The Value of a Random Life: Modelling Survival Probabilities in a Stochastic Environment

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1. Introduction

During the 20th century, human expected lifetimes have increased considerably in many countries. Mortality improvements are naturally viewed as a positive change for individuals and as a substantial social achievement of developed countries. However, this structural change poses a serious challenge for the planning of public retirement systems, the long term risk management of supplemental pension plans as well as for the pricing and reserving for life insurance companies. Historically, actuaries have been calculating premiums and mathematical reserves using a deterministic approach, by considering a deterministic mortality intensity, which is a function of the age only, extracted from available (static) lifetables and by setting a flat ("best estimate") interest rate to discount cash flows over time. Since neither the mortality intensity nor interest rates are actually deterministic, life insurance companies are exposed to both financial and mortality (systematic and unsystematic) risks when pricing and reserving for any kind of long-term living benefits, particularly on annuities. In particular, the calculation of expected present values requires an appropriate mortality projection in order to avoid significant underestimation of future costs.

In order to protect the company from mortality improvements, actuaries have different solutions, among them to resort to projected (dynamic or prospective) lifetables instead of static lifetables (see, e.g., Pitacco (2004), Wong-Fupuy and Haberman (2004) and Bravo (2007) for a detailed review on this subject). Since the future mortality is actually unknown, in this paper we assume that an appropriate description of the demographic (and financial) risks requires a stochastic model, as is it has recently been proposed by several authors, .e.g., Milevsky and Promislow (2001), Dahl (2004), Biffis (2005), Cairns *et al.* (2006), Schrager (2006) and references therein.

By modelling the mortality intensity as a stochastic process, we expect to obtain a more accurate description of the liabilities of life insurance companies. In addition, a stochastic mortality model contributes to a proper quantification of systematic mortality risk (also called longevity risk) faced by insurance companies. Stochastic mortality models also provide an adequate framework for the development longevity risk hedging tools, namely mortality-linked contracts such as longevity bonds or mortality derivatives. In this paper, we use doubly stochastic processes (also known as *Cox processes*) in order to model the random evolution of the stochastic force of mortality of an individual aged x in a manner that is common in the credit risk literature. The model is then embedded into the well know affine term structure framework, widely used in the term structure literature, in order to derive closed-form solution for survival probability. The paper is organized as follows: Section 2 briefly describes the mathematical tools used in this paper. Section 3 develops and new model for the mortality intensity by considering a standard interest rate model, modified in order to include both positive and negative mortality jumps. We derive analytical solutions for the survival probability and calibrate the model to a matrix of data. Finally, Section 4 concludes.

2. Mathematical framework

We are given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and concentrate on an individual aged x at time 0. Following the pioneering work of Artzner and Delbaen (1995) in the credit risk literature and the proposals by Dahl (2004) and Biffis (2005) among others in mortality area, we model his/her random lifetime as an $\mathbb{F}$-stopping time τ_x admitting a random intensity μ_x. Specifically, we consider τ_x as the first jump-time of a nonexplosive $\mathbb{F}$-counting process N recording at each time $t \geq 0$ whether the individual has died $(N_t \neq 0)$ or survived $(N_t = 0)$. The stopping time τ_x is said to admit an intensity μ_x if the compensator of N does, i.e., if μ_x is a nonnegative predictable process such that $\int_0^t \mu_x(s)ds < \infty$ for all $t \geq 0$ and such that the compensated process $M_t = \left\{N_t - \int_0^t \mu_x(s)ds : t \geq 0\right\}$ is a local $\mathbb{F}$-martingale. If the stronger condition $E\left(\int_0^t \mu_x(s)ds\right) < \infty$ is satisfied, then M_t is an $\mathbb{F}$-martingale.

From this, we derive

$$(1) \quad E\left(N_{t+\Delta t} - N_t \middle| \mathcal{F}_t\right) = E\left(\int_t^{t+\Delta t} \mu_x(s)ds \middle| \mathcal{F}_t\right)$$

based on which we can write

$$(2) \quad E\left(N_{t+\Delta t} - N_t \middle| \mathcal{F}_t\right) = \mu_x(t)\Delta t + o(\Delta t)$$

an expression comparable with that of the instantaneous probability of death ${}_{\Delta t}q_{x+t}$ derived in the traditional deterministic context.

By further assuming that N is a Cox (or doubly stochastic) process driven by a subfiltration $\mathbb{G}$ of $\mathbb{F}$, with $\mathbb{F}$-predictable intensity μ it can be shown, by using the law of iterated expectations, that the probability of survival up to time $T \geq t$, on the set $\{\tau > t\}$, is given by

$$(3) \quad \mathbb{P}\left(\tau > T \middle| \mathcal{F}_t\right) = E\left[e^{-\int_t^T \mu_x(s)ds} \middle| \mathcal{F}_t\right]$$

Readers who are familiar with mathematical finance and, in particular, with the interest rate literature, can without difficulty observe that the right-hand-side of equation (3) represents the price at time t of a unitary default-free zero coupon bond with maturity at time $T > t$, if the intensity μ is to represent the short-term interest rate.

One of the main advantages of this mathematical framework is that we can approach the survival probability (3) by using well known affine-jump diffusion processes. In particular, an $\mathbb{R}^n$-valued affine-jump diffusion process X is an $\mathbb{F}$-Markov process whose dynamics is given by

$$(4) \quad dX_t = \delta(t, X_t)dt + \sigma(t, X_t)dW_t + \sum_{h=1}^{m} dJ_t^h$$

where W is a $\mathbb{F}$-standard Brownian motion in $\mathbb{R}^n$ and each component J^h is a pure-jump process in $\mathbb{R}^n$ with jump-arrival intensity $\left\{\eta^h(t, X_t) : t \geq 0\right\}$ and time-dependent jump distribution ν_t^h on $\mathbb{R}^n$. An important requirement of affine processes is that the drift $\delta : D \to \mathbb{R}^n$, the instantaneous covariance matrix $\sigma\sigma^{\mathrm{T}} : D \to \mathbb{R}^{n \times n}$ and the jump-arrival intensity $\eta^h : D \to \mathbb{R}_+$ must all have an affine dependency on X. The convenience of adopting affine processes in modelling the mortality intensity comes from the fact that, for any $a \in \mathbb{C}^n$, for given $T \geq t$ and an affine function R defined by $R(t, X) = \rho_0(t) + \rho_1(t) \cdot X$, under certain technical conditions (see (Duffie et al. (2003)), we have:

$$(5) \quad \phi^{\mathcal{X}}(a, X_t, t, T) \triangleq E\left[e^{-\int_t^T R(s, X_s)ds} e^{a \cdot X_T} \middle| \mathcal{F}_t\right] = e^{\alpha(t) + \beta(t) \cdot X_t}$$

where $\alpha(\cdot) \doteq \alpha(\cdot; a, T)$, $\beta(\cdot) \doteq \beta(\cdot; a, T)$ satisfy generalized Ricatti ordinary differential equations, that can be solved at least numerically and, in some cases, as we will see below, analytically.

3. Mortality intensity as a stochastic process

Turning now to the problem of modelling adequately the dynamic of mortality, we develop a new model for the mortality intensity by considering the classic Ornstein-Uhlenbeck equation (without the mean-reverting component) and by adding a jump component. Formally, the process $\mu_x(t)$ is given by:

$$(6) \quad d\mu_x(t) = -a\mu_x(t)dt + \sigma dW(t) + dJ(t), \quad \mu_x(0) = \bar{\mu}_x$$

where $\bar{\mu}_x > 0$, $a > 0$, $\sigma \geq 0$, $W(t)$ is a standard Brownian motion and where $J(t)$ is a compound Poisson process, independent of W, with constant jump-arrival intensity $\eta \geq 0$ and jump sizes that are random variables double asymmetric exponentially distributed with density (Kou (2002)):

$$(7) \quad f(z) = \pi_1 \left(\frac{1}{\upsilon_1}\right) e^{-\frac{z}{\upsilon_1}} \mathbb{I}_{\{z \geq 0\}} + \pi_2 \left(\frac{1}{\upsilon_2}\right) e^{\frac{z}{\upsilon_2}} \mathbb{I}_{\{z < 0\}}$$

where $\pi_1, \pi_2 \geq 0$, $\pi_1 + \pi_2 = 1$, represent, respectively, the probabilities of a positive (with average size $\upsilon_1 > 0$) and negative (with average size $\upsilon_2 > 0$) jump. By setting $\pi_1 = 0$ we are interested only on the importance of longevity risk (e.g. Biffis (2005)).

In the spirit of (5) let us now assume that the survival probability ${}_{T-t}p_{x+t}(t)$ is represented by an exponentially affine function. By applying the framework described above, we have that:

$$(8) \quad {}_{T-t}p_{x+t}(t) \equiv e^{\mathcal{A}(\tau) + \mathcal{B}(\tau)\cdot \mu_x(t)}$$

where $\tau = T - t$. It can be shown that the solution to this problem admits the following Feynman-Kac representation:

$$\upsilon(t, \mu_x(t)) \left\{ -\dot{\mathcal{A}}(\tau) - \dot{\mathcal{B}}(\tau)\mu_x(t) - a\mu_x(t)\mathcal{B}(\tau) + \frac{\sigma^2}{2}\mathcal{B}^2(\tau) + \right.$$

$$(9) \quad \left. +\eta \left(\frac{\pi_1}{1 - \upsilon_1 \mathcal{B}(\tau)} + \frac{\pi_2}{1 + \upsilon_2 \mathcal{B}(\tau)} - 1\right) - \mu_x(t) \right\} = 0$$

where $\upsilon(t, \mu_x(t)) ={}_{T-t} p_{x+t}(t)$ and where $A(\tau)$ and $B(\tau)$ are solutions to the following system of ODEs':

$$(10) \quad \dot{\mathcal{B}}(\tau) = -a\mathcal{B}(\tau) - 1$$

$$(11) \quad \dot{\mathcal{A}}(\tau) = a\theta\mathcal{B}(\tau) + \eta \left(\frac{\pi_1}{1 - \upsilon_1 \mathcal{B}(\tau)} + \frac{\pi_2}{1 + \upsilon_2 \mathcal{B}(\tau)} - 1\right)$$

with boundary conditions:

$$(12) \quad \mathcal{B}(0) = 0, \ \mathcal{A}(0) = 0.$$

where $\dot{\mathcal{B}}(\tau) = \frac{\partial}{\partial \tau}\mathcal{B}(\tau)$, $\dot{\mathcal{A}}(\tau) = \frac{\partial}{\partial \tau}\mathcal{A}(\tau)$.

By solving the system (10)-(11)-(12), we get the following closed-form solutions for $\mathcal{A}(\tau)$ e $\mathcal{B}(\tau)$:

$$(13) \quad \mathcal{B}(\tau) = \frac{e^{-a(T-t)} - 1}{a}$$

$$(14) \quad \mathcal{A}(\tau) = \left[-\theta + \eta \left(\frac{\pi_1 a}{a + \upsilon_1} + \frac{\pi_2 a}{a - \upsilon_2} - 1\right)\right](T - t) - \theta\mathcal{B}(\tau)$$

$$+\eta \left\{ \frac{\pi_1 \ln(1 - \upsilon_1 \mathcal{B}(\tau))}{a + \upsilon_1} + \frac{\pi_2 \ln(1 + \upsilon_2 \mathcal{B}(\tau))}{a - \upsilon_2} \right\}$$

defined for

$$1 - \upsilon_1 \mathcal{B}(\tau) > 0 \quad \text{and} \quad 1 + \upsilon_2 \mathcal{B}(\tau) > 0$$

We observe that the model stipulates a decreasing (deterministic) trend for the mortality intensity, around which random fluctuations occur due to the stochastic component and due to the jump component. The model assumes that both negative and positive jumps can be registered in mortality, a feature that contrasts with similar models that are interested in sudden improvements in mortality (e.g. due to medical advances) only. We think this gives us a more realistic description of reality, in which unexpected increases in mortality can occur, caused e.g., by natural catastrophes or epidemics. The model offers a nice analytical solution, easy to use in pricing and reserving applications within the life insurance industry. The calibration of this model to different Portuguese mortality tables (not reported here) provided us very good fits of the survival probabilities ${}_{T-t}p_{x+t}$.

4. Concluding remarks

In this paper we have described the evolution of mortality by using Cox processes. The mortality intensity has been described as an affine-jump diffusion process with jump sizes that are random variables double asymmetric exponentially distributed. We derive closed-form solutions for the survival probability. We investigate the applicability of these processes in describing the individual mortality, and provide a first calibration to the Portuguese population.

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