset x_0 + \delta F^0$. Indeed, given $y \in \bar{x} - \beta F^0$, using the inequalities (1.3), we obtain

$$\rho_{F^0}(y - x) \leq \|F\|\|F^0\|\rho_{F^0}(\bar{x} - y) \leq \|F\|\|F^0\|(\rho_{F^0}(\bar{x} - x_0) + \varepsilon),$$

and it follows from (1.36) that

$$\rho_{F^0}(y - x_0) \leq \rho_{F^0}(y - \bar{x}) + \rho_{F^0}(\bar{x} - x_0) < \delta.$$

In particular, $\bar{u}(x) \geq \mu$ for all $x \in \bar{x} - \beta F^0$. Combining this inequality with (1.37) by Theorem 1.2.1 **(i)** we find $\eta > 0$ such that

$$\bar{u}(x) \geq \mu + \varphi(\eta)(\beta - \rho_{F^0}(\bar{x} - x)) \;\; \forall x \in \bar{x} - \beta F^0. \tag{1.38}$$

Applying (1.38) to the point $x_0 \in \bar{x} - \beta F^0$, we obtain finally

$$\bar{u}(x_0) \geq \mu + \varphi(\eta)\varepsilon \geq \mu + a\varepsilon > \mu,$$

which is a contradiction. The equality (1.33) can be extended then to the boundary of $x_0 + \frac{\delta}{\|F\|\|F^0\|+1} F^0$ by continuity of the involved functions. □

From the latter part of the proof of Theorem 1.2.1 **(i)** it is easy to see that in the case $n = 1$, due to the disconnectedness of the annulus $A_{\alpha,\beta}$, estimates like (1.15) and (1.18) hold without symmetry. To be more precise, let us consider a convex (not necessarily even) function $L : \mathbb{R} \to \mathbb{R}^+ \cup \{+\infty\}$ with $L(0) = 0$ and $0 \in \text{int dom} L$, lower semicontinuous on its domain and such that $L(\cdot)$ is not identically equal to zero on both negative and positive half-lines. In what follows the set of functions with these properties will be denoted by $\mathfrak{L}$. Given $L(\cdot) \in \mathfrak{L}$ it is obvious that the function L decreases on $]-\infty, 0[$ and increases on $]0, +\infty[$, that there exist $0 < b^{\pm} \leq +\infty$ with $\text{dom}\, L^* = \{t : -b^- < t < b^+\}$, where one of the signs "$<$" (or both of them) can be replaced to "$\leq$", and that $0 \in \partial L(0)$, $0 \in \partial L^*(0)$. Consequently, for some nonnegative (finite) $k^{\pm}$ and $a^{\pm}$ we have $\partial L(0) = [-k^-, k^+]$ and $\partial L^*(0) = [-a^-, a^+]$. As in the symmetric case, let us introduce the upper semicontinuous nondecreasing function $\varphi : (-b\ \ , b^+) \to \mathbb{R}$ by setting $\varphi(t) := \sup \partial L^*(t)$. Observe that by monotonicity of the subdifferential, one of the numbers k^+ or a^+ (analogously, k^- or a^-) is always equal to zero. The following statement contains the one-sided estimates for solutions of the one-dimensional problem (P_F). For the sake of simplicity we consider here only the case of *local minimum*. One easily makes the respective modifications when the symmetric conditions (of *local maximum*) take place.

Theorem 1.2.2. *Let $L \in \mathfrak{L}$, and let $\bar{u}(\cdot)$ be a continuous solution of (P_F). Assume that a point $\bar{x} \in \Omega$ and numbers $\mu \in \mathbb{R}$, $\beta > 0$ are such that*

$$u(x) \geq \mu \;\; \forall x \in [\bar{x} - \beta, x] \subset \Omega \tag{1.39}$$

and

$$\bar{u}(\bar{x}) > \mu + a^+\beta. \tag{1.40}$$

If L is not affine in a right-hand neighbourhood of zero then there exists $\eta > 0$ such that

$$\bar{u}(x) \geq \mu + \varphi(k^+ + \eta)(\beta + x - \bar{x}) \quad \forall x \in [\bar{x} - \beta, \bar{x}]. \tag{1.41}$$

Analogously, if L is not affine in a left-hand neighbourhood of zero and the relations (1.39), (1.40) are substituted by the following:

$$u(x) \geq \mu \quad \forall x \in [\bar{x}, x + \beta] \subset \Omega;$$

$$u(\bar{x}) > \mu + a^-\beta,$$

then for some $\eta > 0$

$$\bar{u}(x) \geq \mu - \varphi(-k^- - \eta)(\beta + \bar{x} - x) \quad \forall x \in [\bar{x}, \bar{x} + \beta]. \tag{1.42}$$

Proof. We use here the same arguments as in the proof of Theorem 1.2.1. Let us emphasize only some simplifications in the main steps that have a methodical interest. Let us consider the first case ($L(\cdot)$ is not affine on the right side of the origin, and (1.39), (1.40) are fulfilled). We write (1.40) as $u(\bar{x}) > \mu + \varphi(k^+)\beta$ and by the upper semicontinuity of $\varphi(\cdot)$ choose $\eta > 0$ so small that the latter inequality holds with $\varphi(k^+ + \eta)$ in the place of $\varphi(k^+)$. Assume also (see the last part of the proof of Theorem 1.2.1 **(i)**) that the subdifferential $\partial L^*(k^+ + \eta)$ is a singleton, namely, $\partial L^*(k^+ + \eta) = \{\varphi(k^+ + \eta)\}$. Here we use the nonaffinity of $L(\cdot)$ on the right side of zero. Defining on $\overline{\Omega}$ the continuous function

$$R_\delta(x) := \begin{cases} \mu & \text{if } \ x < x - \beta, \\ \mu + \varphi(k^+ + \eta)(\beta + x - \bar{x}) & \text{if } \ \bar{x} - \beta \leq x \leq \bar{x}, \\ \mu + \varphi(k^+ + \eta)\beta & \text{if } \ x > \bar{x}, \end{cases}$$

we wish to prove that $\bar{u}(x) \geq R_\delta(x)$ for all $x \in [\bar{x} - \beta, \bar{x}]$. If this inequality is violated in some (open) set $U \subset (\bar{x} - \beta, \bar{x})$ then by the Newton-Leibnitz formula

$$\int_\Omega (\bar{u}'(x) - w'(x))\, dx = \int_U (\bar{u}'(x) - R_\delta'(x))\, dx = 0, \tag{1.43}$$

where $w(x) := \max\{u(x), R_\delta(x)\}$. Since $k^+ + \eta \in \partial L(R_\delta'(x))$, we have

$$L(\bar{u}'(x)) - L(w'(x)) \geq (k^+ + \eta)(\bar{u}'(x) - w'(x)) \tag{1.44}$$

for a.e. $x \in \Omega$, and we conclude by (1.43) and by the choice of $\bar{u}(\cdot)$ that $w(\cdot)$ is also a solution of (P_F). Therefore, the inequality (1.44) becomes equality almost everywhere on Ω. Consequently, both $\bar{u}'(x)$ and $w'(x)$ belong to $\partial L^*(k^+ + \eta)$, i.e., $\bar{u}'(x) = w'(x) = \varphi(k^+ + \eta)$ for a.e. $x \in U$, contradicting the assumption that $\bar{u}(x) < R_\delta(x)$ on U. Thus, the estimate (1.41) holds. The *right-sided* inequality (1.42), where $-\varphi(-k^- - \eta^-) > 0$, can be obtained similarly by using the behaviour of $L(\cdot)$ on the left side of zero. □

We are ready now to establish various versions of SMP, starting from the traditional one.

1.3 Strong Maximum Principle under the strict convexity assumption

Our main hypothesis here is

$$(\mathbf{H}_1) \qquad \partial L^*(0) = \{0\},$$

which is, obviously, equivalent to $a = 0$ when $L = f \circ \rho_F$. In the case $n > 1$ we also assume the dual hypothesis

$$(\mathbf{H}_2) \qquad \partial L(0) = \{0\},$$

which reduces to $k = 0$ if $L = f \circ \rho_F$. In the asymmetric case ($n = 1$), clearly, $(\mathbf{H}_1) \Leftrightarrow a^+ = a^- = 0$ while $(\mathbf{H}_2) \Leftrightarrow k^+ = k^- = 0$.

Theorem 1.3.1 (Strong Maximum Principle)**.** *Assume that one of the following conditions holds:*

(i) $n = 1$ *and* $L(\cdot) \in \mathfrak{L}$ *is not affine in both left- and right-hand neighbourhoods of zero;*

(ii) $n > 1$ *and the lagrangean* $L(\cdot)$ *satisfies both hypotheses* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$, *being represented as* $L = f \circ \rho_F$, *where* $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ *is a convex lower semicontinuous function with* $f(0) = 0$, *and* $F \subset \mathbb{R}^n$ *is a convex closed bounded set with* $0 \in \operatorname{int} F$.

Then for each open bounded connected region $\Omega \subset \mathbb{R}^n$ there is no continuous admissible nonconstant minimizer of

$$u(\cdot) \mapsto \int_{\Omega} L(\nabla u(x))\, dx \tag{1.45}$$

on $u^0(\cdot) + W_0^{1,1}(\Omega)$ for $u^0(\cdot) \in W^{1,1}(\Omega)$, which admits its minimal (or maximal) value in Ω.

Proof. Observe first that in the framework of the condition **(i)** the hypothesis $(\mathbf{H}_1)$ holds automatically, while $(\mathbf{H}_2)$ can be violated. An open bounded connected set in this case is an interval $\Omega = (A, B)$ with $A < B$. Assuming that $\bar{u}(\cdot)$ is a minimizer of (1.45), and $\bar{x} \in \Omega$ is such that $\bar{u}(\bar{x}) > \mu := \min_{\overline{\Omega}} u(x)$, we put $\beta := \bar{x} - A$ and obtain by Theorem 1.2.2 that $\bar{u}(x) > \mu$ for all $x \in (A, \bar{x}]$ (because in (1.41) we have $\varphi(k^+ + \eta) > 0\ \forall \eta > 0$). Analogously, by using the estimate (1.42) of Theorem 1.2.2, we conclude that $\bar{u}(x) > \mu\ \forall x \in [\bar{x}, B)$. Consequently, there are only two possibilities: either the minimum of $\bar{u}(\cdot)$ is attained in one of the end-points of the segment $[A, B]$ or $\bar{u} \equiv \mu$ on Ω. In the case of maximum the reasoning is similar.

Let us suppose now condition **(ii)**. Take an arbitrary continuous admissible function $\bar{u} : \Omega \to \mathbb{R}$ that minimizes the integral (1.45), and let μ be its minimum on $\overline{\Omega}$. If $\bar{u}(\cdot)$ is not constant then the (open) set $W := \{x \in \Omega : \bar{u}(x) \neq \mu\}$ is nonempty. Since $a = 0$, it follows from Corollary 1.2.1 that $\bar{x} - \beta F^0 \subset W$ whenever $\bar{x} \in W$, $0 < \beta < \mathfrak{r}^-(\bar{x})$ (see (1.8)). Fix now $x^* \in \overline{W}$ (closure of W in Ω). By the continuity let us choose an arbitrary $0 < \varepsilon < \frac{1}{2\|F\|}\mathfrak{r}^-(x^*)$ so small that $\mathfrak{r}^-(x^*) \leq 2\mathfrak{r}^-(x)$ for all $x \in \Omega$ with $\|x - x^*\| \leq \varepsilon$. Let $\bar{x} \in W \cap \left(x^* + \varepsilon \overline{B}\right)$. Then $\bar{x} \in W$ and

$$\rho_{F^0}(\bar{x} - x^*) \leq \varepsilon \|F\| < \frac{1}{2}\mathfrak{r}^-(x^*) \leq \mathfrak{r}^-(\bar{x}).$$

Hence, x^* also belongs to W, and W is closed in Ω, implying that $W = \Omega$ because Ω is connected. Thus, the open region Ω is free from the points of mimimum of $\bar{u}(\cdot)$. Analogously, $\bar{u}(\cdot)$ being non constant can not attain in Ω its maximum, and SMP is proved. Notice that here we use the argument in some sense dual to the classical proof for harmonic functions (see, e.g., [36, p. 15]). □

As an example of the functional, for which the validity of SMP follows from Theorem 1.3.1 but is not covered by earlier results, we consider the integral

$$\int_{\Omega} \left(\sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right| \right)^2 dx.$$

Here $f(t) = t^2$ and $\rho_F(\cdot)$ is the l_1-norm in $\mathbb{R}^n$. Observe that the set

$$F = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} \|x_i\| \le 1\}$$

in this case is neither smooth, nor strictly convex.

The Strong Maximum Principle is not valid if the hypothesis $(\mathbf{H}_1)$ does not hold because, as shown in [19], there are many (Lipschitz) continuous nonnegative (nonpositive) minimizers of (1.45) with the trivial boundary condition $u_0(x) = 0$, which touch the zero level at interior points of Ω as well. If $(\mathbf{H}_2)$ is violated then in the case $n > 1$ one can construct a counterexample to SMP based on the same arguments as those in [16]. Indeed, fix arbitrary $\bar{x} \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ and define

$$\Omega := \{x \in \mathbb{R}^n : \alpha < \rho_{F^0}(\bar{x} - x) < \beta\}$$

for some $\beta > \alpha > 0$. Then for each $\delta > 0$ the function $S_\delta(x) := R_\delta(\rho_{F^0}(\bar{x}-x))$ (see (1.20)) is a solution of (P_F) (here $\partial f(0) = [0, k]$, $k > 0$) with $u^0(x) = S_\delta(x)$ in virtue of the relations (1.28) and (1.23), where $w(\cdot) = S_\delta(\cdot)$, $U = \Omega$, $\bar{u}(\cdot)$ is another minimizer, and the mapping $p(\cdot)$ is given by (1.27). Clearly, $\mu = \min_{x \in \overline{\Omega}} S_\delta(x)$, and it is enough only to choose $\delta > 0$, α and β such that $R_\delta(\alpha) > \mu$ and $R_\delta(r) = \mu$ for $\beta - \varepsilon \le r \le \beta$, where $\varepsilon > 0$ is sufficiently small. Let, for instance, $\beta = 2\alpha$ and $k < \delta < 2^{n-1}k$ (observe the importance of the condition $n > 1$ for this construction). Then for some $0 < \varepsilon < \beta - \alpha$

$$\delta \left(\frac{\alpha}{\alpha + \varepsilon} \right)^{n-1} > k, \quad \delta \left(\frac{\alpha}{2\alpha - \varepsilon} \right)^{n-1} < k$$

and, consequently,

$$R_\delta(\alpha) \ge \mu + \int_{\alpha}^{\alpha+\varepsilon} \varphi \left(\delta \left(\frac{\alpha}{s} \right)^{n-1} \right) ds \ge \mu + \varphi \left(\delta \left(\frac{\alpha}{\alpha + \varepsilon} \right)^{n-1} \right) \varepsilon > \mu,$$

while for $\beta-\varepsilon\le r\le\beta$ we have

$$\mu\le R_\delta(r)\le\mu+\int\limits_{\beta-\varepsilon}^{\beta}\varphi\left(\delta\left(\frac{\alpha}{s}\right)^{n-1}\right)\,ds\le\mu+\varphi\left(\delta\left(\frac{\alpha}{2\alpha-\varepsilon}\right)^{n-1}\right)\varepsilon=\mu.$$

We use here the fact that $\varphi(t)=0$ on $[0,k]$ and $\varphi(t)>0$ for $t>k$. Thus $S_\delta(\cdot)$ is a nonconstant continuous solution of (P_F) admitting its minimum at each point $x\in\Omega$ with $\beta-\varepsilon\le\rho_{F^0}(\bar{x}-x)\le\beta$, which contradicts SMP.

The function $L(\cdot)$ may have a nontrivial slope at zero, which is different from the slopes at all points $x\ne0$. Let us give a simple example of such function, which, moreover, is neither strictly convex nor smooth near the origin.

Example 1. *Fix an arbitrary strictly decreasing sequence* $\{\tau_m\}\subset(0,\pi/2)$ *converging to zero, and define the continuous function* $f:(0,\pi/2)\to\mathbb{R}^+$, *which is equal to* tgt *for* $t=\tau_m$, $m=1,2,\ldots$, *and for* $\tau_1<t<\pi/2$, *and affine on each interval* (τ_{m+1},τ_m). *We set also* $f(t)=+\infty$ *for* $t\ge\pi/2$. *Then* $\partial f(0)=[-1,1]$ *but, nevertheless, the Strong Maximum Principle is valid for the functional* $\int\limits_\Omega f(\|u'(x)\|)\,dx$ *due to Theorem 1.3.1.*

However, even in the case $n=1$ the function $f(\cdot)$ can not be affine near the origin.

Example 2. *If* $f(t)=kt$ *for* $0\le t\le\varepsilon$ *then for each admissible function* $u(\cdot)\in W^{1,1}(0,1)$ *with the boundary values* $u(0)=0$ *and* $u(1)=\varepsilon/2$ *we have by Jensen's inequality*

$$\int\limits_0^1 f(\|u'(x)\|)\,dx\ge f(\|\int\limits_0^1 u'(x)\,dx\|)=\frac{k\varepsilon}{2}=\int\limits_0^1 f(\|u'(x)\|)\,dx,$$

where $u(x):=\frac{\varepsilon}{2}x$, $x\in[0,1]$. *On the other hand, the function* $\tilde{u}(x)$ *equal to* 0 *on* $\left[0,\frac{1}{2}\right]$ *and to* $\varepsilon x-\frac{\varepsilon}{2}$ *on* $\left[\frac{1}{2},1\right]$ *gives the same minimal value to the integral.*

1.4 A one-point extended Strong Maximum Principle

The traditional Strong Maximum Principle is no longer valid if $L = f \circ \rho_F$ with $\partial f^*(0) \neq \{0\}$. However, also in this case we can consider a similar property, which clearifies the structure of minimizers of the functional (P_F) and has the same field of applications as the classical SMP. Let us start with the one-point SMP when a test function admits a unique point of *local minimum (maximum)*. Setting $\hat{u}(x) = \mu + a\rho_{F^0}(x - x_0)$ with $\mu \in \mathbb{R}$ and $x_0 \in \Omega$, it was already shown (see Corollary 1.2.2) that for an arbitrary minimizer $\bar{u}(\cdot), \bar{u}(x_0) = \mu$, of (P_F) the inequality $\bar{u}(x) \geq \hat{u}(x)$ valid on a neighbourhood of x_0 becomes, in fact, the equality on another (may be smaller) neighbourhood. First we show that this property can be extended to the maximal homothetic set $x_0 + rF^0$ contained in Ω.

Theorem 1.4.1. *Assume that $L = f \circ \rho_F$, where $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is a convex lower semicontinuous function, $f(t) = 0$ iff $t \in [0, a]$, $a > 0$, and $F \subset \mathbb{R}^n$ is a convex closed bounded set with $0 \in \text{int}F$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $x_0 \in \Omega$ and $\bar{u}(\cdot)$ be a continuous solution of (P_F). If*

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \tag{1.46}$$

for all $x \in x_0 + \mathfrak{r}^+(x_0)\, F^0$ then the equality

$$\bar{u}\,(x) = \bar{u}\,(x_0) + a\rho_{F^0}\,(x - x_0) \tag{1.47}$$

holds on $x_0 + \mathfrak{r}^+(x_0)\, F^0$. Analogously, if

$$\bar{u}\,(x) \leq \bar{u}\,(x_0) - a\rho_{F^0}\,(x_0 - x) \tag{1.48}$$

for all $x \in x_0 - \mathfrak{r}^-(x_0)\, F^0$, then the equality

$$\bar{u}\,(x) = \bar{u}\,(x_0) - a\rho_{F^0}\,(x_0 - x) \tag{1.49}$$

holds on $x_0 - \mathfrak{r}^-(x_0)\, F^0$.

Proof. Let us prove the first assertion, while the second one is symmetric and can be proved similarly. Let us define

$$R := \sup \{ r > 0 : \bar{u}(x) = \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \\ \text{for all } x \in x_0 + rF^0 \subset \Omega \} . \tag{1.50}$$

By Corollary 1.2.2 we have $R > 0$. Our goal now is to prove that $R = \mathfrak{r}^+(x_0)$. Assuming the contrary, i.e., $R < \mathfrak{r}^+(x_0)$ let us choose $\delta > 0$ so small that

$$R + \delta(\|F\| \|F^0\| + 1) < \mathfrak{r}^+(x_0) \tag{1.51}$$

and $\bar{x} \in \Omega$ with $R < \rho_{F^0}(\bar{x} - x_0) < R + \delta$ such that

$$\bar{u}(\bar{x}) > \bar{u}(x_0) + a\rho_{F^0}(\bar{x} - x_0) \tag{1.52}$$

(see the condition (1.46)). By strictness of the inequalities (1.51) and (1.52) we find also $\varepsilon > 0$ such that

$$R + \delta(\|F\| \|F^0\| + 1) + \varepsilon \|F\| \|F^0\| < \mathfrak{r}^+(x_0) \tag{1.53}$$

and

$$\bar{u}(\bar{x}) > \bar{u}(x_0) + a(\rho_{F^0}(\bar{x} - x_0) + \varepsilon). \tag{1.54}$$

Setting $\mu := \bar{u}(x_0) + aR$ let us define the function $\bar{v}(\cdot)$ equal to

$$\max\{\bar{u}(x), \mu\}$$

on $x_0 + \mathfrak{r}^+(x_0)F^0$ and to $\bar{u}(x)$ on the remainder of $\overline{\Omega}$. It is continuous because $R < \mathfrak{r}^+(x_0)$ and for each $x \in \Omega$ with $\rho_F(x - x_0) \geq R$ we have

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \geq \mu, \tag{1.55}$$

i.e., $\bar{v}(x) = \bar{u}(x)$. We claim that $\bar{v}(\cdot)$ minimizes the functional in (P_F) on $u(\cdot) + W_0^{1,1}(\Omega)$. Indeed, it follows from (1.55) that $\bar{v}(\cdot) \in u(\cdot) + W_0^{1,1}(\Omega)$, and clearly $\nabla \bar{v}(x) = 0$ for each $x \in \Omega$ with $\bar{v}(x) \neq \bar{u}(x)$. Denoting by $\Omega' := \{x \in \Omega : \bar{u}(x) = \bar{v}(x)\}$ we have

$$\begin{aligned} \int_\Omega f(\rho_F(\nabla v(x)))\, dx &= \int_{\Omega'} f(\rho_F(\nabla u(x)))\, dx \leq \\ \leq \int_\Omega f(\rho_F(\nabla \bar{u}(x)))\, dx &\leq \int_\Omega f(\rho_F(\nabla \bar{v}(x)))\, dx. \end{aligned} \tag{1.56}$$

Therefore, $\bar{v}(\cdot)$ is a solution of (P_F) with $u^0(\cdot) = \bar{v}(\cdot)$.

Let x_0' be a unique point from $[x_0, \bar{x}]$ such that $\rho_{F^0}(x_0' - x_0) = R$. Namely, $x_0' = x_0 + \lambda(\bar{x} - x_0)$ where $0 < \lambda := \frac{R}{\rho_{F^0}(\bar{x}-x_0)} < 1$. Setting now $\beta := \rho_{F^0}(\bar{x} - x_0') + \varepsilon$, by (1.54) we obtain

$$\begin{aligned} \bar{v}(\bar{x}) &\geq \bar{u}(\bar{x}) > \bar{u}(x_0) + a\rho_{F^0}(x_0' - x_0) + \\ &+ a(\rho_{F^0}(\bar{x} - x_0') + \varepsilon) = \mu + a\beta. \end{aligned} \tag{1.57}$$

On the other hand, the inequality $\rho_{F^0}(\bar{x} - x) \leq \beta$ implies that

$$\begin{aligned} \rho_{F^0}(x - x_0) &\leq \rho_{F^0}(x - \bar{x}) + \rho_{F^0}(\bar{x} - x_0') + \rho_{F^0}(x_0' - x_0) \leq \\ &\leq \|F\| \left\|F^0\right\| \rho_{F^0}(\bar{x} - x) + \rho_{F^0}(\bar{x} - x_0') + R \leq \\ &\leq \left(\|F\| \left\|F^0\right\| + 1\right) \rho_{F^0}(\bar{x} - x_0') + \\ &+ \varepsilon \|F\| \left\|F^0\right\| + R. \end{aligned} \tag{1.58}$$

Since obviously $\rho_{F^0}(\bar{x} - x_0') < \delta$, we obtain from (1.58) and (1.53) that $\rho_{F^0}(x - x_0) < \mathfrak{r}^+(x_0)$. Hence, by the definition of $\bar{v}(\cdot)$

$$v(x) \geq \mu \quad \forall x \in x - \beta F^0. \tag{1.59}$$

The inequalities (1.59) and (1.57) allow us to apply Theorem 1.2.1 with the solution $\bar{v}(\cdot)$ in the place of $\bar{u}(\cdot)$ and to find $\eta > 0$ such that

$$\bar{v}(x) \geq \mu + \varphi(\eta)(\beta - \rho_{F^0}(\bar{x} - x)) \quad \forall x \in \bar{x} - \beta F^0.$$

In particular, for $x = x_0' \in \bar{x} - \beta F^0$ we have

$$\bar{v}(x_0') \geq \mu + \varphi(\eta)\varepsilon > \mu. \tag{1.60}$$

However, by the definition of R (see (1.50))

$$u(x_0') = \bar{u}(x_0) + a\rho_{F^0}(x_0' - x_0) = \mu.$$

Thus $\bar{v}(x_0') = \mu$ as well, contradicting the strict inequality (1.60). □

In the case when the Minkowski functional $\rho_F(\cdot)$ is differentiable, the latter extremality result can be extended to arbitrary (densely) star-shaped domains in the place of ρ_{F^0}-balls $x_0 \pm rF^0$.

Theorem 1.4.2. *Assume that the lagrangean $L(\cdot)$ is such as in Theorem 1.4.1 with a convex compact and smooth set $F \subset \mathbb{R}^n$, $0 \in \mathrm{int}F$; that $\Omega \subset \mathbb{R}^n$ is an open bounded region and $x_0 \in \Omega$. Then for each continuous solution $\bar{u}(\cdot)$ of (P_F) and each open $\hat{\Omega} \subset \Omega$, which is densely star-shaped w.r.t. x_0, the inequality (1.46) (respectively, (1.48)) holds for all $x \in \hat{\Omega}$ if and only if the equality (1.47) (respectively, (1.49)) is valid on $\hat{\Omega}$.*

Proof. As earlier we prove here only the implication (1.46)$\Rightarrow$(1.47), while the other one ((1.48)$\Rightarrow$(1.49)) can be treated similarly. Moreover, it is enough to prove the equality (1.47) on the star $\mathrm{St}_{\hat{\Omega}}(x_0)$, because to each $x \in \hat{\Omega} \setminus \mathrm{St}_{\hat{\Omega}}(x_0) \subset \overline{\mathrm{St}_{\hat{\Omega}}(x_0)}$ it can be extended by continuity of the involved functions.

Let us assume validity of (1.46) on $\hat{\Omega}$ and fix $\bar{x} \in \mathrm{St}_{\hat{\Omega}}(x_0)$. By the compactness we choose $\varepsilon > 0$ such that

$$[x_0, \bar{x}] \pm \varepsilon F^0 \subset \hat{\Omega}. \tag{1.61}$$

Set

$$\delta := 2\varepsilon \mathfrak{M}_{F^0}\left(\frac{2\varepsilon}{\Delta}; \frac{\varepsilon}{\varepsilon+\Delta}, \frac{\Delta}{\varepsilon+\Delta}\right), \tag{1.62}$$

where $\mathfrak{M}_{F^0}$ is the *rotundity modulus* defined by (1.5) and Δ is the ρ_{F^0}-*diameter* of the domain $\hat{\Omega}$, i.e.,

$$\Delta := \sup_{\xi,\eta \in \hat{\Omega}} \rho_{F^0}(\xi - \eta) > 0. \tag{1.63}$$

It follows from the remarks of Section 1.1 that F^0 is rotund, and therefore $\delta > 0$. Let us consider an uniform partition of the segment $[x_0, \bar{x}]$ by the points

$$x_i := x_0 + ih\frac{x - x_0}{\rho_{F^0}(\bar{x} - x_0)},$$

$i = 0, 1, ..., m$, with

$$h := \frac{\rho_{F^0}(\bar{x} - x_0)}{m} \le \min\left\{\frac{\varepsilon}{\sqrt{M}}, \frac{\delta}{M}\right\}, \tag{1.64}$$

where $M := (\|F\|\,\|F^0\| + 1)^2$. Since the inequality (1.46) holds for all $x \in x_0 + \varepsilon F^0$ (see (1.61)) and $\rho_{F^0}(x_1 - x_0) = h \le \frac{\varepsilon}{\|F\|\|F^0\|+1}$, it follows from Corollary 1.2.2 that

$$u(x_1) = \bar{u}(x_0) + a\rho_{F^0}(x_1 - x_0) = \bar{u}(x_0) + ah.$$

We want to prove by induction in i that

$$\bar{u}(x_i) = \bar{u}(x_0) + iah, \tag{1.65}$$

$i = 1, 2, ..., m$. Then for $i = m$ we will have

$$\bar{u}(\bar{x}) = \bar{u}(x_m) = \bar{u}(x_0) + a\rho_{F^0}(\bar{x} - x_0),$$

and theorem will be proved due to arbitrarity of $\bar{x} \in \mathrm{St}_{\hat{\Omega}}(x_0)$. So, we assume that (1.65) is true for some i with $1 \leq i \leq m-1$ and define the function $\bar{u}_i : \overline{\Omega} \to \mathbb{R}$ as

$$\max\{\bar{u}(x), \min\{\bar{u}(x_i) + a\rho_{F^0}(x - x_i), \mu_i - a\rho_{F^0}(x_i - x)\}\} \tag{1.66}$$

on $\hat{\Omega}$ and as $\bar{u}(x)$ elsewhere. Here

$$\mu_i := \bar{u}(x_0) + a(i + M)h. \tag{1.67}$$

Let us devide the remainder of the proof in three steps.

Step 1. We claim first that for each $x \notin [x_0, \bar{x}] \pm \varepsilon F^0$ the inequality

$$\rho_{F^0}(x - x_0) + \rho_{F^0}(x_i - x) - \rho_{F^0}(x_i - x_0) \geq \delta \tag{1.68}$$

holds. Indeed, given such a point x we have

$$\rho_0 := \rho_{F^0}(x - x_0) \geq \varepsilon \quad \text{and} \quad \rho_i := \rho_{F^0}(x_i - x) \geq \varepsilon.$$

On the other hand, clearly $\rho_0 \leq \Delta$ and $\rho_i \leq \Delta$ (see (1.63)). Hence, by the monotonicity

$$\lambda := \frac{\rho_0}{\rho_0 + \rho_i} \in \left[\frac{\varepsilon}{\Delta + \varepsilon}, \frac{\Delta}{\Delta + \varepsilon}\right]. \tag{1.69}$$

Furthermore,

$$\begin{aligned} & \rho_{F^0}(x - x_0) + \rho_{F^0}(x_i - x) - \rho_{F^0}(x_i - x_0) = \\ = \;& (\rho_0 + \rho_i)\left[1 - \rho_{F^0}\left(\frac{\rho_i}{\rho_0 + \rho_i}\frac{x_i - x}{\rho_i} + \frac{\rho_0}{\rho_0 + \rho_i}\frac{x - x_0}{\rho_0}\right)\right] \geq \\ \geq \;& 2\varepsilon\left[1 - \rho_{F^0}(\xi + \lambda(\eta - \xi))\right], \end{aligned} \tag{1.70}$$

where $\xi := \frac{x_i - x}{\rho_i}$ and $\eta := \frac{x - x_0}{\rho_0}$. Since

$$\xi - \eta = \left(\frac{1}{\rho_0} + \frac{1}{\rho_i}\right)\left(\frac{\rho_0}{\rho_0 + \rho_i}x_i + \frac{\rho_i}{\rho_0 + \rho_i}x_0 - x\right),$$

by the choice of x we obtain

$$\rho_{F^0}(\xi - \eta) \geq \left(\frac{1}{\rho_0} + \frac{1}{\rho_i}\right)\varepsilon \geq \frac{2\varepsilon}{\Delta}. \tag{1.71}$$

Joining together (1.69)-(1.71), (1.62) and the definition of the rotundity modulus (1.5) we immediately arrive at (1.68).

Step 2. Let us show that $\bar{u}_i(\cdot)$ is the continuous minimizer of the functional in (P_F) on $\bar{u}(\cdot) + W_0^{1,1}(\Omega)$. Given $x \in \hat{\Omega}$ with $x \notin [x_0, \bar{x}] \pm \varepsilon F^0$ it folows from (1.68) and (1.64) that

$$\rho_{F^0}(x - x_0) + \rho_{F^0}(x_i - x) \geq \delta + ih \geq (i + M)\,h. \tag{1.72}$$

Since $\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0)$ by the condition, we deduce from (1.72) and (1.67) that $\bar{u}(x) \geq \mu_i - a\rho_{F^0}(x_i - x)$. Consequently, $\bar{u}_i(x) = \bar{u}(x)$ (see (1.66)). Due to (1.61) this means continuity of the function $\bar{u}_i(\cdot)$. Besides that obviously $\bar{u}_i(\cdot) \in \bar{u}(\cdot) + W_0^{1,1}(\Omega)$. We see from (1.66) that for each $x \in \Omega$ with $\bar{u}_i(x) \neq \bar{u}(x)$ the gradient $\nabla\bar{u}_i(x)$ belongs to aF, and therefore $f(\rho_F(\nabla u_i(x))) = 0$. By the same argument as in the proof of Theorem 1.4.1 (see (1.56)) $\bar{u}_i(\cdot)$ gives the minimum to (P_F) among all the functions with the same boundary data.

Step 3. Here we prove that for each x with

$$\rho_{F^0}(x - x_i) \leq \left(\|F\|\left\|F^0\right\| + 1\right)h \tag{1.73}$$

(such x belong to $\hat{\Omega}$ by (1.61) and (1.64)) the inequality

$$\bar{u}_i(x) \geq \bar{u}_i(x_i) + a\rho_{F^0}(x - x_i) \tag{1.74}$$

holds. Indeed, it follows from (1.73) and (1.3) that

$$\rho_{F^0}(x - x_i) + \rho_{F^0}(x_i - x) \leq Mh, \tag{1.75}$$

and taking into account the definition of μ_i (see (1.67)) and the induction hypothesis (1.65) we obtain from (1.75) that

$$\bar{u}(x_i) + a\rho_{F^0}(x - x_i) \leq \mu_i - a\rho_{F^0}(x_i - x),$$

i.e., the minimum in (1.66) is equal to $\bar{u}(x_i) + a\rho_{F^0}(x - x_i)$. This implies (1.74) because

$$\bar{u}_i(x_i) = \max\{\bar{u}(x_i), \min\{\bar{u}(x_i), \mu_i\}\} = \bar{u}(x_i). \tag{1.76}$$

Applying now Corollary 1.2.2 to the solution $\bar{u}_i(\cdot)$ (see Step 2) and to the point $x_i \in \Omega$ we conclude that the inequality (1.74) becomes the equality for each $x \in x_i + hF^0$, in particular, for $x = x_{i+1}$. Thus, by (1.76) and (1.65) we have

$$\bar{u}_i(x_{i+1}) = \bar{u}(x_0) + (i+1)\,ah.$$

On the other hand, by the definition of $\bar{u}_i(\cdot)$ (see (1.66))

$$\bar{u}(x_{i+1}) \leq \bar{u}_i(x_{i+1}) = \bar{u}(x_0) + a\rho_{F^0}(x_{i+1} - x_0).$$

Since the opposite inequality holds by the condition, we finally obtain

$$\begin{aligned} \bar{u}(x_{i+1}) &= \bar{u}(x_0) + a\rho_{F^0}(x_{i+1} - x_0) = \\ &= \bar{u}(x_0) + (i+1)\,ah, \end{aligned}$$

and the induction is complete. □

The following one-point version of SMP is the immediate consequence of the latter result.

Corollary 1.4.1 (one-point Strong Maximum Principle). *Let the lagrangean $L = f \circ \rho_F$ be such as in Theorem 1.4.2 (the set F is supposed to be smooth), and $\Omega \subset \mathbb{R}^n$ be an open bounded and densely star-shaped w.r.t. $x_0 \in \Omega$. Then given a continuous solution $\bar{u}(\cdot)$ of (P_F) the inequality (1.46) (respectively, (1.48)) holds for all $x \in \Omega$ if and only if the respective equality (1.47) (or (1.49)) takes place.*

Notice that the (dense) star-shapedness of the region Ω in the above statement can not be dropped as the following counter-example shows.

Example 3. *Let $\Sigma \subset \mathbb{R}^n$ be an arbitrary open bounded set, which is star-shaped w.r.t. $x_0 \in \Sigma$. Fix $\bar{x} \in \Sigma$, $\bar{x} \neq x_0$, and a real number $0 < r < \min\{\|x_0 - \bar{x}\|, \mathrm{d}_{\partial\Sigma}(\bar{x})\}$, where $\mathrm{d}_{\partial\Sigma}(\cdot)$ means the distance from a point to the boundary of Σ. Let us consider the other domain $\Omega := \Sigma \setminus (\bar{x} + r\overline{B})$, where $\overline{B}$ is the closed unit ball centred in zero. Assuming the function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ and the gauge set $F \subset \mathbb{R}^n$ to be such as in Theorem 1.4.2 we construct a continuous solution $\bar{u}(\cdot)$ of (P_F) with the property*

$$\bar{u}(x) \geq \bar{u}(x_0) + a\rho_{F^0}(x - x_0) \quad \forall x \in \Omega, \tag{1.77}$$

where the inequality is strict on some nonempty (open) set $\Omega' \subset \Omega$.

In what follows by $\mathrm{St}(x_0)$ we intend the star associated with the point x_0 in the domain Ω. It is obvious that $x \in \Omega \diagdown \mathrm{St}(x_0)$ if and only if the segment $[x_0, x]$ meets the ball $\bar{x} + r\overline{B}$, or, in other words, the quadratic equation

$$\begin{aligned} \|x_0 + \lambda (x - x_0) - \bar{x}\|^2 = \lambda^2 \|x - x_0\|^2 - \\ -2\lambda \langle x - x_0, x - x_0 \rangle + \|x - x_0\|^2 = r^2 \end{aligned} \tag{1.78}$$

has (one or two) roots both belonging to the interval $(0,1)$. We write the condition of resolvability of (1.78) as

$$\mathfrak{D}(x) := \langle x - x_0, x - x_0 \rangle^2 - \|x - x_0\|^2 \left(\|x - x_0\|^2 - r^2 \right) \geq 0 \tag{1.79}$$

and denote by

$$\lambda_{\pm}(x) := \frac{\langle x - x_0, \bar{x} - x_0 \rangle \pm \sqrt{\mathfrak{D}(x)}}{\|x - x_0\|^2} \tag{1.80}$$

its roots. Taking into account continuity of all the involved functions we have

$$\begin{aligned} \Omega' := \mathrm{int}\,(\Omega \diagdown \mathrm{St}(x_0)) = \{x \in \Sigma : \mathfrak{D}(x) > 0 \text{ and} \\ 0 < \lambda_-(x) < \lambda_+(x) < 1\} \end{aligned} \tag{1.81}$$

and

$$\begin{aligned} E := \partial (\Omega \diagdown \mathrm{St}(x_0)) \cap \partial (\mathrm{St}(x_0)) = \\ = \left\{ x \in \Sigma : \mathfrak{D}(x) = 0 \text{ and } 0 < \lambda_{\pm}(x) = \tfrac{\langle x - x_0, \bar{x} - x_0 \rangle}{\|x - x_0\|^2} < 1 \right\}. \end{aligned} \tag{1.82}$$

The open set Ω' is clearly nonempty since it contains, e.g., all the points $\bar{x} + (r + \delta) \frac{\bar{x}\ \ x_0}{\|\bar{x} - x_0\|}$ with $\delta > 0$ small enough. For each $x \in E$ let us define the "trace" operator

$$\Phi(x) := x_0 + \lambda_{\pm}(x)(x - x_0), \tag{1.83}$$

which is continuous and satisfies the following "cone" property: if $x \in E$ and $x' = x_0 + \lambda (x - x_0) \in \Omega$ with $\lambda > \lambda_{\pm}(x)$ then also $x' \in E$ and $\Phi(x') = \Phi(x)$. Indeed, we find from (1.79) and (1.80) that $\mathfrak{D}(x') = \lambda^2 \mathfrak{D}(x) = 0$ and $\lambda_{\pm}(x') = \frac{1}{\lambda} \lambda_{\pm}(x) \in (0,1)$. Hence $x' \in E$ and the equality $\Phi(x') = \Phi(x)$ follows directly from (1.83). This property and continuity of $\Phi(\cdot)$ imply that the image $C := \Phi(E)$ is a compact subset of $\{x \in \Omega : \|x - \bar{x}\| = r\}$.

Define the function $\bar{u} : \overline{\Omega} \to \mathbb{R}$ by

$$\bar{u}(x) := \begin{cases} a \inf\limits_{y \in C} \{\rho_{F^0}(x - y) + \rho_{F^0}(y - x_0)\} & \text{if } \ x \in \overline{\Omega} \diagdown \mathrm{St}(x_0), \\ a \rho_{F^0}(x - x_0) & \text{if } \ x \in \mathrm{St}(x_0), \end{cases} \tag{1.84}$$

and show, first, its continuity. To this end it is enough to verify the equality

$$\inf_{y\in C}\left\{\rho_{F^0}(x-y)+\rho_{F^0}(y-x_0)\right\}=\rho_{F^0}(x-x_0) \tag{1.85}$$

for each point $x\in E$. Notice that the inequality "$\geq$" in (1.85) is obvious. In order to prove "$\leq$" let us take an arbitrary $x\in E$ and put $y:=\Phi(x)=x_0+\lambda_{\pm}(x)(x-x_0)$. Then $\rho_{F^0}(y-x_0)=\lambda_{\pm}(x)\rho_{F^0}(x-x_0)$ and $\rho_{F^0}(x-y)=(1-\lambda_{\pm}(x))\rho_{F^0}(x-x_0)$. So that the inequality "$\leq$" in (1.85) immediately follows. The function $\bar{u}(\cdot)$ is, moreover, lipschitzean on $\overline{\Omega}$, and its Clarke subdifferential $\partial^c u(x)$ is always contained in aF (see [20, p.92]).

Since by Rademaher's Theorem the gradient $\nabla\bar{u}(x)$ a.e. exists and $\nabla\bar{u}(x)\in\partial^c\bar{u}(x)$, we have $f(\rho_F(\nabla\bar{u}(x)))=0$ for a.e. $x\in\Omega$, and, consequently, $\bar{u}(\cdot)$ is a solution of (P_F).

The inequality (1.77) is obvious (here $\bar{u}(x_0)=0$). Fix now $x\in\Omega'$ and $y\in C=\Phi(E)$. Then $y=x_0+\lambda_{\pm}(z)(z-x_0)$ with some $z\in E$. Assuming that there exists $\lambda>0$ with $x-y=\lambda(y-x_0)$ and taking into account the representation of y, we have $x=x_0+\mu(z-x_0)$ with $\mu:=(\lambda+1)\lambda_{\pm}(z)>\lambda_{\pm}(z)$. Due to the "cone" property of the "trace" operator (see above) $x\in E$ as well, which is a contradiction. Thus, by the rotundity of the set F^0 the strict inequality

$$\rho_{F^0}(x-y)+\rho_{F^0}(y-x_0)>\rho_{F^0}(x-x_0)$$

holds. Finally, by the compactness of the set C and by arbitrarity of $y\in C$ we conclude that the inequality (1.77) is strict whenever $x\in\Omega'$, and the construction is complete.

The method used in Example 3 can be essentially sharpened in order to cover the case of an arbitrary domain Ω, in which an open subset is not linearily attainable from $x_0\in\Omega$. So, let us formulate the following conjecture.

Conjecture *The condition* $\Omega\subset\overline{\mathrm{St}(x_0)}$ *is necessary for validity of the one-point Strong Maximum Principle w.r.t.* $x_0\in\Omega$ as given by Corollary 1.4.1.

1.5 A multi-point version of SMP

The one-point version of SMP given in the previous section (see Corollary 1.4.1) can be easily extended to the case when the *test function* $\hat{u}(\cdot)$ has a finite number of local minimum (maximum) points. Namely, let $x_i \in \mathbb{R}^n$, $i = 1, ..., m$, be different, $m \in \mathbb{N}$, and arbitrary real numbers $\theta_1, \theta_2, ..., \theta_m$ be such that the *compatibility condition*

$$\theta_i - \theta_j < a\rho_{F^0}(x_i - x_j), \quad i \neq j, \tag{1.86}$$

is fulfilled. Considering the functions

$$\hat{u}^+(x) := \min_{1\leq i\leq m} \{\theta_i + a\rho_{F^0}(x - x_i)\} \tag{1.87}$$

and

$$\hat{u}^-(x) := \max_{1\leq i\leq m} \{\theta_i - a\rho_{F^0}(x_i - x)\}, \tag{1.88}$$

we deduce immediately from (1.86) that $\hat{u}^+(x_i) = \hat{u}^-(x_i) = \theta_i$, $i = 1, ..., m$, and that $\{x_1, x_2, ..., x_m\}$ is the set of all local minimum (maximum) points of the function (1.87) or (1.88), respectively.

Before proving the main statement (Theorem 1.5.1) let us give some property of minimizers in (1.87) (respectively, maximizers in (1.88)), which extends a well-known property of metric projections. Since it will be used in the next sections as well, we formulate the following assertion in the form convenient for both situations.

Lemma 1.5.1. *Let $\Gamma \subset \mathbb{R}^n$ be a nonempty closed set and $\theta(\cdot)$ be a real-valued function defined on Γ. Given $x \in \mathbb{R}^n \setminus \Gamma$ assume that the minimum of the function $y \mapsto \theta(y) + a\rho_{F^0}(x - y)$ (respectively, the maximum of $y \mapsto \theta(y) - a\rho_{F^0}(y - x)$) on Γ is attained at some point $\bar{y} \in \Gamma$. Then $\bar{y}$ continues to be a minimizer of $y \mapsto \theta(y) + a\rho_{F^0}(x_\lambda - y)$ (respectively, a maximizer of $y \mapsto \theta(y) - a\rho_{F^0}(y - x_\lambda)$) for all $\lambda \in [0,1]$, where $x_\lambda := (1 - \lambda)x + \lambda\bar{y}$.*

Proof. Given $\lambda \in [0,1]$ and $y \in \Gamma$ we obviously have $x - y = x_\lambda - y + \lambda(x - \bar{y})$, and by the semilinearity

$$\rho_{F^0}(x - y) \leq \rho_{F^0}(x_\lambda - y) + \lambda\rho_{F^0}(x - \bar{y}). \tag{1.89}$$

Since $\theta(\bar{y}) + a\rho_{F^0}(x - \bar{y}) \leq \theta(y) + a\rho_{F^0}(x - y)$, it follows from (1.89) that

$$\theta(\bar{y}) + a(1 - \lambda)\rho_{F^0}(x - \bar{y}) \leq \theta(y) + a\rho_{F^0}(x_\lambda - y).$$

This proves the first assertion because $(1 - \lambda)(x - \bar{y}) = x_\lambda - \bar{y}$. The second case can be easily reduced to the first one by changing the signs. □

Theorem 1.5.1 (multi-point Strong Maximum Principle). *Assume that the lagrangean $L = f \circ \rho_F$ satisfies our main hypotheses with the smooth gauge set F, and $\Omega \subset \mathbb{R}^n$ is an open bounded convex region containing the points $x_1, ..., x_m$. Then each continuous admissible minimizer $\bar{u}(\cdot)$ in the variational problem (P_F) such that $\bar{u}(x_i) = \theta_i$, $i = 1, ..., m$, and*

$$\bar{u}(x) \geq \hat{u}^+(x) \quad \forall x \in \Omega \tag{1.90}$$

(respectively, $u(x) \leq \hat{u}^-(x) \quad \forall x \in \Omega$) coincides with $\hat{u}^+(x)$ (respectively, with $\hat{u}^-(x)$). Here the functions $\hat{u}^\pm(\cdot)$ are defined by (1.87) and (1.88).

Proof. As usual we prove only the first part of the statement. Denoting by

$$C_i := \left\{x \in \Omega : \hat{u}^+(x) = \theta_i + a\rho_{F^0}(x - x_i)\right\},$$

and by $\Omega_i := \text{int}C_i$, $i = 1, ..., m$, we obviously have $x_i \in \Omega_i$ and deduce from Lemma 1.5.1 that the open set Ω_i is star-shaped w.r.t. x_i. Since on Ω_i the inequality (1.90) admits the form

$$\bar{u}(x) \geq \bar{u}(x_i) + a\rho_{F^0}(x - x_i),$$

applying Theorem 1.4.2 we conclude that

$$u(x) = \bar{u}(x_i) + a\rho_{F^0}(x - x_i) = \hat{u}^+(x) \quad \forall x \in \Omega_i.$$

It is enough now to observe that the union of (disjoint) sets Ω_i, $i = 1, ..., m$, is dense in Ω. Therefore, the equality $\bar{u}(x) = \hat{u}^+(x)$ holds for all $x \in \Omega$ due to the continuity of both functions. □

1.6 Definition and interpretation of the inf- and sup-convolutions

Now we essentially enlarge the class of test functions involving the infinite (continuous) envelopes of the functions $\theta + a\rho_{F^0}(x - x_0)$ (respectively, $\theta - a\rho_{F^0}(x_0 - x)$) by such a way that the generalized SMP (see Section 1.7) gets an *unique extremal extension principle* and unifies both the traditional

Strong Maximum Principle proposed by A. Cellina and the extended Strong Maximum Principle, which we studied in Sections 1.4 and 1.5.

Namely, given an arbitrary real function $\theta(\cdot)$ defined on a closed subset $\Gamma \subset \Omega$ and satisfying a natural *slope condition* w.r.t. F:

$$\theta(x) - \theta(y) \leq a\rho_{F^0}(x - y) \qquad \forall x, y \in \Gamma,. \tag{1.91}$$

we prove that the *inf-convolution*

$$u^+_{\Gamma,\theta}(x) := \inf_{y \in \Gamma} \{\theta(y) + a\rho_{F^0}(x - y)\} \tag{1.92}$$

(respectively, the *sup-convolution*

$$u^-_{\Gamma,\theta}(x) := \sup_{y \in \Gamma} \{\theta(y) - a\rho_{F^0}(y - x)\}\,) \tag{1.93}$$

is the only continuous minimizer $u(\cdot)$ in the problem (P_F) such that $u(x) = \theta(x)$ on Γ and $u(x) \geq u^+_{\Gamma,\theta}(x)$ (respectively, $u(x) \leq u^-_{\Gamma,\theta}(x)$), $x \in \Omega$.

As standing hypotheses in what follows we assume that F is convex closed bounded and smooth with $0 \in \text{int}F$, and that Ω is an open convex bounded region in $\mathbb{R}^n$.

Observe that the function $u^\pm_{\Gamma,\theta}(\cdot)$ is the minimizer in (P_F) with $u^0(x) = u^\pm_{\Gamma,\theta}(x)$. Indeed, it is obviously Lipschitz continuous on Ω, and for its (classic) gradient $\nabla u^\pm_{\Gamma,\theta}$ existing almost everywhere by Rademacher's theorem we have

$$\nabla u^\pm_{\Gamma,\theta}(x) \in \partial^c u^\pm_{\Gamma,\theta}(x) \subset aF$$

for a.e. $x \in \Omega$ (see [20, Theorem 2.8.6]). Here ∂^c stands for the *Clarke's subdifferential*. Consequently,

$$f\left(\rho_F\left(\nabla u^\pm_{\Gamma,\theta}(x)\right)\right) = 0$$

a.e. on Ω, and the function $u^\pm_{\Gamma,\theta}(\cdot)$ gives to (P_F) the minimal possible value zero.

Due to the slope condition (1.91) it follows that $u^\pm_{\Gamma,\theta}(x) = \theta(x)$ for all $x \in \Gamma$. Moreover, $u^\pm_{\Gamma,\theta}(\cdot)$ is the (unique) *viscosity solution* of the *Hamilton-Jacobi equation*

$$\pm(\rho_F(\nabla u(x)) - a) = 0, \qquad u|_\Gamma = \theta,$$

(see, e.g., [11]).

Notice that Γ can be a finite set, say $\{x_1, x_2, ..., x_m\}$. In this case $\theta(\cdot)$ associates to each x_i a real number θ_i, $i = 1, ..., m$, and the condition (1.91) slightly strengthened (by assuming that the inequality in (1.91) is strict for $x_i \neq x_j$) means that all the simplest test functions $\theta_i + a\rho_{F^0}(x - x_i)$ (respectively, $\theta_i - a\rho_{F^0}(x_i - x)$) are essential (not superfluous) in constructing the respective lower or upper envelope. Then the extremal property established in Section 1.7 is reduced to the extended SMP of Section 1.5 (see Theorem 1.5.1).

On the other hand, if $\theta(\cdot)$ is a Lipschitz continuous function defined on a closed convex set $\Gamma \subset \Omega$ with nonempty interior then (1.91) holds iff

$$\nabla\theta(x) \in aF$$

for a.e. $x \in \Gamma$. This immediately follows from Lebourg's mean value theorem (see [20, p. 41]) recalling the properties of the Clarke's subdifferential and from the separability theorem.

Certainly, the mixed (discrete and continuous) case can be considered as well, and all the situations are unified by the hypothesis (1.91).

In the particular case $\theta \equiv 0$ ((1.91) is trivially fulfilled) the function $u^+_{\Gamma,\theta}(x)$ is nothing else than the *minimal time* necessary to achieve the closed set Γ from the point $x \in \Omega$ by trajectories of the differential inclusion with the constant convex right-hand side

$$-a\dot{x}(t) \in F^0, \tag{1.94}$$

while $-u^-_{\Gamma,\theta}(x)$ is, contrarily, the *minimal time*, for which trajectories of (1.94) arrive at x starting from a point of Γ. Furthermore, if $F = \overline{B}$ then the gauge function $\rho_{F^0}(\cdot)$ is the euclidean norm in $\mathbb{R}^n$, and we have

$$u^\pm_{\Gamma,\theta}(x) = \pm a\mathrm{d}_\Gamma(x)$$

where $\mathrm{d}_\Gamma(\cdot)$ means the *distance* from a point to the set Γ.

1.7 Generalized Strong Maximum Principle

Now we are ready to deduce the extremal property of the functions $u^\pm_{\Gamma,\theta}(\cdot)$ announced above.

Theorem 1.7.1. *Under all the standing hypotheses let us assume that a continuous admissible minimizer in* (P_F) *is such that*

(i) $\bar{u}(x) = u^{+}_{\Gamma,\theta}(x) = \theta(x) \qquad \forall x \in \Gamma;$

(ii) $\bar{u}(x) \geq u^{+}_{\Gamma,\theta}(x) \qquad \forall x \in \Omega.$

Then $\bar{u}(x) \equiv u^{+}_{\Gamma,\theta}(x)$ *on* Ω.

Symmetrically, if a continuous admissible minimizer $\bar{u}(\cdot)$ *satisfies the conditions*

(i)$'$ $\bar{u}(x) = u^{-}_{\Gamma,\theta}(x) = \theta(x) \qquad \forall x \in \Gamma;$

(ii)$'$ $\bar{u}(x) \leq u^{-}_{\Gamma,\theta}(x) \qquad \forall x \in \Omega,$

then $\bar{u}(x) \equiv u^{-}_{\Gamma,\theta}(x)$ *on* Ω.

Proof. Let us prove the first part of Theorem only since the respective changements in the symmetric case are obvious.

Given a continuous admissible minimizer $\bar{u}(\cdot)$ satisfying conditions (**i**) and (**ii**) we suppose, on the contrary, that there exists $\bar{x} \in \Omega \setminus \Gamma$ with $\bar{u}(\bar{x}) > u^{+}_{\Gamma,\theta}(\bar{x})$.

Let us denote by

$$\Gamma^{+} := \left\{ x \in \Omega : \bar{u}(x) = u^{+}_{\Gamma,\theta}(x) \right\}$$

and claim that

$$u^{+}_{\Gamma,\theta}(x) = \inf_{y \in \Gamma^{+}} \left\{ u(y) + a\rho_{F^0}(x-y) \right\} \tag{1.95}$$

for each $x \in \Omega$. Indeed, the inequality "$\geq$" in (1.95) is obvious because $\Gamma^{+} \supset \Gamma$ and $\bar{u}(y) = \theta(y)$, $y \in \Gamma$. On the other hand, given $x \in \Omega$ take an arbitrary $y \in \Gamma^{+}$ and due to the compactness of Γ we find $y^{*} \in \Gamma$ such that

$$u(y) = \theta(y^{*}) + a\rho_{F^0}(y - y^{*}). \tag{1.96}$$

Then, by triangle inequality,

$$\begin{aligned} \bar{u}(y) + a\rho_{F^0}(x-y) &\geq \theta(y^{*}) + a\rho_{F^0}(x-y^{*}) \geq \\ &\geq u^{+}_{\Gamma,\theta}(x). \end{aligned} \tag{1.97}$$

Passing to infimum in (1.97) we prove the inequality "$\leq$" in (1.95) as well.

Since for arbitrary $x, y \in \Gamma^+$ and for $y^* \in \Gamma$ satisfying (1.96) we have

$$\begin{aligned} \bar{u}(x) - \bar{u}(y) &= u^+_{\Gamma,\theta}(x) - u^+_{\Gamma,\theta}(y) \leq \\ &\leq a\rho_{F^0}(x - y^*) - a\rho_{F^0}(y - y^*) \leq a\rho_{F^0}(x - y), \end{aligned} \tag{1.98}$$

we can extend the function $\theta : \Gamma \to \mathbb{R}$ to the (closed) set $\Gamma^+ \subset \Omega$ by setting

$$\theta(x) = \bar{u}(x), \; x \in \Gamma^+,$$

and all the conditions on $\theta(\cdot)$ remain valid (see (1.98) and (1.95)). So, without loss of generality we can assume that the strict inequality

$$\bar{u}(x) > u^+_{\Gamma,\theta}(x) \tag{1.99}$$

holds for all $x \in \Omega \setminus \Gamma \neq \emptyset$.

Notice that the *convex hull* $K := \operatorname{co}\Gamma$ is the compact set contained in Ω (due to the convexity of Ω). Let us choose now $\varepsilon > 0$ such that $K \pm \varepsilon F^0 \subset \Omega$ and denote by

$$\delta := 2\varepsilon\, \mathfrak{M}_{F^0}\left(\frac{2\varepsilon}{\Delta}; \frac{\varepsilon}{\varepsilon + \Delta}, \frac{\Delta}{\varepsilon + \Delta}\right) > 0,$$

where as usual $\mathfrak{M}_{F^0}$ is the modulus of rotundity associated to F^0 (see 1.5) and

$$\Delta := \sup_{\xi,\eta \in \Omega} \rho_{F^0}(\xi - \eta).$$

Similarly as in Step 1 in the proof of Theorem 1.4.2 we show that

$$\rho_{F^0}(y_1 - x) + \rho_{F^0}(x - y_2) - \rho_{F^0}(y_1 - y_2) \geq \delta \tag{1.100}$$

whenever $y_1, y_2 \in \Gamma$ and $x \in \Omega \setminus [(K + \varepsilon F^0) \cup (K - \varepsilon F^0)]$. Indeed, we obviously have

$$\varepsilon \leq \rho_1 := \rho_{F^0}(y_1 - x) \leq \Delta \quad \text{and} \quad \varepsilon \leq \rho_2 := \rho_{F^0}(x - y_2) \leq \Delta,$$

and, consequently,

$$\lambda := \frac{\rho_2}{\rho_1 + \rho_2} \in \left[\frac{\varepsilon}{\varepsilon + \Delta}, \frac{\Delta}{\varepsilon + \Delta}\right]. \tag{1.101}$$

Setting $\xi_1 := \frac{y_1 - x}{\rho_1}$ and $\xi_2 := \frac{x - y_2}{\rho_2}$ we can write

$$\xi_1 - \xi_2 = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left(\frac{\rho_2}{\rho_1 + \rho_2} y_1 + \frac{\rho_1}{\rho_1 + \rho_2} y_2 - x\right),$$

and hence

$$\rho_{F^0}(\xi_1 - \xi_2) \geq \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\varepsilon \geq \frac{2\varepsilon}{\Delta}. \tag{1.102}$$

On the other hand,

$$\begin{aligned} & \rho_{F^0}(y_1 - x) + \rho_{F^0}(x - y_2) - \rho_{F^0}(y_1 - y_2) \\ = \ & (\rho_1 + \rho_2)\left[1 - \rho_{F^0}\left(\frac{\rho_1}{\rho_1 + \rho_2}\xi_1 + \frac{\rho_2}{\rho_1 + \rho_2}\xi_2\right)\right] \geq \\ \geq \ & 2\varepsilon\left[1 - \rho_{F^0}(\xi_1 + \lambda(\xi_2 - \xi_1))\right]. \end{aligned} \tag{1.103}$$

Combining (1.103), (1.102), (1.101) and the definition of the rotundity modulus we arrive at (1.100).

Let us fix $\bar{x} \in \Omega \setminus \Gamma$ and $\bar{y} \in \Gamma$ such that

$$u^+_{\Gamma,\theta}(\bar{x}) = \theta(\bar{y}) + a\rho_{F^0}(\bar{x} - y).$$

Then by Lema 1.5.1 the point $\bar{y}$ is also a minimizer on Γ of the function $y \mapsto \theta(y) + a\rho_{F^0}(x_\lambda - y)$, where $x_\lambda := \lambda\bar{x} + (1 - \lambda)\bar{y}$, $\lambda \in [0, 1]$, i.e.,

$$u^+_{\Gamma,\theta}(x_\lambda) = \theta(\bar{y}) + a\rho_{F^0}(x_\lambda - \bar{y}). \tag{1.104}$$

Define now the Lipschitz continuous function

$$\begin{aligned} & \bar{v}(x) := \max\{\bar{u}(x), \min\{\theta(\bar{y}) + a\rho_{F^0}(x - \bar{y}), \\ & \theta(\bar{y}) + a(\delta - \rho_{F^0}(\bar{y} - x))\}\} \end{aligned} \tag{1.105}$$

and claim that $\bar{v}(\cdot)$ minimizes the functional in (P_F) on the set $\bar{u}(\cdot) + W_0^{1,1}(\Omega)$. In order to prove this we observe first that for each $x \in \Omega$, $x \notin K \pm \varepsilon F^0$, and for each $y \in \Gamma$ by the slope condition (1.91) and by (1.100) the inequality

$$\begin{aligned} & \theta(y) + a\rho_{F^0}(x - y) - \theta(\bar{y}) + a\rho_{F^0}(\bar{y} - x) \geq \\ \geq \ & a(\rho_{F^0}(\bar{y} - x) + \rho_{F^0}(x - y) - \rho_{F^0}(\bar{y} - y)) \geq a\delta \end{aligned} \tag{1.106}$$

holds. Passing to infimum in (1.106) for $y \in \Gamma$ and taking into account the basic assumption **(ii)**, we have

$$\begin{aligned} \bar{u}(x) &\geq \inf_{y\in\Gamma} \{\theta(y) + a\rho_{F^0}(x-y)\} \\ &\geq \theta(\bar{y}) + a(\delta - \rho_{F^0}(\bar{y}-x)), \end{aligned}$$

and, consequently, $\bar{v}(x) = \bar{u}(x)$ for all $x \in \Omega \setminus [(K+\varepsilon F^0) \cup (K - \varepsilon F^0)]$. In particular, $\bar{v}(\cdot) \in \bar{u}(\cdot) + W_0^{1,1}(\Omega)$. Furthermore, setting

$$\Omega' := \{x \in \Omega : \bar{v}(x) \neq \bar{u}(x)\},$$

by the well known property of the support function, we have $\nabla \bar{v}(x) \in aF$ for a.e. $x \in \Omega'$, while $\nabla \bar{v}(x) = \nabla \bar{u}(x)$ for a.e. $x \in \Omega \diagdown \Omega'$. Then

$$\begin{aligned} \int_\Omega f(\rho_F(\nabla \bar{v}(x)))\,dx &= \int_{\Omega\diagdown\Omega'} f(\rho_F(\nabla \bar{u}(x)))\,dx \\ &\leq \int_\Omega f(\rho_F(\nabla \bar{u}(x)))\,dx \leq \int_\Omega f(\rho_F(\nabla u(x)))\,dx \end{aligned}$$

for each $u(\cdot) \in \bar{u}(\cdot) + W_0^{1,1}(\Omega)$.

Finally, setting

$$\mu := \min\left\{\varepsilon, \frac{\delta}{(\|F\|\,\|F^0\|+1)^2}\right\},$$

we see that the minimizer $\bar{v}(\cdot)$ satisfies the inequality

$$\bar{v}(x) \geq \theta(\bar{y}) + a\rho_{F^0}(x-\bar{y}) \tag{1.107}$$

on $\bar{y} + \mu(\|F\|\,\|F^0\|+1)F^0$. Indeed, it follows from (1.3) that

$$\rho_{F^0}(x-\bar{y}) + \rho_{F^0}(\bar{y}-x) \leq \mu\left(\|F\|\,\left\|F^0\right\|+1\right)^2 \leq \delta$$

whenever $\rho_{F^0}(x-\bar{y}) \leq \mu(\|F\|\,\|F^0\|+1)$, implying that the minimum in (1.105) is equal to $\theta(\bar{y}) + a\rho_{F^0}(x-y)$. Since, obviously, $\bar{v}(\bar{y}) = \theta(\bar{y})$, applying Corollary 1.2.2 we deduce from (1.107) that

$$\bar{v}(x) = \theta(\bar{y}) + a\rho_{F^0}(x-\bar{y})$$

for all $x \in \bar{y} + \mu F^0 \subset K + \varepsilon F^0 \subset \Omega$. Taking into account (1.105), we have

$$\bar{u}\,(x) \leq \theta(\bar{y}) + a\rho_{F^0}\,(x - \bar{y})\,, \qquad x \in \bar{y} + \mu F^0. \tag{1.108}$$

On the other hand, for some $\lambda_0 \in (0, 1]$ the points x_λ, $0 \leq \lambda \leq \lambda_0$, belong to $\bar{y} + \mu F^0$. Combining now (1.108) for $x = x_\lambda$ with (1.104) we obtain

$$\bar{u}\,(x_\lambda) \leq u^+_{\Gamma,\theta}\,(x_\lambda)$$

and hence (see the hypothesis (**ii**))

$$\bar{u}\,(x_\lambda) = u^+_{\Gamma,\theta}\,(x_\lambda)\,,$$

$0 \leq \lambda \leq \lambda_0$, contradicting (1.99). □

Chapter 2

Lagrangeans linearly depending on u

In Chapter 1 we arrived at various versions of the variational Strong Maximum Principle starting from homogeneous elliptic partial differential equations such as Laplace equation $\Delta u = 0$, which is necessary optimality condition for the convex symmetric integral

$$\int_\Omega \frac{1}{2}\|\nabla u(x)\|^2 \, \mathrm{d}x.$$

Slightly generalizing this problem we are led to the problem with additive linear term depending on the state variable

$$\int_\Omega \left(\frac{1}{2}\|\nabla u(x)\|^2 + \sigma u(x)\right) \, \mathrm{d}x, \tag{2.1}$$

where $\sigma \neq 0$ is a real constant. The Euler-Lagrange equation for (2.1) is the simplest Poisson equation

$$\Delta u(x) = \sigma, \quad x \in \Omega. \tag{2.2}$$

Similarly as in [16, 37, 38] (see also Chapter 1 of this Thesis) for the case $\sigma = 0$, some properties of solutions to (2.2) regarding the SMP can be extended to minimizers of more general functional than (2.1), for which the Euler-Lagrange equation is no longer valid. Certainly, the Strong Maximum (Minimum) Principle in the traditional sense for these functionals (as well as for Poisson equation (2.2) doesn't hold. Nevertheless, using an *a priori* class of minimizers given by A. Cellina [13] one can obtain a comparison result (see Section 2.2) as well as local estimates of solutions to respective variational problems, which approximate to validity of the traditional SMP or of its modifications considered in Chapter 1 as $\sigma \to 0$.

2.1 Preliminaries

Let us fix $\sigma > 0$ and consider two variational problems

$$\min\left\{\int_\Omega [f(\rho_F(\nabla u(x))) + \sigma u(x)]\ \mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega)\right\} \qquad (P_\sigma)$$

and

$$\min\left\{\int_\Omega [f(\rho_F(\nabla u(x))) - \sigma u(x)]\ \mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega)\right\} \qquad (P_{-\sigma})$$

Here $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$, $f(0) = 0$, is a convex lower semicontinuous function, $F \subset \mathbb{R}^n$ is a convex closed bounded set with $0 \in \mathrm{int} F$, $\Omega \subset \mathbb{R}^n$ is an arbitrary open bounded connected region and $u^0(\cdot) \in W^{1,1}(\Omega)$. These are our standing assumptions along with the chapter.

In what follows, as earlier, f^* stands fot the *Legendre-Fenchel transform (conjugate)* of the convex function f, and $b > 0$ is such that $\mathrm{dom} f^* = [0, b]$ (or $[0, b)$). In particular, b can be $+\infty$.

Let us introduce the (index) sets

$$\Gamma_\Omega^\pm := \{\gamma = (x_0, k) \in \Omega \times \mathbb{R} : \Omega \subset x_0 \pm b \cdot \frac{n}{\sigma} F^0\} \qquad (2.3)$$

and the following Lipschitz continuous functions

$$\omega_\gamma^+(x) = \frac{n}{\sigma} f^*\left(\frac{\sigma}{n}\rho_{F^0}(x - x_0)\right) + k, \quad \gamma \in \Gamma_\Omega^+, \qquad (2.4)$$

$$\omega_\gamma^-(x) = -\frac{n}{\sigma} f^*\left(\frac{\sigma}{n}\rho_{F^0}(x_0 - x)\right) + k, \quad \gamma \in \Gamma_\Omega^-, \qquad (2.5)$$

defined on Ω. Further on we will naturally assume $\Omega \subset \mathbb{R}^n$ to be chosen so small that $\Gamma_\Omega^\pm \neq \emptyset$. Sometimes it is convenient to consider only the case when $b = +\infty$ (equivalently, $f(\cdot)$ not affine on some semiline $[b, +\infty)$).

The following result allows us to consider $\omega_\gamma^\pm(\cdot)$ as the possible candidates to comparison functions for the problems (P_σ) and $(P_{-\sigma})$.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected domain with sufficiently regular boundary such that the Divergence Theorem holds (see, e.g., [26]). Then the function $\omega_\gamma^+(\cdot)$ defined by (2.4) is the only solution to (P_σ) with $u^0(\cdot) = \omega_\gamma^+(\cdot)$. Similarly, the function $\omega_\gamma^-(\cdot)$ defined by (2.5) is the only solution to $(P_{-\sigma})$ with the same boundary condition, i.e., $u^0(\cdot) = \omega_\gamma^-(\cdot)$.*

The statement of Theorem follows immediately from Theorem 1 [13], taking into account that the conjugate of the function $\xi \in \mathbb{R}^n \mapsto f(\rho_F(\xi))$ coincides with $f^*(\rho_{F^0}(\cdot))$ and that the domain of $(f \circ \rho_F(\cdot))^*$ is the set bF^0 or its interior.

Observe that although the Cellina's result above holds for general functional (with no symmetry assumption), we believe that such type symmetry is rigorously needed for results below.

2.2 Comparison theorem

Here we obtain an auxiliary comparison result on minimizers of the variational problems $(P_{\pm\sigma})$, which will be used further for proving local estimates on minimizers as well as some local versions of SMP.

Theorem 2.2.1. *Given $x \in \Omega$ let us choose positive numbers α, β with $\alpha < \beta$ such that the closure of the annulus $A_{\alpha,\beta}^+(\bar{x}) := \{x \in \mathbb{R}^n : \alpha < \rho_{F^0}(\bar{x} - x) < \beta\}$ is contained in Ω. Assume that there exist $M > 0$ and a decreasing absolutely continuous function $R : [\alpha, \beta] \mapsto \mathbb{R}$ such that*

$$\frac{M}{r^{n-1}} + \frac{\sigma}{n} r \in \partial f(|R'(r)|) \tag{2.6}$$

for a.e. $r \in [\alpha, \beta]$. If a continuous minimizer $\bar{u}(\cdot)$ in $(P_{-\sigma})$ satisfies the inequality

$$\bar{u}(x) \geq R(\rho_{F^0}(\bar{x} - x)) \tag{2.7}$$

on the boundary $\partial A_{\alpha,\beta}^+(\bar{x})$ then the same inequality takes place for all $x \in A_{\alpha,\beta}^+(\bar{x})$.

Similarly, setting $A_{\alpha,\beta}^-(\bar{x}) := \{x \in \mathbb{R}^n : \alpha < \rho_{F^0}(x - \bar{x}) < \beta\}$, let us assume that the closure $\overline{A_{\alpha,\beta}^-(\bar{x})} \subset \Omega$, and there exist a constant $M > 0$ and an increasing absolutely continuous function $R : [\alpha, \beta] \mapsto \mathbb{R}$ satisfying (2.6) *for a.e. $r \in [\alpha, \beta]$. Then for an arbitrary continuous minimizer $\bar{u}(\cdot)$ in (P_σ) validity of the inequality*

$$\bar{u}(x) \leq R(\rho_{F^0}(x - \bar{x})) \tag{2.8}$$

on the boundary $\partial A^-_{\alpha,\beta}(\bar{x})$ implies its validity for all $x \in A^-_{\alpha,\beta}(\bar{x})$.

Proof. Let us prove the first part of Theorem. Denoting by

$$S(x) = R(\rho_{F^0}(\bar{x} - x)),$$

we assume that the (open) set $U = \{x \in A^+_{\alpha,\beta}(\bar{x}) : \bar{u}(x) < S(x)\}$ is nonempty. Define the function $\omega(\cdot)$ to be equal to $\max(\bar{u}(x), S(x))$ on $A^+_{\alpha,\beta}(\bar{x})$ and to $u(x)$ elsewhere. Obviously, $\omega(\cdot)$ is continuous and equals to $S(x)$ on U. For $x \in \Omega \backslash U$ instead $\omega(x) = \bar{u}(x)$. Now we show that $\omega(\cdot)$ is a solution to $(P_{-\sigma})$ with $u^0(x) = \omega(x) = \bar{u}(x)$, $x \in \partial\Omega$.

Since $\bar{u}(\cdot)$ is a minimizer in $(P_{-\sigma})$, we have

$$\begin{aligned} 0 &\geq \int_\Omega \{[f(\rho_F(\nabla \bar{u}(x))) - \sigma \bar{u}(x)] - [f(\rho_F(\nabla(\omega(x)))) - \sigma \omega(x)]\} \, \mathrm{d}x = \\ &= \int_U [f(\rho_F(\nabla \bar{u}(x))) - f(\rho_F(\nabla S(x))) - \sigma(\bar{u}(x) - S(x))] \, \mathrm{d}x \geq \\ &\geq \int_U [\langle p(x), \nabla(\bar{u}(x) - S(x))\rangle - \sigma(\bar{u}(x) - S(x))] \, \mathrm{d}x, \end{aligned} \tag{2.9}$$

where $p(x)$ is a measurable selection of the mapping

$$x \mapsto \partial(f \circ \rho_F)(\nabla S(x)).$$

We want to construct a selection $p(\cdot)$ such that the last integral in (2.9) equals zero. To this end we define $p(\cdot)$ according to the differential properties of $S(x)$. Observe first that by (2.6) and monotonicity of the function $R(\cdot)$

$$-R'(r) \in \partial f^* \left(\frac{M}{r^{n-1}} + \frac{\sigma}{n} r \right) \tag{2.10}$$

for a.e. $r \in [\alpha, \beta]$. Taking into account that the gradient $\nabla \rho_{F^0}(\bar{x} - x)$ exists almost everywhere, we have

$$\nabla S(x) = -R'(\rho_{F^0}(\bar{x} - x)) \nabla \rho_{F^0}(\bar{x} - x) \tag{2.11}$$

for a.e. $x \in U$. Furthermore, the gradient $\nabla \rho_{F^0}(\bar{x} - x)$ belongs to ∂F (see (1.4)) and, consequently,

$$\rho_F(\nabla S(x)) = |R'(\rho_{F^0}(\bar{x} - x))| \in \partial f^* \left(\frac{M}{\rho_{F^0}^{n-1}(\bar{x} - x)} + \frac{\sigma}{n} \rho_{F^0}(\bar{x} - x) \right)$$

or, equivalently,

$$\frac{M}{\rho_{F^0}^{n-1}(\bar{x} - x)} + \frac{\sigma}{n} \rho_{F^0}(\bar{x} - x) \in \partial f(\rho_F(\nabla S(x)))$$

for a.e. $x \in U$. The subdifferential of the composed function $f \circ \rho_F$ at $\nabla S(x)$ is equal to

$$\partial f(\rho_F(\nabla S(x))) \partial \rho_F(\nabla S(x)),$$

and by (1.4) the gradient $\nabla S(x)$ is normal to F^0 at $\frac{\bar{x}-x}{\rho_{F^0}(\bar{x}-x)}$, or, in dual form,

$$\frac{\bar{x} - x}{\rho_{F^0}(\bar{x} - x)} \in \mathbf{N_F} \left(\frac{\nabla S(x)}{\rho_F(\nabla S(x))} \right) \cap \partial F^0 = \partial \rho_F(\nabla S(x)).$$

So we can define the function

$$\begin{aligned} p(x) &= \left(\frac{M}{\rho_{F^0}^{n-1}(\bar{x} - x)} + \frac{\sigma}{n} \rho_{F^0}(\bar{x} - x) \right) \frac{\bar{x} - x}{\rho_{F^0}(\bar{x} - x)} = \\ &= \frac{M(\bar{x} - x)}{\rho_F^n(\bar{x} - x)} + \frac{\sigma}{n}(\bar{x} - x), \end{aligned}$$

which is a measurable selection of $x \mapsto \partial(f \circ \rho_F)(\nabla S(x))$ on U. Then,

$$\int_U [\langle p(x), \nabla(\bar{u}(x) - S(x)) \rangle - \sigma(\bar{u}(x) - S(x))] \, \mathrm{d}x =$$

$$= \int_U \left[\left\langle \frac{\sigma}{n}(\bar{x} - x), \nabla(\bar{u}(x) - S(x)) \right\rangle - \sigma(\bar{u}(x) - S(x)) \right] \mathrm{d}x + \\ + \int_U \left\langle \frac{M(\bar{x} - x)}{\rho_F^n(\bar{x} - x)}, \nabla(\bar{u}(x) - S(x)) \right\rangle \mathrm{d}x.$$

On one hand, by the Divergence Theorem (applied to the annulus $A^+_{\alpha,\beta}(\bar{x})$)

$$\begin{aligned}
&\int_U \left[\left\langle \frac{\sigma}{n}(\bar{x}-x), \nabla(\bar{u}(x)-S(x))\right\rangle - \sigma(\bar{u}(x)-S(x))\right] \mathrm{d}x = \\
= \; & -\frac{\sigma}{n}\int_U \operatorname{div}\left([\bar{u}(x)-S(x)](x-x)\right) \; \mathrm{d}x = \\
= \; & -\frac{\sigma}{n}\int_{A^+_{\alpha,\beta}(\bar{x})} \operatorname{div}\left([\bar{u}(x)-\omega(x)](x-\bar{x})\right) \; \mathrm{d}x = \\
= \; & \frac{\sigma}{n}\int_{\partial A^+_{\alpha,\beta}(\bar{x})} \langle[\bar{u}(x)-S(x)](\bar{x}-x), \mathbf{n}\rangle \; \mathrm{d}x = 0,
\end{aligned} \tag{2.12}$$

where $\mathbf{n}$ is the outward unit normal to the boundary (recall that $\bar{u}(x) = \omega(x)$ outside $A^+_{\alpha,\beta}(\bar{x})$).

On the other hand, applying the polar coordinates $r = \|x - \bar{x}\|$ and $\omega = (x-\bar{x})/\|x-\bar{x}\|$, and observing that for each ω, $\|\omega\| = 1$, the equality $\bar{u}(x) = S(x)$ holds on the boundary of the (open) linear set $l_\omega := \{r \in (\alpha, \beta) : (r, \omega) \in U\}$, we obtain as earlier (see Section 1.2)

$$\int_U \langle \frac{M(\bar{x}-x)}{\rho^n_{F^0}(\bar{x}-x)}, \nabla(\bar{u}(x)-S(x))\rangle \; \mathrm{d}x =$$

$$= -M\int_{\|\omega\|=1} \frac{d\omega}{\rho^n_{F^0}(-\omega)} \int_{l_\omega} \frac{1}{r^n} < r\omega, \nabla(\bar{u}(r,\omega)-S(r,\omega)) > r^{n-1} \; \mathrm{d}r = 0.$$

Hence

$$\int_U [\langle p(x), \nabla(\bar{u}(x)-S(x))\rangle - \sigma(\bar{u}(x)-S(x))] \; \mathrm{d}x = 0, \tag{2.13}$$

and, recalling (2.9), we have that $\omega(\cdot)$ is a solution to $(P_{-\sigma})$. It follows from the same inequalities that

$$\int_U [f(\rho_F(\nabla\bar{u}(x))) - f(\rho_F(\nabla S(x))) - \sigma(\bar{u}(x)-S(x))] \; \mathrm{d}x = 0. \tag{2.14}$$

Combining (2.13) and (2.14) we find

$$\int_U [f(\rho_F(\nabla u(x))) - f(\rho_F(\nabla S(x))) - \langle p(x), \nabla(\bar{u}(x)-S(x))\rangle] \; \mathrm{d}x = 0.$$

Taking into account the definition of the subdifferential of a convex function, the integrand here is nonnegative, and we conclude that

$$f(\rho_F(\nabla \bar{u}(x))) = f(\rho_F(\nabla S(x))) + \langle p(x), \nabla(\bar{u}(x) - S(x))\rangle$$

a.e. on U. Notice that there is at most a countable family of disjoint open intervals $J_1, J_2, ... \subset [\alpha, \beta]$ such that the real function $f(\cdot)$ is affine on each J_i with a slope $\tau_i > 0$, $i = 1, 2, ...$. Denoting by

$$E_i = \{x : \frac{M}{\rho_{F^0}^{n-1}(\bar{x} - x)} + \frac{\sigma}{n}\rho_{F^0}(\bar{x} - x) = \tau_i\},$$

we have that for all $x \notin \bigcup_{i=1}^{\infty} E_i$ (the set of null measure) $\partial f^*(\rho_{F^0}(p(x)))$ is a singleton. Thus $\nabla u(x) = \nabla S(x)$ for a.e. $x \in U$,which is a contradiction.

The second part of Theorem can be proved similarly with some obvious changes. For instance, we define $S(x)$ as the right-hand side of the inequality (2.8) and $\omega(x)$ to be $\min(\bar{u}(x), S(x))$ on $A^-_{\alpha,\beta}(\bar{x})$. Moreover, the respective signs in (2.9), (2.10), (2.11) and (2.12) will be changed, and the final conclusions remain valid. □

2.3 Local estimates of minimizers

In this section based on the comparison theorem from Section 2.2 we deduce estimates of minimizers in the problems (P_σ) and $(P_{-\sigma})$ in neighbourhoods of points, which are (or not) points of local extremum. First, we prove that if $\bar{x} \in \Omega$ is not a point of local maximum (minimum) of a solution $\bar{u}(\cdot)$ to the problem (P_σ) (respectively, $(P_{-\sigma})$) then the deviation of $\bar{u}(\cdot)$ from the extremal level can be controlled near $\bar{x}$ by a solution $\omega^{\pm}_{\gamma}$ to the respective problem suggested by Theorem 2.1.1.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^n$, the gauge set $F \subset \mathbb{R}^n$ and the function $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ satisfy our standing assumptions.*

(i) *Given a solution $\bar{u}(\cdot)$ to the problem $(P_{-\sigma})$ let us assume that a point $\bar{x} \in \Omega$ is such that for some real k and $\beta > 0$*

$$\bar{u}(x) \geq k \qquad \forall x \in \bar{x} - \beta F^0 \subset \Omega, \tag{2.15}$$

and

$$\bar{u}(\bar{x}) > k + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \beta \right). \tag{2.16}$$

Then for all $x \in x - \beta F^0$ the inequality

$$\bar{u}(x) \geq k + \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \rho_{F^0}(\bar{x} - x) \right) \right]$$

holds.

(ii) *Similarly, if $\bar{u}(\cdot)$ is a solution to (P_σ) and a point $\bar{x} \in \Omega$ satisfies for some k and $\beta > 0$ the conditions*

$$\bar{u}(x) \leq k \qquad \forall x \in \bar{x} + \beta F^0 \subset \Omega, \tag{2.17}$$

and

$$\bar{u}(\bar{x}) < k - \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \beta \right) \tag{2.18}$$

then

$$\bar{u}(x) \leq k - \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \rho_{F^0}(x - \bar{x}) \right) \right]$$

for all $x \in \bar{x} + \beta F^0 \subset \Omega$.

Proof. **(i)** By continuity of $\bar{u}(\cdot)$ we find $\alpha \in (0, \beta)$ such that

$$\bar{u}(x) > k + \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \alpha \right) \right] \tag{2.19}$$

for all x, $\rho_{F^0}(\bar{x} - x) \leq \alpha$. It is obvious that the conjugate function $f^*(\cdot)$ is uniformly continuous on the segment $I := [0, \beta + \eta]$ for some small $\eta > 0$. So, fixed $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$|f^*(t) - f^*(s)| \leq \frac{\sigma \varepsilon}{2n} \tag{2.20}$$

whenever $t, s \in I$ with $|t - s| \leq \delta$. Combining (2.20) with (2.19) we have that

$$\bar{u}(x) > k - \varepsilon + \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta + \delta \right) - f^* \left(\frac{\sigma}{n} \alpha + \delta \right) \right] \tag{2.21}$$

whenever $\rho_{F^0}(\bar{x} - x) \leq \alpha$.

Since $f^*(\cdot)$ is absolutely continuous on I, denoting by $\varphi(\cdot)$ its derivative, which exists almost everywhere, we deduce from (2.21) that

$$\bar{u}(x) > k - \varepsilon + \int_{\alpha}^{\beta} \varphi\left(\frac{\sigma}{n} s + \delta\right) \, \mathrm{d}s \tag{2.22}$$

for all $x \in \Omega$ with $\rho_{F^0}(\bar{x} - x) \leq \alpha$ Let us define the absolutely continuous decreasing function

$$R(r) := k - \varepsilon + \int_{r}^{\beta} \varphi\left(\frac{\sigma}{n} s + \delta \left(\frac{\alpha}{s}\right)^{n-1}\right) \, \mathrm{d}s, \tag{2.23}$$

satisfying obviously the condition (2.6) with $M = \delta\alpha^{n-1}$. It follows from (2.22) and from the monotonicity of $\varphi(\cdot)$ that

$$\bar{u}(x) \geq R(\rho_{F^0}(\bar{x} - x)) \tag{2.24}$$

for all x with $\rho_{F^0}(\bar{x} - x) = \alpha$. On the other hand, for x with $\rho_{F^0}(\bar{x} - x) = \beta$ the estimate (2.24) follows directly from the definition (2.23) and from (2.15). Applying now Theorem 2.2.1 we see that (2.24) holds on the whole annulus $A^{+}_{\alpha,\beta}(\bar{x}) = \{x : \alpha < \rho_{F^0}(\bar{x} - x) < \beta\}$.

Taking now an arbitrary x with $\rho_{F^0}(\bar{x} - x) \leq \beta$, we successively obtain

$$\bar{u}(x) \geq k - \varepsilon + \int_{\rho_{F^0}(\bar{x}-x)}^{\beta} \varphi\left(\frac{\sigma}{n} s + \delta \left(\frac{\alpha}{s}\right)^{n-1}\right) \, \mathrm{d}s \geq$$

$$\geq k - \varepsilon + \int_{\rho_{F^0}(\bar{x}-x)}^{\beta} \varphi\left(\frac{\sigma}{n} s + \delta \left(\frac{\alpha}{\beta}\right)^{n-1}\right) \, \mathrm{d}s \geq$$

$$\geq k - \varepsilon + \frac{n}{\sigma}\left[f^*\left(\frac{\sigma}{n}\beta + \delta\left(\frac{\alpha}{s}\right)^{n-1}\right) - f^*\left(\frac{\sigma}{n}\rho_{F^0}(\bar{x} - x) + \delta\left(\frac{\alpha}{\beta}\right)^{n-1}\right)\right].$$

Since $\alpha < \beta$, recalling (2.20), from the latter inequality we have

$$\bar{u}(x) \geq k - 2\varepsilon + \frac{n}{\sigma}\left[f^*\left(\frac{\sigma}{n}\beta\right) - f^*\left(\frac{\sigma}{n}\rho_{F^0}(\bar{x} - x)\right)\right],$$

and by arbitrarity of ε the assertion of Theorem follows.

The symmetric assertion **(ii)** can be proved similarly with some obvious changes. In this case, e.g., we set

$$R(r) := k + \varepsilon - \int_r^{\beta} \varphi\left(\frac{\sigma}{n}s + \frac{M}{s^{n-1}}\right) \, ds,$$

and use the second part of Theorem 2.2.1. □

In fact, in the above theorem we proved that any solution to the problem (P_σ) (respectively, $(P_{-\sigma})$) admits a lower (respectively, upper) estimate by a minimizer of the respective functional from the Cellina's class (see Theorem 2.1.1) near a nonextremum point $\bar{x}$. For instance, in the case of (P_σ) it is a function $\omega_\gamma^+(\cdot)$ where $\gamma = \left(\bar{x}, k - \frac{n}{\sigma} f^*\left(\frac{\sigma}{n}\beta\right)\right)$, k is local maximum, and the constant $\beta > 0$ caracterizes the "deepness" of nonmaximality of the given solution at $\bar{x}$.

Based on these properties we can obtain local estimates of solutions, on the contrary, near points of (local) extremum, which are in some sense similar to the Strong Maximum and Strong Minimum principles established in Chapter 1. However, for the problems $(P_{\pm\sigma})$ this property has a "cross" effect, i.e., the (lower) estimate of an arbitrary solution $\bar{u}(\cdot)$ to the problem $(P_{\;\sigma})$ is obtained by means of a special minimizer $\omega_\gamma^+(\cdot)$ of the functional in (P_σ) and vice versa. Nevertheless, this effect vanishes as σ tends to zero.

In order to prove the following statement we need to require additionally that the function $f(\cdot)$ is smooth at the origin

$$\partial f(0) = \{0\}. \tag{2.25}$$

This is the standard hypothesis used also for proving SMP in Chapter 1.

Theorem 2.3.2. *Let $\overline{u}(\cdot)$ be a continuous solution to $(P_{-\sigma})$ and $x_0 \in \Omega$ be a point of its local minimum. Namely, assume that there exists $\delta > 0$ such that*

$$\overline{u}(x) \geq \overline{u}(x_0) \tag{2.26}$$

for all $x \in x_0 + \delta F^0 \subset \Omega$. Then on the (slightly smaller) neighbourhood $x_0 + \frac{\delta}{\|F\|\|F^0\|+1} F^0$ we have

$$\overline{u}(x) \leq \omega_\gamma^+(x)$$

where $\gamma = (x_0, \bar{u}(x_0))$.

Similarly, let $\bar{u}(\cdot)$ *be a continuous solution to* (P_σ), $x_0 \in \Omega$ *be a point of its local maximum, and* $\delta > 0$ *be such that*

$$u(x) \leq u(x_0)$$

whenever $x \in x_0 - \delta F^0 \subset \Omega$. *Then*

$$\bar{u}(x) \geq \omega_\gamma^-(x)$$

for all $x \in x_0 - \frac{\delta}{\|F\|\|F^0\|+1} F^0$ *with the same* $\gamma = (x_0, \bar{u}(x_0))$.

Proof. Let us prove the first part of theorem. Assume, on the contrary, that for some $\bar{x} \in \Omega$ with $\rho_{F^0}(\bar{x} - x_0) \leq \frac{\delta}{\|F\|\|F^0\|+1}$ the strict inequality

$$\bar{u}(\bar{x}) > k + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \rho_{F^0}(\bar{x} - x_0) \right)$$

takes place with $k := \bar{u}(x_0)$. Let us choose $\varepsilon > 0$ so small that

$$(\|F\|\|F^0\| + 1)\rho_{F^0}(\bar{x} - x_0) + \varepsilon \|F\|\|F^0\| < \delta \tag{2.27}$$

and

$$\bar{u}(\bar{x}) > k + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} [\rho_{F^0}(\bar{x} - x_0) + \varepsilon] \right).$$

Seting $\beta = \rho_{F^0}(\bar{x} - x_0) + \varepsilon$ we have

$$u(\bar{x}) > k + \frac{n}{\sigma} f^* \left(\frac{\sigma}{n} \beta \right). \tag{2.28}$$

We claim now that $\bar{x} - \beta F^0 \subset x_0 + \delta F^0$. Indeed, given $y \in \overline{x} - \beta F^0$ by (1.3) we have

$$\rho_{F^0}(y - \bar{x}) \leq \|F\|\|F^0\| \rho_{F^0}(\bar{x} - y) \leq \|F\|\|F^0\| (\rho_{F^0}(\bar{x} - x_0) + \varepsilon),$$

and by (2.27)

$$\rho_{F^0}(y - x_0) \leq \rho_{F^0}(y - \bar{x}) + \rho_{F^0}(\bar{x} - x_0) < \delta.$$

In particular, due to the condition (2.26) we have $\bar{u}(x) \geq k$. Then by Theorem 2.3.1 (see also the condition (2.28))

$$\bar{u}(x) \geq k + \frac{n}{\sigma} \left[f^* \left(\frac{\sigma}{n} \beta \right) - f^* \left(\frac{\sigma}{n} \rho_{F^0}(\bar{x} - x) \right) \right]$$

for all $x \in \bar{x} - \beta F^0$. In particular, for x_0 (which obviously belongs to $\bar{x} - \beta F^0$) we obtain

$$k = \bar{u}(x_0) \geq k + \frac{n}{\sigma}\left[f^*\left(\frac{\sigma}{n}(\rho_{F^0}(\bar{x} - x_0) + \varepsilon)\right) - f^*\left(\frac{\sigma}{n}\rho_{F^0}(\bar{x} - x_0)\right)\right] > k.$$

The latter (strict) inequality follows from the fact that $\partial f(0) = \{\xi : f^*(\xi) = 0\}$ is reduced to the singleton $\{0\}$ (see (2.25)), and, consequently, the convex function $f^*(\cdot)$ is strictly increasing on $[0, +\infty)$. Obtained contradiction proves the first part of Theorem, while the second part can be proved similarly. □

Concluding this chapter, let us illustrate estimates given by Theorems 2.3.1 and 2.3.2 by a simple example.

Example 4. *Let $f(\xi) = \frac{1}{2}\xi^2$, $\xi > 0$. In the case $F = \overline{B}$ the special minimizers of the functional*

$$\int_\Omega \left(\frac{1}{2}\|\nabla u(x)\|^2 + \sigma u(x)\right)\ dx \qquad (2.29)$$

with $\sigma \neq 0$ given by Theorem 2.1.1 have the form

$$\omega_\gamma^{sgn\sigma}(x) = k + \frac{1}{2}\frac{n}{\sigma}\left(\frac{\sigma}{n}\|x - x_0\|\right)^2 = k + \frac{\sigma}{2n}\|x - x_0\|^2$$

where $\gamma = (k, x_0) \in \Gamma_\Omega^{sgn\sigma}$.

According to Theorems 2.3.1 and 2.3.2 each continuous minimizer of (2.29) on $u^0(\cdot) \in W^{1,1}(\Omega)$ (with the same boundary condition) is locally contained between two functions

$$k - \frac{\sigma}{2n}\|x - x_0\|^2 \qquad \text{and} \qquad k + \frac{\sigma}{2n}\|x - x_0\|^2 \qquad (2.30)$$

for various values of x_0 and k. If now x^* is a point of (local) extremum of the minimizer then the upper and lower estimates (2.30) are centered at the point $x_0 = x^*$ and both of them converge to a constant as $\sigma \to 0$. So, we arrive at the traditional Strong Maximum Principle. The same property holds of course when the function $f(\cdot)$ is not smooth and (or) some (neither smooth nor rotund) gauge set in the place of the norm is considered.

Chapter 3

Rotationally invariant lagrangeans with a nonlinear term depending on u

In this Chapter we are interested in proving SMP for functionals which are rotationally invariant with respect to the gradient and contain a nonlinear additive term depending on the state variable. Namely, we consider the minimization problem

$$\min\left\{\int_\Omega [f(\|\nabla u(x)\|) + g(u(x))]\,\mathrm{d}x : u(\cdot) \in u^0(\cdot) + W_0^{1,1}(\Omega)\right\} \tag{P_g}$$

where $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ are convex with $f(0) = g(0) = 0$.

Here we use the same notions of Convex Analysis as in the previous chapters. Let us notice only that if a convex function $f : \mathbb{R} \mapsto \mathbb{R} \cup \{+\infty\}$ is differentiable at some point t then, in particular, the *Fenchel identity*

$$f(t) + f^*(s) = ts \qquad \forall s \in \partial f(t)$$

can be written as

$$f^*(f'(t)) = tf'(t) - f(t). \tag{3.1}$$

If the functions $f(\cdot)$ and $g(\cdot)$ in (P_g) are sufficiently regular then we can associate to (P_g) the Euler-Lagrange equation

$$\mathrm{div}\nabla(f \circ \|\cdot\|)(\nabla u(x)) = g'(u(x)),$$

which is a particular case of elliptic partial differential equations studied by P. Pucci, J. Serrin and H. Zou (see [52, 54, 53]). In our terms the authors assumed that

- $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(0) = 0$, admits a strictly increasing continuous derivative on $(0, +\infty)$ such that $\lim_{t \to 0+} f'(t) = 0$;
- $g : \mathbb{R}^+ \to \mathbb{R}^+$ has non-decreasing continuous derivative $g'(t)$, $g'(0) = 0$, on some interval $[0, \delta)$, $\delta > 0$.

Under these standing hypotheses they gave the necessary and sufficient conditions for validity of the SMP in the following form:

- either $g(s) = 0$ on $[0, \delta)$ or
- $g(s) > 0$ on $(0, \delta]$, and the (Riemann) improper integral

$$\int_0^{\delta} \frac{ds}{(f^* \circ f')^{-1}(g(s))} \tag{3.2}$$

  diverges.

To obtain their results the authors used the comparison technique, where the (auxiliary) comparison function, being itself a solution of the partial differential equation, is constructed as a fixed point of some nonlinear compact operator in the space of continuous functions.

We do not suppose the function $g(\cdot)$ to be continuously differentiable that does not allow us to refer directly to the results obtained in the context of Partial Differential Equations. However, we try to adapt the technique by P. Pucci and J. Serrin to the case of the multivalued subdifferential. In particular, we are lead to consider a set-valued upper semicontinuous operator in the functional space and to prove a parametrized version of the Kakutani theorem. It turns out that a fixed point of the so constructed multifunction satisfies all the properties announced in the works by P. Pucci and J. Serrin and can be used as a comparison function in the variational technique developed by A. Cellina for the functionals depending only on the gradient (see Chapter 1). Thus, in the last section of this chapter we conclude the proof of the variational SMP for the problem (P_g). It is interesting to observe that despite of the essentially multivalued character of the proof (that, in our opinion, can not be avoided), the final sufficient condition forvalidity of the SMP is obtained, exactly, in the form above (including divergence of the integral (3.2)) by exploiting the fact that the convex function $g(\cdot)$ admits derivative at almost each point.

3.1 Construction of the auxiliary operator

Similarly as in the works by P. Pucci and J. Serrin ([53, 54]) we search a comparison function $\omega : C([0, \beta-\alpha], \mathbb{R}) \mapsto \mathbb{R}$ such as

$$\begin{aligned}\omega(t) &= m - \int_t^{\beta-\alpha} (f')^{-1}\left(\frac{1}{(\beta-s)^{n-1}} \times\right.\\ &\quad \left.\times \left[\lambda - \int_s^{\beta-\alpha} (\beta-\tau)^{n-1} p(\tau)\, \mathrm{d}\tau\right]\right)\, \mathrm{d}s, \qquad (3.3)\end{aligned}$$

where $p(\cdot)$ is a measurable selection of $t \mapsto \partial g(\omega(t))$ and $\lambda > 0$ is such that $\omega(0) = 0$. In order to prove the existence of such a function we make some constructions as follows.

Let us denote by $\mathfrak{F}$ the class of all strictly convex functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(0) = 0$, admitting continuous derivative $f'(\cdot)$ on $(0, +\infty)$ with $\lim_{t\to 0^+} f'(t) = 0$, and by $\mathfrak{G}$ the family of functions $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g(0) = 0$, which are convex on some interval $[0, \delta)$, $\delta > 0$, such that $\partial g(0) = \{0\}$.

For our convenience we extend the functions $f'(\cdot)$ and $g(\cdot)$ to the whole real line by setting $f'(s) = -f'(-s)$ and $g(s) = 0$ for $s < 0$. Also choosing an arbitrary $m \in (0, \delta)$ (the exact sense of this constant will be clarified later) we redefine $g(z)$ for $z \geq m$ by affine way.

Choose also α, β with $0 < \alpha < \beta$ and set $\bar{p} := \sup\{z : z \in \partial g(u), u \in [0, m]\}$ and

$$\bar{\lambda} := \beta^{n-1}\left[f'\left(\frac{\sigma m}{\beta-\alpha}\right) + \bar{p}(\beta-\alpha)\right]. \qquad (3.4)$$

Given $\omega(\cdot) \in C([0, \beta-\alpha], \mathbb{R})$ and $\sigma \in [0, 1]$ let us consider the multifunction $P_{\omega,\sigma} : [0, \bar{\lambda}] \to \mathbb{R}$ defined by

$$\begin{aligned}P_{\omega,\sigma}(\lambda) = \sigma m - \int_0^{\beta-\alpha} (f')^{-1}((\beta-s)^{1-n}[\lambda -\\ -\sigma \int_s^{\beta-\alpha} (\beta-\tau)^{n-1} \partial g(\omega(\tau))\, \mathrm{d}\tau])\, \mathrm{d}s. \qquad (3.5)\end{aligned}$$

We claim that $P_{\omega,\sigma}(\cdot)$ is upper semicontinuous. Indeed, for each $\lambda \in [0, \lambda]$ and $u \in P_{\omega,\sigma}(\lambda)$ we have

$$u = \sigma m - \int_0^{\beta-\alpha} (f')^{-1}((\beta - s)^{1-n}[\lambda - \\ -\sigma \int_s^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau) \, \mathrm{d}\tau]) \, \mathrm{d}s \tag{3.6}$$

for some measurable selection $p(\cdot)$ of $t \mapsto \partial g(\omega(t))$ on $[0, \beta - \alpha]$. By (3.4) and monotonicity of the derivative $f'(\cdot)$, on one hand, the number (3.6) is not smaller than

$$\begin{aligned} & -\int_0^{\beta-\alpha} (f')^{-1} \left(\frac{\beta^{n-1}}{(\beta - s)^{n-1}} \left[f' \left(\frac{m}{\beta - \alpha} \right) + \bar{p}(\beta - \alpha) \right] \right) \, \mathrm{d}s \geq \\ \geq & -\int_0^{\beta-\alpha} (f')^{-1} \left(\left(\frac{\beta}{\alpha} \right)^{n-1} \left[f' \left(\frac{m}{\beta - \alpha} \right) + \bar{p}(\beta - \alpha) \right] \right) \, \mathrm{d}s = \\ = & -(\beta - \alpha)(f')^{-1} \left(\left(\frac{\beta}{\alpha} \right)^{n-1} \left[f' \left(\frac{m}{\beta - \alpha} \right) + \bar{p}(\beta - \alpha) \right] \right), \end{aligned}$$

while, on the other hand,

$$\begin{aligned} u & \leq \sigma m - \int_0^{\beta-\alpha} (f')^{-1} \left(-\left(\frac{\beta}{\alpha} \right)^{n-1} \bar{p}(\beta - \alpha) \right) \, \mathrm{d}s = \\ & = \sigma m + (\beta - \alpha)(f')^{-1} \left(\left(\frac{\beta}{\alpha} \right)^{n-1} \bar{p}(\beta - \alpha) \right). \end{aligned}$$

Therefore the values of $P_{\omega,\sigma}(\cdot)$ are contained in $[C_1, C_2]$ for some real C_1, C_2, and we need to prove only that $P_{\omega,\sigma}(\cdot)$ has closed graph. Consider a sequence of elements $(\lambda_k, v_k) \in \operatorname{graph} P_{\omega,\sigma}$ that converges to some $(\lambda, v) \in [0, \bar{\lambda}] \times [C_1, C_2]$. We can represent

$$\begin{aligned} v_k & = \sigma m - \int_0^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta - \alpha)^{n-1}} \times \right. \\ & \times \left. \left[\lambda_k - \sigma \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p_k(\tau) \, \mathrm{d}\tau \right] \right) \, \mathrm{d}s \end{aligned}$$

for some measurable selection $p_k(\cdot)$ of $t \mapsto \partial g(\omega(t))$, $k = 1, 2, \ldots.$ It follows from the upper semicontinuity of the latter multifunction that the sequence

$(p_k(\cdot))_k$ is bounded and by the Dunford - Pettis theorem we may assume without loss of generality that it converges weakly to some $p(t) \in \partial g(\omega(t))$ (the subdifferential admits convex values). Then for each $t \in [0, \beta - \alpha]$ the integrals

$$\int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p_k(\tau)\ \mathrm{d}\tau$$

converge to

$$\int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau)\ \mathrm{d}\tau,$$

and using continuity of the functions $(f')^{-1}(\cdot)$, Lebesgue dominated convergence theorem and unicity of the limit we conclude that

$$v = \sigma m - \int_0^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta-\alpha)^{n-1}} \left[\lambda - \sigma \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau)\ \mathrm{d}\tau \right] \right)\ \mathrm{d}s,$$

i.e., $(\lambda, v) \in \operatorname{graph} P_{\omega,\sigma}$. Hence $P_{\omega,\sigma}(\cdot)$ is upper semicontinuous.

Now we prove existence of $c \in [0, \bar{\lambda}]$ such that $0 \in P_{\omega,\sigma}(c)$. Let us start by observing that $P_{\omega,\sigma}(0) \subset [0, +\infty)$ and $P_{\omega,\sigma}(\bar{\lambda}) \subset (-\infty, 0]$. Indeed, let $z_1 \in P_{\omega,\sigma}(0)$. This means that for some measurable selection $p_1(t) \in \partial g(\omega(t))$ accordingly to our convention

$$\begin{aligned} z_1 = \sigma m \ &- \ \int_0^{\beta-\alpha} (f')^{-1} \left(\frac{-\sigma}{(\beta - s)^{n-1}} \times \right. \\ &\times \ \left. \int_s^{\beta-\alpha} (\beta-\tau)^{n-1} p_1(\tau)\ \mathrm{d}\tau \right)\ \mathrm{d}s \geq \sigma m \geq 0, \end{aligned}$$

Similarly, if $z_2 \in P(\bar{\lambda})$ then there exists a measurable selection $p_2(t) \in \partial g(\omega(t))$ with

$$z_2 = \sigma m - \int_0^{\beta-\alpha} (f')^{-1}((\beta - s)^{1-n})[\beta^{n-1} f'(\sigma m/(\beta-\alpha)) +$$

$$+ \bar{p}\beta^{n-1}(\beta - \alpha) - \sigma \int_s^{\beta-\alpha} (\beta-\tau)^{n-1} p_2(\tau)\ \mathrm{d}\tau])\ \mathrm{d}s \leq$$

$$\leq \sigma m - \int_0^{\beta-\alpha} (f')^{-1} \left(f' \left(\frac{\sigma m}{\beta - \alpha} \right) \right)\ \mathrm{d}s = 0.$$

Let us choose a sequence of continuous approximate selections $(f_k)_k$ of the mapping $\lambda \mapsto P_{\omega,\sigma}(\lambda)$, i.e., such that

$$\operatorname{graph} f_k \subset \operatorname{graph} P_{\omega,\sigma} + \frac{1}{k} B, \tag{3.7}$$

$k = 1, 2, \ldots$ (see [4]). These selections, clearly can be chosen passing through some fixed points of graph $P_{\omega,\sigma}$, say $f_k(0) = u_0 \geq 0$ and $f_k(\bar{\lambda}) = u_{\bar{\lambda}} \leq 0$. By Bolzano theorem there exists a sequence $(c_k)_k$ in $(0, \bar{\lambda})$ such that $f_k(c_k) = 0$. Without loss of generality we assume that c_k converges to some $c \in [0, \lambda]$. Therefore, by (3.7) there exist sequences $(\lambda_k)_k \subset [0, \lambda]$ and $(u_k)_k$, $u_k \in P_{\omega,\sigma}(\lambda_k)$ such that $|u_k| + |c_k - \lambda_k| \to 0$, or, in other words, $\lambda_k \to c$ and $u_k \to 0$ as $k \to \infty$. Consequently, $0 \in P_{\omega,\sigma}(c)$ as required.

Let us consider now the set of all zeros

$$\Lambda(\omega, \sigma) := \{c \in [0, \bar{\lambda}] : 0 \in P_{\omega,\sigma}(c)\}$$

and show that the mapping $(\omega, \sigma) \mapsto \Lambda(\omega, \sigma)$ defined on $C([0, \beta-\alpha], \mathbb{R}) \times [0, 1]$ has closed graph. To this end, we choose sequences $(\omega_k(\cdot))_k \subset C([0, \beta-\alpha], \mathbb{R})$, $(\sigma_k)_k \subset [0, 1]$ and $(c_k)_k \subset [0, \bar{\lambda}]$ such that

$$\begin{aligned} \sigma_k m \;\; &= \;\; \int_0^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta - s)^{n-1}} \times \right. \\ &\times \;\; \left. \left[c_k - \sigma_k \int_s^{\beta-\alpha} (\beta - \tau)^{n-1} p_k(\tau) \, \mathrm{d}\tau \right] \right) \, \mathrm{d}s \end{aligned} \tag{3.8}$$

for some measurable selections $p_k(t) \in \partial g(\omega_k(t))$, $t \in [0, \beta - \alpha]$, converging, respectively, to a function $\omega(\cdot) \in C([0, \beta - \alpha], \mathbb{R})$, to $\sigma \in [0, 1]$ and to some $c \in [0, \bar{\lambda}]$. Due to the upper semicontinuity of the subdifferential, given $k = 1, 2, \ldots.$ we choose $i_k \geq 1$ such that

$$\partial g(\omega_{i_k}(t)) \subset \partial g(\omega(t)) + \frac{1}{k} \bar{B} \tag{3.9}$$

for all $t \in [0, \beta - \alpha]$. Notice that i_k does not depend on t by the uniform convergence of functions. It follows from (3.9) that there exist measurable selections $q_{i_k}(t) \in \partial g(\omega(t))$ such that $p_{i_k}(t) - q_{i_k}(t) \to 0$ as $k \to +\infty$ uniformly in $t \in [0, \beta - \alpha]$. Choosing if necessary a subsequence from $(q_{i_k}(\cdot))_k$ we may suppose that $(q_{i_k}(\cdot))_k$ and, consequently, $(p_{i_k}(\cdot))_k$ converges weakly to some measurable selection $p(t) \in \partial g(\omega(t))$. This implies the weak convergence of the integrals $\int_s^{\beta-\alpha}(\beta-\tau)^{n-1}p_{i_k}(\tau)\ \mathrm{d}\tau$ for each $s\in[0,\beta-\alpha]$. Passing now to the limit in the equality (3.8) we conclude that $c\in\Lambda(\omega,\sigma)$.

Let us denote by $H(\omega,\sigma)$ the set of continuous mappings

$$t\mapsto \sigma m \ - \ \int_t^{\beta\ \ \alpha}(f')^{-1}\left(\frac{1}{(\beta-s)^{n-1}}\ \times \right.$$
$$\left.\times\ \left[\lambda-\sigma\int_t^{\beta-\alpha}(\beta-\tau)^{n-1}p(\tau)\ \mathrm{d}\tau\right]\right)\ \mathrm{d}s \qquad (3.10)$$

for all measurable selections $p(t)\in\partial g(\omega(t))$ and all $\lambda\in\Lambda(\omega,\sigma)$. In the next section we prove existence of a function satisfying (3.3) through fixed points of the one-parametric family of the multivalued mappings $\omega\to H(\omega,\sigma)$.

3.2 Existence of a fixed point

We preface the proof of existence theorem with several lemmas.

Lemma 3.2.1. *The multivalued mapping*

$$H: C([0,\beta-\alpha],\mathbb{R})\times[0,1]\to C([0,\beta-\alpha],\mathbb{R})$$

defined above (see (3.10)) has closed graph.

Proof. Let $(\omega_k)_k\subset C([0,\beta-\alpha],\mathbb{R})$, $(\sigma_k)_k\subset[0,1]$ and $(v_k)_k\subset C([0,\beta-\alpha],\mathbb{R})$, $v_k(\cdot)\in H(\omega_k,\sigma_k)$, $k=1,2,...$, be sequences converging, respectively, to $\omega(\cdot)$, to $\sigma\in[0,1]$ and to $v(\cdot)$. Representing each function $v_k(\cdot)$ as

$$v_k(t)=\sigma m \ - \ \int_t^{\beta-\alpha}(f')^{-1}\left(\frac{1}{(\beta-s)^{n-1}}\ \times\right.$$
$$\left.\times\ \left[\lambda_k-\sigma_k\int_t^{\beta-\alpha}(\beta-\tau)^{n-1}p_k(\tau)\ \mathrm{d}\tau\right]\right)\ \mathrm{d}s \qquad (3.11)$$

for some $\lambda_k\in\Lambda(\omega_k,\sigma_k)$ and some measurable selection $p_k(t)\in\partial g(\omega_k(t))$, $t\in[0,\beta-\alpha]$, we assume without loss of generality that $\lambda_k\to\lambda\in\Lambda(\omega,\sigma)$ (see Section 3.1) and that the sequence $(p_k)_k$ converges weakly to a measurable function $p(t) \in \partial g(\omega(t))$, $t \in [0, \beta - \alpha]$. Passing now to the (pointwise) limit in (3.11), we conclude that

$$v(t) = \sigma m - \int_t^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta - s)^{n-1}} \left[\lambda - \sigma \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau) \; \mathrm{d}\tau \right] \right) \; \mathrm{d}s,$$

$t \in [0, \beta - \alpha]$. Thus, $(\omega(\cdot), \sigma, v(\cdot))$ belongs to graph H, and the statement is proved. □

Lemma 3.2.2. *Given $C > 0$ there exists a compact set $K_C \subset C([0, \beta - \alpha], \mathbb{R})$ such that $H(\omega, \sigma) \subset K_C$ for all $\omega(\cdot) \in C([0, \beta - \alpha], \mathbb{R})$ with $\|w(t)\| \le C$, $t \in [0, \beta - \alpha]$, and for all $\sigma \in [0, 1]$.*

Proof. Since the subdifferential $\partial g(\cdot)$ is bounded on the ball $C\bar{B}$, the set of functions $t \mapsto \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau) \; \mathrm{d}\tau$ for all $p(t) \in \partial g(\omega(t))$, $\|\omega(t)\| \le C$, $t \in [0, \beta - \alpha]$, is relatively compact in $C([0, \beta - \alpha], \mathbb{R})$. Furthermore, since the function $(f')^{-1}$ is continuous, the expression under the first integral in (3.10) is contained in some ball $R_C B$. Therefore, $H(\omega, \sigma) \subset K_C$ for all $\omega(\cdot) \in C([0, \beta - \alpha], \mathbb{R})$ with $\|\omega(t)\| \le C$, where

$$K_C := \{x(\cdot) \in C^1([0, \beta - \alpha], \mathbb{R}) : \|\dot{x}(t)\| \le R_C \quad \forall t \in [0, \beta - \alpha]\},$$

is relatively compact in $C([0, \beta - \alpha], \mathbb{R})$ by Ascoli-Arzelà theorem. □

Lemma 3.2.3. *The set*

$$A := \{(\omega, \sigma) \in C([0, \beta - \alpha], \mathbb{R}) \times [0, 1] : \omega \in H(\omega, \sigma)\}.$$

is nonempty and bounded.

Proof. Nonemptiness of A is obvious ($(0, 0) \in A$). In order to show the boundedness let us represent each $\omega(\cdot)$ with $\omega \in H(\omega, \sigma)$ in the form

$$\begin{aligned} \omega(t) = \sigma m \quad - \quad & \int_t^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta - s)^{n-1}} \times \right. \\ \times \quad & \left. \left[\lambda - \sigma \int_s^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau) \; \mathrm{d}\tau \right] \right) \; \mathrm{d}s, \qquad (3.12) \end{aligned}$$

with some $\lambda \in \Lambda(\omega, \sigma) \subset [0, \bar{\lambda}]$, and some measurable selection $p(t) \in \partial g(\omega(t))$. Since the subdifferential $\partial g(z)$ is bounded on $[0, m]$ and single-valued for $z \ge m$, we immediately obtain boundedness of the right-hand side of (3.12). □

Now we are ready to prove a fixed point theorem. For this we adopt the Leray-Schauder theorem (see, e.g., [36]) to multivalued operators by using one of the versions of Kakutani theorem, proved by Bohnenblust and Karlin (see, e.g., [5]).

Theorem 3.2.1. *The multivalued mapping $\omega \mapsto H(\omega, 1)$ has a fixed point $\omega(\cdot)$ such that $\omega(0) = 0$.*

Proof. In accordance with Lemma 3.2.3, there exists $M > 0$ such that $\|\omega\| \leq M$ whenever $(\omega, \sigma) \in A$, $\sigma \in [0, 1]$. Consider the linear space

$$\mathcal{B} = \{\omega \in C([0, \beta - \alpha], \mathbb{R}) : \omega(0) = 0\}$$

and given $\varepsilon > 0$ define the multivalued operator $F_\varepsilon : \mathcal{B} \mapsto \mathcal{B}$,

$$F_\varepsilon(\omega) = \begin{cases} H\left(\frac{M\omega}{M-\varepsilon}, 1\right) \cap \mathcal{B} & \text{if } \|\omega\| \leq M - \varepsilon, \\ H\left(\frac{M\omega}{\|\omega\|}, \frac{M-\|\omega\|}{\varepsilon}\right) \cap \mathcal{B} & \text{if } M - \varepsilon < \|\omega\| < M, \\ \{0\} & \text{if } \|\omega\| \geq M. \end{cases}$$

Due to upper semicontinuity of $(\omega, \sigma) \mapsto H(\omega, \sigma)$, by the construction it follows that the set-valued mapping $F_\varepsilon : \mathcal{B} \mapsto \mathcal{B}$ is upper semicontinuous as well. Moreover, it admits nonempty values due to the definition of $\Lambda(\omega, \sigma)$.

The values of $F_\varepsilon(\cdot)$ are, obviously, closed and by Lemma 3.2.2 contained in the compact set K_M. Then, applying Bohnenblust-Karlin theorem (see [5]) we find a fixed point $\omega_\varepsilon(\cdot) \in \mathcal{B}$, $\omega_\varepsilon(\cdot) \in F_\varepsilon(\omega_\varepsilon)$.

Setting $\varepsilon = \frac{1}{k}$, $k = 1, 2, ...$, and denoting by

$$\sigma_k = \begin{cases} 1 & \text{if } \|\omega_{\frac{1}{k}}\| \leq M - \frac{1}{k}, \\ k(M - \|\omega_{\frac{1}{m}}\|) & \text{if } M - \varepsilon < \|\omega_{\frac{1}{k}}\| < M, \\ 0 & \text{if } \|\omega_{\frac{1}{k}}\| \geq M, \end{cases}$$

we observe first that

$$\omega_{\frac{1}{k}} \in H(\omega_{\frac{1}{k}}, \sigma_k) \tag{3.13}$$

for each $k = 1, 2, ...$, and, consequently, $(\omega_{\frac{1}{k}}, \sigma_k) \in A$. Since $\omega_{\frac{1}{k}}(\cdot) \in K_M$ and $\sigma_k \in [0, 1]$, without loss of generality we can assume that the sequence $(\omega_{\frac{1}{k}})_k$ converges uniformly to some $\omega(\cdot) \in \mathcal{B}$, $\omega(0) = 0$, and $\sigma_k \to \sigma \in [0, 1]$. Passing to the limit in (3.13) as $k \to \infty$, by closedness of graph H we obtain $\omega \in H(\omega, \sigma)$.

Let us show finally that $\sigma = 1$. Assuming $\sigma < 1$, we have $\sigma_k < 1$ for all $k \geq 1$ large enough and consequently $\|\omega_{\frac{1}{k}}\| > M - \frac{1}{k}$. Passing to the limit we obtain $\|\omega\| \geq M$, contradicting the choice of $M > 0$. Theorem is proved. $\square$

Thus, there exist $\lambda \in \Lambda(\omega, 1) \subset [0, \bar{\lambda}]$ and a measurable selection $p(\cdot)$ of the mapping $t \mapsto \partial g(\omega(t))$ on $[0, \beta - \alpha]$, such that

$$\begin{aligned} \omega(t) = m \quad - & \quad \int_t^{\beta-\alpha} (f')^{-1} \left(\frac{1}{(\beta - s)^{n-1}} \times \right. \\ \times & \quad \left. \left[\lambda - \int_s^{\beta-\alpha} (\beta - \tau)^{n\ \ 1} p(\tau)\ \mathrm{d}\tau \right] \right)\ \mathrm{d}s \end{aligned}$$

and $\omega(0) = 0$.

3.3 Properties of the comparison function

In the previous section we proved existence of a function $\omega(\cdot) \in C([0, \beta - \alpha], \mathbb{R})$ such that for some measurable selection $p(t) \in \partial g(\omega(t))$ the equality (3.3) holds. Fixed such a function we study now some of its properties.

Lemma 3.3.1. *The function $\omega(\cdot)$ is continuously differentiable on $[0, \beta - \alpha]$ and*

(**a**) $\omega(t) \geq 0 \qquad \forall t \in [0, \beta - \alpha]$;

(**b**) $\omega'(t) \geq 0 \qquad \forall t \in [0, \beta - \alpha]$;

(**c**) *there exists $\bar{t} \in [0, \beta - \alpha)$ such that $\omega(t) \equiv 0$ on $[0, \bar{t}]$ and $\omega(t) > 0$, $\omega'(t) > 0$ on $(\bar{t}, \beta - \alpha]$.*

Proof. The continuous differentiability of $\omega(\cdot)$ follows from (3.3) and continuity of $(f')^{-1}$ ($(f')^{-1}$ is continuous since f' is continuous and strictly increasing).

If the statement (a) is not true then, taking into account that $\omega(0) = 0$ and $\omega(\beta - \alpha) = m > 0$, one can choose two different points $t_0, t_1 \in [0, \beta - \alpha)$,

$t_0 < t_1$, such that $\omega(t_0) = \omega(t_1) = 0$ and $\omega(t) < 0$ for $t \in (t_0, t_1)$. It follows from (3.3) that

$$[(\beta - t)^{n-1} f'(\omega'(t))]' = (\beta - t)^{n-1} p(t) \tag{3.14}$$

for all $t \in [0, \beta - \alpha]$. Multiplying both sides of (3.14) by $\omega(t)$ and integrating by parts on $[t_0, t_1]$ yields

$$\int_{t_0}^{t_1} (\beta - t)^{n-1} f'(\omega'(t))\omega'(t)\ \mathrm{d}t = -\int_{t_0}^{t_1} (\beta - t)^{n-1} p(t)\omega(t)\ \mathrm{d}t.$$

Since $p(t) \in \partial g(\omega(t))$, $\omega(t) < 0$ for $t \in [t_0, t_1]$, and by the convention $g(s) = 0$ for $s \leq 0$, we have $p(t) = 0$ for all $t \in [t_0, t_1]$. Hence,

$$\int_{t_0}^{t_1} (\beta - t)^{n-1} f'(\omega'(t))\omega'(t)\ \mathrm{d}t = 0. \tag{3.15}$$

Recall that $xf'(x) > 0$ for all $x \neq 0$. Then the equality (3.15) means that $\omega'(t) \equiv 0$ on $[t_0, t_1]$, and, hence, $\omega(t) \equiv 0$ because $\omega(t_0) = \omega(t_1) = 0$, which is a contradiction.

In order to prove items (b) and (c), let us define the set

$$J = \{t \in (0, \beta - \alpha) : \omega'(t) > 0\},$$

which is nonempty (because $\omega(0) = 0$ and $\omega(\beta - \alpha) = m > 0$), and open by continuity of the derivative $\omega'(\cdot)$.

Set $t = \inf J \in [0, \beta - \alpha)$. Then $\omega(t) = \int_0^t \omega'(s)\ \mathrm{d}s \leq 0$ for each $t \in [0, t]$. Consequently, by (a) we conclude that $\omega \equiv 0$ on $[0, \bar{t}]$. Furthermore, for each $t \in (\bar{t}, \beta - \alpha]$ there exists $\hat{t}$, $\bar{t} < \hat{t} < t$ such that $\hat{t} \in J$. Then, by monotonicity of the function $s \mapsto (f')^{-1}(s)$ we have

$$\begin{aligned} \omega'(t) &= (f')^{-1}\left(\frac{1}{(\beta - t)^{n-1}}\left[\lambda - \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau)\ \mathrm{d}\tau\right]\right) \geq \\ &\geq (f')^{-1}\left(\frac{1}{(\beta - \hat{t})^{n-1}}\left[\lambda - \int_{\hat{t}}^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau)\ \mathrm{d}\tau\right]\right) = \omega'(\hat{t}) > 0. \end{aligned}$$

Thus, $\omega'(t) > 0$ and

$$\omega(t) = \int_{\bar{t}}^{t} \omega'(s)\ \mathrm{d}s > 0$$

for all $t \in (\bar{t}, \beta - \alpha]$. □

The important step in proving the validity of SMP consists in the establishment of a connection between properties of the function $\omega(\cdot)$ and divergence of the improper integral (3.2). Namely, through the following estimates we prove that this divergence implies that $\omega'(0) > 0$. In other words, in terms of the previous lemma $\hat{t} = 0$.

Lemma 3.3.2. *Assuming that $\omega'(0) = 0$, the inequality*

$$f'(\omega'(t)) \leq \frac{p(t)}{(\beta - t)^{n-1}} \left(\frac{\beta^n}{n} - \frac{(\beta - t)^n}{n} \right) \tag{3.16}$$

holds for $t \in [0, \beta - \alpha]$.

Proof. By integrating the equality (3.14) (which is equivalent to the representation of $\omega(\cdot)$, see (3.3)) on a segment $[\tau, t]$, $0 \leq \tau \leq t \leq \beta - \alpha$, we have

$$(\beta - t)^{n-1} f'(\omega'(t)) - (\beta - \tau)^{n-1} f'(\omega'(\tau)) = \int_\tau^t (\beta - s)^{n-1} p(s) \, \mathrm{d}s.$$

Letting $\tau \to 0^+$, due to monotonicity of the function $\omega(\cdot)$ and of the subdifferential $\partial g(\cdot)$, taking into account that $f'(\cdot)$ and $\omega'(\cdot)$ are continuous and $\omega'(0) = 0$, we obtain that

$$(\beta - t)^{n-1} f'(\omega'(t)) \leq p(t) \int_0^t (\beta - s)^{n-1} \, \mathrm{d}s,$$

and the estimate (3.16) follows. □

In the lemma above we proved an estimate on f' along with $\omega'(\cdot)$. Now we treat the same property for the composed function $f^* \circ f'$.

Lemma 3.3.3. *Assume that $\omega(\cdot)$ is such as in Lemma 3.3.2 . Then,*

$$f^*(f'(\omega'(t))) \leq L(t) \int_0^t p(s)\omega'(s) \, ds$$

for all $t \in [0, \beta - \alpha]$ where

$$L(t) := 1 + \frac{n-1}{n} \left(\left(\frac{\beta}{\beta - t} \right)^n - 1 \right). \tag{3.17}$$

Proof. By (3.1) we find

$$f^*(f'(\omega'(t))) = \omega'(t)f'(\omega'(t)) - f(\omega'(t)) =$$
$$= \omega'(t)f'(\omega'(t)) - \int_0^{\omega'(t)} f'(s)\ \mathrm{d}s, \tag{3.18}$$

$t \in [0, \beta - \alpha]$. Integrating (3.18) by parts, we obtain that

$$f^*(f'(\omega'(t))) = \int_0^{\omega'(t)} s\ \mathrm{d}f'(s). \tag{3.19}$$

Since $f'(\cdot)$ is strictly increasing, we can make the change of variables $y = f'(s)$ in the Stieltjes integral (3.19) and arrive at the formula

$$f^*(f'(\omega'(t))) = \int_0^{f'(\omega'(t))} (f')^{-1}(y)\ \mathrm{d}y. \tag{3.20}$$

It follows from (3.20) that the function $f^* \circ (f' \circ \omega')$ is almost everywhere differentiable on $[0, \beta - \alpha]$, and

$$[f^*(f'(\omega'(t)))]' = \omega'(t)[f'(\omega'(t))]' \tag{3.21}$$

for a.e. $t \in [0, \beta - \alpha]$. Derivating in the left-hand side we can rewrite the equality (3.14) in the form

$$(f'(\omega'(t))' = \frac{n-1}{\beta - t} f'(\omega'(t)) + p(t).$$

Then, substituting to (3.21) and integrating on the segment $[0, t]$ we have

$$f^*(f'(\omega'(t))) = \int_0^t \frac{n-1}{\beta - s} f'(\omega'(s))\omega'(s)\ \mathrm{d}s + \int_0^t p(s)\omega'(s)\ \mathrm{d}s.$$

Hence, applying the estimate (3.16), we obtain

$$\begin{aligned} f^*(f'(\omega'(t))) &\leq \int_0^t \left(\frac{n-1}{\beta - s} \cdot \frac{p(s)}{(\beta - s)^{n-1}} \times \right. \\ &\times \left. \left(\frac{\beta^n}{n} - \frac{(\beta - s)^n}{n} \right) \omega'(s) + p(s)\omega'(s) \right)\ \mathrm{d}s = \\ &= \int_0^t \left(1 + \frac{n-1}{(\beta - s)^n} \left(\frac{\beta^n}{n} - \frac{(\beta - s)^n}{n} \right) \right) p(s)\omega'(s)\ \mathrm{d}s \leq \\ &\leq L(t) \int_0^t p(s)\omega'(s)\ \mathrm{d}s, \end{aligned}$$

and Lemma is proved. $\square$

We are ready now to establish the main property of the function $\omega(\cdot)$, which will be used in the sequel.

Theorem 3.3.1. *Assume that $g \in \mathfrak{G}$ is strictly increasing. If $\omega'(0) = 0$ then for each $\delta > 0$ the integral*

$$\int_0^\delta \frac{ds}{(f^* \circ f')^{-1}(g(s))}$$

converges.

Proof. By Lemma 3.3.1 there exists $\bar{t} \in [0, \beta - \alpha)$ such that $\omega(t) = 0$ for $t \in [0, \bar{t}]$ while $\omega(t) > 0$ in $(\bar{t}, \beta - \alpha)$. By the monotonicity of f' and f^* the limit (finite or infinite)

$$D := \lim_{t \to +\infty} f^*(f'(t))$$

exists. Observe that the derivative $f'(\cdot)$ can be upper bounded by a constant $\mu \in \operatorname{dom} f^*$, and the latter domain not necessarily coincides with $\mathbb{R}$. Due to this fact the limit D can be finite as well. Let us fix now $\delta > 0$ and t_1 and t_2, $0 < \bar{t} < t_1 < t_2 < \beta - \alpha$ such that

$$\omega(t_1) L(t_2) < \delta$$

and

$$\int_0^{\omega(t_1) L(t_2)} p(\omega^{-1}(y)) \, \mathrm{d}y < D,$$

where $p(\cdot)$ is the measurable selection of $\partial g(\omega(\cdot))$ associated to $\omega(\cdot)$, and $L(\cdot)$ is defined by (3.17).

By Lemma 3.3.3,

$$f^*(f'(\omega'(t))) \leq L(t) \int_0^t p(s)\omega'(s) \, \mathrm{d}s \leq$$

$$\leq L(t_2) \int_{\bar{t}}^t p(s)\omega'(s) \, \mathrm{d}s \tag{3.22}$$

for all $t \in [\bar{t}, t_2]$ (notice that the function $L(\cdot)$ is nondecreasing). Taking into account the strict monotonicity of the function $\omega(\cdot)$ on $[\bar{t}, t_2]$ we can apply to the integral in (3.22) the change of variables $y = L(t_2)\omega(s)$ and obtain

$$\int_{\bar{t}}^t p(s)\omega'(s) \, \mathrm{d}s = \frac{1}{L(t_2)} \int_0^{L(t_2)\omega(t)} p\left(\omega^{-1}\left(\frac{y}{L(t_2)}\right)\right) \, \mathrm{d}y. \tag{3.23}$$

Since $p(\cdot)$ and $\omega^{-1}(\cdot)$ are nondecreasing and $L(t_2) \geq 1$, we have from (3.23)

$$\int_{\bar{t}}^{t} p(s)\omega'(s)\ \mathrm{d}s \leq \frac{1}{L(t_2)} \int_0^{L(t_2)\omega(t)} p(\omega^{-1}(y))\ \mathrm{d}y,$$

which together with (3.22) gives

$$f^*(f'(\omega'(t))) \leq \int_0^{L(t_2)\omega(t)} p(\omega^{-1}(y))\ \mathrm{d}y. \tag{3.24}$$

Notice that $y \mapsto p(\omega^{-1}(y))$ is a measurable selection of $y \mapsto \partial g(y)$. Being the function $g(\cdot)$ absolutely continuous it admits a finite derivative $g'(\cdot)$ at almost each point. Therefore, $p(\omega^{-1}(y)) = g'(y)$ for almost every $y > 0$. Hence, from (3.24) by monotonicity we find

$$\omega'(t) \leq (f^* \circ f')^{-1}(g(L(t_2)\omega(t))). \tag{3.25}$$

Finally, returning to the old variables through the substitution $y = L(t_2)\omega(t)$ and using the inequality (3.25) we conclude

$$\begin{aligned} &\int_0^{\omega(t)L(t_2} \frac{dy}{(f^* \circ f')^{-1}(g(y))} = \\ = \ & L(t_2) \int_{\bar{t}}^{t_1} \frac{\omega'(s)ds}{(f^* \circ f')^{-1}(g(L(t_2)\omega(s)))} \leq \\ \leq \ & L(t_2)(t_1 - \bar{t}) < +\infty, \end{aligned}$$

and Theorem is proved. □

3.4 Proof of the Strong Maximum Principle

Our first task in this section is to associate to the function $\omega(\cdot)$ defined in the previous sections a continuous minimizer in the problem (P_g) for some special domain. Let $f \in \mathfrak{F}$ and $g \in \mathfrak{G}$. Given $x^0 \in \mathbb{R}^n$ and $\beta > \alpha > 0$, let us define the open annulus

$$A_{\alpha,\beta}(x^0) := \{x \in \mathbb{R}^n : \alpha < \|x - x^0\| < \beta\}. \tag{3.26}$$

Theorem 3.4.1. *If $\Omega = A_{\alpha,\beta}(x^0)$ and $\omega(\cdot) \in C([0, \beta-\alpha], \mathbb{R})$ satisfies the equality (3.3) for some (fixed) measurable $p(t) \in [0, \beta-\alpha]$ then the function $V : \overline{\Omega} \to \mathbb{R}$, $V(x) = \omega(\beta - \|x - x^0\|)$ is a solution to (P_g) with $u^0(x) = V(x)$.*

Proof. Denoting by $q(x) := p(\beta - \|x - x^0\|)$ and taking arbitrary $u(\cdot) \in V(\cdot) + W_0^{1,1}(\Omega)$, we sucessively obtain

$$\begin{aligned} & \int_\Omega (f(\|\nabla u(x)\|) + g(u(x)) - [f(\|\nabla V(x)\|) + g(V(x))]) \, \mathrm{d}x \geq \\ \geq \ & \int_\Omega [\langle \nabla(f \circ \|\cdot\|)(\nabla V(x)), \nabla u(x) - \nabla V(x)\rangle + \\ + \ & q(x)(u(x) - V(x))] \, \mathrm{d}x = \int_\Omega [f'(\|\nabla V(x)\|)\langle -(x-x^0)/\|x-x^0\|, \\ & \nabla u(x) - \nabla V(x)\rangle + q(x)(u(x) - V(x))] \, \mathrm{d}x. \end{aligned} \tag{3.27}$$

Here we used the definition of the subdifferential $\partial g(\cdot)$ and convexity of the function f.

Let us recall the definition of $\omega(\cdot)$ (see (3.3)), which can be written in the form

$$f'(\omega'(t)) = \frac{1}{(\beta - t)^{n-1}} \left[\lambda - \int_t^{\beta-\alpha} (\beta - \tau)^{n-1} p(\tau) \, \mathrm{d}\tau\right], \tag{3.28}$$

or

$$f'(-v'(r)) = \frac{1}{r^{n-1}} \left[\lambda + \int_r^\alpha s^{n-1} p(\beta - s) \, \mathrm{d}s\right], \tag{3.29}$$

where $v(r) := \omega(\beta - r)$, $\alpha \leq r \leq \beta$. Pass now to the polar coordinates $r = \|x - x^0\|$ and $\theta = \frac{x - x^0}{\|x - x^0\|}$ in (3.27), taking into account that $\nabla V(x) = v'(r)\theta$ and $\|\nabla V(x)\| = -v'(r)$. So, we can represent the last integral in (3.27) as

$$\begin{aligned} & \int_{\|\theta\|=1} \mathrm{d}\theta \left(\int_\alpha^\beta \left[-f'(-v'(r)) \frac{d}{dr}(u(\theta, r) - v(r))\right] r^{n-1} \, \mathrm{d}r + \right. \\ + \ & \left. \int_\alpha^\beta [p(\beta - r)(u(\theta, r) - v(r))] \, r^{n-1} \, \mathrm{d}r \right). \end{aligned} \tag{3.30}$$

Further, substituting the value of $f'(-v'(r))$ from (3.29) and integrating by parts the first integral in (3.30) with respect to the variable $r \in [\alpha, \beta]$, we reduce (3.30) to

$$\int_{\|\theta\|=1}\Big[-\int_{\alpha}^{\beta}((u(r,\theta)-v(r))r^{n-1}p(\beta-r)\ \mathrm{d}r+$$
$$+\quad\int_{\alpha}^{\beta}((u(r,\theta)-v(r))r^{n-1}p(\beta-r)\ \mathrm{d}r\Big]\ \mathrm{d}\theta=0. \qquad (3.31)$$

We used here the fact that the functions $u(\cdot)$ and $v(\cdot)$ admit the same values on the boundary of Ω. Combining (3.31) and (3.27) we conclude

$$\int_{\Omega}[f(\|\nabla V(x)\|)+g(V(x))]\ \mathrm{d}x\le\int_{\Omega}[f(\|\nabla u(x)\|)+g(u(x))]\ \mathrm{d}x,$$

and Theorem is proved. □

We know from the results of the previous section that $\omega(t)>0$ on $(0,\beta-\alpha]$ and $\omega'(t)>0$ on $[0,\beta-\alpha]$ under the assumption (3.2) . In order to prove the Srong Maximum Principle we need the following comparison result.

Theorem 3.4.2. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded connected region, $x^0\in\Omega$ and α,β, $0<\alpha<\beta$, be such that $A_{\alpha,\beta}(x^0)\subset\Omega$, and $\omega(\cdot)$ be a continuous function defined by (3.3). Set $v(r):=\omega(\beta-r)$, $r\in[\alpha,\beta]$, and suppose that $u(\cdot)$ is a continuous solution of (P_g) with*

$$u(x)\ge v(\|x-x^0\|)\qquad\forall x\in\partial A_{\alpha,\beta}(x^0).$$

Then the same inequality holds for all $x\in A_{\alpha,\beta}(x^0)$.

Proof. Let us denote $V(x)=v(\|x-x^0\|)$ and assume, on contrary, that the (open) set

$$E^-=\{x\in A_{\alpha,\beta}(x^0):u(x)<V(x)\}$$

is non empty. Define the function $\eta^-(x)$ equal to

$$\min\{u-V,0\}=\begin{cases}u(x)-V(x) & \text{if } x\in E^-,\\ 0 & \text{if } x\in A_{\alpha,\beta}(x^0)\setminus E^-,\end{cases}$$

and to zero on $\Omega\setminus A_{\alpha,\beta}(x^0)$. Clearly, $\eta^-(\cdot)\in W_0^{1,1}(\Omega)$ is continuous,

$$(u-\eta^-)(x)=\begin{cases}V(x) & \text{if } x\in E^-,\\ u(x) & \text{if } x\in\Omega\setminus E^-,\end{cases}\qquad (3.32)$$

and (see [41])

$$\nabla(u-\eta^-)(x) = \begin{cases} \nabla V(x) & \text{for a.e. } x \in E^-, \\ \nabla u(x) & \text{for a.e. } x \in \Omega \setminus E^-. \end{cases} \tag{3.33}$$

Since $u(\cdot)$ is a solution of (P_g), it follows from (3.32) and (3.33) that

$$\begin{aligned} 0 \ &\geq \ \int_\Omega (f(\|\nabla u(x)\|) + g(u(x)) - \\ &- \ [f(\|\nabla(u-\eta^-)(x)\|) + g((u-\eta^-)(x))]) \ \mathrm{d}x = \\ &= \ \int_{E^-} [f(\|\nabla u(x)\|) - f(\|\nabla V(x)\|) + g(u(x)) - g(V(x))] \ \mathrm{d}x \geq \\ &\geq \ \int_{E^-} [\langle \nabla(f \circ \|\cdot\|)(\nabla V(x)), \nabla u(x) - \nabla V(x)\rangle + q(x)(u(x) - V(x))] \ \mathrm{d}x = \\ &= \ \int_{E^-} [f'(\|\nabla V(x)\|)\langle -(x-x^0)/\|x-x^0\|, \nabla u(x) - \nabla V(x)\rangle + \\ &+ \ q(x)(u(x) - V(x))] \ \mathrm{d}x, \end{aligned} \tag{3.34}$$

where $q(x) := p(\beta - \|x - x^0\|)$, and $p(t)$ as usual is the measurable selection of $t \mapsto \partial g(\omega(t))$, associated to $\omega(\cdot)$. Introducing the polar coordinates $r = \|x - x^0\|$ and $\theta = \frac{x-x^0}{\|x-x^0\|}$ similarly as in the proof of Theorem 3.4.1 we reduce the last integral in (3.34) to

$$\begin{aligned} &\int_{\|\theta\|=1} \left(\int_{l_\theta} \left[f'(-v'(r)) \frac{d}{dr}(u(\theta, r) - v(r)) \right] r^{n-1} \ \mathrm{d}r + \right. \\ &+ \ \left. \int_{l_\theta} p(\beta - r)(u(\theta, r) - v(r)) r^{n-1} \ \mathrm{d}r \right) \ \mathrm{d}\theta, \end{aligned} \tag{3.35}$$

where $l_\theta := \{r \in (\alpha, \beta) : (r, \theta) \in E^-\}$.

Recalling (3.29), integrating (3.35) by parts and taking into account that the functions $u(\theta, r)$ and $v(r)$ coincide on the boundary of the linear set l_θ, $\theta \in [\alpha, \beta]$, similarly as in the proof of Theorem 3.4.1, we we deduce from (3.34) that

$$\int_{E^-} [f(\|\nabla u(x)\|) - f(\|\nabla V(x)\|) + g(u(x)) - g(V(x))] \ \mathrm{d}x = 0. \tag{3.36}$$

It follows also (see (3.34)) that

$$\begin{aligned} &\int_E \ [\langle \nabla(f \circ \|\cdot\|)(\nabla V(x)), \nabla u(x) - \nabla V(x)\rangle + \\ &+ \ q(x)(u(x) - V(x))] \ \mathrm{d}x = 0. \end{aligned} \tag{3.37}$$

On the other hand, the integral

$$\int_{E^-} [f(\|\nabla u(x)\|) - f(\|\nabla V(x)\|) + q(x)(u(x) - V(x))] \, \mathrm{d}x$$

is contained between the integrals (3.36) and (3.37), and so it is equal to zero as well. Hence,

$$\begin{aligned} &\int_{E^-} [f(\|\nabla u(x)\|) - f(\|\nabla V(x)\|) - \\ - \ & \langle \nabla(f \circ \|\cdot\|)(\nabla V(x)), \nabla u(x) - \nabla V(x)\rangle \, \mathrm{d}x = 0. \end{aligned} \tag{3.38}$$

By convexity, the integrand in (3.38) is nonnegative for a.e. $x \in E^-$ and we have the pointwise equality

$$\begin{aligned} & f(\|\nabla u(x)\|) = f(\|\nabla V(x)\|) + \\ + \ & \langle \nabla(f \circ \|\cdot\|)(\nabla V(x)), \nabla u(x) - \nabla V(x)\rangle \end{aligned}$$

for a.e. $x \in E^-$. In other words, $(\nabla u(x), f(\|\nabla u(x)\|))$ belongs to the same face of the epigraph of the convex function $z \mapsto f(\|z\|)$ as $(\nabla V(x), f(\|\nabla V(x)\|))$. Since f is strictly convex, we conclude that $\nabla u(x) = \nabla V(x)$ a.e. in E^-, i.e. $\eta^- = 0$ on Ω, which is a contradiction. □

Finally, the following theorem gives sufficient conditions for the validity of the Strong Maximum Principle.

Theorem 3.4.3. *In addition to our standing assumptions let us suppose that either $g(s) = 0$, $s \in [0, \delta]$, for some $\delta > 0$, or $g(s) > 0$ near zero and for some $\delta > 0$ the Riemann improper integral (3.2) diverges. Then the Strong Maximum Principle holds for the problem (P_g).*

Proof. In the first case ($g(s) = 0 \, \forall s \in (0, \delta]$) the Strong Maximum Principle is proved in [16].

Let us consider the second hypothesis, i.e.,

$$\int_0^\delta \frac{ds}{(f^* \circ f')^{-1}(g(s))} = +\infty.$$

Let $\Omega \subset \mathbb{R}^n$ be any open bounded connected domain and $u(\cdot)$ be a continuous nonnegative solution of (P_g). Suppose that both sets

$$E = \{x \in \Omega : u(x) = 0\} \text{ and } \Omega \setminus E = \{x \in \Omega : u(x) > 0\}$$

are nonempty. Due to connectedness of Ω there exists $x^* \in E \cap \overline{\Omega \setminus E}$. Choose $\beta > 0$ such that the ball $\bar{B}_{2\beta}(x^*) \subset \Omega$, and let $x^0 \in \Omega \setminus E$ be such that $\|x^* - x^0\| < \beta$. From continuity of $u(\cdot)$ one can find $m \in (0, \delta)$ and $\alpha \in (0, \|x^* - x^0\|)$ such that $u(x) \geq m$ on $\bar{B}_\alpha(x^0)$. We associate to the numbers α, β and m a function $\omega(\cdot) \in C([0, \beta - \alpha], \mathbb{R})$ studied in the previous sections, which is defined by the formula (3.3) with some measurable selection $p(t) \in \partial g(\omega(t))$, and consider the annulus

$$A_{\alpha,\beta}(x^0) := \{x \in \mathbb{R}^n : \alpha < \|x - x^0\| < \beta\}.$$

Let us set $v(r) = \omega(\beta - r)$, $r \in [\alpha, \beta]$, and apply the comparison theorem. Namely, by the choice of the constants m and α we obviously have $u(x) \geq v(\|x - x_0\|)$ whenever $\|x - x_0\|$ is equal to α or to β. By Theorem 2.4.2 the same inequality holds also inside the annulus $A_{\alpha,\beta}(x^0)$. In particular, $u(x^*) \geq v(\|x^* - x_0\|)$. But by Lemma 2.3.1 and Theorem 2.3.1 the derivative $v'(r)$ is negative on $[\alpha, \beta]$, i.e., the function $v(\cdot)$ is strictly decreasing. Then $u(x^*) \geq v(\|x^* - x_0\|) > v(\beta) = 0$, which is a contradiction.

□

Example 5. *Consider the problem of minimizing the integral functional*

$$\int_\Omega [\|\nabla u(x)\|^p + (u(x))^p] \; dx$$

with $p > 1$. *Here* $f(\xi) = \xi^p$ *for* $\xi > 0$, *its derivative* $f'(\xi) = p\xi^{p-1}$, *and*

$$f^*(v) = v\left(\frac{v}{p}\right)^{\frac{1}{p-1}} - \left(\frac{v}{p}\right)^{\frac{p}{p-1}}.$$

Thus,

$$(f^* \circ f')(t) = (p-1)t^p;$$

$$(f^* \circ f')^{-1}(t) = \left(\frac{t}{p-1}\right)^{\frac{1}{p}},$$

and

$$\int_0^\delta \frac{ds}{(f^* \circ f')^{-1}(g(s))} = \int_0^\delta \frac{ds}{\left(\frac{s^p}{p-1}\right)^{\frac{1}{p}}} = +\infty.$$

So, the Strong Maximum Principle holds by the above theorem.

Conclusion

Let us emphasize the main results obtained in Thesis. We considered the minimization problems for the convex integral functionals, depending by some symmetric way on the gradient and containing, possibly, and additive term depending on the state variable.

1. In the case when this additive term is equal to zero and the lagrangean depends on the gradient through a gauge function we
   (a) proved some local estimates for minimizers close to their nonextremum points;
   (b) proved validity of the Strong Maximum Principle in the traditional sense;
   (c) extended the Strong Maximum Principle to the case when the lagrangean is affine near the origin;
   (d) proved a "multi-point" version of the Strong Maximum Principle;
   (e) proved an uniqueness extremal extension principle.
2. In the case when the additive term is linear with respect to u we
   (a) proved some local estimates for minimizers close to their nonextremum points;
   (b) obtained estimates of minimizers in neighbourhoods of points of local maximum (minimum) which can be interpreted as an (approximate) version of the Strong Maximum Principle.

3. Finally when the lagrangean is rotationally invariant w.r.t. the gradient and is nonlinear w.r.t. u we:

    (a) proved the classic Strong Maximum Principle under the hypothesis that some Riemann improper integral, involving all the data of the problem, diverges.

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