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ON HIGHER ORDER NONLINEAR IMPULSIVE BOUNDARY VALUE PROBLEMS*

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Abstract. This work studies some two point impulsive boundary value problems composed by a fully differential equation, which higher order contains an increasing homeomorphism, by two point boundary conditions and impulsive effects. We point out that the impulsive conditions are given via multivariate generalized functions, including impulses on the referred homeomorphism. The method used apply lower and upper solutions technique together with fixed point theory. Therefore we have not only the existence of solutions but also the localization and qualitative data on their behavior. Moreover a Nagumo condition will play a key role in the arguments.

1. Introduction.

In this article we study the following two point boundary value problem composed by the one-dimensional ϕ-Laplacian equation

$$(\phi(u''(t)))' + q(t)f(t,u(t),u'(t),u''(t)) = 0, t \in J', \tag{1}$$

where

(A_1) ϕ is an increasing homeomorphism such that $\phi(0) = 0$ and $\phi(\mathbb{R}) = \mathbb{R}$,

(A_2) $q \in C([0,1])$ with $q > 0$ and $\int_0^1 q(s)ds < \infty$, $f \in C([0,1] \times \mathbb{R}^3, \mathbb{R})$,

together the boundary conditions

$$u(0) = A,\ u'(0) = B,\ u''(1) = C, \quad A, B, C \in \mathbb{R}, \tag{2}$$

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and the impulsive conditions

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(u(t_k), u'(t_k)),\ k = 1, 2, ...n,\ \Delta u^{(i)}(t_k) = u^{(i)}(t_k^+) - u^{(i)}(t_k^-), \\ \Delta u'(t_k) &= I_{2k}(u(t_k), u'(t_k)), \\ \Delta \phi(u''(t_k)) &= I_{3k}(u(t_k), u'(t_k), u''(t_k)), \end{aligned} \tag{3}$$

being $I_{ik} \in C(\mathbb{R}^2, \mathbb{R})$, $i = 1, 2$, and $I_{3k} \in C(\mathbb{R}^3, \mathbb{R})$, with t_k fixed points such that $0 < t_1 < t_2 < ... < t_n < 1$.

The theory of impulsive problems have become more important due to the applications of real processes, in which sudden and discontinuous jump occurs. Such examples can be found in population dynamics, control and optimization theory, ecology, biology and biotechnology, economics, pharmacokinetics and other physics and mechanics problems.

For the classical approach to impulsive differential equations we can refer, as example, [1, 2, 7, 10, 12, 13, 14] and the references therein. Most of the arguments apply critical point theory and variational techniques ([15, 16]), fixed point results in cones ([6, 17]), bifurcation theory ([9, 11]), and lower and upper solutions method ([3, 8]).

In this work we consider a nonlinear third order fully differential equation together with generalized impulsive conditions. As far as we know, it is the first time where it is allowed to the impulsive effects the dependence on the unknown variable and its derivative, and even on its second derivative for the impulses on the homeomorphism ϕ, which includes the Laplacian or p-Laplacian cases. The paper is organized as it follows: Section 2 contains an uniqueness result for an associated problem to (1)-(3) and the definition of lower and upper solutions, with strict inequalities in some boundary and impulsive conditions. In Section 3 the main existence and localization result is obtained via an truncation and perturbation methods (suggested in [4, 5]) lower and upper solution technique and fixed point theory. Last section provide an example where the impulses depend on the function and on its variation.

2. **Definitions and auxiliary results.** Let

$$PC[0,1] = \left\{ \begin{array}{c} u : u \in C([0,1], \mathbb{R}) \text{ continuous for } t \neq t_k, u(t_k) = u(t_k^-), u(t_k^+) \\ \text{exists for } k = 1, 2, ..., n \end{array} \right\}$$

and $PC^2[0,1] = \{u : u''(t) \in PC[0,1]\}$. Then $PC^2[0,1]$ is a Banach Space with norm

$$||u(t)|| = \max\{||u||_\infty, ||u'||_\infty, ||u''||_\infty\},$$

where

$$||w||_\infty = \sup_{0 \le t \le 1} |w(t)|.$$

Defining $J := [0,1]$ and $J' = J \backslash \{t_1, ..., t_n\}$, for a solution u of problem (1)-(3) one should consider $u(t) \in E$, where $E := PC[0,1] \cap C^2(J')$.

Next lemma provides an uniqueness result an adequate problem related to (1)-(3).

Lemma 2.1. *The problem composed by the differential equation*

$$(\phi(u''(t)))' + v(t) = 0 \tag{4}$$

and conditions (2), (3), has a unique solution given by

$$\begin{aligned} u(t) &= A + Bt + \sum_{t<t_k} I_{1k}(u(t_k), u'(t_k)) + \sum_{t<t_k} I_{2k}(u(t_k), u'(t_k))\ t \\ &\quad + \int_0^t \int_0^\mu \phi^{-1} \left[\phi(C) + \int_\zeta^1 v(s)ds - \sum_{\zeta<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\zeta d\mu. \end{aligned}$$

Proof. For $t \in (t_n, 1]$, integrating (4) from t to 1, we have

$$u''(t) = \phi^{-1} \left(\phi(C) + \int_t^1 v(s)ds \right). \tag{5}$$

For $t \in (t_{n-1}, t_n]$, with $t_0 := 0$, by integration of (4), it is obtained by (5),

$$\begin{aligned} u''(t) &= \phi^{-1} \left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^-\right)\right) \right] \\ &= \phi^{-1} \left[\int_t^{t_n} v(s)ds + \phi\left(u''\left(t_n^+\right)\right) - I_{3n}(u(t_n), u'(t_n), u''(t_n)) \right] \\ &= \phi^{-1} \left[\phi(C) + \int_{t_n}^1 v(s)ds + \int_t^{t_n} v(s)ds - I_{3n}(u(t_n), u'(t_n), u''(t_n)) \right] \\ &= \phi^{-1} \left[\phi(C) + \int_t^1 v(s)ds - I_{3n}(u(t_n), u'(t_n), u''(t_n)) \right]. \end{aligned}$$

Therefore by induction, for $t \in (0, 1)$, we get

$$u''(t) = \phi^{-1} \left[\phi(C) + \int_t^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right]. \tag{6}$$

By integration of (6) in $[0, t_1]$,

$$u''(t_1^-) = B + \int_0^{t_1} \phi^{-1} \left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\mu. \tag{7}$$

Integrating (6) for $t \in (t_1, t_2]$, and applying (3) and (7),

$$\begin{aligned} u'(t) &= u'(t_1^+) + \int_{t_1}^t \phi^{-1} \left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\mu \\ &= u'(t_1^-) + I_{21}(u(t_1), u'(t_1)) \\ &\quad + \int_{t_1}^t \phi^{-1} \left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\mu \\ &= I_{21}(u(t_1), u'(t_1)) + B \\ &\quad + \int_0^t \phi^{-1} \left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k)) \right] d\mu. \end{aligned}$$

So, for $t \in [0,1]]$,

$$u'(t) = \sum_{t<t_k} I_{2k}(u(t_k), u'(t_k)) + B + \int_0^t \phi^{-1}\left[\phi(C) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu \tag{8}$$

Integrating (8) for $t \in [0, t_1]$,

$$u(t_1^-) = A + \left(\sum_{t<t_k} I_{2k}(u(t_k), u'(t_k)) + B\right) t_1 + \int_0^{t_1}\int_0^r \phi^{-1}\left[\phi(c) + \int_\mu^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\mu dr \tag{9}$$

Integrating (8) for $t \in (t_1, t_2]$, by (9),

$$\begin{aligned} u(t) &= u(t_1^-) + \left(\sum_{t<t_k} I_{2k}(u(t_k), u'(t_k)) + B\right)(t - t_1) \\ &\quad + \int_{t_1}^t\int_0^\mu \phi^{-1}\left[\phi(C) + \int_\tau^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\tau d\mu \\ &= A + \left(\sum_{t<t_k} I_{2k}(u(t_k), u'(t_k)) + B\right) t \\ &\quad + \int_0^t\int_0^\mu \phi^{-1}\left[\phi(C) + \int_\tau^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\tau d\mu \end{aligned}$$

Finally, for $t \in [0,1]$,

$$u(t) = A + \sum_{t<t_k} I_{1k}(u(t_k), u'(t_k)) + \left(\sum_{t<t_k} I_{2k}(u(t_k), u'(t_k)) + B\right) t + \int_0^t\int_0^\mu \phi^{-1}\left[\phi(C) + \int_\tau^1 v(s)ds - \sum_{\mu<t_k} I_{3k}(u(t_k), u'(t_k), u''(t_k))\right] d\tau d\mu. \tag{10}$$

□

Lower and upper functions will be defined as it follows:

Definition 2.2. A function $\alpha(t) \in E$ with $\phi(\alpha''(t)) \in PC^1[0,1]$ is a lower solution of problem (1), (2), (3) if

$$\begin{cases} (\phi(\alpha''(t)))' + q(t)f(t, \alpha(t), \alpha'(t), \alpha''(t)) \geq 0 \\ \Delta\alpha(t_k) \leq I_{1k}(\alpha(t_k), \alpha'(t_k)) \\ \Delta\alpha'(t_k) > I_{2k}(\alpha(t_k), \alpha'(t_k)) \\ \Delta\phi(\alpha''(t_k)) > I_{3k}(\alpha(t_k), \alpha'(t_k), \alpha''(t_k)) \\ \alpha(0) \leq A, \\ \alpha'(0) \leq B, \\ \alpha''(1) < C. \end{cases} \tag{11}$$

A function $\beta(t) \in E$ such that $\phi(\beta''(t)) \in PC^2[0,1]$ and satisfies the opposite inequalities above, is an upper solution of (1)-(3).

The Nagumo condition is an important tool to control second derivatives:

Definition 2.3. A function f satisfies a Nagumo condition related to a pair of functions $\gamma, \Gamma \in PC[0,1] \cap C^2(J')$, with $\gamma' \leq \Gamma'$, if exists a function $\psi : C([0,+\infty),]0,+\infty))$ such that:

$$|f(t,x,y,z)| \leq \psi(|z|), \text{ for all } (t,x,y,z) \in F \tag{12}$$

with

$$F = \{(t,x,y,z) \in [0,1] \times \mathbb{R}^3 : \gamma(t) \leq x \leq \Gamma(t), \gamma'(t) \leq y \leq \Gamma'(t)\}$$

and such that

$$\int_{\phi(\mu)}^{+\infty} \frac{ds}{\psi(\phi^{-1}(s))} > \int_0^1 q(s)ds,$$

where

$$\mu := \max_{k=0,1,2,\dots,n} \left\{ \left| \frac{\Gamma'(t_{k+1}) - \gamma'(t_k)}{t_{k+1} - t_k} \right|, \left| \frac{\gamma'(t_{k+1}) - \Gamma'(t_k)}{t_{k+1} - t_k} \right| \right\}.$$

3. **Main result.** The main result is an existence and localization theorem, as it provides not only the existence of solutions but also some qualitative properties on it.

Theorem 3.1. *Suppose that assumptions* (A_1), (A_2) *hold and there are* α *and* β *lower and upper solutions, respectively, of problem (1)-(3) such that* $\alpha \leq \beta$ *and* $\alpha' \leq \beta'$ *in* $[0,1]$.
Assume that the continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ *satisfies a Nagumo condition, and verifies*

$$f(t,\alpha(t),y,z) \leq f(t,u(t),y,z) \leq f(t,\beta(t),y,z), \tag{13}$$

for $\alpha \leq u \leq \beta$, *and fixed* $(y,z) \in \mathbb{R}^2$. *Moreover, if the impulsive functions satisfy*

$$I_{1k}(\alpha(t_k),\alpha'(t_k)) \leq I_{1k}(u(t_k),u'(t_k)) \leq I_{1k}(\beta(t_k),\beta'(t_k)), \tag{14}$$

and

$$I_{2k}(\alpha(t_k),y) \geq I_{2k}(u(t_k),y) \geq I_{2k}(\beta(t_k),y), \tag{15}$$

for $k = 1,...,n$, $\alpha(t_k) \leq u(t_k) \leq \beta(t_k)$, $\alpha'(t_k) \leq u'(t_k) \leq \beta'(t_k)$ *and fixed* $y \in \mathbb{R}$, *then problem (1)-(3) has at least one solution* $u \in E$, *such that*

$$\alpha(t) \leq u(t) \leq \beta(t), \ \alpha'(t) \leq u'(t) \leq \beta'(t) \ \text{ and } \ -N \leq u''(t) \leq N, \text{ for } t \in [0,1].$$

To prove this theorem we need some preliminary results:
Define the continuous functions $\delta_i(t,u^{(i)}(t))$, for $i = 0,1$, such that

$$\delta_i(t,u^{(i)}) = \begin{cases} \beta^{(i)}(t), & u^{(i)}(t) \geq \beta^{(i)}(t) \\ u^{(i)}(t), & \alpha^{(i)}(t) \leq u^{(i)}(t) \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & u^{(i)}(t) \leq \alpha^{(i)}(t) \end{cases}$$

and consider the following modified and perturbed equation

$$(\phi(u''(t)))' + q(t)f(t,\delta_0(t,u(t)),\delta_1(t,u'(t)),\frac{d}{dt}\delta_1(t,u'(t)) \tag{16}$$
$$+\frac{\delta_1(t,u'(t)) - u'(t)}{1 + |u'(t) - \delta_1(t,u'(t))|} = 0,$$

coupled with the truncated impulsive conditions

$$\begin{aligned} \Delta u(t_k) &= I_{1k}(\delta_0(t_k,u(t_k),\delta_1(t_k,u'(t_k)), \\ \Delta u'(t_k) &= I_{2k}(\delta_0(t_k,u(t_k)),\delta_1(t_k,u'(t_k)), \\ \Delta \phi(u''(t_k)) &= I_{3k}(\delta_0(t_k,u(t_k)),\delta_1(t_k,u'(t_k)),\frac{d}{dt}\delta_1(t_k,u'(t_k))). \end{aligned} \tag{17}$$

and boundary conditions (2).

Next lemma will prove the equivalence between problem (1)-(3) and problem (16), (17), (2):

Lemma 3.2. *Assume that $\alpha(t)$ and $\beta(t)$ are lower and upper solutions of problem (1)-(3), respectively, with $\alpha'(t) \le \beta'(t)$, the continuous function f satisfies (13) and the impulsive functions I_{ik} satisfy (14) and (15), then every $u(t)$ solution of problem (16), (17), (2) verifies*

$$\alpha(t) \le u(t) \le \beta(t), \text{ and } \alpha'(t) \le u'(t) \le \beta'(t), \text{ for } t \in [0,1].$$

To prove the second inequalities suppose, by contradiction, that there is $t \in [0,1]$ such that $u'(t) > \beta'(t)$. Therefore

$$\sup_{t\in[0,1]} (u'(t) - \beta'(t)) := u'(\bar{t}_0) - \beta'(\bar{t}_0) > 0. \tag{18}$$

As by boundary conditions, $u'(0) - \beta'(0) \le 0$, then $\bar{t}_0 \ne 0$. In the same way $u''(1^-) - \beta''(1^-) < 0$ and then $\bar{t}_0 \ne 1$.

Let $t_0 = 0$ and $t_{n+1} = 1$. As the $\max_{t\in[0,1]} (u' - \beta')(t)$ can not be achieved for $t = 1$ because of boundary conditions, only two cases must be considered:

Case 1: Assume that there is $p \in \{1,2,...,n\}$ such that $\bar{t}_0 \in (t_p, t_{p+1})$.

Define

$$\bar{t}_1 = \max_{t\in(t_p,\bar{t}_0)} \{t : (u' - \beta')(t) \le 0\}$$

and

$$\bar{t}_2 = \min_{t\in(\bar{t}_0,t_p)} \{t : (u' - \beta')(t) \le 0\}$$

If $(u'-\beta')(t) > 0$, $\forall t \in (t_p, \bar{t}_0)$ then consider $\bar{t}_1 = t_p$. Analogously for $(u'-\beta')(t) > 0$, $\forall t \in (\bar{t}_0, t_{p+1})$ then define $\bar{t}_2 = t_{p+1}$.

Therefore, by (13), for all $t \in (\bar{t}_1, \bar{t}_2)$,

$$\begin{aligned} (\phi(u''(t)))' - (\phi(\beta''(t))' &\ge -q(t)\ f\left(t,\delta_0(t,u),\delta_1(t,u'),\frac{d}{dt}\delta_1(t,u')\right) \\ &\quad -\frac{\delta_1(t,u') - u'(t)}{1 + |u'(t) - \delta_1(t,u')|} + q(t)f\left(t,\beta(t),\beta'(t),\beta''(t)\right) \\ &= -q(t)f(t,\delta_0(t,u),\beta'(t),\beta''(t)) \\ &\quad -\frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} + q(t)f((t,\beta(t),\beta'(t),\beta''(t)) \\ &\ge -q(t)f(t,\beta(t),\beta'(t),\beta''(t)) - \frac{\beta'(t) - u'(t)}{1 + |u'(t) - \beta'(t)|} \\ &\quad +q(t)f((t,\beta(t),\beta'(t),\beta''(t)) \\ &= \frac{u'(t) - \beta'(t)}{1 + |u'(t) - \beta'(t)|} > 0. \end{aligned}$$

So $\phi(u''(t)) - \phi(\beta''(t)$ is increasing for all $t \in (\bar{t}_1, \bar{t}_2)$.

For $t \in]\bar{t}_0, \bar{t}_2]$,

$$0 = \phi(u''(\bar{t}_0)) - \phi(\beta''(\bar{t}_0)) < \phi(u''(t)) - \phi(\beta''(t))$$

and $u''(t) > \beta''(t)$. Therefore $(u' - \beta')(t)$ is increasing in $]\bar{t}_0, t_2]$, which contradicts (18).

Case 2: Suppose that there is $p \in \{0, 1, 2, ..., n-1\}$ such that

$$\sup_{t\in[0,1]} (u'(t) - \beta'(t)) := u'(t_p) - \beta'(t_p) > 0 \tag{19}$$

or

$$\max_{t\in[0,1]} (u'(t) - \beta'(t)) := u'(t_{p+1}) - \beta'(t_{p+1}) > 0. \tag{20}$$

If (19) happens then

$$u''(t_p^+) - \beta''(t_p^+) \le 0$$

and, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} u''(t) - \beta''(t) &\le 0, \\ u'(t) - \beta'(t) &> 0, \ \forall t \in (t_p, t_p + \varepsilon). \end{aligned} \tag{21}$$

So, for $t \in (t_p, t_p + \varepsilon) \subset [t_p, t_{p+1}]$,

$$\begin{aligned} (\phi(u''(t)))' - (\phi(\beta''(t)))' &\ge -q(t) f\left(t, \delta_0(t,u), \delta_1(t,u'), \frac{d}{dt}\delta_1(t,u')\right) \\ &\quad - \frac{\beta'(t) - u'(t)}{1 + |u'(t)} + q(t) f(t, \beta(t), \beta'(t), \beta''(t)) \\ &\ge \frac{u'(t) - \beta'(t)}{1 + |u'(t)} > 0. \end{aligned}$$

There is $\varepsilon > 0$ such that by integration on $t \in (t_p, t_p + \varepsilon)$ we get that $u''(t) > \beta''(t)$, which contradicts (21).

Assuming (20), we have

$$\max_{t\in[0,1]} (u'(t) - \beta'(t)) = u'(t_{p+1}) - \beta'(t_{p+1}) = u'(t_{p+1}^-) - \beta''(t_{p+1}^-) > 0$$

and, by (17) and (15), we achieve to the contradiction.

$$\begin{aligned} 0 &\ge u'(t_{p+1}^+) - \beta'(t_{p+1}^+) - \left[u'(t_{p+1}^-) - \beta'(t_{p+1}^-)\right] \\ &> I_{2,p+1}(\delta_0(t_{p+1}, u), \delta_1(t_{p+1}, u')) - I_{2,p+1}(\beta(t_{p+1}), \beta'(t_{p+1})) \\ &= I_{2,p+1}(\delta_0(t_{p+1}, u), \beta'(t_{p+1})) - I_{2,p+1}(\beta(t_{p+1}), \beta'(t_{p+1})) \ge 0. \end{aligned}$$

Therefore $u'(t) \le \beta'(t)$, for $t \in [0,1]$. By similar arguments it can be proved the remaining inequality and therefore

$$\alpha'(t) \le u'(t) \le \beta'(t), \ \text{for } t \in [0,1]. \tag{22}$$

By integration of (22) for $t \in [0, t_1]$,

$$\alpha(t) \le u(t) - u(0) + \alpha(0) \le u(t). \tag{23}$$

Integrating (22) for $t \in]t_1, t_2]$, we have, by (14) and (23),

$$\begin{aligned} \alpha(t) &\le u(t) - u(t_1^+) + \alpha(t_1^+) \\ &\le u(t) - I_{11}\left(\delta_0(t_1, u), \delta_1(t_1, u')\right) - u(t_1^-) + I_{11}\left(\alpha(t_1), \alpha'(t_1)\right) + \alpha(t_1^-) \\ &\le u(t). \end{aligned}$$

By recurrence, it can be proved analogously, that

$$\alpha(t) \le u(t),\ \forall t \in [t_k, t_{k+1}],\ \text{ for } k = 1, 2, ..., n.$$

So $\alpha(t) \le u(t)$, $\forall t \in [0,1]$. Applying similar arguments it can be proved the remaining inequality and, therefore,

$$\alpha(t) \le u(t) \le \beta(t),\ \text{ for } t \in [0,1].$$

Lemma 3.3. *Let α and β be lower and upper solutions of problem (1)-(3) such that $\alpha \le \beta$ and $\alpha' \le \beta'$ in $[0,1]$. If the continuous function $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies a Nagumo condition in the set F, referred to α and β, then there is $N \ge \mu > 0$ such that every solution u of the differential equation (1) verifies $||u''||_\infty \le N$.*

Proof. Let $u(t)$ be a solution of (1) such that

$$\alpha(t) \le u(t) \le \beta(t) \text{ and } \alpha'(t) \le u'(t) \le \beta'(t),\ \text{ for } t \in [0,1].$$

By the Mean Value Theorem, there exists $\eta_0 \in (t_k, t_{k+1})$ with

$$u''(\eta_0) = \frac{u'(t_{k+1}) - u'(t_k)}{t_{k+1} - t_k},\ \text{ with } k = 0, 1, 2, ..., n.$$

Moreover,

$$-N \le -\mu \le \frac{\alpha'(t_{k+1}) - \beta'(t_k)}{t_{k+1} - t_k} \le u''(\eta_0) \le \frac{\beta'(t_{k+1}) - \alpha'(t_k)}{t_{k+1} - t_k} \le \mu \le N.$$

If $|u''(t)| \le N$ in $[0,1]$, the proof is complete.

Assume that there is $\tau \in [0,1]$ such that $|u''(\tau)| > N$.

Consider the case where $u''(\tau) > N$. Therefore there is η_1 such that $u''(\eta_1) = N$.

If $\eta_0 < \eta_1$, suppose, without loss of generality, that

$$u''(t) > 0 \text{ and } u''(\eta_0) \le u''(t) \le N,\ \text{ for } t \in [\eta_0, \eta_1].$$

So

$$|\phi(u''(t))| = |q(t) f(t, u(t), u'(t), u''(t))| \le q(t)|\psi(u''(t))|,\ \text{ for } t \in [\eta_0, \eta_1],$$

and, by (12),

$$\begin{aligned}\int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} &\le \int_{\eta_0}^{\eta_1} \frac{|(\phi(u''(t))'|}{\psi(u''(t))} dt = \int_{\eta_0}^{\eta_1} \frac{|q(t) f(t, u(t), u'(t), u''(t))|}{\psi(u''(t))} dt \\ &\le \int_{\eta_0}^{\eta_1} q(t) dt < \int_0^1 q(t) dt.\end{aligned}$$

As $u''(\eta_0) \le \mu < N$, by the monotony of ϕ,

$$\phi(u''(\eta_0)) \le \phi(\mu)$$

and

$$\int_{\phi(u''(\eta_0))}^{\phi(N)} \frac{ds}{\psi(\phi_p^{-1}(s)} \ge \int_{\phi(\mu)}^{\phi(N)} \frac{ds}{\psi(\phi^{-1}(s)} > \int_0^1 q(t) dt$$

which leads to a contradiction.

The other cases, that is, $u''(\tau) > N$ with $\eta_1 < \eta_0$, and $u''(\tau) < -N$ with $\eta_0 < \eta_1$ or $\eta_1 < \eta_0$, follow the same arguments to obtain a contradiction.

Therefore $|u''(t)| \le N,\ \text{ for } t \in [0,1]$. ∎

Proof of Theorem 3.1:

Consider the modified and perturbed problem (16), (17), (2).

Obtain a solution for problem (16), (17), (2) is equivalent to find a function $u \in E$ such that

$$
\begin{aligned}
u(t) \quad &= \quad A + Bt + \sum_{t<t_k} I_{1k}^*(u(t_k), u'(t_k)) + \sum_{t<t_k} I_{2k}^*(u(t_k), u'(t_k))\, t \\
&\quad + \int_0^t \int_0^\mu \phi^{-1}\left[\phi(c) + \int_\zeta^1 F_u(s)ds - \sum_{\zeta<t_k} I_{3k}^*(u(t_k), u'(t_k), u''(t_k))\right] d\zeta d\mu.
\end{aligned}
$$

where

$$
\begin{aligned}
F_u(s) \quad &: \quad = q(s) f(s, \delta_0(s, u(s)), \delta_1(s, u'(s)), \frac{d}{ds}\delta_1(s, u'(s)) \\
&\qquad + \frac{\delta_1(s, u'(s)) - u'(s)}{1 + |u'(s) - \delta_1(s, u'(s))|}, \\
I_{ik}^*(u(t_k), u'(t_k)) \quad &: \quad = I_{ik}(\delta_0(t_k, u(t_k), \delta_1(t_k, u'(t_k)),\ i = 1, 2, \\
I_{3k}^*(u(t_k), u'(t_k), u''(t_k)) \quad &: \quad = I_{3k}(\delta_0(t_k, u(t_k)), \delta_1(t_k, u'(t_k)), \frac{d}{dt}\delta_1(t_k, u'(t_k))).
\end{aligned}
$$

Define the operator $T : E \to E$ by

$$
\begin{aligned}
T(u)(t) \quad &: \quad = A + Bt + \sum_{t<t_k} I_{1k}^*(u(t_k), u'(t_k)) + \sum_{t<t_k} I_{2k}^*(u(t_k), u'(t_k))\, t \\
&\quad + \int_0^t \int_0^\mu \phi^{-1}\left[\phi(c) + \int_\zeta^1 F_u(s)ds - \sum_{\zeta<t_k} I_{3k}^*(u(t_k), u'(t_k), u''(t_k))\right] d\zeta d\mu.
\end{aligned}
$$

As T is completely continuous, by Schauder's fixed point theorem, T has a fixed point $u \in E$ which is a solution of (16), (17), (2).

By Lemma 2.1, this function $u \in E$ is also a solution of the problem (1)-(3). □

4. **Example.** Consider the problem composed by the differential equation

$$
\frac{u'''(t)}{1 + (u''(t))^2} + \arctan(u) - 6\,(u'(t))^3 - 2\sqrt[3]{u''(t) + 1} = 0, \quad \text{in } [0,1] \setminus \left\{\frac{1}{2}\right\}, \tag{24}
$$

the impulses given, for $t_1 = \frac{1}{2}$, by

$$
\begin{cases}
\Delta u(\frac{1}{2}) = u(\frac{1}{2}) + u'(\frac{1}{2}), \\
\Delta u'(\frac{1}{2}) = -u(\frac{1}{2}) + u'(\frac{1}{2}) \\
\Delta\phi(u''(\frac{1}{2})) = u(\frac{1}{2}),
\end{cases} \tag{25}
$$

and the boundary conditions (2).

Problem (24), (25), (2) is a particular case of problem (1)-(3) with

$$
\begin{aligned}
\phi(w) \quad &= \quad \arctan(w),\ q(t) \equiv 1, \\
f(t, x, y, z) \quad &= \quad \arctan(x) - 6y^3 - 2\sqrt[3]{z + 1} \\
I_{11}(x, y) \quad &= \quad x + y,\ I_{21}(x, y) = -x + y,\ I_{31}(x, y, z) = x.
\end{aligned}
$$

For $A \in [-1, 0]$, $B \in [-2, 1]$ and $C \in]0, 6[$, the functions

$$
\alpha(t) = \begin{cases} -2t - 1 & , \quad 0 \le t \le \frac{1}{2} \\ -t - 6 & , \quad \frac{1}{2} < t \le 1 \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} t^3 + t & , \quad 0 \le t \le \frac{1}{2} \\ t^3 + 3t + 2 & , \quad \frac{1}{2} < t \le 1, \end{cases}
$$

are, respectively, lower and upper solutions of problem (24), (25), (2), considering

$$\alpha'(t)=\begin{cases}-2 & , \quad 0\le t\le \frac{1}{2}\\ \\ -1 & , \quad \frac{1}{2}<t\le 1,\end{cases} \quad \text{and} \quad \beta'(t)=\begin{cases}3t^2+1 & , \quad 0\le t\le \frac{1}{2}\\ \\ 3t^2+3 & , \quad \frac{1}{2}<t\le 1,\end{cases}$$

$\alpha''(t)\equiv 0$ and $\beta''(t)=6t$, in $[0,1]$.

As the assumptions of Theorem 3.1 are fulfilled, therefore there is a solution of problem (24), (25), (2), for $A\in[-1,0]$, $B\in[-2,1]$ and $C\in]0,6[$, such that

$$\alpha(t)\le u(t)\le\beta(t), \alpha'(t)\le u'(t)\le\beta'(t), \text{ in } [0,1].$$

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